OLIMPIADA MASALALARINI YECHISHDA MATEMATIK ANALIZ METODLARIDAN FOYDALANISH

Asosiy savollar:

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Tayanch iboralar: sonli ketma-ketlik tushunchasi, monoton o‘suvchi (kamayuvchi) ketma-ketliklar, ketma-ketlikning limiti tushunchasi, yaqinlashuvchi ketma-ketliklar, ajoyib limitlar, Teplits va Shtols teoremalari, funksiya hosilasi va differensiali tushunchalari, hosilaning geometrik va mexanik ma’nolari, differensial hisobning asosiy teoremalari, Ferma, Roll, Lagranj va Koshi teoremalari, xususiy yechim, umumiy yechim.

Annotatsiya: Ushbu mavzuda matematikadan olimpiada masalalarini yechishda matematik analiz ba’zi metodlaridan foydalanishning ba’zi usullarini ko‘rsatib beriladi. Bunda asosiy diqqat sonli ketma-ketliklarning Teplits va Shtols teoremalari va differensial hisobning asosiy teoremalariga qaratiladi.

1.1 §. Sonli ketma-ketliklarning Teplits va Shtols teoremalari yordamida yechiladigan masalalar

Akademik litsey va kasb-hunar maktablariga sonli ketma-ketlik tushunchasi, monoton o‘suvchi (kamayuvchi) ketma-ketliklar, ketma-ketlikning limiti tushunchasi, yaqinlashuvchi ketma-ketliklar haqidagi teoremalar o‘rganilganligi uchun biz bu paragrafda qisqa ma’lumotlar keltirdik. Bu paragrafdagi masalalar talabani ketma-ketlikni limitini topish bo‘yicha malakasini oshirishga qaratilgan.

Bizga $\{x_n\}$-ketma-ketlik berilgan bo‘lsin.

1-ta’rif. Ketma-ketlik yuqoridan (quyidan) chegaralangan deyiladi, agar shunday $M \in R$ soni topilib, barcha $n \in N$ larda $x_n \le M$ ($x_n \ge M$) tengsizlik bajarilsa.

Quyidan va yuqoridan chegaralangan ketma-ketlik chegaralangan ketma-ketlik deyiladi.

1-misol. Ushbu

$$x_n = \frac{2020^{\,n}}{n!}$$

ketma-ketlikni chegaralanganligini isbotlang.

Isbot. Ma’lumki, $\frac{x_{n+1}}{x_n} = \frac{2020^{\,n+1}}{(n+1)!} \cdot \frac{n!}{2020^{\,n}} = \frac{2020}{n+1}$ tenglik o‘rinli. Agar $n \ge 2019$ $(1 \le n \le 2018)$ bo‘lsa, u holda $x_{n+1} \le x_n$ ($x_{n+1} \ge x_n$) bo‘lib, $\{x_n\}$ ketma-ketlik kamayuvchi (o‘suvchi) bo‘ladi. Bu esa istalgan $n \in N$ larda $0 \le x_n \le x_{2019} = \frac{2020^{\,2019}}{2019\,!}$ tengsizlik o‘rinli bo‘lishini bildiradi, ya’ni $\{x_n\}$ ketma-ketlik chegaralangan.

Endi ketma - ketlikning limiti tushinchasini keltiramiz.

2-ta’rif. $a \in R$ soni $\{x_n\}$ ketma-ketlikning limiti deyiladi, agar har qanday $\varepsilon > 0$ son olinganda ham shunday $n_0(\varepsilon) \in N$ son topilib, barcha $n > n_0(\varepsilon)$ larda

$$|x_n - a| < \varepsilon$$

tengsizlik bajarilsa.

$\{x_n\}$ ketma-ketlikning limiti odatda $\lim\limits_{n\to\infty} x_n = a$ kabi belgilanadi.

Agar ketma-ketlik chekli limitga ega bo‘lsa, u yaqinlashuvchi ketma-ketlik deyiladi.

3-ta’rif. Agar ixtiyoriy a son va ixtiyoriy natural n_0 son olganda ham shunday musbat ε_0 soni va shunday natural $n > n_0$ son topilsaki,

$$|x_n - a| \ge \varepsilon$$

bo‘lsa, $\{x_n\}$ ketma-ketlik limitga ega emas deb ataladi.

Agar ketma-ketlik limitga ega bo‘lmasa, u uzoqlashuvchi ketma-ketlik deyiladi.

2-misol. Ushbu $x_n = \frac{a}{n}$ $(n \in N, a \in R)$ ketma-ketlikning limiti 0 ga teng bo‘lishini ko‘rsating.

Ixtiyoriy $\varepsilon > 0$ soniga ko‘ra, $n_0 = \left[\frac{|a|}{\varepsilon}\right] + 1$ ni tanlaymiz.

U holda barcha $n > n_0$ larda

$$|x_n - a| = \left|\frac{a}{n} - 0\right| = \frac{|a|}{n} < \frac{|a|}{n_0} = \frac{|a|}{\left[\frac{|a|}{\varepsilon}\right] + 1} < \varepsilon$$

tengsizlik o‘rinli bo‘ladi. Demak, $\lim\limits_{n\to\infty} x_n = 0$ bo‘ladi.

3-misol. Ushbu $x_n = (-1)^n$ ketma-ketlikning limiti mavjud emasligini isbotlang.

Isbot. Faraz qilamiz, $\{x_n\}$ ketma-ketlik chekli a limitga ega bo‘lsin. U holda ixtiyoriy ε $(0 < \varepsilon < 1)$ soni uchun shunday $n_0(\varepsilon) \in N$ son mavjudki, barcha $n > n_0(\varepsilon)$ larda $|x_n - a| < \varepsilon$ tengsizlik bajariladi. $x_n = (-1)^n$ bo‘lgani uchun $n > n_0(\varepsilon)$ da $|1 - a| < \varepsilon$, $|1 + a| < \varepsilon$ tengsizlik o‘rinli va bundan

$$2 = |1 - a + 1 + a| \le |1 - a| + |1 + a| < 2\varepsilon$$

tengsizlikni hosil qilamiz. Bu esa mumkin emas. Demak, $\{(-1)^n\}$ ketma-ketlik limitga ega emas.

Endi yaqinlashuvchi ketma-ketliklarning ba’zi xossalarini keltiramiz.

1^0. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty} x_n = a$ bo‘lib, $a > p$ ($a < q$) bo‘lsa, u holda ketma-ketlikning biror hadidan keyingi barcha hadlari ham p sondan katta (q sondan kichik) bo‘ladi.

2^0. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, u chegaralangan bo‘ladi.

3^0. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, uning limiti yagona bo‘ladi.

4^0. Agar $\{x_n\}$ ketma-ketlik o‘suvchi (kamayuvchi) bo‘lib, yuqoridan (quyidan) chegaralangan bo‘lsa, u yaqinlashuvchi bo‘ladi.

5^0. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lib, $\lim\limits_{n\to\infty} x_n = a$ bo‘lsa, u holda istalgan $k \in N$ uchun $\lim\limits_{n\to\infty} x_{n+k} = a$ bo‘ladi.

1-teorema. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lsa, $\{x_n \pm y_n\}$, $\{x_n \cdot y_n\}$ ketma-ketliklar ham yaqinlashuvchi va

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$$

formulalar o‘rinli bo‘ladi.

2-teorema. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lib, $\forall n \in N$ uchun $y_n \neq 0$ va $\lim\limits_{n\to\infty} y_n \neq 0$ bo‘lsa, $\left\{\dfrac{x_n}{y_n}\right\}$ ketma-ketlik ham yaqinlashuvchi hamda

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$$

formulalar o‘rinli bo‘ladi.

Keltirilgan teoremalarni isbotini keltirmadik. Uni mustaqil ravishda [8-15], [23] adabiyotlardan o‘rganishingizni tavsiya qilamiz.

Endi ketma-ketliklar nazariyasining eng ajoyib misollarini isbotlari bilan keltiramiz.

4-misol. $x_n = \dfrac{a^n}{n!}$ $(a > 0)$ ketma-ketlikni limiti 0 ga teng bo‘lishini isbotlang.

Isbot. $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1}$. Agar $n \geq [a]+1$ bo‘lsa, u holda $x_{n+1} \leq x_n$ bo‘lib, $\{x_n\}$ ketma-ketlik monoton kamayuvchi va quyidan ($x_n > 0$) chegaralangan. Yuqorida keltirilgan 4^0 xossaga ko‘ra $\lim\limits_{n\to\infty} x_n = c$ mavjud va chekli. 5^0 xossaga ko‘ra $\lim\limits_{n\to\infty} x_{n+1} = c$ o‘rinli bo‘ladi. Bundan

$$c = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(x_n \cdot \frac{a}{n+1} \right).$$

1-teoremaga asosan

$$\lim_{n\to\infty}\left(x_n \cdot \frac{a}{n+1} \right) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} \frac{a}{n+1} = c \cdot 0 = 0$$

Demak, $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$ bo‘ladi.

5-misol. Ushbu

$$x_n = \sqrt{a + \sqrt{a + \sqrt{a + ... + \sqrt{a}}}}\ , \quad (a > 0)$$

ketma-ketlikning limitini toping.

Yechish: Har qanday $a > 0$ soni uchun $\sqrt{a+\sqrt{a}} > \sqrt{a}$ tengsizlikni o‘rinli bo‘lishini e’tiborga olsak, istalgan $n \in N$ larda $x_{n+1} > x_n$ tengsizlik bajariladi, ya’ni $\{x_n\}$ ketma-ketlik monoton o‘suvchi. $\sqrt{a+\sqrt{a}} < \sqrt{a}+1$ va $\sqrt{a+\sqrt{a}+1} < \sqrt{a}+1$ tengsizliklardan foydalansak, istalgan $n \in N$ larda $x_n < \sqrt{a}+1$ tengsizlikka ega bo‘lamiz. 4^0 xossaga ko‘ra $\lim\limits_{n\to\infty} x_n = c$ mavjud va chekli. 5^0 xossaga ko‘ra,

$$c = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{a + x_n} = \sqrt{a + c}\ .$$

$c = \sqrt{a+c}$ tenglamadan c ni topsak, $c = \frac{1+\sqrt{1+4a}}{2}$ bo‘ladi.

6-misol. Agar $\{x_n\}$ ketma-ketlik $x_{n+1} - 2x_n + x_{n-1} = A \quad (n \geq 2,\ A \in R)$ rekurrent munosabat orqali berilgan bo‘lsa, $\lim\limits_{n\to\infty} \frac{x_n}{n^2}$ ni toping.

Yechish. Belgilash kiritamiz: $x_n - x_{n-1} = a_n$. Bundan

$$a_{n+1} - a_n = x_{n+1} - x_n - (x_n - x_{n-1}) = A$$

bo‘ladi. Bu esa $\{a_n\}$ ketma-ketlik arifmetik progressiya ekanini bildiradi. Arifmetik progressiyani umumiy hadi formulasiga asosan $a_n = a_2 + (n-2)A$ bo‘ladi.

$$x_n - x_{n-1} = a_2 + (n-2)A$$

$$x_{n-1} - x_{n-2} = a_2 + (n-3)A$$

$$\cdots\cdots\cdots\cdots$$

$$x_3 - x_2 = a_2 + A$$

Bu tengliklar hadlab qo‘shsak,

$$x_n - x_2 = (n-2)a_2 + \frac{(n-1)(n-2)}{2}A$$

tenglikni hosil qilamiz. Bu tenglikka asosan $\lim\limits_{n\to\infty} \frac{x_n}{n^2} = \frac{1}{2}A$ bo‘ladi.

7-misol. Istalgan $0 \leq x \leq 1$ uchun

$$\ln(1+x) \geq x - \frac{1}{2}x^2$$

tengsizlik o‘rinli bo‘lishini isbotlang.

Isbot. Funktsiya kiritamiz:

$$f(x) = \ln(1+x) - x + \frac{1}{2}x^2.$$

Uni hosilasini hisoblaymiz.

$$f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x}.$$

$f'(x) = \frac{x^2}{1+x} \geq 0$ bo‘lgani uchun $f(x)$ funksiya $0 \leq x \leq 1$ da o‘suvchi va

$$f(x) \geq f(0) = 0 .$$

Bundan esa $\ln(1+x) \geq x - \frac{1}{2}x^2$ tengsizlik kelib chiqadi.

8-misol. Istalgan $0 \leq \varepsilon \leq \frac{1}{4}$ da

$$f_\varepsilon(x) = \left(1 + \frac{1}{x}\right)^{x+\varepsilon}, \; x \geq 2$$

funksiyani monoton o‘suvchi bo‘lishini isbotlang.

Isbot. Funktsiya hosilasini nomanfiy qiymatlar qabul qilishini isbotlaymiz.

$$f_\varepsilon'(x) = \left(1 + \frac{1}{x}\right)^{x+\varepsilon} \left(\ln\left(1 + \frac{1}{x}\right) - \left(\frac{\varepsilon}{x} + \frac{1-\varepsilon}{x+1}\right)\right).$$

Agar $\ln\left(1 + \frac{1}{x}\right) \geq \frac{1}{x} - \frac{1}{2x^2}$, $x \geq 2$ tengsizlikdan foydalansak,

$$f_\varepsilon'(x) \geq \left(1 + \frac{1}{x}\right)^{x+\varepsilon} \left(\frac{1}{x} - \frac{1}{2x^2} - \left(\frac{\varepsilon}{x} + \frac{1-\varepsilon}{x+1}\right)\right) = \left(1 + \frac{1}{x}\right)^{x+\varepsilon} \frac{(1-2\varepsilon)x - 1}{2x^2(x+1)} \geq 0$$

bo‘ladi. Bu esa $f_\varepsilon(x)$ funksiyani o‘suvchi ekanini bildiradi.

1 – natija. $x = n \in N$, $n \geq 2$ bo‘lsa,

$$x_n = \left(1 + \frac{1}{n}\right)^{n+\varepsilon}, \; 0 \leq \varepsilon < \frac{1}{4}$$

ketma – ketlik monoton o‘suvchi .

2 – natija. Istalgan $0 \leq \varepsilon < \frac{1}{2}$ da

$$f_\varepsilon(x) = \left(1 + \frac{1}{x}\right)^{x+\varepsilon}, \; x \geq \frac{1}{1-2\varepsilon}$$

funksiya monoton o‘suvchi bo‘ladi.

3 – natija. Istalgan $\frac{1}{2} \leq \varepsilon \leq 1$ da

$$f_\varepsilon(x) = \left(1 + \frac{1}{x}\right)^{x+\varepsilon}, \; x \geq 1$$

funksiya monoton kamayuvchi bo‘ladi.

Agar $\varepsilon = 1$, $x = n \in N$ qilib tanlasak,

$$x_n = f_1(x) = \left(1 + \frac{1}{n}\right)^{n+1}$$

ketma – ketlik monoton kamayuvchi va quyidan chegaralangan ($x_n > 0, n \in N$).

Agar $\varepsilon = 0$, $x = n \in N$ qilib tanlasak,

$$x_n = f_0(x) = \left(1 + \frac{1}{n}\right)^{n}$$

ketma – ketlik monoton o‘suvchi va yuqoridan chegaralangan (Istalgan $n \in N$ larda $\left(1 + \frac{1}{n}\right)^{n} < \left(1 + \frac{1}{n}\right)^{n+1} < \left(1 + \frac{1}{1}\right)^{2} = 4$).

Shuning uchun 4^0 xossaga ko‘ra $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n}$ mavjud va chekli.

4-ta’rif. Ushbu $\{x_n\} = \left\{\left(1 + \frac{1}{n}\right)^{n}\right\}$ ketma – ketlikning limiti e soni deb ataladi:

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} = e.$$

9 – misol. Teplits teoremasini isbotlang: Agar 1) $P_{nk} \geq 0$; 2) $\sum\limits_{k=1}^{n} P_{nk} = 1$; 3) har bir tayinlangan k larda $\lim\limits_{n\to\infty} P_{nk} = 0$; 4) $\lim\limits_{n\to\infty} x_n = a$ bo‘lsa, u holda hadlari $t_n = \sum\limits_{k=1}^{n} P_{nk} x_k$ bo‘lgan ketma-ketlik yaqinlashadi va $\lim\limits_{n\to\infty} t_n = a$.

Isbot. 4) shartga ko‘ra, shunday $n_0 = n_0(\varepsilon) \in N$ son topilib, barcha $n > n_0(\varepsilon)$ larda

$$|x_n - a| < \frac{\varepsilon}{2}$$

tengsizlik o‘rinli bo‘ladi, va shu shartga asosan shunday $M > 0$ son mavjud bo‘lib barcha $n \in N$ larda

$$|x_n| \le M \quad \text{va} \quad |x_n - a| \le 2M$$

tengsizliklar o‘rinli bo‘ladi. 3) shartga asosan $n_1 = n_1(\varepsilon) > n_0(\varepsilon)$ son topilib, barcha $n > n_1$ larda

$$P_{nk} < \frac{\varepsilon}{4n_0 M},\ k = \overline{1, n_0},$$

o‘rinli bo‘ladi.

Bu tengsizliklar va teoremaning 1)-2) shartlariga asosan quyidagini hosil qilamiz:

$$\left|\sum_{k=1}^{n} P_{nk} x_k - a\right| = \left|\sum_{k=1}^{n} P_{nk} x_k - \sum_{k=1}^{n} P_{nk} a\right| = \left|\sum_{k=1}^{n} P_{nk} (x_k - a)\right| \le \sum_{k=1}^{n} P_{nk} |x_k - a| = P_{n1}|x_1 - a| +$$

$$+ P_{n2}|x_2 - a| + P_{nn_0}|x_{n_0} - a| + P_{nn_0+1}|x_{n_0+1} - a| + \ldots + P_{nn}|x_n - a| \le$$

$$\le n_0 \cdot \frac{\varepsilon}{4n_0 M} \cdot 2M + \frac{\varepsilon}{2}\left(P_{nn_0+1} + \ldots + P_{nn}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

barcha $n > n_1$ larda. Demak, $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} P_{nk} x_k = a$.

10 – misol. a) Agar $\{x_n\}$ ketma-ketlik yaqinlashsa, u holda o‘rta arifmetik ketma-ketlik $\{\xi_n\}$

$$\xi_n = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$$

ham yaqinlashishini va $\lim\limits_{n\to\infty} \xi_n = \lim\limits_{n\to\infty} x_n$ bo‘lishini isbotlang.

b) Agar $\{y_n\}$ ketma-ketlik yaqinlashuvchi va $\forall n \in N$ larda $y_n > 0$ bo‘lsa, u holda o‘rta garmonik ketma-ketlik $\{\gamma_n\}$

$$\gamma_n = \frac{n}{\frac{1}{y_1} + \frac{1}{y_2} + \ldots + \frac{1}{y_n}}$$

ham yaqinlashishini va $\lim\limits_{n\to\infty}\gamma_n = \lim\limits_{n\to\infty} y_n$ bo‘lishini isbotlang.

v) Agar $\lim\limits_{n\to\infty} y_n = +\infty$ bo‘lsa, u holda $\lim\limits_{n\to\infty}\gamma_n = +\infty$ va $\lim\limits_{n\to\infty}\xi_n = +\infty$ bo‘lishini isbotlang.

Isbot. a) Agar $P_{nk} = \frac{1}{n}\ \left(k = \overline{1,n},\ \ n \in N\right)$ deb olsak, u holda P_{nk} va x_n 9-misolning barcha shartlarini qanoatlantiradi, shuningdek $t_n = \sum\limits_{k=1}^{n} P_{nk}\, x_k = \xi_n$. Demak, $\lim\limits_{n\to\infty}\xi_n = \lim\limits_{n\to\infty} x_n$.

b) Aytaylik,

$$P_{nk} = \frac{\dfrac{1}{y_k}}{\dfrac{1}{y_1} + \dfrac{1}{y_2} + \ldots + \dfrac{1}{y_n}},\ \left(k = \overline{1,n}\right),\quad x_n = y_n$$

bo‘lsin. U holda 9-misolning barcha shartlari o‘rinli bo‘ladi. Shuningdek $t_n = \sum\limits_{k=1}^{n} P_{nk}\, y_k = \xi_n$. Demak, $\lim\limits_{n\to\infty}\gamma_n = \lim\limits_{n\to\infty} y_n$.

v) Agar $\lim\limits_{n\to\infty}\frac{1}{y_n} = 0$ bo‘lsa, $\lim\limits_{n\to\infty}\frac{1}{\gamma_n} = 0$ bo‘lishini ko‘rsatamiz. Bu esa $\lim\limits_{n\to\infty}\gamma_n = +\infty$ ga ekvivalent. 9-misoldan foydalanib,

$$P_{nk} = \frac{1}{n}\ \left(k = \overline{1,n},\ \ n \in N\right) \text{ va } x_n = \frac{1}{y_n}$$

deb olsak, $t_n = \sum\limits_{k=1}^{n} P_{nk}\, x_k = \frac{1}{\gamma_n}$ hamda $\lim\limits_{n\to\infty}\frac{1}{\gamma_n} = \lim\limits_{n\to\infty}\frac{1}{y_n} = 0$ ga ega bo‘lamiz.

Ushbu $\lim\limits_{n\to\infty}\xi_n = +\infty$ tasdiq esa $\gamma_n \le \xi_n$ tengsizlik va $\lim\limits_{n\to\infty}\gamma_n = +\infty$ ekanligidan kelib chiqadi.

11 – misol. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi va $x_n > 0$ bo‘lsa, u holda

$$\lim_{n\to\infty}\sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = \lim_{n\to\infty} x_n$$

bo‘lishini isbotlang.

Isbot. Ma’lumki,

$$\gamma_n = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}} \le \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \le \frac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right) = \xi_n$$

tengsizlik o‘rinli. 10-misolga asosan esa $\lim\limits_{n\to\infty} \gamma_n = \lim\limits_{n\to\infty} \xi_n = \lim\limits_{n\to\infty} x_n$, u holda

$$\lim_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = \lim_{n\to\infty} x_n .$$

12 – misol. Agar $\forall n \in N$ larda $x_n > 0$ bo‘lsa, u holda

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_n}{x_{n-1}}$$

bo‘lishini isbotlang.

Isbot. 11-misolga asosan, bu tasdiqning isboti quyidagicha bo‘ladi:

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{x_1 \cdot \frac{x_2}{x_1} \cdot \frac{x_3}{x_2} \cdot \ldots \cdot \frac{x_n}{x_{n-1}}} = \lim_{n\to\infty} \frac{x_n}{x_{n-1}} .$$

13-misol. Isbotlang: $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}} = e$.

Isboti. Agar $x_n = \dfrac{n^n}{n!}$ deb olsak, unda

$$\frac{n}{\sqrt[n]{n!}} = \sqrt[n]{\frac{n^n}{n!}} = \sqrt[n]{x_n}$$

bo‘ladi. Shuningdek, $\lim\limits_{n\to\infty} \dfrac{x_n}{x_{n-1}} = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n-1}\right)^{n-1} = e$. 12-misolga asosan,

$\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}} = e$. Isbot tugadi.

14-misol. Shtols teoremasini isbotlang: agar a) $\forall n \in N$ larda $y_{n+1} > y_n$; b) $\lim\limits_{n\to\infty} y_n = +\infty$; v) $\lim\limits_{n\to\infty} \dfrac{x_n - x_{n-1}}{y_n - y_{n-1}}$ mavjud bo‘lsa, u holda

$\lim\limits_{n\to\infty}\dfrac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim\limits_{n\to\infty}\dfrac{x_n}{y_n}$ bo‘ladi.

Isbot. Aytaylik, $\lim\limits_{n\to\infty}\dfrac{x_n - x_{n-1}}{y_n - y_{n-1}} = a$ (a – chekli) bo‘lsin. Unda, agar

$y_0 = 0$, $x_0 = 0$ va $P_{nk} = \dfrac{y_k - y_{k-1}}{y_n}$, $k = \overline{1,n}$, $X_n = \dfrac{x_n - x_{n-1}}{y_n - y_{n-1}}$ deb olsak, P_{nk} va X_n uchun 9-misol, ya’ni Teplits teoremasining barcha shartlari bajariladi, shuningdek, $t_n = \dfrac{x_n}{y_n}$.

Demak, $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = \lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} X_n = \lim\limits_{n\to\infty}\dfrac{x_n - x_{n-1}}{y_n - y_{n-1}} = a$.

15-misol. Agar p – natural son bo‘lsa, quyidagilarni isbotlang:

a) $\lim\limits_{n\to\infty}\dfrac{1^p + 2^p + \ldots + n^p}{n^{p+1}} = \dfrac{1}{p+1}$; b) $\lim\limits_{n\to\infty}\left(\dfrac{1^p + 2^p + \ldots + n^p}{n^p} - \dfrac{n}{p+1}\right) = \dfrac{1}{2}$;

v) $\lim\limits_{n\to\infty}\left(\dfrac{1^p + 3^p + \ldots + (2n+1)^p}{n^{p+1}} - \dfrac{n}{p+1}\right) = \dfrac{2^p}{p+1}$.

Isbot. Isbotlash uchun Shtols teoremasidan foydalanamiz. b) bandni isbotini keltiramiz (qolgan a) va b) bandlar ham xuddi shunday isbotlanadi).

b) Agar $x_n = (p+1)\left(1^p + 2^p + \ldots + n^p\right) - n^{p+1}$, $y_n = (p+1)n^p$ deb olsak, unda

$$\lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty}\frac{(p+1)(n+1)^p - (n+1)^{p+1} + n^{p+1}}{(p+1)\left((n+1)^p - n^p\right)} =$$

$$= \lim_{n\to\infty}\left(\frac{(p+1)\left(n^p + pn^{p-1} + \dfrac{p(p-1)}{2}n^{p-2} + \ldots + 1\right)}{(p+1)\left(n^p + pn^{p-1} + \dfrac{p(p-1)}{2}n^{p-2} + \ldots + 1 - n^p\right)} + \right.$$

$$+\frac{-n^{p+1}-(p+1)n^{p}-\frac{(p+1)p}{2}n^{p-1}-\ldots-1+n^{p+1}}{(p+1)\left(n^{p}+pn^{p-1}+\frac{p(p-1)}{2}n^{p-2}+\ldots+1-n^{p}\right)}\Bigg).$$

Limit ichidagi ifodada bir xil darajali n larning koeffitsientlarini yig‘ib chiqamiz. Keyin esa kasrning surat va maxrajini n^{p-1} ga bo‘lamiz va darajasi -1 dan oshmagan barcha hadlar yig‘indisini $o\left(\frac{1}{n}\right)$ bilan belgilaymiz. Natijada ushbuni hosil qilamiz:

$$\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}=\lim_{n\to\infty}\frac{\frac{p(p+1)}{2}+o\left(\frac{1}{n}\right)}{p(p+1)+o\left(\frac{1}{n}\right)}=\frac{1}{2}.$$

16 – misol. Agar $\{x_n\}$ ketma-ketlik quyidagicha aniqlangan bo‘lsa, uning limitini hisoblang.

$$x_1=a,\ x_2=b,\ x_n=\frac{x_{n-1}+x_{n-2}}{2},\ (n=3,4,\ldots)$$

Isbot. Shartga asosan

$$x_k-x_{k-1}=\frac{x_{k-1}-x_{k-2}}{2}-x_{k-1}=-\frac{x_{k-1}-x_{k-2}}{2}$$

munosabat o‘rinli. Bu ifodani ushbu tenglikka qo‘yamiz:

$$x_n=x_1+(x_2-x_1)+(x_3-x_2)+\ldots+(x_n-x_{n-1}).$$

Natijada quyidagi yig‘indiga ega bo‘lamiz:

$$x_n=a+(b-a)-\frac{b-a}{2}+\frac{b-a}{4}+\ldots+(-1)^n\frac{b-a}{2^{n-2}}.$$

Ushbu yig‘indining hadlari ikkinchi hadidan boshlab geometrik progressiyani tashkil etadi va uning yig‘indisi

$$x_n=a+\frac{2(b-a)}{3}+\frac{b-a}{3}\cdot\frac{(-1)^n}{2^{n-2}}$$

ga teng. Bundan esa

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(a + \frac{2(b-a)}{3} + \frac{b-a}{3}\cdot\frac{(-1)^n}{2^{n-2}} \right) = \frac{a+2b}{3}$$

ekanligi kelib chiqadi.

17 – misol. Ikkita $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar quyidagicha aniqlangan:

$$x_1 = a\,,\ y_1 = b\,,\ x_{n+1} = \sqrt{x_n y_n}\,,\ y_{n+1} = \frac{x_n + y_n}{2}$$

Bu ketma-ketliklar umumiy limitga ega bo‘lishini isbotlang:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n\,.$$

Isbot. Misolning shartidan, $\forall n \in N$ larda $x_n \geq 0$, $y_n \geq 0$ ekanligi kelib chiqadi. Ushbu

$$\sqrt{ab} \leq \frac{a+b}{2},\ a \geq 0\,,\ b \geq 0$$

tengsizlikka asoslanib

$$y_{n+1} = \frac{x_n + y_n}{2} \geq \sqrt{x_n y_n} = x_{n+1}$$

munosabatni va bundan esa $x_{n+1} = \sqrt{x_n y_n} \geq \sqrt{x_n^{\,2}} = x_n$, $y_{n+1} = \frac{x_n + y_n}{2} \leq y_n$ munosabatni hosil qilamiz. Natijada esa $x_n \leq y_n \leq y_1$, $y_n \geq x_n \geq x_1$ munosabatlar hosil bo‘ladi. Demak, $\{x_n\}$ va $\{y_n\}$ monoton va chegaralangan ketma-ketliklar. Ular chekli A va B limitlarga ega. Ushbu

$$y_{n+1} = \frac{x_n + y_n}{2}$$

tenglikda limitga o‘tib, $A = B$ ekanligini topamiz.

Bu ketma-ketliklar limitining umumiy qiymatiga a va b sonlarining arifmetik-geometrik o‘rtachasi deb ataladi va $\mu(a,b)$ kabi belgilanadi:

$$\mu(a,b) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n\,.$$

1.2§. Differensial hisobning asosiy teoremalari yordamida yechiladigan masalalar

Biz ushbu paragrafda funksiya hosilasi va differensiali tushunchalarini, hosilaning geometrik va mexanik ma’nolarini keltirmadik. Ushbu paragrafda differensial hisobning asosiy teoremalari bo‘lgan Ferma, Roll, Lagranj va Koshi teoremalari keltiriladi. Funksiya hosilasini bir nechta sodda masalalarga tatbiqi keltiriladi.

1-teorema (Ferma). $f(x)$ funksiya $x \in (a,b)$ oraliqda aniqlangan va $c \in (a,b)$ nuqtada o‘zining eng katta (kichik) qiymatga erishsin. Agar $f(x)$ funksiya c nuqtada chekli $f'(c)$ hosilaga ega bo‘lsa, u holda $f'(c) = 0$ bo‘ladi.

2-teorema (Roll). $f(x)$ funksiya $[a,b]$ segmentda aniqlangan, uzluksiz va $f(a) = f(b)$ bo‘lsin. Agar bu funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo‘lsa, uchun holda shunday $c \in (a,b)$ nuqta topiladiki $f'(c) = 0$ bo‘ladi.

3-teorema (Lagranj). $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz bo‘lsin. Agar bu funksiya (a,b) oraliqda $f'(x)$ hosilaga ega bo‘lsa, u holda shunday $c \in (a,b)$ nuqta mavjudki, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ bo‘ladi.

4-teorema. $f(x)$ funksiya (a,b) oraliqda chekli hosilaga ega bo‘lsin. Bu funksiya shu oraliqda o‘suvchi (kamayuvchi) bo‘lishi uchun $f'(x) \geq 0$ ($f'(x) \leq 0$) bo‘lishi zarur va yetarli.

Endi yuqoridagi teoremalarni ba’zi masalalarga tadbiq qilamiz.

1-masala. $\forall x \in R$ uchun $e^x \geq x + 1$ bo‘lishi isbotlang.

Isboti. Funksiya tuzamiz. $f(x) = e^x - x - 1,\ x \in R$.

Ravshanki, $f'(x) = e^x - 1$, $f'(x) = f(0) = 0$.

1). $x \geq 0$ bo‘lsin. U holda $f'(x) = e^x - 1 \geq e^0 - 1 = 0$. Bundan esa $f(x)$ funksiyani o‘suvchi ekanligi kelib chiqadi.

Shuning uchun $f(x) \geq f(0) \Rightarrow e^x \geq x + 1$.

2). $x \leq 0$ bo‘lsin. U holda $f'(x) = e^x - 1 \leq e^0 - 1 = 0$. Bundan esa $f(x)$ funksiyani kamayuuvchi ekanligini bildiradi.

Shuning uchun $f(x) \geq f(0) \Rightarrow e^x \geq x + 1$.

Demak, $\forall x \in R$ uchun $e^x \geq x+1$ bo‘ladi.

2-masala. Agar $a \geq 0$, $b \geq 0$, $p \geq 1$ bo‘lsa, $\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}$ bo‘lishini isbotlang.

Isboti. Agar $a = 0$ yoki $b = 0$ bo‘lsa, tengsizlikni tenglik sharti bajariladi. Shuning uchun $a \geq b > 0$ faqat holatni qaraymiz va

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2} \Leftrightarrow \left(\frac{\frac{a}{b}+1}{2}\right)^p \leq \frac{\left(\frac{a}{b}\right)^p + 1}{2}$$

ekanini inobatga olamiz. Ushbu funksiyani tuzamiz:

$$f(x) = \frac{x^p + 1}{2} - \left(\frac{x+1}{2}\right)^p,\ x \geq 1,\ p \geq 1.$$

U holda $f(0) = 0$, $f'(x) = \frac{p}{2}\left(x^{p-1} - \left(\frac{x+1}{2}\right)^p\right) \geq 0$ bo‘ladi. $f'(x) \geq 0$ bo‘lgani uchun $f(x)$ funksiya o‘suvchi. Shuning uchun $f(x) \geq f(0)$.

Bundan $\frac{x^p + 1}{2} \geq \left(\frac{x+1}{2}\right)^p$ bo‘ladi. $x = \frac{a}{b}$ qilib tanlasak

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}$$

tengsizlikka ega bo‘lamiz.

3-masala. Agar $x \in \left[0; \frac{\pi}{2}\right)$ bo‘lsa, $2^{tgx} + 2^{\sin x} \geq 2^{x+1}$ bo‘lishini isbotlang.

Isboti. Koshi tengsizligiga ko‘ra:

$$2^{tgx} + 2^{\sin x} \geq 2\sqrt{2^{tgx} \cdot 2^{\sin x}} = 2 \cdot 2^{\frac{tgx + \sin x}{2}}.$$

Endi $tgx + \sin x \geq 2x$ tengsizlikni o‘rinli bo‘lishini ko‘rsatamiz. Quyidagi funksiyani tuzamiz:

$$f(x) = tgx + \sin x - 2x,\ x \in \left[0; \frac{\pi}{2}\right).$$

Bundan, $f(0) = 0$, $f'(x) = \frac{1}{\cos^2 x} + \cos x - 2$.

Koshi tengsizligiga ko‘ra,

$$f'(x) = \frac{1}{\cos^2 x} + \cos x - 2 \geq \frac{1}{\cos x} + \cos x - 2 \geq 2\sqrt{\frac{1}{\cos x} \cdot \cos x} - 2 = 0,$$

ya’ni $f'(x) \geq 0$. Bu esa $f(x)$ funksiyani $\left[0; \frac{\pi}{2}\right)$ oraliqda o‘suvchi ekanligini bildiradi.

Bundan, $f(x) \geq f(0)$, ya’ni $tgx + \sin x \geq 2x$ bo‘ladi.

Demak, ixtiyoriy $x \in \left[0; \frac{\pi}{2}\right)$ uchun $2^{tgx} + 2^{\sin x} \geq 2^{x+1}$ tengsizlik o‘rinli.

4-masala. c ning hech bir qiymatida $x(x^2 - 1)(x^2 - 10) = c$ tenglama beshta butun yechimga ega bo‘la olmasligini isbotlang.

Isboti. Ushbu funksiyani tuzamiz: $f(x) = x(x^2 - 1)(x^2 - 10)$.

Bu funksiya butun sonlar o‘qida aniqlangan va $f'(x) = 5x^4 - 3x^2 + 10$.

1). $f'(x) = 0$ tenglamani echamiz. Bundan

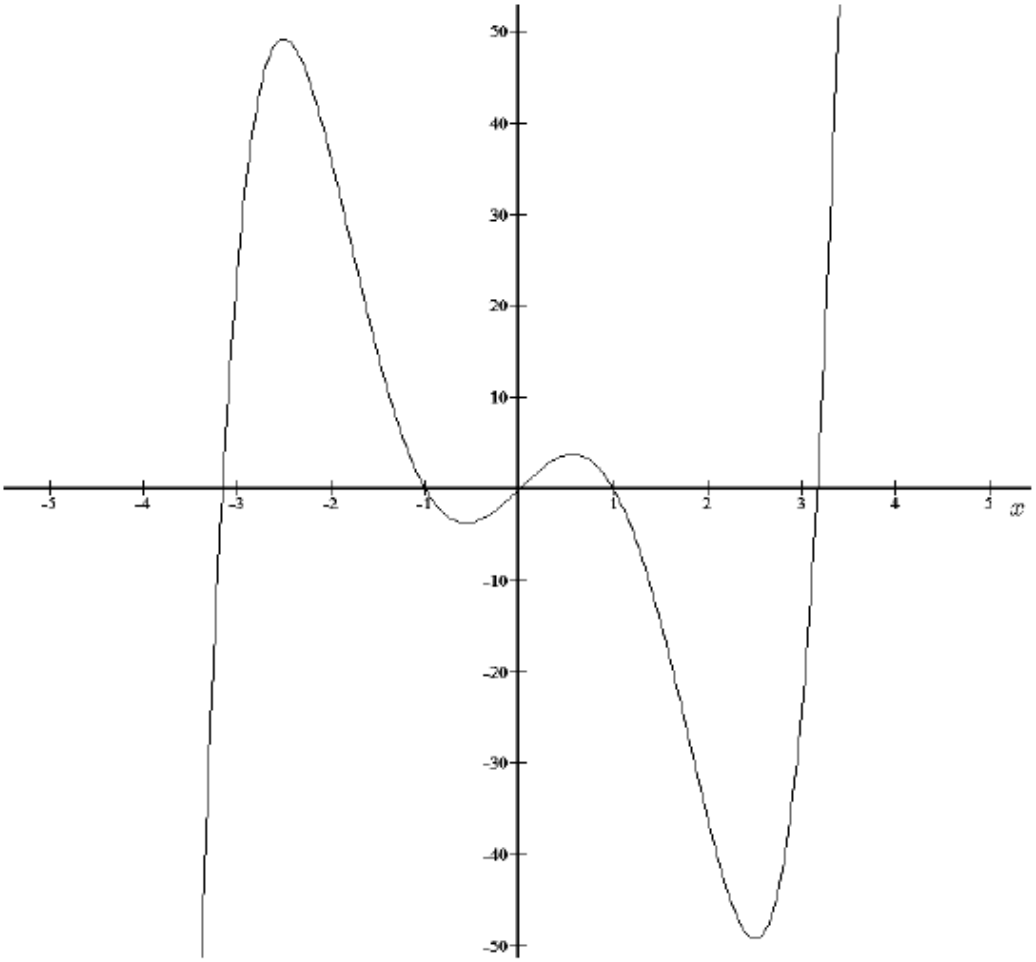
$$x_1, x_4 = \pm\sqrt{\frac{33 + \sqrt{889}}{10}} \quad \text{va} \quad x_2, x_3 = \pm\sqrt{\frac{33 - \sqrt{889}}{10}}$$

kelib chiqadi.

2). $f(x) = 0$ tenglamani echamiz.

$$x(x^2 - 1)(x^2 - 10) = 0 \ \Rightarrow\ x = 0,\ x = \pm 1 \text{ va } x = \pm\sqrt{10}.$$

3). Funktsiya grafigini yasaymiz.



Funktsiyaning monotonlik oraliqlari beshta:

1) $(-\infty; x_1]$, 2) $[x_1; x_2]$ 3) $[x_2; x_3]$, 4) $[x_3; x_4]$, 5) $[x_4; +\infty)$.

Shuning uchun $y = c$ to‘g‘ri chiziq $f(x)$ funksiyani ko‘pi bilan beshta nuqtada kesishi mumkin. $[x_1; x_2]$ ($[x_1; x_2] \subset (-1;1)$) monotonlik oralig‘ida yagona $x = 0$ butun son bor.

Demak, $c = 0$ bo‘lgandagi $f(x) = c$ tenglama ko‘pi bilan beshta butun yechimga ega bo‘lishi mumkin.

$f(x) = 0$ tenglama esa $x = 0$ va $x = \pm 1$ butun yechimlarga ega. Bu holat $f(x) = c$ tenglama beshta butun yechimga ega emasligini bildiradi.

4-masala. Agar $a > 0$ $(a \neq 1)$, $p > q > 0$ bo‘lsa, u holda

$$\frac{a^p - 1}{p} > \frac{a^q - 1}{q} \qquad (1)$$

tengsizlik o‘rinli bo‘ladi.

Isbot. Ushbu funksiyani qaraymiz:

$$f(x)=\frac{a^x-1}{x},\ x>0.$$

Bu funksiyani $(0;\infty)$ oraliqda o‘suvchi ekanini ko‘rsatamiz. Shu maqsadda uning hosilasini hisoblaymiz:

$$f'(x)=\frac{xa^x\ln a-a^x+1}{x^2}.$$

Bu hosilani musbatligini ko‘rsatish uchun quyidagi

$$g(x)=xa^x\ln a-a^x+1,\ x\geq 0$$

funksiyaning nomanfiyligini ko‘rsatamiz. Buning uchun avvalo uning hosilasini hisoblaymiz:

$$g'(x)=a^x\ln a+xa^x(\ln a)^2-a^x\ln a=xa^x(\ln a)^2\geq 0.$$

$g(x)$ funksiya hosilasi nomanfiy bo‘lgani uchun u o‘suvchi bo‘ladi. Bunga ko‘ra $g(x)\geq g(0)$, ya’ni $g(x)\geq 0$ bo‘ladi. Bu esa $x>0$ bo‘lganda $f(x)>0$ tengsizlik o‘rinli ekanini bildiradi, ya’ni $f(x)$ funksiya o‘suvchi.

Demak, $p>q>0$ bo‘lgani uchun (1) tengsizlik o‘rinli.

5-masala. e^π va π^e sonlarini taqqoslang.

Yechish. Ushbu funksiyani qaraymiz:

$$f:[e;+\infty)\to R:\quad f(x)=\frac{\ln x}{x}.$$

Uning hosilasi barcha $x\in(e;+\infty)$ larda

$$f'(x)=\frac{1-\ln x}{x^2}<0$$

manfiy qiymat qabul qiladi va $f(x)$ funksiya $[e;+\infty)$ da uzluksiz, shunday qilib, $f(x)$ funksiya $[e;+\infty)$ da qat’iy kamayuvchi. Bundan $e<\pi$ ekandigini hisobga olib

$$f(e)<f(\pi)\ \Rightarrow\ \frac{\ln e}{e}<\frac{\ln\pi}{\pi}\ \Rightarrow\ \pi\ln e<e\ln\pi.$$

Demak, $e^\pi>\pi^e$ deb ayta olamiz.

1.3 §. Funktsiya hosilasini ba’zi murakkab masalalariga tadbiqlari

Ushbu paragrafda funksiya hosilasi ba’zi murakkab masalalarni yechishga tadbiq qilingan.

1-teorema. $ABCD$ to‘g‘ri to‘rtburchakda ixtiyoriy M nuqta olingan bo‘lib, $|AB| = a$, $|AD| = b$, $\lambda \geq 1$ bo‘lsa, u holda

a) $\max \left\{ |MA|^{\lambda} + |MB|^{\lambda} + |MC|^{\lambda} + |MD|^{\lambda} \right\} = a^{\lambda} + b^{\lambda} + \left(\sqrt{a^2 + b^2} \right)^{\lambda}$;

b) $\min \left\{ |MA|^{\lambda} + |MB|^{\lambda} + |MC|^{\lambda} + |MD|^{\lambda} \right\} = 4 \left(\frac{a^2 + b^2}{4} \right)^{\lambda}$

tengliklar o‘rinli bo‘ladi.

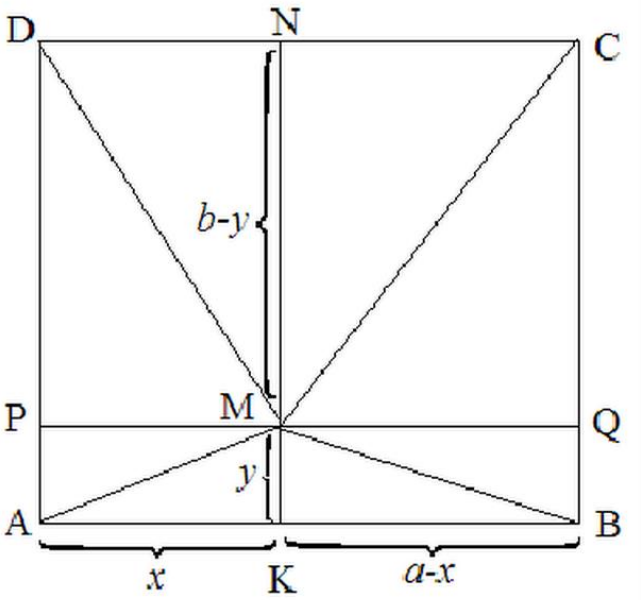
Isbot. M nuqtadan $KN \perp AB$, $PQ \perp AD$ kesmalarni o‘tkazamiz (1-chizma). Aytaylik, $|AK| = x$, $|MK| = y$ bo‘lsin. M nuqta to‘rtburchakda bo‘lganligi uchun $0 \leq x \leq a$, $0 \leq y \leq b$ bo‘ladi. Pifagor teoremasiga ko‘ra:

$$|MA|^{\lambda} + |MB|^{\lambda} + |MC|^{\lambda} + |MD|^{\lambda} = \left(x^2 + y^2\right)^{\frac{\lambda}{2}} + \left((x-a)^2 + y^2\right)^{\frac{\lambda}{2}} +$$

$$+ \left((x-a)^2 + (y-b)^2\right)^{\frac{\lambda}{2}} + \left(x^2 + (y-b)^2\right)^{\frac{\lambda}{2}}.$$

Ushbu belgilashni kiritib olib, quyidagi ikki holni ko‘rib chiqamiz.

$$P(x,y) = \left(x^2 + y^2\right)^{\frac{\lambda}{2}} + \left((x-a)^2 + y^2\right)^{\frac{\lambda}{2}} + \left((x-a)^2 + (y-b)^2\right)^{\frac{\lambda}{2}} + \left(x^2 + (y-b)^2\right)^{\frac{\lambda}{2}}$$



1-chizma

1-hol. $y = 0$ yoki $y = b$ bo‘lsin. Bu holda

$$P(x,0) = P(x,b) = x^{\lambda} + (a-x)^{\lambda} + \left((x-a)^2 + b^2\right)^{\frac{\lambda}{2}} + \left(x^2 + b^2\right)^{\frac{\lambda}{2}}$$

bo‘lganligi sababli faqat $P(x,0)$ ni o‘rganish etarli. $P(x,0)$ funksiyani $(0,\, a)$ oraliqda x bo‘yicha birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$P'(x,0) = \lambda\left(x^{\lambda-1} - (a-x)^{\lambda-1} + (x-a)\left((x-a)^2 + b^2\right)^{\frac{\lambda}{2}-1} + x\left(x^2 + b^2\right)^{\frac{\lambda}{2}-1} \right)$$

$$P''(x,0) = \lambda\left((\lambda-1)\left(x^{\lambda-2} - (a-x)^{\lambda-2}\right) + \left((x-a)^2 + b^2\right)^{\frac{\lambda}{2}-2}\left((\lambda-1)(x-a)^2 + b^2\right) \right) +$$

$$+ \lambda\left(x^2 + b^2\right)^{\frac{\lambda}{2}-2}\left((\lambda-1)x^2 + b^2\right).$$

$P''(x,0) \geq 0$ bo‘lgani uchun $P'(x,0)$ funksiya $(0,a)$ oraliqda o‘suvchi. Shuning uchun $P'(x,0) = 0$ tenglama $(0,a)$ oraliqda ko‘pi bilan bitta yechimga ega bo‘lishi mumkin. $P'\left(\frac{a}{2}, 0\right) = 0$ bo‘lgani sababli, yagona yechim $x = \frac{a}{2}$ dan iborat bo‘ladi. Demak, $P(x,0)$ funksiya $\left[0, \frac{a}{2}\right]$ kesmada kamayuvchi, $\left[\frac{a}{2}, a\right]$ kesmada esa o‘suvchi bo‘ladi. Bunga asosan quyidagi tengliklarga ega bo‘lamiz:

$$\max_{0\le x\le a} P(x,0)=P(0,0)=P(a,0)=a^{\lambda}+b^{\lambda}+\left(\sqrt{a^2+b^2}\right)^{\lambda},$$

$$\min_{0\le x\le a} P(x,0)=P\left(\frac{a}{2},0\right)=2\left(\frac{a}{2}\right)^{\lambda}+2\left(\left(\frac{a}{2}\right)^2+b^2\right)^{\frac{\lambda}{2}}.$$

2-hol. $0<y<b$ bo‘lsin. Bu holda y o‘zgaruvchini tayinlab, $P(x,y)$ funksiyani x bo‘yicha birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$P'(x,y)=\lambda x\left(x^2+y^2\right)^{\frac{\lambda}{2}-1}+\lambda(x-a)\left((x-a)^2+y^2\right)^{\frac{\lambda}{2}-1}+$$

$$+\lambda(x-a)\left((x-a)^2+(y-b)^2\right)^{\frac{\lambda}{2}-1}+\lambda x\left(x^2+(y-b)^2\right)^{\frac{\lambda}{2}-1},$$

$$P''(x,y)=\lambda\left(x^2+y^2\right)^{\frac{\lambda}{2}-2}\left(x^2(\lambda-1)+y^2\right)+$$

$$+\lambda\left((x-a)^2+y^2\right)^{\frac{\lambda}{2}-2}\left((x-a)^2(\lambda-1)+y^2\right)+$$

$$+\lambda\left((x-a)^2+(y-b)^2\right)^{\frac{\lambda}{2}-2}\left((\lambda-1)(x-a)^2+(y-b)^2\right)+$$

$$+\lambda\left(x^2+(y-b)^2\right)^{\frac{\lambda}{2}-2}\left(x^2(\lambda-1)+(y-b)^2\right).$$

Yuqorida qilingan ishlarni takrorlab, ushbu

$$\max_{0\le x\le a} P(x,y)=P(0,y)=P(a,y)=y^{\lambda}+\left(a^2+y^2\right)^{\frac{\lambda}{2}}+\left(a^2+(y-b)^2\right)^{\frac{\lambda}{2}}+(b-y)^{\lambda}$$

$$\min_{0\le x\le a} P(x,y)=P\left(\frac{a}{2},y\right)=2\left(\left(\frac{a}{2}\right)^2+y^2\right)^{\lambda}+2\left(\left(\frac{a}{2}\right)^2+(y-b)^2\right)^{\frac{\lambda}{2}}$$

tengliklarni hosil qilamiz.

Endi ushbu

$$f(y)=y^{\lambda}+\left(a^2+y^2\right)^{\frac{\lambda}{2}}+\left(a^2+(y-b)^2\right)^{\frac{\lambda}{2}}+(b-y)^{\lambda},$$

$$g(y)=2\left(\left(\frac{a}{2}\right)^2+y^2\right)^{\frac{\lambda}{2}}+2\left(\left(\frac{a}{2}\right)^2+(y-b)^2\right)^{\frac{\lambda}{2}}$$

yordamchi funksiyalarni kiritib olamiz. Bu funksiyalar uchun $\max\limits_{0\le x\le b} f(y)$ va $\min\limits_{0\le x\le b} g(y)$ larni hisoblaymiz. Shu maqsadda, $(0,b)$ oraliqda, $f(y)$ va $g(y)$ funksiyalarning birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$f'(y)=\lambda y^{\lambda-1}+\lambda y\left(a^2+y^2\right)^{\frac{\lambda}{2}-1}+\lambda(y-b)\left(a^2+(y-b)^2\right)^{\frac{\lambda}{2}-1}-\lambda(b-y)^{\lambda-1},$$

$$f''(y)=\lambda(\lambda-1)y^{\lambda-2}+\lambda\left(a^2+y^2\right)^{\frac{\lambda}{2}-2}\left(a^2+(\lambda-1)y^2\right)+$$

$$+\lambda\left(a^2+(y-b)^2\right)^{\frac{\lambda}{2}-2}\left(a^2+(\lambda-1)(y-b)^2\right)+\lambda(\lambda-1)(b-y)^{\lambda-2},$$

$$g'(y)=2\lambda y\left(\left(\frac{a}{2}\right)^2+y^2\right)^{\frac{\lambda}{2}-1}+2\lambda(y-b)\left(\left(\frac{a}{2}\right)^2+(y-b)^2\right)^{\frac{\lambda}{2}-1},$$

$$g''(y)=2\lambda\left(\left(\frac{a}{2}\right)^2+y^2\right)^{\frac{\lambda}{2}-2}\left(\left(\frac{a}{2}\right)^2+(\lambda-1)y^2\right)+$$

$$+2\lambda\left(\left(\frac{a}{2}\right)^2+(y-b)^2\right)^{\frac{\lambda}{2}-2}\left(\left(\frac{a}{2}\right)^2+(\lambda-1)(y-b)^2\right).$$

Bu ifodalardan $0\le y\le b$ bo‘lganida $f''(y)\ge 0$, $g''(y)\ge 0$ va $f'\left(\frac{b}{2}\right)=0$, $g'\left(\frac{b}{2}\right)=0$ kelib chiqadi. Demak, $f(y)$ va $g(y)$ funksiyalar funksiyalar $y=\frac{b}{2}$ nuqtada o‘zining eng kichik qiymatlariga erishadi, kesmaning chetki nuqtalarida esa eng katta qiymatlarga erishadi, xususan

$$\max_{0\le x\le b} f(y)=f(0)=f(b)=a^{\lambda}+b^{\lambda}+\left(\sqrt{a^2+b^2}\right)^{\lambda},$$

$$\min_{0\le x\le b} g(y)=g\left(\frac{b}{2}\right)=4\left(\frac{a^2+b^2}{4}\right)^{\frac{\lambda}{2}}$$

bo‘ladi. Birinchi va ikkinchi hollarni hisobga olsak, a) va b) tengliklar kelib chiqadi. Teorema isbotlandi.

Natija. Agar $\lambda=1$ bo‘lsa, ushbu

$$\max\left\{|MA|+|MB|+|MC|+|MD|\right\}=a+b+\sqrt{a^2+b^2},$$

$$\min\left\{|MA|+|MB|+|MC|+|MD|\right\}=2\sqrt{a^2+b^2}$$

Tengliklar, $\lambda=2$ bo‘lsa, quyidagi

$$\max\left\{|MA|^2+|MB|^2+|MC|^2+|MD|^2\right\}=2\left(a^2+b^2\right),$$

$$\min\left\{|MA|^2+|MB|^2+|MC|^2+|MD|^2\right\}=a^2+b^2$$

tengliklar o‘rinli bo‘ladi.

1-masala. Yuzasi S bo‘lgan $ABCD$ – qavariq to‘rtburchakning AB, BC, CD, DA tomonlarida mos ravishda M, N, P, Q nuqtalar shunday olinganki, bunda

$$\frac{|AM|}{|MB|}=\frac{|BN|}{|NC|}=\frac{|CP|}{|PD|}=\frac{|DQ|}{|QA|}.$$

$MNPQ$ to‘rtburchak yuzasining eng kichik qiymatini toping.

Yechish. Aytaylik, $\frac{|AM|}{|MB|}=\lambda$ bo‘lsin. U holda

$$|AB|=|AM|+|MB|=(1+\lambda)|MB|,$$

$$|BC|=|BN|+|NC|=\frac{\lambda+1}{\lambda}|BN|,$$

$$|AD|=|AQ|+|QD|=\frac{\lambda+1}{\lambda}|QD|,$$

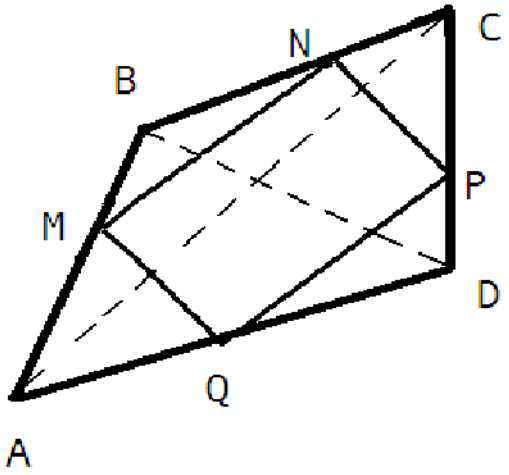
$$|CD|=|CP|+|pD|=\frac{\lambda+1}{\lambda}|PD|$$

bo‘ladi. Bularga asosan quyidagilarga ega bo‘lamiz (2-chizma):

$$S_{PQD}=\frac{\lambda}{(1+\lambda)^2}\cdot S_{ACD},\quad S_{MBN}=\frac{\lambda}{(1+\lambda)^2}\cdot S_{ABC}.$$

Bundan ushbu tenglik kelib chiqadi: $S_{PQD}+S_{MBN}=\frac{\lambda}{(1+\lambda)^2}\cdot S$.

Xuddi shuningdek, $S_{QAM}+S_{NCP}=\frac{\lambda}{(1+\lambda)^2}\cdot S$ tenglikka ega bo‘lamiz.



2-chizma

Demak,

$$S_{MNPQ} = S_{ABCD} - \left(S_{QAM} + S_{NCP} + S_{MBN} + S_{PDQ}\right) =$$

$$= S - \frac{2\lambda}{(1+\lambda)^2} \cdot S = \frac{\lambda^2+1}{(1+\lambda)^2} \cdot S .$$

Oxirgi ifodaning eng kichik qiymatini topish maqsadida ushbu

$$f(\lambda) = \frac{\lambda^2+1}{(1+\lambda)^2}, \quad \lambda > 0$$

funksiyani kiritib olamiz va uning eng kichik qiymatini topamiz. Buning uchun avvalo uning hosilasini hisoblaymiz:

$$f'(\lambda) = \frac{2\lambda \cdot (\lambda^2+1) - (\lambda^2+1) \cdot 2(\lambda+1)}{(1+\lambda)^4} = \frac{2(\lambda-1)}{(1+\lambda)^3}.$$

Bu ifodaga binoan $f'(\lambda) = 0$ tenglamaning yagona yechimi $\lambda = 1$ bo‘lib, $(0;1)$ oraliqda $f'(\lambda) < 0$ va $(1;+\infty)$ oraliqda $f'(\lambda) > 0$ bo‘ladi. Bu esa $f(\lambda)$ funksiyaning eng kichik qiymati $f(1) = \frac{1}{2}$ ga teng ekanini bildiradi. Demak, $MNPQ$ to‘rtburchak yuzasining eng kichik qiymati $\frac{1}{2}S$ ga teng bo‘lib, bu qiymat yuqoridagi nisbat 1 ga teng bo‘lganda, ya’ni M , N , P , Q nuqtalar to‘rtburchak tomonlarining o‘rtalarida bo‘lganida erishiladi.

2-masala. Agar a, b, c – musbat sonlar bo‘lsa, u holda ixtiyoriy λ, μ, $(\lambda \geq \mu \geq 0)$ sonlari uchun ushbu

$$\frac{a^{\lambda}}{b^{\lambda}+c^{\lambda}}+\frac{b^{\lambda}}{a^{\lambda}+c^{\lambda}}+\frac{c^{\lambda}}{a^{\lambda}+b^{\lambda}} \geq \frac{a^{\mu}}{b^{\mu}+c^{\mu}}+\frac{b^{\mu}}{a^{\mu}+c^{\mu}}+\frac{c^{\mu}}{a^{\mu}+b^{\mu}} \qquad (1)$$

tengsizlik o‘rinli bo‘ladi.

Isbot. Quyidagi funksiyani qaraymiz:

$$f(\lambda)=\frac{p^{\lambda}}{q^{\lambda}+1}+\frac{q^{\lambda}}{p^{\lambda}+1}+\frac{1}{p^{\lambda}+q^{\lambda}}, \quad \lambda \geq 0.$$

Bunda p, q $(p \geq q \geq 1)$ – o‘zgarmas sonlar. Bu funksiyaning hosilasini hisoblab, uni quyidagi tarzda yozib olamiz:

$$f'(\lambda)=\frac{p^{\lambda}\ln p\left(q^{\lambda}+1\right)-p^{\lambda}q^{\lambda}\ln q}{\left(q^{\lambda}+1\right)^2}+\frac{q^{\lambda}\ln q\left(p^{\lambda}+1\right)-p^{\lambda}q^{\lambda}\ln p}{\left(p^{\lambda}+1\right)^2}-$$

$$-\frac{p^{\lambda}\ln p+q^{\lambda}\ln q}{\left(p^{\lambda}+q^{\lambda}\right)^2}=(pq)^{\lambda}\ln\frac{p}{q}\left(\frac{1}{\left(q^{\lambda}+1\right)^2}-\frac{1}{\left(p^{\lambda}+1\right)^2}\right)+$$

$$+p^{\lambda}(\ln p)\left(\frac{1}{\left(q^{\lambda}+1\right)^2}-\frac{1}{\left(p^{\lambda}+q^{\lambda}\right)^2}\right)+q^{\lambda}(\ln q)\left(\frac{1}{\left(p^{\lambda}+1\right)^2}-\frac{1}{\left(p^{\lambda}+q^{\lambda}\right)^2}\right).$$

Bu ifodaga muvofiq, $p \geq q \geq 1$ bo‘lgani uchun $f'(\lambda) \geq 0$, ya’ni $f(\lambda)$ funksiya o‘suvchi bo‘ladi. Demak, har qanday λ, μ, $(\lambda \geq \mu \geq 0)$ – nomanfiy sonlar uchun $f(\lambda) \geq f(\mu)$, ya’ni

$$\frac{p^{\lambda}}{q^{\lambda}+1}+\frac{q^{\lambda}}{p^{\lambda}+1}+\frac{1}{p^{\lambda}+q^{\lambda}} \geq \frac{p^{\mu}}{q^{\mu}+1}+\frac{q^{\mu}}{p^{\mu}+1}+\frac{1}{p^{\mu}+q^{\mu}} \qquad (2)$$

tengsizlik o‘rinli. Umumiylikka zid ish qilmagan holda $a \geq b \geq c$ deb olib, $p=\frac{a}{c}$, $q=\frac{b}{c}$ deb tanlasak, u holda (2) tengsizlik quyidagi ko‘rinishga keladi:

$$\frac{\left(\frac{a}{c}\right)^{\lambda}}{\left(\frac{b}{c}\right)^{\lambda}+1}+\frac{\left(\frac{b}{c}\right)^{\lambda}}{\left(\frac{a}{c}\right)^{\lambda}+1}+\frac{1}{\left(\frac{a}{c}\right)^{\lambda}+\left(\frac{b}{c}\right)^{\lambda}} \geq \frac{\left(\frac{a}{c}\right)^{\mu}}{\left(\frac{b}{c}\right)^{\mu}+1}+\frac{\left(\frac{b}{c}\right)^{\mu}}{\left(\frac{a}{c}\right)^{\mu}+1}+\frac{1}{\left(\frac{a}{c}\right)^{\mu}+\left(\frac{b}{c}\right)^{\mu}}.$$

Bu tengsizlikning chap va o‘ng tomonlarida shakl almashtirishlar bajarib, (1) tengsizlikni hosil qilamiz.

Endi (1) tengsizlikni ba’zi xususiy hollarni keltirib o‘tamiz.

1. Agar $\lambda = 1$, $\mu = 0$ bo‘lsa, ushbu

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

tengsizlik hosil bo‘ladi.

2. Agar $\lambda \geq 0$, $\mu = 0$ bo‘lsa, ushbu

$$\frac{a^{\lambda}}{b^{\lambda}+c^{\lambda}} + \frac{b^{\lambda}}{a^{\lambda}+c^{\lambda}} + \frac{c^{\lambda}}{a^{\lambda}+b^{\lambda}} \geq \frac{3}{2}$$

tengsizlik hosil bo‘ladi.

3. Agar $\lambda = 2$, $\mu = 1$ bo‘lsa, ushbu

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2} \geq \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$$

tengsizlik hosil bo‘ladi.

3-masala. Agar $a \geq 0$, $b \geq 0$, $p \geq 1$ bo‘lsa, quyidagi

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2} \qquad (3)$$

tengsizlik o‘rinli bo‘lishini isbotlang.

Isboti. Agar $a = 0$ yoki $b = 0$ bo‘lsa, bu tengsizlikning tenglik sharti bajarilishi, ravshan. Umumiylikka zid ish qilmagan holda faqat $a \geq b > 0$ holatni qarashimiz mumkin va

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2} \Leftrightarrow \left(\frac{\frac{a}{b}+1}{2}\right)^p \leq \frac{\left(\frac{a}{b}\right)^p + 1}{2}$$

ekanini inobatga olamiz. Ushbu funksiyani qaraymiz:

$$f(x) = \frac{x^p + 1}{2} - \left(\frac{x+1}{2}\right)^p, \; x \geq 1, \; p \geq 1.$$

U holda

$$f(0)=0,\ f'(x)=\frac{p}{2}\left(x^{p-1}-\left(\frac{x+1}{2}\right)^{p}\right)\geq 0$$

bo‘ladi. Bu ifodadan $x\geq 1$ ekanini e’tiborga olsak, $f'(x)\geq 0$ bo‘ladi. Shuning uchun $f(x)$ funksiya $[1;+\infty)$ oraliqda o‘suvchi. Natijada

$$f(x)\geq f(1),\ f(x)\geq f(0),$$

ya’ni

$$\frac{x^{p}+1}{2}\geq\left(\frac{x+1}{2}\right)^{p}$$

bo‘ladi. Agar $x=\frac{a}{b}$ qilib tanlasak, (3) tengsizlikka ega bo‘lamiz.

4-masala. R radiusli doirada ichki chizilgan to‘g‘ri to‘rtburchaklardan yuzasi eng kattasini toping.

Yechish. Doiraga $ABCD$ to‘g‘ri to‘rtburchak ichki chizilgan bo‘lib, $AB=x$, $BC=y$ bo‘lsin (3-chizma). Ravshanki,

$$x^2+y^2=4R^2 \text{ va } S_{ABCD}=S=x\cdot y$$

Chizmaga asosan,

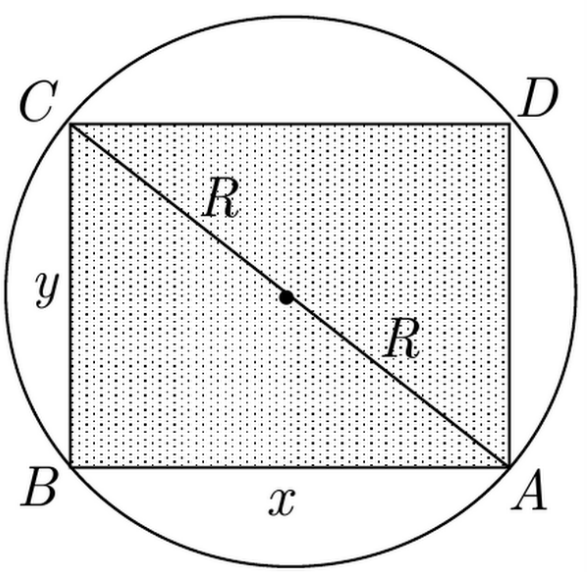
$$S_{ABCD}=x\cdot\sqrt{4R^2-x^2}=S(x),\quad 0\leq x\leq 2R.$$

Uning hosilasi

$$S'(x)=\sqrt{4R^2-x^2}+x\cdot\frac{-2x}{2\sqrt{4R^2-x^2}}=\frac{4R^2-2x^2}{\sqrt{4R^2-x^2}}$$

ga teng. Ushbu funksiya hosilasi qatnashgan tenglamani echamiz:

$$S'(x)=0\ \Rightarrow\ x=\sqrt{2}R.$$



3-chizma

Bundan ko‘rinadiki, funksiya hosilasi $\left(0;\sqrt{2}R\right]$ da musbat, $\left[\sqrt{2}R;2R\right)$ da manfiy bo‘lgani uchun $x=\sqrt{2}R$ nuqtada funksiya o‘zining eng katta qiymatiga ega bo‘ladi:

$$S\left(\sqrt{2}R\right)=\sqrt{2}R\cdot\sqrt{4R^2-\left(\sqrt{2}R\right)^2}=2R^2.$$

Demak, $AB=BC=\sqrt{2}R$, $S_{ABCD}=2R^2$.