LECTURE 2: INTERSECTION OF GEOMETRIC SOLIDS WITH A PLANE

Lecture outline

1. Introduction. Importance and aims of the topic
2. Basic concepts and definitions
3. Specifying a cutting plane in a drawing
4. Sectioning polyhedra by a plane
5. Sectioning solids of revolution by a plane
6. Sectioning by a plane in general position
7. Determining the true size of the section figure
8. Surface development
9. Solving intersection problems in computer graphics
10. Conclusions, review questions and references

Key terms: cutting plane, section figure, section line, projecting plane, collective property, characteristic points, true size, development, conic sections, intersection of an edge with a plane.

GLOSSARY (ГЛОССАРИЙ)

№	O'zbekcha	Русский	English	Mazmuni (Content)
1	Kesuvchi tekislik	Секущая плоскость	Cutting plane	Jismni kesib o'tuvchi tekislik; α, β, γ harflari yoki izlari bilan belgilanadi.
2	Kesim chizig'i	Линия сечения	Section line	Jism sirti bilan kesuvchi tekislikning umumiy nuqtalari to'plami.
3	Kesim figurasi	Фигура сечения	Section figure	Kesim chizig'i bilan chegaralangan tekis shakl.
4	Kesishuv nuqtasi	Точка пересечения	Intersection point	Chiziq (qirra, yasovchi) bilan tekislikning umumiy nuqtasi.
5	Xarakterli (tayanch) nuqtalar	Характерные (опорные) точки	Characteristic (key) points	Kesim chizig'ining o'ziga xos nuqtalari: eng yuqori va eng past, kontur yasovchilardagi, ellips o'qlari uchlaridagi nuqtalar.
6	Oraliq nuqtalar	Промежуточные точки	Intermediate points	Egri chiziqni yetarli aniqlikda yasash uchun olinadigan qo'shimcha nuqtalar.
7	Monj usuli	Метод Монжа	Monge's method	Jismni o'zaro perpendikulyar proyeksiya tekisliklariga ortogonal proyeksiyalash usuli.
8	Frontal proyeksiya (V)	Фронтальная проекция	Frontal projection	Jismning frontal proyeksiya tekisligidagi (oldidan ko'rinish) tasviri.
9	Gorizontal proyeksiya (H)	Горизонтальная проекция	Horizontal projection	Jismning gorizontal tekislikdagi (ustidan ko'rinish) tasviri.
10	Profil proyeksiya (W)	Профильная проекция	Profile projection	Jismning profil tekislikdagi (yon tomondan ko'rinish) tasviri.

1. Introduction

In engineering practice, parts rarely have the shape of a pure geometric solid (prism, cylinder, cone, sphere). More often they are shapes obtained by **cutting such solids with planes**: a

keyway in a bushing, a flat on the end of a shaft, an oblique cut on a pipe, a flat face on a flange, and so on. To produce drawings of such parts, one must be able to construct accurately the line of intersection of a solid's surface with a plane.

In addition, section drawings (sections and cut views) are one of the principal views of mechanical drafting and show the internal structure of a part. In modern CAD systems (AutoCAD, KOMPAS-3D, SolidWorks and others) the "section" commands are based on exactly these geometric principles.

Aim of the lecture: to develop the student's skill in solving problems of sectioning solids by a plane in orthogonal projections, in determining the true size of the section figure, and in understanding the computer algorithms behind this process.

After mastering this lecture the student should know:

the types of cutting planes and their properties in projections;
the algorithm for constructing sections of polyhedra and solids of revolution;
methods for determining the true size of a section figure;
the analytic (computer) solution of the intersection problem.

2. Basic concepts and definitions

Cutting plane — a plane that cuts through a solid; it may be given or auxiliary. It is usually denoted by Greek letters (α, β, γ) or by its traces.

Section line — the set of points common to the surface of the solid and the cutting plane. On a polyhedron it is a broken line; on a solid of revolution it is a curve (or segments of straight lines).

Section figure — the plane figure bounded by the section line.

Points of the section are of two kinds:

characteristic (key) points — the distinctive places on the section line: the highest and lowest points, the extreme left and right points, points on the outline generatrices (limits of visibility), the ends of the axes of an ellipse, the vertices of a broken line (points where edges meet the plane);

intermediate (additional) points — points taken to construct a curve with sufficient accuracy.

General idea of an intersection problem. Any intersection problem is solved as follows: *lines lying on the surface of the solid (edges, generatrices, parallels) are intersected with the plane, the resulting points are connected in order, and visibility (visible and hidden parts) is determined.*

The form of the section depends on the type of solid and the position of the plane (Table 1).

Table 1.

Shapes of sections

Solid	Position of cutting plane	Section figure
Prism	Parallel to the bases	Polygon equal to the base
Prism	Parallel to the lateral edges	Rectangle (parallelogram)
Prism	Inclined	Polygon (number of vertices ≤ number of faces)
Pyramid	Parallel to the base	Polygon similar to the base

Solid	Position of cutting plane	Section figure
Pyramid	Through the apex	Triangle
Cylinder	Parallel to the axis	Rectangle
Cylinder	Perpendicular to the axis	Circle
Cylinder	Oblique to the axis	Ellipse
Cone	Through the apex	Triangle
Cone	Perpendicular to the axis	Circle
Cone	Inclined plane cutting all generatrices	Ellipse
Cone	Parallel to one generatrix	Parabola
Cone	Parallel to two generatrices (axis-parallel)	Hyperbola
Sphere	Any plane	Circle

3. Specifying a cutting plane in a drawing

In Monge's method (orthogonal projections) a spatial solid is projected onto three mutually perpendicular planes: the frontal plane **V (π_2)**, the horizontal plane **H (π_1)** and the profile plane **W (π_3)**. A cutting plane can be specified by:

its traces (α_H, α_V, α_W);

two intersecting or two parallel straight lines;

three points, or a straight line and a point not on it.

3.1. Planes in special positions

a) Projecting planes — planes perpendicular to one of the projection planes:

horizontally projecting (⊥ H): appears as a straight line in the horizontal projection;

frontally projecting (⊥ V): appears as a straight line in the frontal projection;

profile projecting (⊥ W): appears as a straight line in the profile projection.

b) Level planes — planes parallel to one of the projection planes (horizontal, frontal and profile level planes). They are perpendicular to two projection planes and appear in true size in one projection.

3.2. Collective property of a projecting plane

Collective (accumulating) property: the projection of any point, line or figure lying in a projecting plane lies on the collective (single-line) projection of that plane.

This property greatly simplifies intersection problems. If the cutting plane is projecting, **one projection of the section line is already known** — it coincides with the collective trace of the plane. It remains only to find the second projection by treating the points as lying on the surface of the solid.

3.3. Plane in general position

A plane inclined to all the projection planes is called a **plane in general position**. None of its projections has the collective property, so the intersection problem is harder (see Section 6).

4. Sectioning polyhedra by a plane

The faces of a polyhedron (prism, pyramid) are planes, so its section by a plane is a **polygon**. Each side of the polygon is the line of intersection of the cutting plane with a face; each vertex is the point where an edge meets the cutting plane.

Two methods are used to construct the section:

1. **Edge method** — each edge of the solid is intersected with the plane, and the points found are connected in order.
2. **Face method** — each face of the solid is intersected with the plane, and each time the segment of the line of intersection is found.

In practice the **edge method** is usually more convenient, because it reduces to finding points.

4.1. Sectioning a prism by a projecting plane — construction procedure

Given: a right triangular or quadrangular prism and a frontally projecting plane α.

1. The three projections of the solid are drawn and the frontal trace α_V of the plane is marked.
2. The points where α_V crosses the frontal projections of the lateral edges are marked: 1′, 2′, 3′, …
3. By the collective property these points are already given in the frontal projection. Their horizontal projections 1, 2, 3, … are found on the horizontal projections of the lateral edges (or on the collective projections of the lateral faces) using projection lines.
4. The profile projections (1″, 2″, 3″, …) are found by projection lines and coordinate differences.
5. The points are connected in order face by face (only points lying on the same face are joined).
6. Visible parts are drawn with solid lines and hidden parts with dashed lines; the outline of the removed part is restored with thin lines.
7. The section figure is hatched.

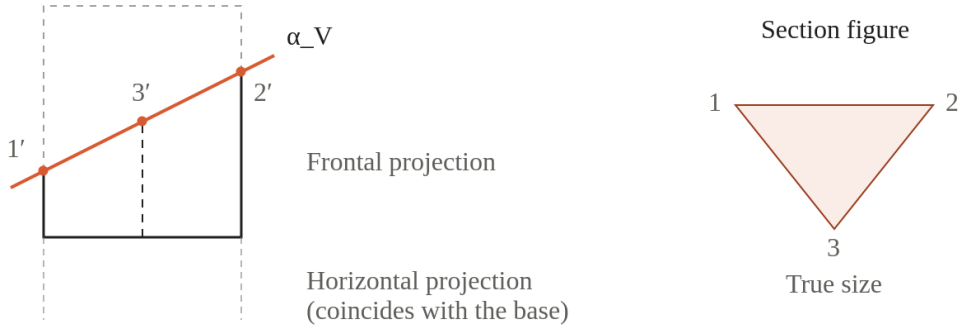


Figure 1. Section of a triangular prism by a frontally projecting plane α

4.2. Sectioning a pyramid

On a pyramid the section points are found by intersecting the lateral edges (or generatrices) with the plane. If an edge does not appear in a collective view, the second projection of a point is found by the **condition of incidence with the edge**: if one projection of a point is known, its second projection lies on the corresponding projection of that edge.

If the cutting plane is parallel to the base of the pyramid, the section is a figure similar to the base, and the similarity ratio equals the ratio of distances from the apex:

$$k = \frac{h_1}{h}$$

where h is the height of the pyramid and h_1 is the distance from the apex to the cutting plane. The area of the section is $S_1 = k^2 S$.

4.3. Characteristic cases

The plane may meet an edge **on its extension**: then the point is found on the extension of the edge, and the part lying within the limits of the face is taken.

The plane may pass through the **apex** of the solid: the section figure is then a triangle.

If the plane is **parallel** to a face, there is no intersection with that face.

5. Sectioning solids of revolution by a plane

The surface of a solid of revolution (cylinder, cone, sphere) is a curved surface, so the section line is usually a curve. To construct it, generatrices or parallels (circles) are drawn on the solid's surface and the points where they meet the plane are found.

5.1. Sectioning a cylinder

Consider a right circular cylinder with a vertical axis (⊥ H) cut by a frontally projecting plane.

The horizontal projection of the cylinder is a circle. Points of the surface lying on it are located on this circle, so the horizontal projection of the section curve coincides with this circle.

In the frontal projection the section is a segment of the straight line α_V.

In the profile projection the section is an ellip**se** (if the plane makes an angle other than 90° with the axis).

Axes of the ellipse. If the plane makes an angle φ with the axis of the cylinder and the radius of the cylinder is R:

semi-minor axis: $b = R$;

semi-major axis: $a = \frac{R}{\sin\varphi}$.

When φ = 90°, $a = b = R$ and the section is a circle; when φ = 45°, $a = R\sqrt{2}$.

Construction procedure (ellipse in the profile projection):

1. The circle in the horizontal projection of the cylinder is divided into equal parts (for example, 12) and the points are numbered 1, 2, 3, …

2. From each point, projection lines are drawn to the frontal projection and intersected with α_V (1′, 2′, 3′, …).
3. In the profile projection, for each point the *height* is taken from the frontal projection and the *depth (y coordinate)* from the horizontal projection.
4. The points 1″, 2″, 3″, … found are joined with a smooth curve using a French curve.
5. The characteristic points (intersections with the outline generatrices and the ends of the major and minor axes of the ellipse) must always be constructed.

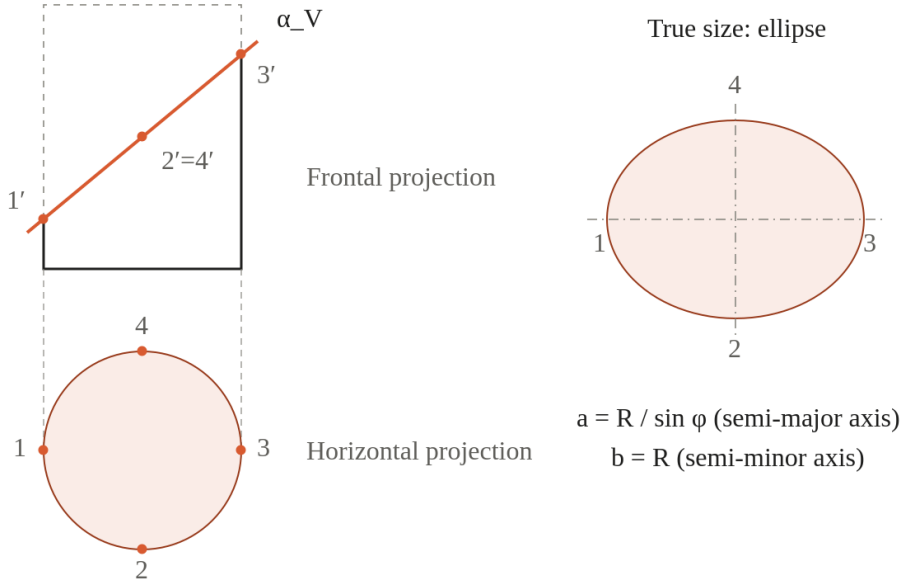


Figure 2. Section of a cylinder by an inclined plane: the section is an ellipse

5.2. Sectioning a cone — conic sections

The curves obtained when a cone is cut by a plane are called **conic sections** (see Table 1). Let α be the angle between the axis of the cone and its generatrix, and φ the angle between the cutting plane and the axis:

Condition	Section
φ = 90°	circle
φ > α (plane cuts all generatrices)	ellipse
φ = α (parallel to one generatrix)	parabola
φ < α (parallel to two generatrices, including parallel to the axis)	hyperbola (two branches)
plane passes through the apex	triangle or straight line / point

There are two ways to find points of the curve on a cone:

Method 1. Generatrix method. The base circle of the cone is divided into equal parts and each point is joined to the apex. The point where each generatrix meets the plane is found.

Method 2. Parallels (auxiliary level planes) method. The cone is cut by horizontal level planes parallel to the base. Each level plane cuts the cone in a circle whose radius is proportional to the distance from the apex. The intersection of this circle with the plane α gives points of the section line.

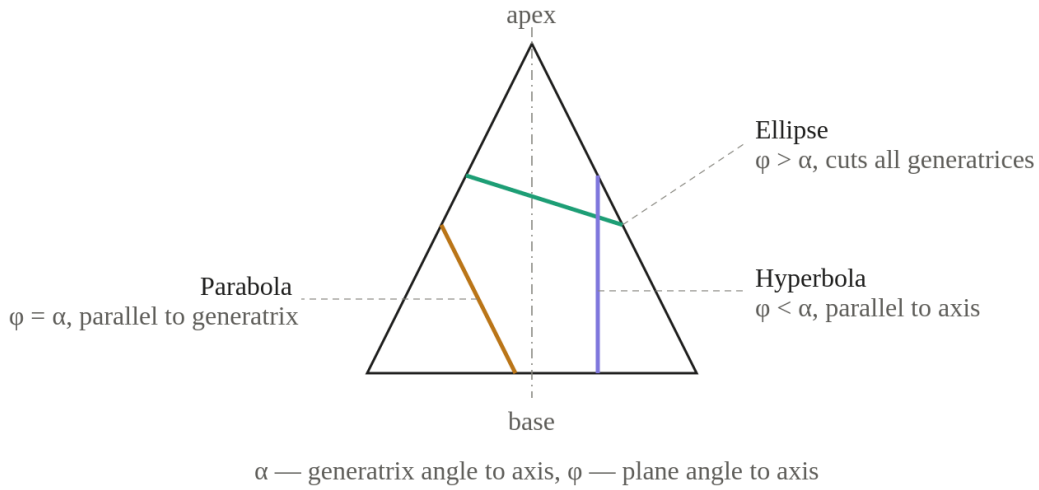


Figure 3. Conic sections: ellipse, parabola and hyperbola

5.3. Sectioning a sphere

For a sphere **every plane section is a circle**. The radius of the section circle is:

$$r = \sqrt{R^2 - d^2}$$

where *R* is the radius of the sphere and *d* is the distance from the center of the sphere to the cutting plane. If the plane is projecting, the circle is a straight-line segment in one projection and an **ellipse** in the other two. The major axis of the ellipse equals the diameter of the circle (2r) and its size does not change; the minor axis is determined by the angle of inclination of the circle's plane to the projection plane.

For construction, the sphere is cut by the parallels method (level circles): the radius of each parallel is taken from the sphere's section at that height.

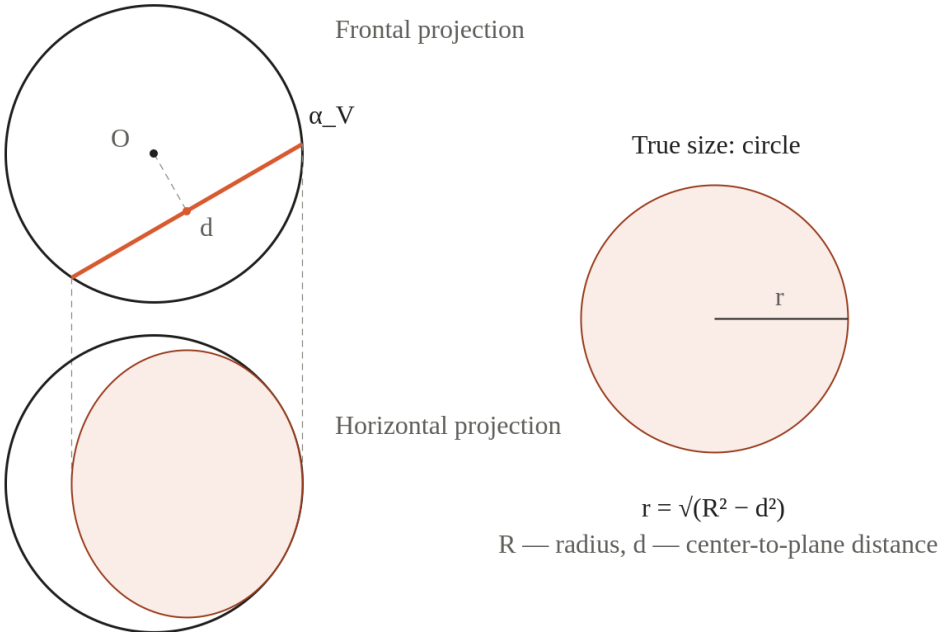


Figure 4. Section of a sphere by a plane: the section is a circle

6. Sectioning by a plane in general position

If the cutting plane is in general position, none of its projections is collective. There are two ways to proceed.

6.1. Auxiliary cutting planes method

The solid and the given plane are cut by **auxiliary level or projecting planes**:

1. An auxiliary plane β is chosen (it should give a simple section of the solid — a circle or a straight line — and cut the given plane α along a simple line; for example, a level plane).
2. The line of intersection (m) of β with the solid is found.
3. The line of intersection (n) of β with α is found — it is a level line of α (a horizontal or a frontal).
4. The points where m and n intersect are the required points of the section.
5. A sufficient number of such auxiliary planes (at least 6–8) is taken and the section line is constructed.

6.2. Change of projection planes method

In this method a new projection plane is chosen so that the cutting plane is **perpendicular** to it, that is, the plane becomes **projecting**. After that, the problem is solved using the collective property, and the result is returned to the original projections by projection lines.

The new axis is drawn perpendicular to a level line of the plane α (for example, its horizontal): $x_1 \perp h'$ (for a frontal level line) or $x_1 \perp h$ (for a horizontal one).

7. Determining the true size of the section figure

When the cutting plane is inclined to the projection planes, the section figure appears distorted (reduced) in the projections. The true size is found by the following methods:

1. **Change of projection planes.** The new plane is taken parallel to the cutting plane. The new axis is drawn parallel to the trace α_V (or α_H). From each point of the figure a perpendicular to the new axis is drawn, and the new coordinates (distances) are measured from the old projection:
   - the figure is then obtained in true size.
2. **Plane-parallel displacement.** The figure is moved into the required position (parallel to a projection plane) without changing its shape relative to the projection plane.
3. **Rotation.** The cutting plane is rotated about its trace until it lies in the projection plane (**revolving into the plane**).

Relation between the area of the projection and the true area:

$$S_{pr} = S \cdot \cos\varphi$$

where S is the true area of the figure and φ is the angle between the cutting plane and the projection plane. This formula can be used as a check: the area of the projection of a section is always less than or equal to the true area.

8. Surface development

Developing the surface of a cut solid onto a plane is important when making articles from sheet material (pipes, air ducts, funnels). The development procedure:

Development of a prism. The lateral surface is a sequence of rectangles. A straight line equal in length to the perimeter of the base is drawn, the length of each edge (face width) is laid off, the heights cut off along the edges are taken from the section points, and these are joined. The true shape of the section figure is added to the development.

Development of a cylinder. The lateral surface of a right circular cylinder develops into a rectangle of length $2\pi R$ and width equal to the height. The circle is divided into n equal parts, the length of the rectangle is also divided into n equal parts, and the length of each generatrix up to the section point is laid off. When the plane cuts the cylinder obliquely, the line on the development is a **sinusoid** (cosine curve).

Development of a cone. The lateral surface of a right circular cone develops into a circular sector. The sector radius is the generatrix length l, and the central angle is:

$$\beta = 360^\circ \cdot \frac{R}{l}$$

where R is the radius of the cone's base. The section points are laid off on the development along the generatrix from the apex; the true length of the generatrix (up to the section point) is determined separately — for example, by rotating the generatrix to the outline generatrix.

9. Solving intersection problems in computer graphics

In computer graphics (CAD, game engines, 3D modeling) the sectioning of a solid by a plane is solved by means of **analytic geometry**. This approach is the algorithmic form of the constructions considered above.

9.1. Equation of a plane

A plane in space is given by the general equation:

$$Ax + By + Cz + D = 0$$

where $\vec{n} = (A, B, C)$ is the **normal vector** of the plane. If the plane is given by three points P_1, P_2, P_3:

$$\vec{n} = (P_2 - P_1) \times (P_3 - P_1), \qquad D = -\vec{n} \cdot P_1$$

9.2. Position of a point relative to a plane

For any point $P = (x, y, z)$ the **signed distance** is:

$$s(P) = \frac{Ax + By + Cz + D}{\sqrt{A^2 + B^2 + C^2}}$$

$s > 0$ — the point is on one side of the plane;

$s < 0$ — on the other side;

$s = 0$ — the point lies in the plane.

This sign is used to determine whether an edge of a polyhedron cuts the plane: if for the ends of the edge $s_1 \cdot s_2 < 0$, the edge cuts the plane.

9.3. Intersection of an edge (segment) with a plane

Let the segment be $P(t) = P_0 + t\,\vec{d},\ t \in [0,1]$, where $\vec{d} = P_1 - P_0$. Substituting into the plane equation gives:

$$t = -\frac{\vec{n} \cdot P_0 + D}{\vec{n} \cdot \vec{d}}$$

if $\vec{n} \cdot \vec{d} = 0$, the segment is parallel to the plane (there is no intersection, or the segment lies in the plane);

if $0 \le t \le 1$, the intersection point $P(t)$ lies within the segment.

9.4. Sectioning algorithm for a polyhedron (pseudocode)

```
input:  set of triangles (mesh), plane (n, D)
output: list of section segments

for each triangle T = (A, B, C):
    sA = s(A); sB = s(B); sC = s(C)
    if all three signs are the same -> skip
    otherwise:
        for each edge (AB, BC, CA):
            if the signs of its ends differ:
                t = -(n·P0 + D) / (n·d)
                point = P0 + t*d     // intersection point
        store the two points found as a segment
join the segments into a closed contour (polygon)
```

This algorithm is used in 3D printing (slicing), in the **SECTION/SLICE** commands of CAD, in SolidWorks *Section View*, and in the visualization of tomographic sections in medicine.

9.5. Intersection with analytic surfaces

To cut a cylinder, cone or sphere by a plane analytically, the parametric expression of a point is substituted into the plane equation. For example, the intersection of the sphere $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$ with a plane is a circle whose center is the projection of the sphere's center onto the plane and whose radius is $r = \sqrt{R^2 - d^2}$ (in agreement with the rule in Section 5.3).

9.6. Practical workflow in CAD systems

1. The 3D model of the solid is created (Extrude, Revolve).
2. The cutting plane is selected (a principal plane or a created auxiliary plane).
3. The "Section/Slice/Cut" command is executed and the required part is removed.
4. Drawing views and section hatching are produced automatically.
5. The true size of the section figure is obtained by looking along the normal to its plane (*Normal To*).

Even so, to understand and check the CAD result one must know the classical construction rules: a wrong setting or the choice of a wrong plane in the program can be detected only by geometric analysis.

10. Conclusions

1. The intersection of a solid with a plane produces a section line and a section figure; they are constructed by finding the points where lines on the surface of the solid meet the plane.
2. When the plane is projecting, the collective property simplifies the problem: one projection of the section line is known in advance.
3. The section of a polyhedron is a polygon whose vertices are the points where the edges meet the plane.
4. On solids of revolution the section of a cylinder or cone may be a straight line, circle, ellipse, parabola or hyperbola; the section of a sphere is always a circle.
5. The problem for a plane in general position is solved by auxiliary planes or by the change of projection planes.
6. The true size of the section is determined by changing projection planes, by rotation or by revolving into the plane.
7. In computer graphics these problems are solved with the plane equation, the signed distance and a parametric segment.

Common mistakes

omitting characteristic points (on outline generatrices, at the ends of axes);

drawing hidden parts with solid lines;

leaving the section line as a broken line between points (it should be a smooth curve);

not restoring the outlines of the removed part;

forgetting to hatch the section figure.

Review questions

1. What are a cutting plane, a section line and a section figure?

2. Explain the projecting plane and its collective property.
3. State the procedure for sectioning a prism by a frontally projecting plane.
4. What is the difference between characteristic and intermediate points?
5. What figure is obtained when a cylinder is cut by an inclined plane? How are its semi-axes found?
6. Conic sections: under what conditions are an ellipse, a parabola and a hyperbola obtained?
7. Why is every plane section of a sphere a circle? Write the formula for its radius.
8. How is the intersection problem solved for a plane in general position?
9. List the methods for determining the true size of a section figure.
10. Write the formula for the intersection of a segment with a plane; what does $\vec{n} \cdot \vec{d} = 0$ mean?

Recommended references

1. Gordon V.O., Semencov-Ogievskiy M.A. *A Course of Descriptive Geometry*.
2. Textbooks and manuals on "Descriptive Geometry" and "Engineering Graphics" specified in the course syllabus.
3. Official user guides of AutoCAD, KOMPAS-3D and SolidWorks (section and cut-view commands).

MAIN CONCLUSIONS

1. **Fundamental concept:** The intersection of a geometric solid with a cutting plane produces a section whose shape depends on both the type of solid and the position/orientation of the cutting plane.
2. **Polyhedra:** Sections of prisms, pyramids, and other polyhedra are polygons. Their vertices lie on the solid's edges, and their sides lie on the solid's faces. The construction requires finding where each intersected edge meets the plane and connecting the points in correct order.
3. **Solids of revolution:** Sections are usually conic sections or other curves:

- Cylinder → circle, ellipse, rectangle, or parallel lines.
- Cone → circle, ellipse, parabola, hyperbola, triangle, point, or line.
- Sphere → always a circle in true shape; appears as an ellipse in oblique projection.
- More complex surfaces require point-by-point construction using auxiliary cutting planes.

4. **General-position planes:** An oblique cutting plane does not appear as an edge in any principal view. It must be converted to an edge view by an auxiliary projection before intersection points can be found directly.

Independent work topics

1. **Classification of plane sections of polyhedra** — Conditions for obtaining a triangle, quadrilateral, pentagon, or hexagon when cutting a prism or pyramid.
2. **Conic sections as plane cuts of a cone** — Circle, ellipse, parabola, and hyperbola; relationship between cutting-plane angle and conic type.
3. **Plane sections of a sphere** — Proof that every plane section is a circle, and how it appears in orthographic projection.
4. **General-position cutting plane** — Comparison of the trace method and the auxiliary projection method.
5. **True shape by change of projection planes** — Mathematical basis and step-by-step drawing procedure.

Information and methodological support:

1. **Gordon, V. O., Sementsov, O. G., & Ogievskii, K. N.** — *A Course in Descriptive Geometry* (Mir Publishers, 1980). A comprehensive course covering orthogonal and axonometric projections, intersection of polygonal surfaces by a plane, curved surfaces, and surface intersection. Contains test questions and historical notes.
2. **Woolf, S.** — *An Elementary Course in Descriptive Geometry* (J. Wiley & Sons, 1893). Includes problems on intersecting a surface of revolution by a given plane, requirements for plane intersections, and development of ruled surfaces.
3. **Low, D. A.** — *Text-book on Practical Solid Or Descriptive Geometry*, Vol. 2 (Longmans, Green, 1899). Covers sections of solids, projection of plane figures, curved surfaces and tangent planes, developments, and intersection of surfaces.
4. **Paré, E. G., Loving, R. O., Hill, I. L., & Pare, R. C.** — *Descriptive Geometry* (6th ed., Macmillan, 1982). A standard modern reference with graded problems on sections and developments.
5. **Hawk, M. C.** — *Schaum's Outline of Theory and Problems of Descriptive Geometry* (1962). Useful for independent problem-solving practice.

SELF-CONTROL TESTS:

Test samples

1. The figure obtained by the intersection of a cutting plane with a solid is called:
a) a projection
b) a section
c) a development
d) an auxiliary view
2. A cutting plane appears as a straight line in a view when it is:
a) parallel to the projection plane
b) perpendicular to the projection plane (edge view)
c) oblique to all principal planes
d) coincident with the axis of the solid
3. Which of the following is NOT a principal projection plane?
a) Horizontal plane H*H*
b) Vertical plane V*V*
c) Profile plane W*W*
d) Tangent plane T*T*
4. The boundary of a section figure is called:
a) the section contour
b) the trace
c) the edge view
d) the development line
5. A cutting plane may be specified by:
a) three points not in a straight line
b) two intersecting lines
c) a line and a point
d) all of the above