LECTURE 4.

PLANAR FORCE SYSTEMS AND THEIR EQUILIBRIUM CONDITIONS. PARALLEL FORCES IN A PLANE

Subject: Theoretical Mechanics (Statics)

Outline:

1. Moment of a force about a point. Varignon’s theorem.
2. Couples and their properties.
3. Parallel transfer of a force.
4. Reduction of a planar force system to a given centre.
5. Equilibrium conditions of a planar force system.
6. Parallel forces in a plane and their resultant.
7. Distributed loads.
8. Support reactions of beams (example).
9. Applications in occupational safety.

GLOSSARY (ГЛОССАРИЙ)

Uzbek	Russian	English	Meaning
Kuch	Сила	Force	A measure of the mechanical action of bodies on one another; a vector defined by magnitude, direction and point of application.
Kuchlar sistemasi	Система сил	Force system	The set of forces acting on one body.
Kesishuvchi kuchlar	Сходящиеся силы	Concurrent forces	Forces whose lines of action meet at one point.
Teng ta’sir etuvchi	Равнодействующая	Resultant	A single force that replaces a given force system with the same effect.
Muvozanat	Равновесие	Equilibrium	A state of rest or uniform straight-line motion of a body; R = 0.
Bog‘lanish	Связь	Constraint	A body that restricts the displacement of another body.
Bog‘lanish reaksiyasi	Реакция связи	Constraint reaction	The force with which a constraint acts on the body.
Kuch momenti	Момент силы	Moment of a force	The product of the force magnitude and its arm, $M_O(F) = \pm F \cdot h$; a measure of the force’s turning effect.
Kuch yelkasi	Плечо силы	Moment arm	The shortest distance from the moment centre to the line of action of the force.

Uzbek	Russian	English	Meaning
Varinon teoremasi	Теорема Вариньона	Varignon’s theorem	The moment of the resultant equals the algebraic sum of the moments of the component forces.
Juft kuch	Пара сил	Couple	Two forces equal in magnitude, parallel and opposite in direction; it has no resultant.
Juft yelkasi	Плечо пары	Couple arm	The shortest distance d between the lines of action of the couple’s forces.
Juft momenti	Момент пары	Moment of a couple	The product of the force magnitude and the couple arm: $M = \pm F \cdot d$.
Puanso lemmasi	Лемма Пуансо	Poinsot’s lemma	Moving a force parallel to itself requires adding an additional couple.
Keltirish markazi	Центр приведения	Reduction centre	The chosen point to which the forces of a system are transferred.
Bosh vektor	Главный вектор	Principal vector	The geometric sum of all forces of the system, $R' = \Sigma F_i$; independent of the centre.
Bosh moment	Главный момент	Principal moment	The sum of the moments of all forces about the reduction centre, $M_O = \Sigma M_O(F_i)$.
Parallel kuchlar	Параллельные силы	Parallel forces	Forces whose lines of action are parallel.
Parallel kuchlar markazi	Центр параллельных сил	Centre of parallel forces	The point fixing the line of action of the resultant: $x_C = \Sigma F_i x_i / \Sigma F_i$.

1. Moment of a force about a point. Varignon’s theorem

Concurrent forces only tend to translate a body. Forces whose lines of action do not meet at one point also tend to **rotate** it. To measure this rotating effect, the **moment of a force** is introduced.

The **moment of a force about point O** (the moment centre) is the product of the force magnitude and the shortest distance from that point to the line of action of the force (the moment arm h). The moment is a signed quantity (Fig. 1):

$$M_O(F) = \pm F \cdot h$$

The moment is positive (+) if the force tends to rotate the body about O counterclockwise, and negative (–) if clockwise. The unit is N·m or kN·m. The moment is zero when F = 0 or when h = 0, i.e. when the line of action passes through O. Sliding a force along its line of action does not change its moment.

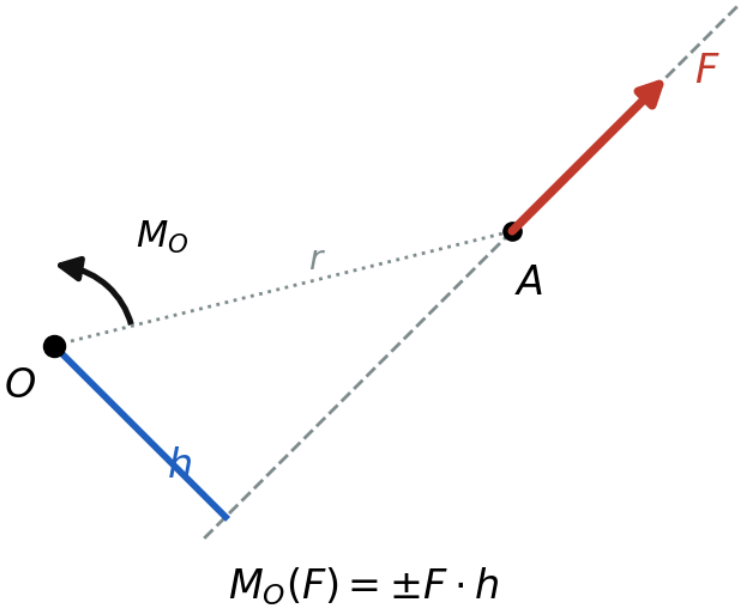


Fig. 1. Moment of a force about a point

Varignon's theorem: the moment of the resultant of a system of concurrent coplanar forces about any point equals the algebraic sum of the moments of the component forces about that point:

$$M_O(R) = \Sigma M_O(F_i)$$

This is useful when the arm of a force is hard to find: the force is split into two components and their moments are added. If the force is applied at point A(x, y), then $M_O(F) = x \cdot F_y - y \cdot F_x$.

2. Couples and their properties

A system of two forces that are equal in magnitude, parallel and opposite in direction is called a **couple**. A couple only rotates a body and has no resultant. The shortest distance d between the lines of action is the **couple arm** (Fig. 2). The **moment of a couple** is $M = \pm F \cdot d$ and does not depend on the position of the moment centre.

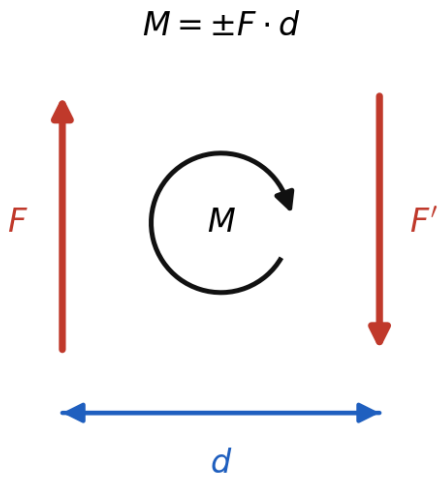


Fig. 2. A couple and its moment

A couple may be moved and rotated anywhere in its plane without changing its effect.

F and d may be changed as long as the product M = F·d is preserved.

Couples in one plane are equivalent to a **single couple** whose moment is the algebraic sum of the moments: $M = \Sigma M_i$.

Couples are in equilibrium if and only if the sum of their moments is zero: $\Sigma M_i = 0$.

3. Parallel transfer of a force (Poinsot's lemma)

To move a force F applied at point A to another point O off its line of action, parallel to itself, one must apply at O a force F′ equal and parallel to F, and add an **additional couple** whose moment equals the moment of the original force about the new point: $M = M_O(F)$. This lemma is the basis for simplifying any planar force system.

4. Reduction of a planar force system to a centre

If all forces $F_1, F_2, \ldots, F_n$ of a planar system are transferred to a chosen point O (the **reduction centre**) using Poinsot's lemma, a concurrent force system at O and a system of additional couples are obtained (Fig. 3). Adding them gives two quantities:

The principal vector $R' = \Sigma F_i$ – the geometric sum of all forces; it does not depend on the centre. Its projections: $R'_x = \Sigma F_{ix}$, $R'_y = \Sigma F_{iy}$; magnitude $R' = \sqrt{(R'^2_x + R'^2_y)}$.

The principal moment $M_O = \Sigma M_O(F_i)$ – the sum of the moments of all forces about O; it depends on the choice of centre.

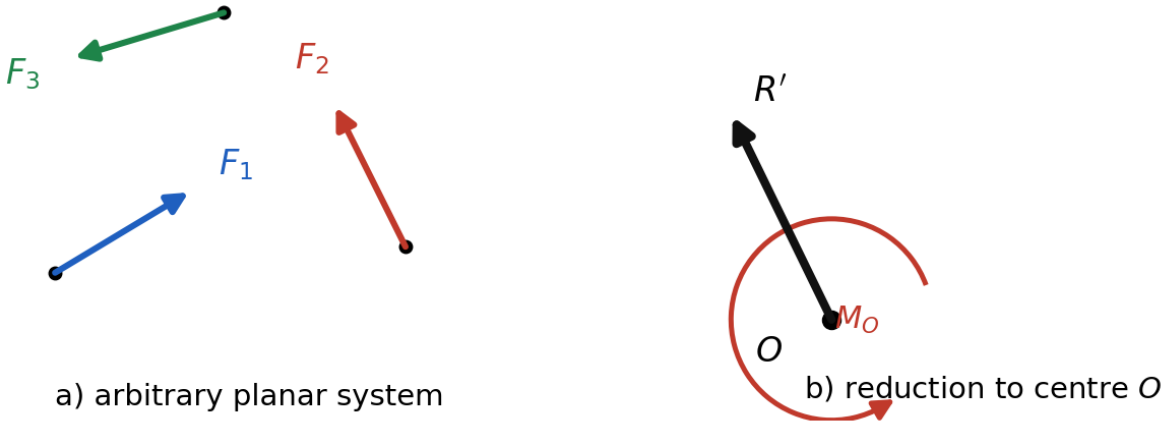


Fig. 3. Reducing a planar force system to centre O: principal vector R' and principal moment M_O

The possible results of reduction are:

Principal vector	Principal moment	Final form of the system
$R' = 0$	$M_O = 0$	Equilibrium
$R' = 0$	$M_O \neq 0$	A single couple ($M = M_O$)
$R' \neq 0$	$M_O = 0$	Resultant $R = R'$ through point O
$R' \neq 0$	$M_O \neq 0$	Resultant $R = R'$ at distance $d = \lvert M_O \rvert / R'$ from O

5. Equilibrium conditions of a planar force system

A system is in equilibrium if and only if both the principal vector and the principal moment are zero: $R' = 0$, $M_O = 0$. This gives the **first (basic) form of the equilibrium equations**:

$$\Sigma F_{ix} = 0, \quad \Sigma F_{iy} = 0, \quad \Sigma M_O(F_i) = 0$$

That is, the sums of the projections of all forces on two axes and of their moments about any point are zero. Two other forms exist:

- **Second form** (one projection and two moments): $\Sigma F_{ix} = 0$, $\Sigma M_A(F_i) = 0$, $\Sigma M_B(F_i) = 0$. Condition: line AB must not be perpendicular to the x-axis.
- **Third form** (three moments): $\Sigma M_A(F_i) = 0$, $\Sigma M_B(F_i) = 0$, $\Sigma M_C(F_i) = 0$. Condition: points A, B, C must not lie on one straight line.

A planar problem has three equations, so at most **three** unknowns can be found; with more unknowns the problem is statically indeterminate. To simplify the calculation, take the moment centre at the point where the lines of action of unknown

forces intersect: their moments vanish and only one unknown remains in the equation.

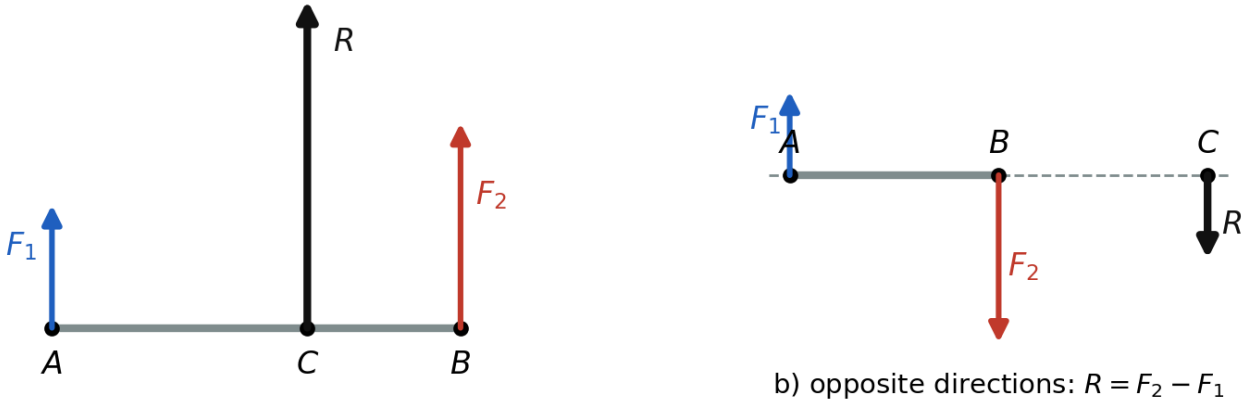
6. Parallel forces in a plane and their resultant

Forces whose lines of action are parallel form a **parallel force system**, a special case of the general planar system. If the y-axis is chosen parallel to the forces, $\Sigma F_{ix} \equiv 0$ and only two equations remain.

a) Two parallel forces in the same direction (Fig. 4a). The resultant is parallel to them, has the same direction and magnitude $R = F_1 + F_2$. It divides segment AB internally into parts **inversely proportional** to the forces:

$$F_1/BC = F_2/AC = R/AB, \quad i.e. \quad F_1 \cdot AC = F_2 \cdot BC$$

b) Two unequal parallel forces in opposite directions (Fig. 4b). The resultant has the direction of the larger force and magnitude $R = F_2 - F_1$ ($F_2 > F_1$); its line of action lies **outside** the segment, on the side of the larger force, and satisfies $F_1 \cdot AC = F_2 \cdot BC$. If the forces are equal in magnitude, they form a couple and have no resultant.



a) same direction: $R = F_1 + F_2$

$F_1 \cdot AC = F_2 \cdot BC$

b) opposite directions: $R = F_2 - F_1$

Fig. 4. Adding parallel forces: a) same direction, b) opposite directions

c) Many parallel forces. The resultant is $R = \Sigma F_i$ (algebraic sum, signs according to direction). Its line of action is found with Varignon's theorem: if the forces are at distances x_i from O, $R \cdot x_C = \Sigma F_i x_i$, so the coordinate of the **centre of parallel forces** is:

$$x_C = \Sigma F_i \cdot x_i / \Sigma F_i$$

The **equilibrium equations** of a parallel force system (forces parallel to the y-axis) are two:

$$\Sigma F_{iy} = 0, \quad \Sigma M_O(F_i) = 0$$

or two moment equations $\Sigma M_A(F_i) = 0$, $\Sigma M_B(F_i) = 0$ (AB must not be parallel to the forces). Hence at most **two** unknowns can be found for parallel forces.

7. Distributed loads

In practice loads are often spread along a length rather than concentrated: self-weight of a structure, snow load, weight of boards or stored materials. A distributed load is described by its **intensity** q (kN/m). In calculations it is replaced by its **resultant** Q (Fig. 5):

Uniform load (q = const): Q = q·l, applied at the middle of the beam, l/2 from the end.

Triangular load (from 0 to q_{max}): Q = ½·q_{max}·l, applied at a distance l/3 from the end with the larger intensity.



a) uniform load: $Q = ql$ at midspan

b) triangular: Q at $l/3$ from larger end

Fig. 5. Distributed loads and their resultants

8. Support reactions of a beam

Three basic supports are used for beams: a **roller (movable hinge) support** (reaction perpendicular to the supporting surface, one unknown); a **pin (fixed hinge) support** (reaction of unknown direction, given by two components); and a **fixed (clamped) support** (two reaction components and a reaction moment, three unknowns). A beam on two supports (one pin, one roller) has three unknowns, equal to the number of equations, so the problem is **statically determinate**.

Example. Beam AB of length l = 6 m rests on a pin support at A and a roller support at B. A vertical force P = 20 kN acts 2 m from A, and a uniform load q = 4 kN/m acts over the whole length (Fig. 6). Find the support reactions. Neglect the beam's weight.

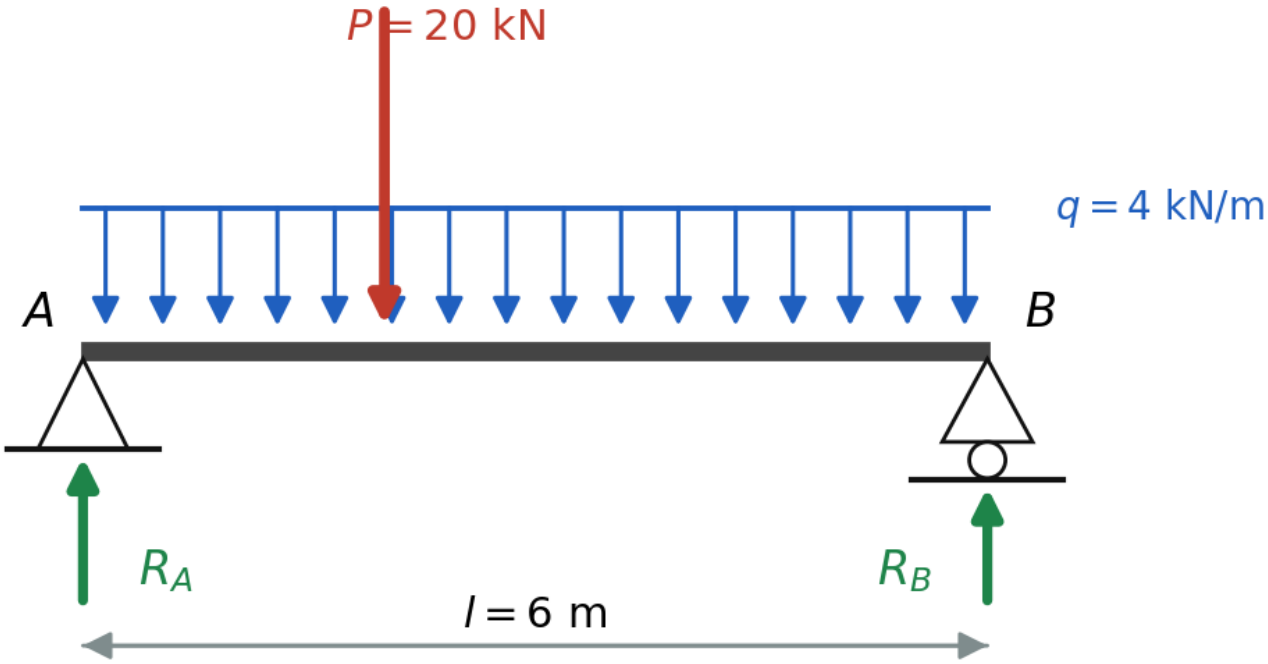


Fig. 6. Beam diagram and support reactions

Solution. Replace the distributed load by its resultant: Q = q·l = 4·6 = 24 kN, acting 3 m from A. Since all loads are vertical, the horizontal reaction at A is $X_A = 0$; the vertical reactions are R_A and R_B. Equilibrium equations:

$$\Sigma M_A = 0: \quad R_B \cdot 6 - P \cdot 2 - Q \cdot 3 = 0$$

$$\Sigma F_{iy} = 0: \quad R_A + R_B - P - Q = 0$$

From the first equation $R_B = (20 \cdot 2 + 24 \cdot 3)/6 = 112/6 \approx$ **18.67 kN**. From the second $R_A = P + Q - R_B = 44 - 18.67 \approx$ **25.33 kN**. **Check:** $\Sigma M_B = 0$ gives $-R_A \cdot 6 + P \cdot 4 + Q \cdot 3 = -152.0 + 80 + 72 = 0$. The result is correct.

9. Applications in occupational safety

The theory of planar and parallel force systems is used in: assessing the **stability** of lifting cranes against overturning; calculating loads on the supports of scaffolds, suspended cradles and working platforms; locating the centre of gravity of a load on lifting beams and spreader bars; and designing wall-mounted brackets, walkways, viewing platforms and unloading ramps.

For example, the stability factor of a crane about the tipping edge A is $k = M_{stab}/M_{over}$, where M_{stab} is the restoring moment of the crane's own weight and counterweight, and M_{over} is the overturning moment of the lifted load and forces such as wind. Regulations require k to be not less than a permitted value (usually about 1.4 with load); the exact value must be checked against current standards. When boards

under a distributed load rest on supports, remember that the support reactions are not equal ($R_A \neq R_B$ in the example of Section 8): the closer the load is to a support, the larger the force on it.

Key safety rules: place loads near the middle of a platform or beam; do not pile loads at the end of a cantilever; calculate the overturning moment in advance and provide a counterweight; never exceed rated loads; check the strength of supports.

Conclusion

An arbitrary planar force system is characterized by the principal vector R' and the principal moment M_O; equilibrium requires $R' = 0$ and $M_O = 0$, i.e. $\Sigma F_{ix} = 0$, $\Sigma F_{iy} = 0$, $\Sigma M_O = 0$. For parallel forces two equations suffice. Distributed loads are replaced by resultants, and support reactions follow from the equations of statics. This knowledge is necessary for an occupational safety specialist when assessing the safe operation of structures and lifting equipment.

Review questions

1. What is the moment of a force about a point? How is its sign determined?
2. State Varignon’s theorem.
3. What is a couple and what are its properties?
4. Explain Poinsot’s lemma.
5. What are the principal vector and principal moment? What are the possible results of reduction?
6. Write the three forms of the equilibrium equations of a planar force system.
7. How is the resultant of parallel forces found?
8. Where is the resultant of a uniform and of a triangular load applied?

References

1. Targ S.M. A Short Course of Theoretical Mechanics (Statics).
2. Butenin N.V., Lunts Ya.L., Merkin D.R. Course of Theoretical Mechanics, Vol. 1: Statics and Kinematics.
3. Yablonsky A.A., Nikiforova V.M. Course of Theoretical Mechanics, Part 1: Statics.
4. Current regulatory documents on the safety of lifting equipment.

TOPICS FOR INDEPENDENT WORK

The student chooses one of the topics below and prepares a report of 8–12 pages: a theoretical part, drawings and at least one worked calculation example.

1. Moment of a force and Varignon’s theorem in occupational safety (wrench, lever, warehouse racks).

2. Calculating the stability of lifting cranes against overturning.
3. Determining support reactions of scaffolds and suspended cradles.
4. Centre of parallel forces and the centre of gravity of a load; lifting spreader beams.
5. Distributed loads (snow, wind, self-weight) and calculation of their resultants.
6. Cantilever structures (canopy, balcony, loading platform): calculation.
7. Types of supports and their reactions in practical structures.
8. Statically determinate and indeterminate systems: the difference and its effect on safety.
9. Graphical method of equilibrium for planar force systems.
10. Sling angle and cable tension: rules for safe lifting.

SELF-CHECK QUESTIONS

TEST QUESTIONS

1. A force F = 20 kN has an arm h = 0.5 m about point O. If it tends to rotate counterclockwise, what is the moment?

A) –10 kN·m
B) +40 kN·m
C) +10 kN·m
D) +20.5 kN·m

2. A couple has F = 15 kN and arm d = 0.4 m. What is its moment?

A) 37.5 kN·m
B) 6 kN·m
C) 15.4 kN·m
D) 0 (the moment of a couple is zero)

3. Which is the basic form of the equilibrium equations of an arbitrary planar force system?

A) $\Sigma F_{ix} = 0$, $\Sigma F_{iy} = 0$ only
B) $\Sigma M_A = 0$, $\Sigma M_B = 0$ only
C) $\Sigma F_{ix} + \Sigma F_{iy} = 0$
D) $\Sigma F_{ix} = 0$, $\Sigma F_{iy} = 0$, $\Sigma M_O = 0$

4. Parallel forces F_1 = 30 kN and F_2 = 60 kN in the same direction act at points A and B, AB = 0.9 m. What are the resultant and its distance from A?

A) R = 90 kN, 0.6 m from A
B) R = 30 kN, 0.6 m from A
C) R = 90 kN, 0.3 m from A
D) R = 90 kN, 0.45 m from A

5. A triangular load has length l = 9 m and maximum intensity q_{max} = 6 kN/m. What are the resultant Q and its point of application?

A) Q = 54 kN, at midspan

B) Q = 27 kN, 6 m from the end of maximum intensity

C) Q = 27 kN, 3 m from the end of maximum intensity

D) Q = 54 kN, 3 m from the end of maximum intensity

Answer key: 1 – C; 2 – B; 3 – D; 4 – A; 5 – C.

Notes: 1) M = 20·0.5 = 10 kN·m. 2) M = 15·0.4 = 6 kN·m. 4) R = 30 + 60 = 90 kN; from $F_1 \cdot AC = F_2 \cdot BC$ and AC + BC = 0.9 m, AC = 0.6 m. 5) Q = ½·6·9 = 27 kN, at l/3 = 3 m from the end of maximum intensity.