O'ZBEKISTON RESPUBLIKASI OLIY VA
O'RTA MAXSUS TA'LIM VAZIRLIGI
GULISTON DAVLAT UNIVERSITETI

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VARIATSION HISOB VA OPTIMALLASHTIRISH USULLARI

fanidan

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(Uslubiy qo‘llanma)

Matematika, amaliy matematika va informatika bakalavriat yo‘nalishlarida ta’lim oluvchilar uchun

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Qo'llanma variatsion hisobning asosiy masalasida birinchi va ikkinchi variatsiyalarni tekshirish, variatsion hisob asosiy masalasining ba'zi umumlashmalari, optimal boshqaruv masalalari mavzularini o'z ichiga oladi. Amaliy topshiriqlar bir necha masalalardan iborat va har bir masalaning oxirgi 21-nomeri batafsil yechib ko'rsatilgan va talaba uchun ko'rsatma rolini bajaradi.

Qo'llanma 4-kursning 7-8 semestrlarida ajratilgan soatlarda bajarilishi lozim bo'lgan amaliy mashg'ulot mavzularini qamrab olgan.

Qo'llanmadan bakalavriatning matematika, amaliy matematika va informatika yo'nalishida o'qitiladigan «Variatsion hisob va optimallashtirish usullari» fani bo'yicha amaliy mashg'ulotlarda ham foydalanish mumkin.

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KIRISH

Uslubiy qoʻllanma “Variatsion hisob va optimallashtirish usullari fanidan oʻqish jarayonining 7-8 semestrida ajratilgan soatlarda oʻtiladigan amaliy mashgʻulot mavzularini qamrab olgan.

1-paragraf variatsion hisobning asosiy masalasida birinchi variatsiyani tekshirishga bagʻishlangan boʻlib, bunda kuchli va kuchsiz ekstremumning zaruriy shartlari, Eyler tenglamasi va uning xususiy hollariga doir misollar qaraladi.

2-paragraf variatsion hisobning asosiy masalasida ikkinchi variatsiyani tekshirishga bagʻishlangan. Bunda ekstremumning zaruriy shartlari, Lejandr sharti, Yakobi sharti va ekstremumning yetarli shartlarini tekshirishga doir misollar qaralgan.

3-paragraf variatsion hisob asosiy masalasining ba’zi umumlashmalariga bagʻishlangan boʻlib, unda Eyler tenglamalari sistemasi, Eyler – Puasson tenglamasi, Eyler - Ostrogradskiy tenglamasi kabilarga doir misollar qaralgan.

4-paragrafda optimal boshqaruv masalalari qaralgan. Bunda Pontryagin-Gamilton funksiyasi, terminal boshqaruv masalasi uchun Pontryaginning maksimum printsipiga doir misollar qaralgan.

Qoʻllanmaning har bir paragrafida mavzuga oid asosiy ta’rif va xossalar (teoremalar) toʻla keltirilgan boʻlib, amaliy mashgʻulotda bajariladigan ishning oʻzi uch qismdan iborat. Birinchi qismda mavzuga oid nazariy savollar, ikkinchi qismda nazariy (muammoli) topshiriqlar, uchinchi, oxirgi qismda esa, amaliy topshiriqlar berilgan. Amaliy topshiriqlar bir necha masalalardan iborat va ularning har biridan bittadan namunaviy masala yechib koʻrsatilgan.

1§. Variatsion hisobning asosiy masalasi. Birinchi variatsiyani tekshirish.

1. Variatsion hisob va uning predmeti.

Funksionallar, odatda, cheksiz oʻlchovli funksional fazolarda berilgan boʻladi. Ularning eng katta (maksimal) va eng kichik (minimal) qiymatlarini topish haqidagi masalalar cheksiz oʻlchovli ekstremal masalalar boʻlib, bunday masalalarni oʻrganish variatsion hisob predmetini tashkil etadi.

XVII asrning oxiridan XX asr oʻrtalarigacha boʻlgan davr klassik variatsion hisobning paydo boʻlishi va rivojlanishini oʻz ichiga oladi. Bu davrda dastlabki fundamental tadqiqotlar L.Eyler va J.Lagranj tomonidan bajarildi. XVIII asrning oxirlarida Eyler, Lagranj va Lejandrlar ilmiy tadqiqotlari natijasida variatsion hisob birinchi variatsiyani tekshirish qismi boʻyicha tugallangan shaklga ega boʻldi.

XIX asrda esa, avval ma’lum boʻlgan variatsion masalalarni umumlashtirish boshlandi va variatsion hisobning tadbiqlari boʻyicha natijalar olindi (M.I.Ostrogradskiy tomonidan 1834 yilda karrali integralli variatsion masalalar uchun zaruriy shartlar olindi, variatsion hisobning mexanikaga tatbiqlari qaraldi).

XIX asrning ikkinchi yarmida funksionallar ekstremumlarining yetarli shartlari olindi (K.Veyershtrass tomonidan, 1879 yilda).

XX asrda variatsion hisobning toʻgʻri usullari yuzaga keldi. Ular variatsion masalalarni taqribiy yechish uchun, hamda ularda yechimning mavjudligini isbotlash uchun juda muhimdir.

XX asrning boshlarida matematikada yangi yoʻnalish – funksional analiz yuzaga keldi va aniq tabiatshunoslikning turli sohalarida, jumladan, kvant mexanikasida keng qo‘llanila boshlandi. Variatsion hisob «chiziqli bo‘lmagan» funksional analizning tarkibiy qismiga aylandi.

XX asrnig ikkinchi yarmiga kelib optimal boshqaruvning matematik nazariyasiga asos solinishi va uning jadal rivojlanishi variatsion hisob taraqqiyotida yangi davrni boshlab berdi. Bu yangi yo‘nalishda akademik L.S.Pontryaginning «maksimum prinsipi» va R. Bellmanning dinamik programmalashtirish usuli asosiy natijalar hisoblanadi.

1. Variatsion hisob asosiy masalasining qo‘yilishi.

Quyidagilar berilgan bo‘lsin:

a) $Q-R^3$ dagi biror ochiq to‘plam (soha);

b) $S=\{(x,y)\in R^2:\ (x,y,z)\in Q\}-Q$ to‘plamning R^2 ga proyeksiyasi;

v) $P_0(x_0,y_0),\ P_1(x_1,y_1)-S$ to‘plamning belgilangan nuqtalari, $x_0<x_1$;

g) $F(x,y,z):\ Q\to R^1$ – uzluksiz funksiya.

$C^1[x_0,x_1]$ fazoning

$$V=\{y(x)\in C^1[x_0,x_1]:\ y(x_0)=y_0,\ \ y(x_1)=y_1,(x,y(x),\ y'(x))\in Q,x\in[x_0,x_1]\} \qquad (1)$$

to‘plamida aniqlangan

$$J[y]=\int_{x_0}^{x_1}F(x,y,y')dx \qquad (2)$$

funksionalning ekstremumini topish masalasini qaraymiz. Bu masalaga *variatsion hisobning asosiy masalasi* deyiladi va u

$$J[y]=\int_{x_0}^{x_1}F(x,y,y')dx\to\min(\max),\ y(x_0)=y_0,\ \ y(x_1)=y_1,\ y(x)\in C^1[x_0,x_1] \qquad (3)$$

ko‘rinishda belgilanadi. (1) ko‘rinishdagi to‘plamga (3) masalaning joiz funksiyalari (chiziqlari) to‘plami deyiladi. (3) masalada joiz chiziqlarning uchlari berilgan P_0 va P_1 nuqtalarda mahkamlangan, ya'ni qo'zg'olmasdir.

Variatsion hisobning asosiy masalasi - chegaralari qo'zg'olmas eng sodda variatsion masaladir.

2. Variatsion hisob asosiy masalasi funksionalining birinchi variatsiyasi.

Funksionalning birinchi variatsiyasi ta'rifini berishdan oldin chiziqli funksional tushunchasini eslatib o'tamiz.

Agar biror chiziqli W fazoda aniqlangan *J[u]* funksional bir jinsli va additiv bo'lsa, ya'ni:

1) $J[cu] = cJ[u],\ \forall u \in W,$ c-ixtiyoriy o'zgarmas;

2) $J[u_1 + u_2] = J[u_1] + J[u_2],\ \forall u_1, u_2 \in W;$

shartlar bajarilsa, *J[u]*-chiziqli funksional deyiladi. Masalan, agar *p(x)* va *q(x)* lar $[x_0, x_1]$ kesmada uzluksiz funksiyalar bo'lsa,

$$J[y] = \int_{x_0}^{x_1} [p(x)y(x) + q(x)y'(x)]dx$$

tenglik bilan aniqlanuvchi *J[y]* funksional $W = C^1[x_0, x_1]$ da chiziqli funksional bo'ladi.

1-ta'rif. Agar *J[y]* funksional W chiziqli normalangan fazoda berilgan bo'lsa,

$$\Delta J = J[y+h] - J[y],\quad h \in W$$

ayirmaga *J[y]* funksionalning orttirmasi deyiladi.

2-ta'rif. Agar W chiziqli normalangan fazoda berilgan *J[y]* funksionalning ΔJ orttirmasi uchun

$$J[y+h] - J[y] = L[y,h] + \beta[y,h] \qquad (4)$$

yoyilma o‘rinli bo‘lib, bunda $L[y,h]$- h ga nisbatan chiziqli funksional, $\beta[y,h]$ esa $\|h\| \to 0$ da $\beta[y,h]/\|h\| \to 0$ munosabatni qanoatlantirsa , *J[y]* funksional $y \in W$ nuqtada differensiallanuvchi, yoki birinchi variatsiyaga ega deyiladi.

(4) yoyilmaning bosh qismidan iborat *L[y,h]* ga esa, *J[y]* funksionalning birinchi variatsiyasi deyiladi va u $\delta J = \delta J[y,h]$ kabi belgilanadi: $\delta J = \delta J[y,h]$.

Keltirilgan ta’rif bo‘yicha variatsiyaga ega funksionallarga adabiyotda *Freshe ma’nosida (yoki kuchli ma’noda) differensiallanuvchi* funksionallar ham deyiladi.

3-ta’rif. *J[y]* funksionalning $y \in W$ nuqtada Lagranj bo‘yicha birinchi variatsiyasi deb, $\varphi(\alpha) = J[y+\alpha h]$ funksiyaning $\alpha = 0$ nuqtadagi hosilasiga aytiladi:

$$\varphi J = \varphi'(0) = \frac{d}{d\alpha} J[y+\alpha h]\Big|_{\alpha=0}.$$

Variatsion hisob asosiy masalasi uchun birinchi variatsiya quyidagi ko‘rinishda bo‘ladi:

$$\delta J = \varphi'(\alpha)\Big| = \int_{x_0}^{x_1} \frac{d}{d\alpha} F(x, y(x) + \alpha h(x),\ y(x) + \alpha h'(x))dx\Big|_{\alpha=0} =$$

$$= \int_{x_0}^{x_1} \left[F_y(x, y(x), y'(x))h(x) + F_{y'}(x, y(x), y'(x))h'(x)\right]dx.$$

3. Variatsion hisob asosiy masalasida ekstremumning zaruriy sharti.

Cheksiz o‘lchovli W fazoning biror V to‘plamida aniqlangan *J[y]* funksional berilgan bo‘lsin.

4-ta'rif. Agar ixtiyoriy $y \in V$ uchun $J[y^*] \le J[y]$ $(J[y^*] \ge J[y])$ tengsizlik bajarilsa, $y^* \in V$ nuqta *J[y]* funksionalning V to'plamdagi global minimum (maksimum) nuqtasi, $J[y^*]$ esa, funksionalning minimal (maksimal) qiymati deyiladi:

$$J[y^*] = \min_{y \in V} J[y] \quad (J[y^*] = \max_{y \in V} J[y]).$$

Funksionalning minimum va maksimum nuqtalarini umumiy nom bilan ekstremum nuqtalari deb ataymiz.

Masalan, $W = C[0,1]$ fazoda aniqlangan

$$J[y] = \int_{x_2}^{x_1} [1 - y(x)]^2 dx$$

funksionalning global minimum nuqtasi $y^*(x) \equiv 1$, $x \in [0,1]$ funksiyadan iborat, chunki

$$J[y] \ge 0 = J[y^*], \quad \forall y \in C[0,1].$$

Endi W – chiziqli normalangan fazo, *J[y]* funksional $V \subset W$ to'plamda aniqlangan bo'lsin, deb faraz qilamiz.

5-ta'rif. *C[0,1]* fazodan olingan ikkita $y_1 = y_1(x)$ va $y_2 = y_2(x)$ funksiyalar orasidagi nolinchi tartibli masofa deb,

$$P_0(y_1, y_2) = \max_{a \le x \le b} |y_1(x) - y_2(x)|$$

tenglik bilan aniqlanadigan ρ_0 songa aytiladi. Demak, ikkita funksiya orasidagi nolinchi tartibli masofa – ular ayirmasining normasiga tengdir.

Nolinchi tartibli metrikaga asoslangan holda,

$$V_0(y^0, \varepsilon) = \{y \in C[0,1] : \rho_0(y_0, y) < \varepsilon\}$$

tenglik bilan, markazi $y^0 \in C[0,1]$ elementda bo'lgan nolinchi tartibli ε atrofni qarash mumkin.

6-ta'rif. Ikkita, $y_1 = y_1(x)$ va $y_2 = y_2(x)$ funksiyalar orasidagi birinchi tartibli masofa deb, quyidagi

$$p_1(y_1, y_2) = \max_{a \le x \le b} |y_2(x) - y_1(x)| + \max_{a \le x \le b} |y_2^{/}(x) - y_1^{/}(x)|$$

tenglik bilan aniqlanadigan r_1 songa aytiladi.

Birinchi tartibli masofa tushunchasi orqali birinchi tartibli ε- atrof ushbu

$$V_1(y^0, \varepsilon) = \{y \in C[0,1] : \rho_1(y_0, y) < \varepsilon\}$$

tenglik yordamida kiritiladi, bunda y^0 – atrofning markazi.

7-ta’rif. $y^0 = y^0(x)$ – joiz funksiya bo‘lsin $(y^0 \in V)$. Agar y^0 ning shunday $V_0(y^0, \varepsilon)$ nolinchi tartibli ε – atrofi mavjud bo‘lib, shu atrofga tegishli barcha $y = y(x)$ joiz funksiyalar uchun

$$J[y^0] \le J[y] \quad (J[y^0] \ge J[y])$$

munosabat bajarilsa, $y^0(x)$ funksiya (2) funksionalning kuchli lokal minimum (maksimum) nuqtasi deyiladi.

8-ta’rif. Agar $y^0 = y^0(x)$ joiz funktsiyaning shunday $V_1(y^0, \varepsilon)$ birinchi tartibli ε - atrofi mavjud bo‘lsaki,

$J[y^0] \le J[y] \quad (J[y^0] \ge J[y]) \quad \forall y \in V_1(y^0, \varepsilon) \cap V$ munosabat bajarilsa, $y^0(x)$ funksiya - (2) funksionalning kuchsiz lokal minimum (maksimum) nuqtasi deyiladi.

(2) funktsionalning kuchli (kuchsiz) lokal ekstremum nuqtalariga variatsion hisob asosiy masalasida kuchli (kuchsiz) ekstremallar ham deyiladi.

Yuqorida keltirilgan ta’riflardan funksionalning global ekstremumi uning lokal ekstremumi ham bo‘lishi kelib chiqadi. Bu tasdiqning aksinchasi esa, o‘rinli emas.

$J[y]$ funksionalning cheksiz o‘lchovli W fazoning V qism to‘plamidagi minimumini (yoki maksimumini) topish haqidagi masala cheksiz o‘lchovli ekstremal masaladir. Bu masalani variatsion masala dyb ataymiz va

$$J[y]\to \min\ (\max),\quad y\in V$$

yoki

$$J[y]\to extr,\qquad y\in V \qquad (5)$$

ko‘rinishda belgilaymiz.

Odatda V to‘plam funksiyalar (yoki ularning geometrik talqini sifatida chiziqlar, sirtlar) to‘plamidan iborat bo‘ladi.

Shuning uchun, (5) ekstremal masalada V to‘plamning elementlarini joiz funksiyalar (chiziqlar, sirtlar) deb ataymiz.

Chiziqli normalangan W fazoning biror V to‘plamida aniqlangan *J[y]* funksional berilgan bo‘lsin (V=W bo‘lishi ham mumkin). V – chiziqli qism fazo, yoki biror $y_0\in V$ uchun qurilgan $M(y_0)=\{h\in W: y+h\in V\}$ to‘plam chiziqli qism fazodan iborat bo‘lsin.

Shu farazlarda (5) masalada ekstremumning birinchi tartibli zaruriy sharti quyidagi teoremada ifodalangan.

<u>**1-teorema.**</u> Agar $y_0\in V$ nuqta *J[y]* funksionalning lokal minimum (maksimum) nuqtasi bo‘lsa, u holda shu nuqtada hisoblangan birinchi variatsiya nolga teng bo‘ladi, ya’ni

$$\delta J=0. \qquad (6)$$

Bu teoremaga ko‘ra, agar y_0- variatsion hisobning asosiy masalasida kuchsiz minimum nuqtasi bo‘lsa, u holda,

$$\delta J(y_0,h)=\int_{x_0}^{x_1}[F_y(x,y^0(x),y^0{}'(x))h(x)+F_{y'}(x,y^0(x),y^0{}'(x))h'(x)]dx=0 \qquad (7)$$

$$u\in C^{(1)}[x_0,x_1],h(x_0)=h(x_1)=0$$

bo‘ladi.

5. Eyler tenglamasi.

$F(x,y,y')$ funksiyaning xususiy hosilalari uchun quyidagi belgilashlardan foydalanamiz:

$$F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_{y'} = \frac{\partial F}{\partial y'},$$

$$F_{xy'} = \frac{\partial^2 F}{\partial x \partial y'}, \quad F_{yy} = \frac{\partial^2 F}{\partial y^2}, \quad F_{yy'} = \frac{\partial^2 F}{\partial y \partial y'}, \quad F_{y'y'} = \frac{\partial^2 F}{\partial y'^2}$$

(7) dan quyidagi teorema kelib chiqadi.

2-teorema. $F(x,y,y') \in C^{(1)}(Q)$ bo‘lsin. Agar $y^0(x) \in C^{(1)}[x_0, x_1]$ joiz funksiya (3) masalada kuchsiz ekstremal bo‘lsa, u

$$F_y(x,y,y') - \frac{d}{dx} F_y(x,y,y') = 0 \qquad (8)$$

tenglamani qanoatlantiradi. (8) tenglamaga Eyler tenglamasi deyiladi.

$F(x,y,y') \in C^{(2)}(Q)$ bo‘lganda, (8) differensial tenglama,

$$F_{y'y'}y'' + F_{yy'}y' + F_{xy'} - F_y = 0 \qquad (9)$$

ko‘rinishni oladi.

9-ta’rif. Eyler tenglamasini qanoatlantiruvchi $y = y(x)$ joiz funksiyalarga (2) funksionalning joiz statsionar funksiyalari deyiladi.

6. Eyler tenglamasining xususiy hollari.

$F(x,y,y')$ integrant, o‘z argumentlarining «to‘liq» funksiyasi bo‘lmagan hollarda Eyler tenglamasini soddalashtirish yoki uning birinchi integralini aniqlash mumkin.

a) F funksiya faqat y' ga boђliq, ya’ni $F = F(y')$ bo‘lsin. Bu holda Eyler tenglamasi, $\frac{d}{dx} F_{y'}(y') = 0$, yoki $F_{y''} = 0$ ko‘rinishda bo‘ladi. Buyerdan $y'' = 0$ yoki $F_{y''} = 0$ bo‘ladi. Agar $y'' = 0$ bo‘lsa, $y = C_1 x + C_2$ - ikki paramyetrli to‘g‘ri chiziqlar oilasiga ega bo‘lamiz. Agar $F_{y'y'}(y) = 0$ tenglama bir yoki bir necha $y'=k_i$ ildizga ega bo‘lsa, $y=k_i x+C$ - bir parametrli to‘g‘ri chiziqlar oilasiga ega bo‘lamiz. Shunday qilib, $F=F(y')$ bo‘lganda, Eyler tenglamasining yechimlari $y=C_1x+C_2$ chiziqli funksiyalardan iboratdir.

b) $F=F(x,y')$, ya’ni F funksiya faqat x va y' ga bog‘liq bo‘lsin. Bu holda $F_y=0$ bo‘ladi va Eyler tenglamasi $\frac{d}{dx}F_{y'}=0$ ko‘rinishni oladi. Bu yerdan uning $F_y(x,y')=C$ birinchi intyegraliga ega bo‘lamiz. Bu olingan tenglama Eyler tenglamasiga teng kuchlidir. Uni y' ga nisbatan yechish yoki biror parametr kiritish yo‘li bilan integrallash mumkin.

v) F funksiya faqat y va y' ga bog‘liq bo‘lsin: $F=F(y,y')$. Bu holda $F\in C^{(2)}$ deb faraz qilib, (9) Eyler tenglamasini yozamiz: $F_{y'y'}y''+F_{y'y}y'-F_y=0$ $(F_{xy'}=0)$. Agar bu tenglamaning ikkala tomonini y'ga ko‘paytirsak, uni $\frac{d}{dx}(F-y'F_{y'})=0$ ko‘rinishda yozish mumkin. Natijada Eyler tenglamasi,

$$F-y'F_{y'}=C \qquad (10)$$

ko‘rinishdagi birinchi integralga ega bo‘ladi. Eyler tenglamasining har qanday yechimi (10) tenglamani qanoatlantirishi ravshan. Aksincha, agar (10) tenglama faqat chekli sondagi nuqtalarda hosilasi nolga teng bo‘lgan $y=y[x]\in C^{(2)}[x_0,x_1]$ yechimga ega bo‘lsa, bu yechim Eyler tyenglamasini ham qanoatlantiradi. (10) tenglamani y' ga nisbatan yechish yoki parametr kiritish yo‘li bilan integrallash mumkin.

II.Nazariy savollar.

1. Funksionalning ta’rifi. Chiziqli funksional.
2. Funksional fazolar. Misollar.
3. Variatsion hisob asosiy masalasining qo‘yilishi.
4. Funksionalning birinchi variatsiyasi.
5. Funksionalning kuchli va kuchsiz ekstremallari.
6. Kuchsiz ekstremumning birinchi tartibli zaruriy sharti.
7. Variatsion hisob asosiy masalasida kuchsiz ekstremumning zaruriy sharti.
8. Eyler tenglamasi.
9. Eyler tenglamasining xususiy hollari.

III. Amaliy topshiriqlar.

1 – masala. Quyidagi funksionalning birinchi variatsiyasini toping.

1. $J(y)=\int_0^1 (y^2-y'^2+5yx^2)dx$ 2. $J(y)=\int_{-1}^1 (y^2-4yy'+y'^2e^{2x})dx$

3. $J(y)=\int_0^\pi (y'^2-4y^2+3y'Sinx)dx$ 4. $J(y)=\int_0^1 (y^2+3y'^2+yx^3)dx$

5. $J(y)=\int_{-1}^1 (3y^2+2yy'-y'^2e^x)dx$ 6. $J(y)=\int_0^\pi (2y'^2+y^2+y'Sin3x)dx$

7. $J(y)=\int_0^1 (2y^2-4y'^2+yx^4)dx$ 8. $J(y)=\int_{-1}^1 (y^2-5yy'-y'^2e^{3x})dx$

9. $J(y)=\int_0^\pi (y'^2+3y^2+y'Sin2x)dx$ 10. $J(y)=\int_0^1 (y^2+7y'^2-2yx^2)dx$

11. $J(y)=\int_{-1}^1 (y^2+2yy'+5y'^2e^x)dx$ 12. $J(y)=\int_0^\pi (2y'^2-6y^2-y'Sinx)dx$

13. $J(y)=\int_0^1 (y^2+2y'^2+3yx^3)dx$ 14. $J(y)=\int_{-1}^1 (5y^2-3yy'+y'^2e^{2x})dx$

15. $J(y)=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(y'^2-4y^2+y'\cos 2x)dx$ 16. $J(y)=\int\limits_0^1(3y^2-y'^2+5yx^4)dx$

17. $J(y)=\int\limits_{-1}^1(y^2+4yy'+3y'^2e^{3x})dx$ 18. $J(y)=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(y'^2+2y^2-5y'\cos x)dx$

19. $J(y)=\int\limits_0^1(y^2-2y'^2+6yx^2)dx$ 20. $J(y)=\int\limits_{-1}^1(4y^2+2yy'+y'^2e^x)dx$

21. $J(y)=\int\limits_0^1(y^2-2y'^2+yx^2)dx$ funksionalning birinchi variatsiyasini toping.

Yechilishi: $\delta J(y,h)=\int\limits_a^b[F_y h(x)+F_{y'}h'(x)]dx$ birinchi variatsiyani topish uchun integral ostidagi $F=y^2-2y'^2+yx^2$ funksiyaning F_y va $F_{y'}$ xususiy hosilalarini topamiz. $F_y=2y+x^2$, $F_{y'}=-4y'$ bo‘ladi. Natijada berilgan funksionalning birinchi variatsiyasi quyidagi ko‘rinishda bo‘ladi:

$$\delta J(y,h)=\int\limits_0^1[(2y+x^2)h(x)-4y'h'(x)]dx\,.$$

2 – masala. Quyidagi variatsion hisob asosiy masalasida Eyler tenglamasini tuzing.

1. $J(y)=\int\limits_0^2(y'^2-yy'+9y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

2. $J(y)=\int\limits_{-1}^1\left(y'^2-9y^2+2yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

3. $J(y)=\int\limits_0^1\left(y'^3-5ye^x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

4. $J(y)=\int\limits_0^2(y'^2+3yy'+4y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

5. $J(y)=\int\limits_{-1}^1\left(y'^2-y^2+6yx^3\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

6. $J(y)=\int_{0}^{1}\left(y'^3-ye^{2x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

7. $J(y)=\int_{0}^{2}(4y'^2+5yy'-y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

8. $J(y)=\int_{-1}^{1}\left(y'^2-4y^2+3yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

9. $J(y)=\int_{0}^{1}\left(y'^3+6ye^{x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

10. $J(y)=\int_{0}^{2}(y'^2-2yy'-4y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

11. $J(y)=\int_{-1}^{1}\left(y'^2+2y^2+3yx^3\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

12. $J(y)=\int_{0}^{1}\left(y'^3+3ye^{2x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

13. $J(y)=\int_{0}^{2}(y'^2-3yy'+8y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

14. $J(y)=\int_{-1}^{1}\left(y'^2-6y^2+yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

15. $J(y)=\int_{0}^{1}\left(y'^3-4ye^{x}+x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

16. $J(y)=\int_{0}^{2}(y'^2+4yy'-y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

17. $J(y)=\int_{-1}^{1}\left(y'^2+y^2-4yx^3\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

18. $J(y)=\int_{0}^{1}\left(y'^3+ye^{x}+2x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

19. $J(y)=\int_{0}^{2}(y'^2-6yy'+3xy^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

20. $J(y)=\int_{-1}^{1}\left(y'^2-2xy^2+3y\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

21. $J(y)=\int_0^2 (y'^2-5yy'+4y^2)dx \to \min, y(0)=0, y(2)=1, y\in C^{(1)}[0;2]$ masala uchun Eyler tenglamasini tuzing.

Yechilishi: Eyler tenglamasi $F_y-\frac{d}{dx}F_{y'}=0$ ni tuzish uchun F_y, $F_{y'}$ $\frac{d}{dx}F_{y'}$ hosilalarni topamiz. Integral ostidagi funksiya $F=y'^2-5yy'+4y^2$ ga teng. Bu funksiyadan y bo‘yicha xususiy hosila olsak, $F_y=-5y'+8y$ ga teng bo‘ladi. Endi integral ostidagi funksiyadan y' bo‘yicha xususiy hosila olib, $F_{y'}=2y'-5y$ ga ega bo‘lamiz. Bu funksiyadan x bo‘yicha to‘la hosila olib, $\frac{d}{dx}F_{y'}=2y''-5y'$ ga ega bo‘lamiz. Yuqoridagi tenglamaga qo‘yib,

$$-5y'+8y-2y''+5y'=0,\ \ 8y-2y''=0,\quad y''-4y=0$$

Eyler tenglamasiga kelamiz.

<u>**3 – masala.**</u> Quyidagi variatsion hisob asosiy masalasida joiz statsionar funksiyani toping.

1. $J(y)=\int_0^2 \left(y'^2-8xyy'\right)dx \to \min;\ y(0)=0, y(2)=1, y\in C^{(1)}[0;2]$

2. $J(y)=\int_0^{\pi} \left(y'^2+2y'Cosx\right)dx \to \min;\ y(0)=1, y(\pi)=0, y\in C^{(1)}[0;\pi]$

3. $J(y)=\int_0^1 \left(y'^2+12xy\right)dx \to \min;\ y(0)=0, y(1)=2, y\in C^{(1)}[0;1]$

4. $J(y)=\int_0^{\frac{\pi}{\sqrt{2}}} \left(y'^2+4xyy'\right)dx \to \min;\ y(0)=0, y(\frac{\pi}{\sqrt{2}})=1, y\in C^{(1)}\left[0;\frac{\pi}{\sqrt{2}}\right]$

5. $J(y)=\int_0^{\frac{\pi}{2}} \left(y'^2-y'Cos2x\right)dx \to \min;\ y(0)=0, y(\frac{\pi}{2})=0, y\in C^{(1)}\left[0;\frac{\pi}{2}\right]$

6. $J(y)=\int_0^1 \left(y'^2-6xy'\right)dx \to \min;\ y(0)=0, y(1)=2, y\in C^{(1)}[0;1]$

7. $J(y)=\int\limits_{0}^{\frac{\pi}{4}}\left(y'^2+2xyy'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{4})=1,\ y\in C^{(1)}\left[0;\frac{\pi}{4}\right]$

8. $J(y)=\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(y'^2-y'\sin x\right)dx\to\min;\ y(\frac{\pi}{4})=1,\ y(\frac{\pi}{2})=0,\ y\in C^{(1)}\left[\frac{\pi}{4};\frac{\pi}{2}\right]$

9. $J(y)=\int\limits_{0}^{1}\left(y'^2+3x^2y\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

10. $J(y)=\int\limits_{0}^{2}\left(y'^2-xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

11. $J(y)=\int\limits_{\frac{\pi}{2}}^{\pi}\left(y'^2-4y'\sin 2x\right)dx\to\min;\ y(\frac{\pi}{2})=1,\ y(\pi)=0,\ y\in C^{(1)}\left[\frac{\pi}{2};\pi\right]$

12. $J(y)=\int\limits_{0}^{1}\left(y'^2-2x^2y'\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

13. $J(y)=\int\limits_{0}^{2}\left(y'^2-18xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

14. $J(y)=\int\limits_{0}^{1}\left(y'^2-3y'Cosx\right)dx\to\min;\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

15. $J(y)=\int\limits_{0}^{1}\left(y'^2-7x^3y\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

16. $J(y)=\int\limits_{0}^{\sqrt{2}\pi}\left(y'^2+16xyy'\right)dx\to\min;\ y(0)=0,\ y(\sqrt{2}\pi)=0,\ y\in C^{(1)}[0;\sqrt{2}\pi]$

17. $J(y)=\int\limits_{-1}^{0}\left(y'^2-5y'Cos2x\right)dx\to\min;\ y(-1)=0,\ y(0)=1,\ y\in C^{(1)}[-1;0]$

18. $J(y)=\int\limits_{0}^{1}\left(y'^2+2x^3y'\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

19. $J(y)=\int\limits_{0}^{2}\left(y'^2+3x^2y'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

20. $J(y)=\int\limits_{0}^{1}\left(y'^2+y'\sin x+yy'\right)dx\to\min;\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[1;3]$

21. $J(y)=\int_0^{\frac{\pi}{4}}\left(y'^2+8xyy'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{4})=1,\ y\in C^{(1)}\left[0;\frac{\pi}{4}\right]$ masalada joiz statsionar funksiyani toping.

Yechilishi: Statsionar funksiyani topish uchun oldin $F_y-\frac{d}{dx}F_{y'}=0$ Eyler tenglamasini tuzib, uning umumiy yechimini topamiz. Integral ostidagi funksiya $F=y'^2+8xyy'$ ga teng. Bu funksiyadan y boʻyicha xususiy hosila olsak, u $F_y=8xy'$ boʻladi. Endi integral ostidagi funksiyadan y' boʻyicha xususiy hosila olib, $F_{y'}=2y'+8xy$ ga ega boʻlamiz. $F_{y'}=2y'+8xy$ funksiyadan x boʻyicha toʻla hosila olib, $\frac{d}{dx}F_{y'}=2y''+8xy'+8y$ ni hosil qilamiz. Bu funksiyalarni yuqoridagi tenglamaga qoʻyib,

$$8xy'-2y''-8xy'-8y=0,\qquad y''+4y=0$$

Eyler tenglamasiga kelamiz. Bu tenglamaning umumiy yechimini $y=e^{\lambda x}$ koʻrinishda izlaymiz. $y''=\lambda^2e^{\lambda x}$ boʻlgani uchun, Eyler tenglamasidan,

$$\lambda^2e^{\lambda x}+4e^{\lambda x}=0\qquad \text{yoki}\quad \lambda^2+4=0$$

xarakteristik tenglamaga kelamiz. Uning yechimi $\lambda_{1,2}=\pm 2i$ boʻladi. U holda Eyler tenglamasining umumiy yechimi

$$y(x)=c_1\sin 2x+c_2\cos 2x$$

boʻladi. Endi joiz statsionar funksiyani topish uchun Eyler tenglamasining yuqoridagi chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini izlaymiz.

$$y(0)=c_2=0,\quad y(\frac{\pi}{4})=c_1=1$$

boʻlgani uchun, $y^0(x)=\sin 2x$ joiz statsionar funksiya boʻladi.

2§. Variatsion hisobning asosiy masalasi. Ikkinchi variatsiyani tekshirish

1. Variatsion hisob asosiy masalasida funksionalning ikkinchi variatsiyasini hisoblash

W – chiziqli normalangan fazo, $J=J[u,v]$ funksional har bir oʻzgaruvchisi boʻyicha chiziqli boʻlsin. Agar $u=v$ deb olsak, hosil boʻlgan $J[u,u]$ funksionalga kvadratik funksional deyiladi. Masalan, agar $a(x)$-$[x_0,x_1]$ oraliqda aniqlangan uzluksiz funksiya boʻlsa,

$$J[u,v]=\int_{x_0}^{1} a(x)\,u(x)\,v(x)\,dx$$

funksional $W=C[x_0,x_1]$ fazoda har bir $u=u(x)$ *va* $v=v(x)$ elementlar boʻyicha chiziqli funksionaldir. Bu yerda $u=v$ deb olib, $C[x_0,x_1]$ da aniqlangan

$$J[u,u]=\int_{x_0}^{x_1} a(x)u^2(x)\,dx$$

kvadratik funksionalga ega boʻlamiz.

1**-ta'rif.** *W* chiziqli normalangan fazoning y elementi va uning ixtiyoriy $h\in W$ elementi uchun funksionalning ΔJ orttirmasi

$$J[y+h]-J[y]=L_1[y,h]+\frac{1}{2}L_2[y,h]+\beta_1(y,h) \qquad (1)$$

koʻrinishdagi yoyilmaga ega boʻlsin, bu yerda $L_1[y,h]-h$ ga nisbatan chiziqli funksional, $L_2[y,h]$ esa h ga nisbatan kvadratik funksional, $\beta_1(y,h)/\|h\|^2\to 0,\ \|h\|\to 0.$ U holda $J[y]$ funksional $y\in W$ nuqtada ikkinchi variatsiyaga ega deyiladi. h ga nisbatan kvadratik funksional $L_2[y, h$ esa, $J[y]$ funksionalning Freshe boʻyicha ikkinchi variatsiyasi deyiladi hamda bu variatsiya $\delta^2 J=\delta^2 J[y,h]$ kabi belgilanadi: $\delta^2 J=L_2[y,h]$.

W chiziqli normalangan fazoning biror V to'plamida aniqlangan *J[y]* funksional berilgan bo'lsin. V to'plam, yoki $M(y)=\{h\in W: y+h\in V\}$ to'plam Wning chiziqli qism fazosi bo'lsin.

2-**ta'rif.** $\varphi(\alpha)=J(y+\alpha h)$ funksiyaning $\alpha=0$ nuqtada ikkinchi tartibli hosilasiga *J[y]* funksionalning *Lagranj bo'yicha ikkinchi variatsiyasi* deyiladi:

$$\delta^2 J=\varphi'(0)=\frac{d^2}{d\alpha^2}J[y+\alpha h]\Big|_{\alpha=0}.$$

1-teorema. Agar $y_0\in V$ nuqta *J[y]* funksionalning kuchsiz lokal minimali (maksimali) bo'lsa, u holda shu nuqtada hisoblangan ikkinchi variatsiya manfiymas (musbatmas) bo'ladi:

$$\delta^2 J\ge 0\quad (\delta^2 J\le 0).$$

2-**teorema.** Agar *J[y]* funksional $y_0\in V$ nuqtada birinchi va ikkinchi variatsiyalarga ega bo'lib, ular

$$\delta J=0,\quad \delta^2 J\ge \alpha h^2\quad (\delta^2 J\le -\alpha h^2),\quad \forall h\in V \qquad (2)$$

(bu yerda α>0 – biror o'zgarmas) shartlarni qanoatlantirsa, u_0 – lokal minimum (lokal maksimum) nuqtasi bo'ladi.

2. Lejandr sharti (ikkinchi tartibli zaruriy shart).

$y^0=y^0(x)$ joiz funksiya bo'lsin ($y^0\in$V). Shu nuqtada (1) funksionalning ikkinchi variatsiyasini hisoblaymiz. Ta'rifga ko'ra, bu variatsiya

$$\delta^2 J[y^2,h]=\frac{d^2J[y^0+\alpha h]}{d\alpha^2}\Big|_{\alpha=0}$$

formula bo'yicha hisoblanadi, bu yerda

$$h=h(x)\in C^{(1)}[x_0,x_1],\qquad h(x_0)=h(x_1)=0.$$

Agar $F(x,y,y')\in C^{(2)}(Q)$ deb faraz qilsak, $\varphi(\alpha)=J[y^0+\alpha h]$ funksiya $\alpha=0$ nuqta atrofida uzluksiz ikkinchi tartibli hosilaga ega. Demak,

$$\delta^2 J[y^0,h]=\frac{d^2}{d\alpha^2}\int_{x_0}^{x_1}F(x,y^0(x)+\alpha h(x),y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0}=$$

$$=\int_{x_0}^{x_1}\frac{d^2}{d\alpha^2}F(x,y^0(x)+\alpha h(x),y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0}=$$

$$=\int_{x_0}^{x_1}[F_{yy}(x,y^0(x),y^{0'}(x)h^2(x)+2F_{yy}(x,y^0(x),y^{0'}(x)h(x)h'(x)+$$

$$+F_{y'y}(x,y^0(x),y^{0'}(x)h(x)h'(x)]dx,$$

$$h=h(x)\in C^{(1)}[x_0,x_1],\ h(x_0)=h(x_1)=0\quad (3)$$

3-**teorema(Lejandr).** $F(x,y,y')\in C^{(2)}(Q)$ bo‘lsin. Agar $y^0(x)\in C^{(1)}[x_0,x_1]-(1)$ funksionalning (2) to‘plamdagi kuchsiz minimali (maksimali) bo‘lsa,

$$F_{y'y'}(x,y^0(x),y^0{}'(x))\geq 0\quad (\leq 0),\quad \forall x\in(x_0,x_1)\quad (4)$$

tengsizlik bajariladi. (4) munosabatga Lejandr sharti deyiladi.

3.**Yakobi sharti (ikkinchi tartibli zaruriy shart).Yakobi tenglamasi.** Lejandr sharti, lokal minimum (maksimum)ning funksional ikkinchi variatsiyasi yordamida ifodalanadigan, $\delta^2 J[y^0,h\ ]\geq 0\ \ (\leq 0)$ shartidan foydalanib, keltirib chiqariladi. Funksional ikkinchi variatsiyasining ekstremum nuqtasida ishorasini saqlashini ifodalovchi bu shartdan yana bitta ikkinchi tartibli zaruriy shartni - Yakobi shartini keltirib chiqarish mumkin.

$F(x,y,y')\in C^{(2)}(Q)$ deb hisoblab, $y^0(x)$ joiz funksiya uchun

$$\omega(x,h,h')=F_{y'y'}(x,y^0(x),y^{0'}(x)h^2+$$
$$+2F_{yy}(x,y^0(x),y^{0'}(x))hh'+F_{yy}(x,y^0(x),y^{0'}(x))h^2$$

funksiyani qaraymiz. U vaqtda (3) formulaga ko‘ra,

$$\delta^2 J[y^0, h\ ] = \int_{x_2}^{x_1} \omega(x, h, h')dx \qquad (5)$$

bo‘ladi. Agar *$y^0(x)$* –kuchsiz minimal (maksimal) bo‘lsa, $\delta^2 J[y^0, h\ ] \geq 0 \quad (\leq 0)$ shart barcha $h(x) \in C^{(1)}[x_0, x_1]$, $h(x_0) = h'(x_1) = 0$ funksiyalar uchun bajariladi. *$h^0(x)$*=0 uchun esa, $\delta^2 J[y^0, h^0] = 0$ bo‘lishi ravshan.Demak, qaralayotgan variatsion hisob asosiy masalasiga *qo‘shib olingan ekstremal masala* deb ataluvchi,

$$\left.\begin{aligned} &\delta^2 J[y^0, h] = \int_{x_0}^{x_1} \omega(x, h, h')dx \to \min(\max) \\ &\mathrm{h}(\mathrm{x}_0) = \mathrm{h}(\mathrm{x}_1) = 0, \ \ \mathrm{h}(\mathrm{x}) \in \mathrm{C}^{(1)}[\mathrm{x}_0, \mathrm{x}_1] \end{aligned}\right\} \qquad (6)$$

masala *$h^0(x)=0$* yechimga ega.

Faraz qilaylik, $F(x, y, y') \in C^{(3)}(Q), y^0(x) \in C^{(2)}[x_0 x_1]$-joiz statsionar funksiya, $F_{y'y'}(x, y^0(x), y^{0'}(x)) \neq 0 \ \forall x \in [x_0 x_1]$ bo‘lsin. U vaqtda (6) masala uchun tuzilgan

$$\omega_h(x, h, h') - \frac{d}{dx}\omega_{h'}(x, h, h') = 0$$

Eyler tenglamasiga variatsion hisob asosiy masalasi uchun *Yakobi tenglamasi* deyiladi. $\omega(x, h, h')$ funksiyaning ko‘rinishini hisobga olib, Yakobi tenglamasini

$$A(x)h'' + B(x)h' + C(x)h = 0 \qquad (7)$$

ikkinchi tartibli bir jinsli differyensial tenglama ko‘rinishida yozish mumkin, bu yerda

$$A(x) = F_{y'y'}(x, y^0(x), y^{0'}(x)), \quad B(x) = \frac{d}{dx}F_{y'y'}(x, y^0(x), y^{0'}(x)),$$

$$C(x) = \frac{d}{dx}F_{y'y'}(x, y^0(x), y^{0'}(x)) - F_{yy}(x, y^0(x), y^{0'}(x))$$

Differensial tenglamalar kursidan ma'lumki, (7) tenglama $h(x_0)=0,\, h'(x_0)=1$ boshlang'ich shartlarni qanoatlantiruvchi (aynan noldan farqli) yagona yechimga ega. Shu yechimning x_0 dan farqli nollariga x_0 nuqtaga *qo'shma nuqta* deyiladi.Qo'shma nuqtaga quyidagi ekvivalent ta'rifni ham berish mumkin.

3-ta'rif. Agar (7) Yakobi tenglamasi $h(x_0)=0,\, h(x^*)=0 \quad x^*\neq x_0$ shartlarni qanoatlantiruvchi trivial(aynan nol) bo'lmagan $h(x),\ x\in[x_0,x_1]$ yechimga ega bo'lsa, x^* nuqta - $y^0(x)$ joiz chiziq bo'ylab, x_0 *nuqtaga qo'shma nuqta* deyiladi.

4-teorema (Yakobi). Faraz qilaylik:

a) $F(x,y,y')\in C^{(3)}(Q), \quad$ б) $y^0(x)\in C^{(2)}[x_0x_1]$-kuchsiz minimal(maksimal)

$F_{y'y'}(x,y^0(x),{y^0}'(x))>0 \quad (<0) \quad \forall x\in[x_0x_1]$ bo'lsin. U holda $y^0(x)$ funksiya Yakobi shartini qanoatlantiradi: (x_0,x_1) oraliqda $y^0(x)$ chiziq bo'ylab x_0 nuqtaga qo'shma nuqta mavjud emas.

4. Kuchsiz ekstremumning yetarli shartlari.

Shunday qilib, variatsion hisob asosiy masalasida ekstremumning birinchi tartibli zaruriy sharti joiz funksiyaning statsionarligi (Eyler tenglamasining bajarilishi) bo'lsa, Lejandr va Yakobi shartlari –ikkinchi tartibli zaruriy shartlardir.

Sodda misollar ko'rsatadiki, bu uchala shartning birortasi ham alohida olinganda ekstremumning yetarli sharti bo'la olmaydi. Ammo ular birgalikda kuchsiz ekstremumning yetarli shartiga yaqinroqdir.

Quyida Lejandr va Yakobi shartlarini kuchaytirish natijasida kelib chiqadigan yetarli shartlarni keltiramiz.

5-teorema. Faraz qilaylik:

a) $F(x,y,y') \in C^{(3)}(Q), y^0(x) \in C^{(2)}[x_0 x_1]$ joiz statsionar funksiya bo‘lsin;

b) kuchaytirilgan Lejandr sharti bajarilsin:

$$F_{y'y'}(x, y^0(x), y^{0'}(x)) > 0 \quad (< 0) \quad \forall x \in [x_0 x_1];$$

v) kuchaytirilgan Yakobi sharti o‘rinli bo‘lsin: $y^0(x)$ joiz chiziq bo‘ylab $(x_0, x_1]$ da x_0 nuqtaga qo‘shma x^* nuqta mavjud emas.

U holda $y^0(x)$- variatsion hisobning asosiy masalasida kuchsiz lokal minimal (maksimal) bo‘ladi.

5. Kuchli ekstremumning zaruriy va yetarli shartlari. Veyershtrass shartlari.

Bu yerda kuchli ekstremumning zaruriy va yetarli shartlarini keltiramiz.

$Q = S \times R^2, S \subset R^2$ - berilgan ochiq to‘plam, $F(x,y,y') \in C'(Q)$ bo‘lsin. Quyidagi

$$E(x,y,y',u,) = F(x,y,u) - F(x,y,y') - (u - y')F_{y'}(x,y,y'), (x,y,y',u) \in Q \times R^1$$

funksiyani qaraymiz. $E(x,y,y',u)$ funksiyaga Veyershtrass funksiyasi deyiladi.

6-teorema. Agar $y^0(x) \in C^1[x_0, x_1]$ (1) funksionalning

$$\tilde{V} = \{y(x) \in C^1[x_0, x_1] : y(x_0) = y_0, y(x_1) = y_1\}$$

to‘plamdagi kuchli lokal minimum (maksimum) nuqtasi bo‘lsa, $y^0{}'(x)$ mavjud bo‘lgan barcha $x \in [x_0, x_1]$ nuqtalarda

$$E(x, y^0(x), y^{0\prime}(x), u) \geq 0 (\leq C) \, \forall u \in R^1 \qquad (8)$$

Veyershtrass sharti bajariladi. $y^0(x)$ ning ξ burchak nuqtalarida esa, (8) shart

$$E(\xi, y^0(\xi), y^0{}'(\xi \pm 0), u) \geq 0 (\leq 0) \; \forall u \in R \qquad (9)$$

ko‘rinishda bo‘ladi.

7-teorema. Quyidagi shartlar bajarilsin:

1). $F(x,y,y')\in C^{(4)}(Q), Q=S\times R^1,$

2). $F_{y'y'}(x,y,y')\ge 0 \ \ (\le 0), \ \forall (x,y,y')\in Q;$

3). $y^0(x)\in C^{(3)}[x_0,x_1]-$ (1) funksionalning joiz statsionar funksiyasi;

4). $y^0(x)$ funksiya uchun kuchaytirilgan Lejandr va Yakobi shartlari o‘rinli.

U holda $y^0(x)$-(1) funksionalning (2) to‘plamdagi kuchli lokal minimum (maksimum) nuqtasi bo‘ladi. 6, 7– teoremalar Veyershtrass shartlari deyiladi.

II. Nazariy savollar.

1. Funksionalning ikkinchi variatsiyasi.
2. Variatsion hisob asosiy masalasi funksionalining ikkinchi variatsiyasini hisoblash formulasini keltirib chiqaring.
3. Kuchsiz ekstremumning ikkinchi tartibli zaruriy sharti va yetarli sharti.
4. Lejandr sharti (tyeoryema).
5. Qo‘shib olingan variatsion masala.
6. Yakobi sharti (teorema).
7. Yakobi tenglamasi. Qo‘shma nuqta.
8. Veyershtrass funksiyasi.
9. Kuchli ekstremumning zaruriy sharti (Veyershtrass sharti).
10. Kuchli ekstremumning yetarli sharti (Veyershtrass sharti).
11. Kvadratik funksionalning ekstremumi.

III.Amaliy topshiriqlar.

1 – masala. Quyidagi funksionalning ikkinchi variatsiyasini hisoblang.

1. $J(y)=\int_0^1 (y'^2+3y^2-y'(y+x^2))dx$ 2. $J(y)=\int_{-1}^1 (y'^3-6xyy')dx$

3. $J(y)=\int_\pi^{2\pi} (y'^2+3y(y'+2yCosx))dx$ 4. $J(y)=\int_0^1 (y'^2+y^2-2y'(y+x^3))dx$

5. $J(y)=\int_{-1}^1 (y'^3+3x^2yy')dx$ 6. $J(y)=\int_\pi^{2\pi} (y'^2-y(y'+5yCos2x))dx$

7. $J(y)=\int_0^1 (y'^2-4y^2-y(y'+2x^2))dx$ 8. $J(y)=\int_{-1}^1 (y'^3-2xyy'+x^3)dx$

9. $J(y)=\int_\pi^{2\pi} (y'^2+2y(y'-y\sin x))dx$ 10. $J(y)=\int_0^1 (y'^2-3y^2-y(2y'+x^3))dx$

11. $J(y)=\int_{-1}^1 (y'^3+3xyy'-ye^x)dx$ 12. $J(y)=\int_\pi^{2\pi} (y'^2-5y(y'-2y\sin 2x))dx$

13. $J(y)=\int_0^1 (y'^2-5y^2+2y'(y+x^4))dx$ 14. $J(y)=\int_{-1}^1 (y'^3-4xyy'+y'x^2)dx$

15. $J(y)=\int_\pi^{2\pi} (y'^2+3y(y'-4yCos2x))dx$ 16. $J(y)=\int_0^1 (y'^2+3y^2+2y(y'-x^4))dx$

17. $J(y)=\int_{-1}^1 (y'^3-3xyy'+4x^3)dx$ 18. $J(y)=\int_\pi^{2\pi} (y'^2+4y(y'+3y\sin 2x))dx$

19. $J(y)=\int_0^1 (y'^2-y^2-yy'(1+x^2))dx$ 20. $J(y)=\int_{-1}^1 (y'^3+2xyy'-y^2)dx$

21. $J(y)=\int_{-1}^1 (y'^3+5xyy'-x^2y)dx$ funksionalning ikkinchi variatsiyasini hisoblang.

Yechilishi: Funksionalning ikkinchi variatsiyasi

$$\delta^2 J(y,h)=\int_a^b [F_{yy}h^2(x)+2F_{yy'}h(x)h'(x)+F_{y'y'}h'^2(x)]dx$$

formula yordamida topiladi. Ikkinchi variatsiyani topish uchun integral ostidagi funksiya $F=y'^3+5xyy'-x^2y$ dan y bo‘yicha ikki marta xususiy hosila olamiz va bu hosilalar $F_y = 5xy' - x^2, \quad F_{yy} = 0$ larga teng bo‘ladi. Bundan keyin y bo‘yicha olingan xususiy hosiladan y' bo‘yicha xususiy hosila olib, $F_{yy'} = 5x$ ga ega bo‘lamiz. Endi integral ostidagi funksiyadan y' bo‘yicha ikki marta xususiy hosila olib, $F_{y'} = 3y'^2 + 5xy, \quad F_{y'y'} = 6y'$ ega bo‘lamiz. Natijada berilgan funksionalning ikkinchi variatsiyasi qu’yidagi ko‘rinishda bo‘ladi:

$$\delta^2 J(y,h) = \int_{-1}^{1} [10xh(x)h'(x) + 6y'h'^2(x)]dx\,.$$

<u>2 – masala.</u> Quyidagi variatsion masalaning kuchsiz lokal ekstremalida Lejandr shartining bajarilishini ko‘rsating.

1. $J(y) = \int_1^3 \left(y'^2 - 3y'Cosx\right)dx \to \min;\ y(1) = 0,\ y(3) = 1,\ y \in C^{(1)}[1;3]$

2. $J(y) = \int_0^1 \left(y'^2 - 7xy\right)dx \to \min;\ y(0) = 0,\ y(1) = 2,\ y \in C^{(1)}[0;1]$

3. $J(y) = \int_0^2 \left(y'^2 + 5yy'\right)dx \to \min;\ y(0) = 0,\ y(2) = 1,\ y \in C^{(1)}[0;2]$

4. $J(y) = \int_1^3 \left(y'^2 - 5y'Cosx\right)dx \to \min;\ y(1) = 0,\ y(3) = 1,\ y \in C^{(1)}[1;3]$

5. $J(y) = \int_0^1 \left(y'^2 + 2x^2y\right)dx \to \min;\ y(0) = 0,\ y(1) = 2,\ y \in C^{(1)}[0;1]$

6. $J(y) = \int_0^2 \left(y'^2 + 3xy'\right)dx \to \min;\ y(0) = 0,\ y(2) = 1,\ y \in C^{(1)}[0;2]$

7. $J(y) = \int_1^3 \left(y'^2 + y'Cosx + y\right)dx \to \min;\ y(1) = 0,\ y(3) = 1,\ y \in C^{(1)}[1;3]$

8. $J(y) = \int_{-1}^1 \left(y'^2 - 6y^2 + y'x^2\right)dx \to \min;\ y(-1) = 0,\ y(1) = 1,\ y \in C^{(1)}[-1;1]$

9. $J(y) = \int_0^1 \left(y'^2 - 4ye^x + x^3\right)dx \to \min;\ y(0) = 0,\ y(1) = 1,\ y \in C^{(1)}[0;1]$

10. $J(y)=\int_0^2 (y'^2+3yy'-y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

11. $J(y)=\int_{-1}^1 \left(y'^2+y^2-4yx^2\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

12. $J(y)=\int_0^1 \left(y'^2-ye^x+2x\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

13. $J(y)=\int_0^2 (y'^2-6yy'+3xy)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

14. $J(y)=\int_{-1}^1 \left(y'^2-2xy+3y\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

15. $J(y)=\int_0^2 \left(y'^2+8xyy'\right)dx \to \min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

16. $J(y)=\int_1^3 \left(y'^2-2y'Cosx\right)dx \to \min;\ y(1)=0,\ y(3)=1,\ y\in C^{(1)}[1;3]$

17. $J(y)=\int_0^1 \left(y'^2+12xy\right)dx \to \min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

18. $J(y)=\int_0^2 (y'^2-5yy'+4y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

19. $J(y)=\int_{-1}^1 \left(y'^2-9y^2+3yx^2\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

20. $J(y)=\int_0^1 \left(y'^2-5ye^x\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

21. $J(y)=\int_0^3 \left(y'^2-2y'x\right)dx \to \min;\ y(0)=0,\ y(3)=1,\ y\in C^{(1)}[0;3]$ masalaning joiz statsionar funksiyasida Lejandr shartining bajarilishini ko‘rsating.

Yechilishi: Eyler tenglamasini tuzamiz: $F_y-\frac{d}{dx}F_{y'}=0$. Integral ostidagi funksiya $F=y'^2-2y'x$ dan y va y' bo‘yicha xususiy hosilalar olib, $F_y=0$ va $F_{y'}=2y'-2x$ larga ega bo‘lamiz. $F_{y'}=2y'-2x$ funksiyadan x

bo‘yicha to‘la hosila olib, $\frac{d}{dx}F_{y'} = 2y''-2$ ega bo‘lamiz. Bu funksiyalarni yuqoridagi tenglamaga keltirib qo‘yib,

$$2y''-2=0 \text{ yoki } y''=1$$

Eyler tenglamasiga kelamiz. Bu tenglamaning ikkala tomonini x bo‘yicha ikki marta integrallab, Eyler tenglamasining quyidagi:

$$y' = x + c_1,$$
$$y = \frac{x^2}{2} + c_1 x + c_2$$

umumiy yechimiga ega bo‘lamiz. Endi yuqoridagi chegaraviy shartlardan foydalanib, c_1 va c_2 larni topamiz:

$$\begin{cases} y(0) = \frac{0^2}{2} + c_1 0 + c_2 = 0 \\ y(3) = \frac{3^2}{2} + 3c_1 + c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ 3c_1 = 1 - \frac{9}{2} \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ 3c_1 = -\frac{7}{2} \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ c_1 = -\frac{7}{6} \end{cases}$$

Demak, joiz statsionar funksiya $y = \frac{x^2}{2} - \frac{7}{6}x$ ko‘rinishda bo‘ladi.

Endi $F_{y'} = 2y'-2x$ funksiyadan y' bo‘yicha xususiy hosila olib, $F_{y'y'} = 2$ ifodaga ega bo‘lamiz. $F_{y'y'} = 2 > 0$ bo‘lgani uchun kuchaytirilgan Lejandr sharti bajariladi (kuchsiz lokal minimum uchun).

<u>3–masala.</u> Quyidagi variatsion masalaning $y^0(x)=x$ joiz funksiyalar bo‘ylab Yakobi tenglamasini tuzing.

1. $J(y)=\int_0^1 \left(y'^2-9y^2+2y'e^x\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

2. $J(y)=\int_{-1}^1 \left(y'^2-3yy'+4y^2-yx^3\right)dx \to \min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

3. $J(y)=\int_{\pi}^{2\pi} \left(y'^2-2y'(y+Sinx)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

4. $J(y)=\int_0^1 \left(y'^2+y^2+y'e^{2x}\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

5. $J(y)=\int_{-1}^1 \left(y'^2+yy'+4y^2-2yx^3\right)dx \to \min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

6. $J(y)=\int_{\pi}^{2\pi} \left(y'^2+4y'(y-Sinx)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

7. $J(y)=\int_0^1 \left(y'^2+9y^2+y'e^{3x}\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

8. $J(y)=\int_{-1}^1 \left(y'^2-2yy'+y^2-3yx^3\right)dx \to \min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

9. $J(y)=\int_{\pi}^{2\pi} \left(y'^2-3y'(y+\cos x\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

10. $J(y)=\int_0^1 \left(y'^2+3y^2+y'e^{4x}\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

11. $J(y)=\int_{-1}^1 \left(y'^2-4yy'+9y^2-yx^3\right)dx \to \min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

12. $J(y)=\int_{\pi}^{2\pi} \left(y'^2+2y'(y-\cos x)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

13. $J(y)=\int_0^1 \left(y'^2-4y^2-y'e^{2x}\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

14. $J(y)=\int_{-1}^1 \left(y'^2+3yy'+16y^2-2yx^3\right)dx \to \min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

15. $J(y)=\int\limits_{\pi}^{2\pi}\left(y'^2+3y'(y+Sin2x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

16. $J(y)=\int\limits_{0}^{1}\left(y'^2-6y^2+y'e^x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

17. $J(y)=\int\limits_{-1}^{1}\left(y'^2-5yy'-4y^2-yx^3\right)dx\to\min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

18. $J(y)=\int\limits_{\pi}^{2\pi}\left(y'^2-4y(y'-\cos 2x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

19. $J(y)=\int\limits_{0}^{1}\left(y'^2-y^2+2y'e^{3x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

20. $J(y)=\int\limits_{-1}^{1}\left(y'^2-6yy'+4y^2+3yx^3\right)dx\to\min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

21. $J(y)=\int\limits_{0}^{1}\left(y'^2+9y^2+y'e^{4x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$ masalada $y^0(x)=x$ joiz funksiyalar bo‘ylab Yakobi tenglamasini tuzing.

Yechilishi: Yakobi tenglamasi $A(x)h''+B(x)h'+C(x)h=0$ ko‘rinishda bo‘lib, bu yerda $A(x)=F_{y'y'}\big|_{y=y_0}$, $B(x)=\frac{d}{dx}F_{y'y'}\big|_{y=y_0}$, $C(x)=\frac{d}{dx}F_{yy'}-F_{yy}\big|_{y=y_0}$. Bu tenglamani tuzish uchun integral ostidagi funksiya $F=y'^2+9y^2+y'e^{4x}$ dan y va y' bo‘yicha xususiy hosilalar olamiz va bu hosilalar $F_y=18y,\quad F_{y'}=2y'+e^{4x}$ lardan iborat bo‘ladi. Bundan keyin $F_y=18y$ dan y va y' bo‘yicha xususiy hosilalar olamiz: $F_{yy}\big|_{y=y_0}=18$, $F_{yy'}=0$. Bundan $\frac{d}{dx}F_{yy'}\big|_{y=y_0}=0$. Endi $F_{y'}$ dan y' bo‘yicha xususiy hosila olib, $F_{y'y'}=2$ ga ega bo‘lamiz. Bu funksiyadan x bo‘yicha to‘la hosila olsak,

$\frac{d}{dx}F_{y'y'}\big|_{y=y_0}=0$ bo‘ladi. Natijada, $A(x)=2,\quad B(x)=0,\quad C(x)=-18$

va berilgan masala uchun y_0 bo‘ylab Yakobi tenglamasi,

$$2h''-18h=0$$
$$h''-9h=0$$

ko‘rinishda bo‘ladi.

<u>**4 – masala.**</u> Quyidagi variatsion masalada kuchsiz lokal ekstremalni toping.

1. $J(y)=\int_0^1 \left(y'^2-4y^2+2y(y'-x^3)\right)dx \to \min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

2. $J(y)=\int_\pi^{2\pi} \left(y'^2+y^2-3y'(y+Sinx)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

3. $J(y)=\int_0^1 \left(y'^2-6y'x^2\right)dx \to \min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

4. $J(y)=\int_0^1 \left(y'^2-y^2-3y(y'-x^3)\right)dx \to \min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

5. $J(y)=\int_\pi^{2\pi} \left(y'^2+4y^2-y'(y+Sinx)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

6. $J(y)=\int_0^1 \left(y'^2+y'x^2\right)dx \to \min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

7. $J(y)=\int_0^1 \left(y'^2-9y^2+y(y'-x^3)\right)dx \to \min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

8. $J(y)=\int_\pi^{2\pi} \left(y'^2+16y^2-5y'(y+Sinx)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

9. $J(y)=\int_0^1 \left(y'^2-3y'x^2\right)dx \to \min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

10. $J(y)=\int_0^1 \left(y'^2+4y^2-2y(y'+x^3)\right)dx \to \min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

11. $J(y)=\int_\pi^{2\pi} \left(y'^2+9y^2+3y'(y+Sinx)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

12. $J(y)=\int_0^1 \left(y'^2+2y'x^2\right)dx \to \min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

13. $J(y)=\int_0^1 \left(y'^2+y^2+4y(y'-x^3)\right)dx \to \min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

14. $J(y)=\int\limits_{\pi}^{2\pi}\left(y'^2+y^2-y'(y+\cos x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

15. $J(y)=\int\limits_{0}^{1}\left(y'^2+4y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

16. $J(y)=\int\limits_{0}^{1}\left(y'^2+9y^2-3y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

17. $J(y)=\int\limits_{\pi}^{2\pi}\left(y'^2+4y^2-5y'(y-\cos x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

18. $J(y)=\int\limits_{0}^{1}\left(y'^2-5y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

19. $J(y)=\int\limits_{0}^{1}\left(y'^2-16y^2+2y(y'+x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

20. $J(y)=\int\limits_{\pi}^{2\pi}\left(y'^2+9y^2-4y'(y-\cos x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

21. $J(y)=\int\limits_{0}^{1}(y'^2+4y^2-3y'x^2)dx\to\min(\max),\quad y(0)=0,\ y(1)=1$ masalada kuchsiz lokal ekstremalni toping.

Yechilishi: 1) Avvalo joiz statsionar funksiyani, ya'ni

$$F_y-\frac{d}{dx}F_{y'}=0$$

Eyler tenglamasining masaladagi chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini topamiz.

$$F=y'^2+4y^2-3y'x^2$$

bo'lgani uchun, $F_y=8y,\quad F_{y'}=2y'-3x^2,\quad \frac{d}{dx}F_{y'}=2y''-6x$ va Eyler tenglamasi

$$8y-2y''+6x=0$$

yoki

$$y''-4y=3x$$

ko'rinishga ega bo'ladi. Uning umumiy yechimini $y = y_1 + \tilde{y}$ ko'rinishda izlaymiz, bunda y_1 - mos bir jinsli tenglamaning umumiy yechimi, $\tilde{y}$ esa qaralayotgan tenglamaning birorta xususiy yechimi. Endi

$$y_1'' - 4y_1 = 0.$$

bir jinsli tenglamani qaraymiz.Bu tenglamaning yechimini $y_1 = e^{\lambda x}$ ko'rinishda izlaymiz:

$$\lambda^2 e^{\lambda x} - 4e^{\lambda x} = 0,$$
$$\lambda^2 - 4 = 0$$
$$\lambda_{1,2} = \pm 2.$$

Demak, $y_1(x) = c_1 e^{2x} + c_2 e^{-2x}$ bo'ladi. $\tilde{y}$ ni $\tilde{y} = ax + b$ ko'rinishda izlaymiz. $\tilde{y}'' = 0$

$$0 - 4(ax + b) = 3x.$$

Bundan $a = -\frac{3}{4}, \quad b = 0.$ Demak, $\tilde{y} = -\frac{3}{4}x$ va Eyler tenglamasining umumiy yechimi

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{3}{4}x$$

bo'ladi. $y(0) = 0, \quad y(1) = 1$ shartlarga ko'ra,

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^2 + c_2 e^{-2} - \frac{3}{4} = 1 \end{cases} \Rightarrow \begin{cases} c_1 = -c_2 \\ c_1 e^2 - c_1 e^{-2} - \frac{3}{4} = 1 \end{cases} \Rightarrow \begin{cases} c_1 = -c_2 \\ c_1 (e^2 - e^{-2}) = \frac{7}{4} \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = -c_2 \\ c_1 = \dfrac{7}{4(e^2 - e^{-2})} \end{cases} \Rightarrow \begin{cases} c_2 = -\dfrac{7}{4(e^2 - e^{-2})} \\ c_1 = \dfrac{7}{4(e^2 - e^{-2})} \end{cases}$$

Demak, $y^0(x) = \dfrac{7\left(e^{2x} - e^{-2x}\right)}{4\left(e^2 - e^{-2}\right)} - \dfrac{3}{4}x$ joiz statsionar funksiya bo'ladi.

2) $y^0(x)$ bo'ylab Lejandr shartini tekshiramiz: $F_{y'y'}\Big|_{y=y^0} = 2 > 0, \quad x \in [0;1].$

Demak, $y^0(x)$ bo'ylab kuchsiz minimum uchun kuchaytirilgan Lejandr sharti bajariladi.

3) Endi $y^0(x)$ bo'ylab Yakobi shartining bajarilishini tekshiramiz. Buning uchun, oldin $A(x)h'' + B(x)h' + C(x)h = 0$ Yakobi tenglamasini tuzamiz:

$$A(x) = F_{y'y'}\Big|_{y=y^0} = 2, \quad B(x) = \frac{d}{dx}\left(F_{y'y'}\right)\Big|_{y=y^0} = 0.$$

$$C(x) = \left(\frac{d}{dx}F_{yy'} - F_{yy}\right)\Big|_{y=y^0} = -8.$$

Natijada Yakobi tenglamasi

$$2h'' - 8h = 0, \quad h'' - 4h = 0.$$

ko'rinishda bo'ladi. Uning yechimi $h = e^{\mu x}$ ko'rinishda izlanadi va $h'' = \mu^2 e^{\mu x}$. U holda

$$\mu^2 e^{\mu x} - 4e^{\mu x} = 0$$
$$\mu^2 - 4 = 0$$
$$\mu = \pm 2.$$

Shunday qilib, Yakobi tenglamasining umumiy yechimi

$$h(x) = d_1 e^{2x} + d_2 e^{-2x}$$

ko'rinishga ega. Boshlang'ich shartdan, $d_1 \neq 0$ bo'lganda

$$h(0) = 0 \Rightarrow \; d_1 + d_2 = 0 \Rightarrow d_2 = -d_1 \Rightarrow h(x) = d_1\left(e^{2x} - e^{-2x}\right) \neq 0,$$

bo'lishi kelib chiqadi.

Agar $h(x^*) = 0$ desak, u holda

$$e^{2x_*} - e^{-2x_*} = 0, \; e^{2x_*} = e^{-2x_*}, \; x_* = 0.$$

Demak, $y^0(x)$ bo'ylab $(0,1]$ kesmada nol nuqtaga qo'shma nuqta yo'q, ya'ni kuchaytirilgan Yakobi sharti bajariladi.

Shunday qilib, $y^0(x)$ - kuchsiz lokal minimal bo'ladi.

3§. Variatsion hisob asosiy masalasining ba'zi umumlashmalari

1. Bir necha funksiyalarga bog'liq funksionalning ekstremumi.

Variatsion hisob asosiy masalasining umumlashmasi sifatida, dastlab, bir necha, $y_1 = y_1(x),..., y_n = y_n(x)$ funksiyalarga bog'liq funksionalning ekstremumi haqidagi masalani qaraymiz.

Faraz qilaylik, $Q \in R^{2n+1}$ – biror ochiq to'plam (soha), $F(x, y_1,..., y_n, z_1,..., z_n) - Q$ da aniqlangan uzluksiz funksiya, $P_0(x_0, y_{01},..., y_{0n})$ va $P_1(x_1, y_{11},..., y_{1n})$ lar $S = \{(x, y_1,..., y_n) : (x, y_1,..., y_n, z_1,..., z_n) \in Q\}$ to'plamning belgilangan nuqtalari, $x_0 < x_1$ bo'lsin.

Qabul qilingan belgilashlar asosida, quyidagi

$$J[y_1,..., y_n] = \int_{x_0}^{x_1} F(x, y_1..., y_n, y_1'..., y_n')dx \to \min(\max) \qquad (1)$$

$$\left.\begin{array}{c} y_1(x_0) = y_{01}, y_2(x_0) = y_{02},..., y_n(x_0) = y_{0n}, \\ y_1(x_1) = y_{11}, y_2(x_1) = y_{12},..., y_n(x_1) = y_{2n}, \\ (x, y_1(x),..., y_n(x), y_1'(x),..., y_n'(x)) \in Q, x \in [x_0, x_1] \\ y_i(x) \in C^1[x_0, x_1], i = 1,2,..., n. \end{array}\right\} \qquad (2)$$

ekstremal masalani qaraymiz.

Qaralayotgan masalani ixchamroq shaklda yozish uchun quyidagi belgilashlarni kiritamiz:

$$y = (y_1,..., y_n),\ \ y' = (y_1',..., y_n'),\ y_0 = (y_{01},..., y_{0n}), y_1 = (y_{11}, y_{12},..., y_{1n}),$$

$C_n^{(1)}[x_0, x_1] -$ $[x_0, x_1]$ kesmada uzluksiz differensiallanuvchi $y(x) = (y_1(x),..., y_n(x)) - n -$ vektor funksiyalar fazosi. U holda (1), (2) masalani

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')dx \to \min(\max), \qquad (1')$$

$$y(x_0) = y_0, y(x_1) = y_1,\ y(x) \in C_n^{(1)}[x_0, x_1] \qquad (2')$$

ko'rinishda yozish mumkin. (2') munosabatlarni qanoatlantiruvchi $y(x)=(y_1(x),...,y_n(x))$ funksiyalarga (1)-(2) masalaning joiz funksiyalari (chiziqlari) deyiladi. Qaralayotgan masalada joiz chiziqlarning uchlari R^{n+1} fazoning $P_0(x_0,y_0)$ va $P_1(x_1,y_1)$ nuqtalarida mahkamlangan.

Biz $C_n^{(1)}[x_0,x_1]$ bilan bir qatorda, $[x_0,x_1]$ kesmada uzluksiz $y(x)-n-$vektor funktsiyalar fazosi $C_n[x_0,x_1]$ dan ham foydalanamiz. Ma'lumki, $C_n[x_0,x_1]$ va $C_n^1[x_0,x_1]$ fazolar chiziqli normalangan fazolar bo'lib, ularda normalar, mos ravishda,

$$\|y\|_{C_n[x_0,x_1]}=\max_{1\le i\le n}\max_{x\in[x_0,x_1]}|y_i(x)|,$$

$$\|y\|_{C_n[x_0,x_1]}=\max_{1\le i\le n}\left[\max_{x\in[x_0,x_1]}|y_i(x)|+\max_{x\in[x_0,x_1]}|y_i'(x)|\right]$$

kabi aniqlanishi mumkin. Shuning uchun $y^0(x)=(y_1^0(x),...,y_n^0(x))$ joiz chiziqning nolinchi va birinchi tartibli $\varepsilon-$ atroflarini, mos ravishda, quyidagicha aniqlaymiz:

$$V_0(y^0,\varepsilon)=\left\{y(x)\in C_n[x_0,x_1]:\|y-y^0\|_{C_n[x_0,x_1]}<\varepsilon\right\}=$$

$$=\left\{(y_1(x),...,y_n(x)):\|y_i-y_i^0\|_{C_n[x_0,x_1]}<\varepsilon,\ i=1,2,...,n\right\};$$

$$V_1(y^0,\varepsilon)=\left\{y(x)\in C_n^1[x_0,x_1]:\|y-y^0\|_{C_n^1[x_0,x_1]}<\varepsilon\right\}=$$

$$=\left\{(y_1(x),...,y_n(x)):\|y_i-y_i^0\|_{C_n^1[x_0,x_1]}<\varepsilon,\ i=1,2,...,n\right\}.$$

1-ta'rif. Agar shunday $\varepsilon>o$ mavjud bo'lib, $y^0(x)$ joiz funksiyaning $V_0(y^0,\varepsilon)$ nolinchi tartibli $\varepsilon-$ atrofiga tegishli barcha $y=y(x)$ joiz funksiyalar uchun

$$J[y^0]\le J[y]\quad \left(J[y^0]\ge J[y]\right) \qquad (3)$$

munosabat bajarilsa, $y^0(x)$ funksiya (1) funksionalning *kuchli lokal minimum (maksimum) nuqtasi* deyiladi.

2-ta’rif. Agar (3) munosabat $y^0(x)$ joiz funksiyaning biror $V_1(y^0,\varepsilon)$ birinchi tartibli ε- atrofiga tegishli barcha $y=y(x)$ joiz funksiyalar uchun bajarilsa, $y^0(x)-$ (1) funksionalning *kuchsiz lokal minimum (maksimum) nuqtasi* deyiladi.

Demak, (1),(2) masala uchun kuchli va kuchsiz ekstremumlar variatsion hisobning asosiy masalasidagiga o‘xshash aniqlanadi.

Keltirilgan ta’riflardan ravshanki, kuchli ekstremum nuqtasi kuchsiz ekstremum nuqtasi ham bo‘ladi. Buning teskarisi esa, hamisha ham to‘g‘ri emas. Shuning uchun, avvalo kuchsiz ekstremumning zaruriy shartlarini keltiramiz.

1-teorema. $F(x,y,y')\in C^1(Q)$ bo‘lsin. Agar (1) funksional $y^0(x)=(y_1{}^0(x),...,y_n{}^0(x))\in C_n{}^{(1)}[x_0,x_1]$ joiz funksiyada kuchsiz lokal ekstremumga erishsa, $[x_0,x_1]$ kesmada

$$F_{y_i}\left(x,y^0(x),y^{0'}(x)\right)-\frac{d}{dx}F_{y_i'}\left(x,y^0(x),y^{0'}(x)\right)=,\ \ i=\overline{1,n} \qquad (4)$$

tengliklar bajariladi.

Bu teorema ko‘rsatadiki, (1), (2) masalada $y^0(x)=(y_1{}^0(x),...,y_n{}^0(x))$ kuchsiz ekstremallar,

$$F_{y_i}(x,y,y')-\frac{d}{dx}F_{y'_i}(x,y,y')=0,\ \ i=\overline{1,n} \qquad (5)$$

Eyler tenglamalari sistemasini qanoatlantirar ekan.

Bu yerda, xususiy holda, $n=1$ bo‘lganda variatsion hisobning asosiy masalasi uchun olingan natija, ya’ni Eyler tenglamasiga ega bo‘lamiz.

Agar $F_{yi}(x,y,y')\in C^{(2)}(Q)$ bo‘lsa, (5) dan

$$\begin{gathered}\sum_{j=1}^{n} F_{y_i' y_j'}(x,y,y') y_j'' + \sum_{j=1}^{n} F_{y'_i y_j}(x,y,y') y_j' + \\ + F_{x y_i'}(x,y,y') - F_{y_i}(x,y,y') = 0\end{gathered} \qquad i=\overline{1,n} \qquad (6)$$

sistemaga ega bo‘lamiz. Bu esa, n noma’lumli ikkinchi tartibli n ta differensial tenglamalar sistemasidir.

Bundan buyon quyidagi belgilashlardan foydalanamiz:

$$F^0_{y_i y_j} = F_{y_i y_j}\left(x, y_1^0(x),..., y_n^0(x), y_1^{0'}(x),..., y_n^{0'}(x)\right),$$

$$F^0_{y_i y'_j} = F_{y_i y_j'}\left(x, y_1^0(x),..., y_n^0(x), y_1^{0'}(x),..., y_n^{0'}(x)\right),$$

$$F^0_{y_i' y_j'} = F_{y_i' y_j'}\left(x, y_1^0(x),..., y_n^0(x), y_1^{0'}(x),..., y_n^{0'}(x)\right), \; i,j=\overline{1,n}.$$

Elementlari $F^0_{y_i' y_j'}(x)$ lardan tuzilgan $n \times n$ matritsani $F^0_{y'y'}(x)$ deb belgilaymiz. Faraz qilaylik, $F(x,y,y') \in C^{(2)}(Q)$ bo‘lsin. Agar $y^0(x) = (y_1{}^0(x),..., y_n{}^0(x))$ kuchsiz lokal ekstremal uchun $\det F^0_{y'y'}(x) \neq 0, \; \forall x \in [x_0, x_1]$ bo‘lsa, $y^0(x)$ funksiya $[x_0,x_1]$ kesmada (6) tenglamalar sistemasini qanoatlantiradi.

3-ta’rif. Eyler tenglamalar sistemasini qanoatlantiruvchi $y(x) = (y_1(x),..., y_n(x))$ joiz funksiyalarga (1) funksionalning *statsionar funksiyalari* deyiladi.

Statsionar funksiyalar ekstremumga shubhali funksiyalardir.

Endi (1), (2) masala uchun ekstremumning ikkinchi tartibli zaruriy shartlari va yetarli shartlarini keltiramiz.

Agar $F(x,y,y') \in C^{(2)}(Q)$ bo‘lsa, (1) funksional har bir $y^0(x) = (y_1{}^0(x),..., y_n{}^0(x)) \in C_n^1[x_0, x_1]$ nuqtada ikkinchi variatsiyaga ega va u quyidagi formula bo‘yicha hisoblanadi:

$$\delta^2 J[y^0, h] = \int_{x_0}^{x_1} \sum_{i,j=1}^{n} \left[F^0_{y_i y_j} h_i(x) h_j(x) + 2\sum_{i,j=1}^{n} F^0_{y_i y'_j} h_i(x) h'_j(x) + \sum_{i,j=1}^{n} F^0_{y_i' y_j'} h_i'(x) h'_j(x) \right] dx, \qquad (7)$$

bu yerda $h = h(x) = (h_1(x),..., h_n(x)),\ h_i(x) \in C^1[x_0, x_1],\ h_i(x_0) = h_i(x_1) = 0.$

2-teorema. $F(x, y, y') \in C^{(2)}(Q)$ bo‘lsin. Agar (1) funksional $y^0(x) \in C_n^{(1)}[x_0, x_1]$ joiz funksiyada kuchsiz lokal minimum (maksimum) ga erishsa, quyidagi:

$$F^0_{y'y'}(x) \ge 0 \quad (\le 0),\ \forall x \in (x_0, x_1) \qquad (8)$$

Lejandr sharti bajariladi.

Eslatamizki, $n \times n$ - o‘lchovli $F^0_{y'y'}(x) = (F^0_{y_i y_j}(x))$ matrisa uchun yozilgan (8) shart shu matrisaga mos keluvchi kvadratik formaning nomanfiy (nomusbat) ishorali ekanligini, ya’ni

$$\sum_{i,j=1}^{n} F^0_{y'_i y'_j}(x)\xi_i \xi_j \ge 0 \ (\le 0),\ \forall \xi = (\xi_1,..., \xi_n) \in R^n.$$

munosabat o‘rinli bo‘lishini anglatadi.

Agar bu munosabatda tenglik faqat $\xi = 0$ bo‘lganda bajarilsa, u $F^0_{y'y'}(x) > 0 \ \ (< 0)$ kabi yoziladi.

(7) formula bo‘yicha hisoblanadigan $\delta^2 J[y^0, h]$ ikkinchi variatsiya $h(x) \in C_n^{(1)}[x_0, x_1]$ ga nisbatan kvadratik funksionaldir. Endi $F(x, y, y') \in C^{(3)}(Q),$ $y^0(x) \in C_n^{(2)}[x_0, x_1]$ deb hisoblab, shu kvadratik funksional uchun Eyler tenglamalari sistemasini yozamiz:

$$\sum_{i,j=1}^{n} F^0_{y_i y_j}(x)h_j + \sum_{j=1}^{n} F^0_{y_i y_j}(x)h_j' - \left[\frac{d}{dx}\sum_{j=1}^{n} F^0_{y_i' y_j'} h_j' + \sum_{j=1}^{n} F^0_{y_i' y_j}(x)h_j\right] = 0,\ i = \overline{1,n}. \qquad (9)$$

(9) sistema – (1), (2) masala uchun Yakobi tenglamalari sistemasi deyiladi.

4-ta’rif. Agar (9) sistema $h_i(x_0) = h_i(x_*) = 0,\ i = \overline{1,n},$ shartlarni qanoatlantiruvchi trivial (aynan nol) bo‘lmagan yechimga ega bo‘lsa, x_* nuqta - $y^0(x)$ joiz chiziq bo‘ylab, x_0 *nuqtaga qo‘shma nuqta* deyiladi.

3-teorema. Faraz qilaylik, $F(x,y,y')\in C^{(3)}(Q),\quad y^0(x)\in C_n^{(2)}[x_0,x_1]$ – (1) funksionalga kuchsiz lokal minimum (maksimum) beruvchi joiz funksiya, $F^0_{y'y'}(x)\geq 0\ \ (\leq 0),\ \forall x\in[x_0,x_1]$ bo‘lsin. U holda Yakobi sharti bajariladi, ya’ni *(x_0,x_1)* intervalda $y^0(x)$ chiziq bo‘ylab x_0 nuqtaga qo‘shma bo‘lgan nuqta mavjud emas.

4-teorema. Quyidagi shartlar bajarilsin:

1) $F(x,y,y')\in C^{(3)}(Q)$; 2) $y^0(x)\in C_n^2[x_0,x_1]$ – joiz statsionar funksiya;

3) kuchaytirilgan Lejandr sharti: $F^0_{y'y'}(x)>0\ \ (<0)\ \forall x\in[x_0,x_1]$;

4) kuchaytirilgan Yakobi sharti: $(x_0,x_1]$ intervalda x_0 nuqtaga qo‘shma x_* nuqta mavjud emas. U holda (1) funksional $y^0(x)$da kuchsiz lokal minimum (maksimum)ga erishadi.

Agar $F(x,y,y')\in C^{(4)}(Q),\quad (Q=S\times R^n,\ S\subset R^{n+1}$ – ochiq to‘plam) $F^0_{y'y'}(x)\geq 0\ \ (\leq 0),\ \forall(x,y,y')\in Q$ bo‘lsa va 2), 3), 4) shartlar bajarilsa, $y^0(x)$ funksiya (1), (2) masalada kuchli lokal minimal (maksimal) bo‘ladi.

Keltirilgan bu teoremani (1) funksional

$$J[y]=\int_{x_0}^{x_1}\left[\frac{d}{dx}\sum_{i,j=1}^{n}p_{ij}(x)y_iy_j+2\sum_{j=1}^{n}q_{ij}(x)y_iy_j'+\sum_{i,j=1}^{n}r_{ij}(x)y_i'y_j'\right]dx. \qquad (10)$$

ko‘rinishdagi kvadratik funksional bo‘lgan holda quyidagi tasdiq bilan to‘ldirish mumkin.

5-teorema. $p_{ij}(x)\in C[x_0,x_1],\ q_{ij}(x)\in C^{(1)}[x_0,x_1],\ r_{ij}(x)\in C^{(1)}[x_0,x_1]$, $j,i=\overline{1,n}$ $r(x)=\left(r_{ij}(x),\ i,j=\overline{1,n}\right)>0\ \ (<0)\ \ \forall x\in[x_0,x_1]$ bo‘lsin. Agar Yakobi sharti bajarilmasa, $\inf J(x)=-\infty(+\infty)$, ya’ni masala yechimga ega emas. Agar kuchaytirilgan Yakobi sharti bajarilsa, yagona statsionar funksiya mavjud va bu funksiya (10) funksional uchun global minimal (maksimal) bo‘ladi.

2. Yuqori tartibli hosilalarga bog‘liq bo‘lgan funksionalning ekstremumi.

Faraz qilaylik, $Q \subset R^{n+2}$ - berilgan ochiq to‘plam (soha),

$$S = \{(x, y, z_1, ..., z_{n-1}) : (x, y, z, ..., z_n) \in Q\}, \quad F(x, y, z_1, ..., z_n) - Q$$

sohada uzluksiz funksiya,

$$P_0 = (x_0, y_{00}, y_{01}, ..., y_{0,n-1}), \ P_1(x_1, y_{10}, y_{11}, ..., y_{1,n-1})$$

S to‘plamning belgilangan nuqtalari, $x_0 < x_1$ bo‘lsin.

Quyidagi

$$J[y] = \int_{x_0}^{x_1} F(x, y, y', ..., y^{(n)})dx \to \min(\max), \tag{11}$$

$$\left. \begin{array}{c} y(x_0) = y_{00}, \ y'(x_0) = y_{01}, ..., \ y^{(n-1)}(x_0) = y_{0.n-1}, \\ y(x_0) = y_{10}, \ y'(x_1) = y_{11}, ..., \ y^{(n-1)}(x_1) = y_{1.n-1}, \\ y(x) \in C^{(n)}[x_0, x_1], \ (x, y(x), y^1(x), ..., y^{(n)}(x)) \in Q, \ x \in [x_0, x_1] \end{array} \right\} \tag{12}$$

ekstremal masalani qaraymiz.

Yuqori tartibli hosilalar qatnashgan (11), (12) masala ham chegaralari qo‘g‘olmas variatsion masaladir. Bu masalada joiz funksiyalar (chiziqlar) (12) shartlar bilan aniqlanadi, ya’ni ularning uchlari berilgan P_0 va P_1 nuqtalarda mahkamlangan.

5-ta’rif. Agar biror $\varepsilon > 0$ son topilib, $\|y - y^0\|_{C^{(n)}[x_0, x_1]} < \varepsilon$ shartni qanoatlantiruvchi barcha $y=y(x)$ joiz chiziqlar uchun $J[y^0] \le J[y]$ $(J[y^0] \ge J[y])$ tengsizlik bajarilsa, (11) funksional $y^0=y^0(x)$ joiz chiziqda kuchsiz lokal minimumga (maksimumga) erishadi, deyiladi. Bunda $y^0(x)$ - (12) *masalaning kuchsiz minimali (maksimali)* deyiladi.

Keltirilgan ta’rifda $C^{(n)}[x_0, x_1]$ fazodagi norma o‘rniga $C^{(n-1)}[x_0, x_1]$ fazodagi normadan foydalansak, kuchli ekstremal ta’rifiga ega bo‘lamiz.

Avvalgi qaralgan variatsion masaladagi kabi bu yerda ham har bir kuchli ekstremalning kuchsiz ekstremal bo‘lishi ravshan.

Kuchsiz ekstremumning birinchi tartibli zaruriy sharti quyidagi teoremada berilgan.

6-teorema. $F(x,y,z_n)\in C^{(n+1)}(Q)$ bo‘lsin. Agar $y^0(x)$ – (11), (12) masalada kuchsiz ekstremal bo‘lsa, barcha $x\in[x_0,x_1]$ uchun

$$F_y^0(x)-\frac{d}{dx}F_{y'}^0(x)+\frac{d^2}{dx^2}F_{y''}^0(x)-...+(-1)\frac{d^n}{dx^n}F_{y^{(n)}}^0(x)=0 \quad (13)$$

tenglik bajariladi, bu yerda

$$F_{y^{(i)}}^0(x)=F_{y^{(i)}}\left(x,y^0(x),y^{0'}(x),...,y^{0^{(n)}}(x)\right),\quad i=\overline{1,n}. \qquad (14)$$

Noma’lum $y=y(x)\in C^{(n)}[x_0,x_1]$ funksiyaga nisbatan

$$F_y-\frac{d}{dx}F_{y'}+\frac{d^2}{dx^2}F_{y''}+...+(-1)\frac{d^n}{dx^n}F_{y^{(n-1)}}=0 \qquad (15)$$

tenglamaga *Eyler–Puasson tenglamasi* deyiladi. $F(x,y,z_1,...,z_n)\in C^{(n+1)}(Q)$, bo‘lganda Eyler-Puasson tenglamasi *2n*– tartibli oddiy differensial tenglamadan iborat.

6-ta’rif. Eyler-Puasson tenglamasini qanoatlantiruvchi $y^0=y^0(x)$ joiz funksiyaga (11), (12) masalaning ***statsionar funksiyasi*** deyiladi.

Endi qaralayotgan masala uchun ekstremumning ikkinchi tartibli zaruriy shartlari va yetarli shartlari haqida to‘xtalamiz.

Agar $F(x,y,z_1,...,z_n)\in C^{(2)}(Q)$ bo‘lsa, (11) funksional har bir $y^0=y^0(x)\in C^{(n)}[x_0,x_1]$ nuqtada ikkinchi variatsiyaga ega va bu variatsiya

$$\delta^2 J\left[y^0,h\right]=\int_{x_0}^{x_1}\sum_{i,j=1}^{n}F_{y^{(i)}y^{(j)}}^0(x)h^{(i)}(x)h^{(j)}(x)dx, \qquad (16)$$

$$h=h(x)\in C^{(n)}\left[x_0,x_1\right],\quad h^{(i)}(x_0)=h^{(i)}(x_1)=0,\quad i=\overline{0,n-1}.$$

formula bo‘yicha hisoblanadi, bu yerda

$$F^0_{y^{(i)}y^{(j)}}(x) = F^0_{y^{(i)}y^{(j)}}\left(x, y^0(x), y^{0'}(x), ..., y^{0^{(n)}}(x)\right).$$

h=h(x) ga bog‘liq bo‘lgan (16) funksional uchun tuzilgan Eyler-Puasson tenglamasiga, (11), (12) masala uchun Yakobi tenglamasi deyiladi.

7-ta’rif. Agar Yakobi tenglamasi $h^{(i)}(x_0) = h^{(i)}(x_*) = 0, \ i = \overline{0, n-1}$, shartlarni qanoatlantiruvchi trivial (aynan nol) bo‘lmagan yechimga ega bo‘lsa, x_* nuqta - *y⁰(x)* joiz chiziq bo‘ylab, x_0 *nuqtaga qo‘shma nuqta* deyiladi.

7-teorema. $F(x, y, z_1, ..., z_n) \in C^{(n+2)}(Q)$ bo‘lsin. Agar $y^0(x) \in C^{(2n)}[x_0, x_1]$ (11), (12) masalada kuchsiz minimal (maksimal) bo‘lsa, quyidagi shartlar bajariladi:

a) Lejandr sharti: $F^0_{y^{(n)}y^{(n)}}(x) \geq 0 \ \ (\leq 0), \ \forall x \in (x_0, x_1)$

b) Yakobi sharti: (x_0, x_1) intervalda *$y^0(x)$* chiziq bo‘ylab x_0 nuqtaga qo‘shma nuqta mavjud emas.

8-teorema. Faraz qilaylik:

a) $F(x, y, z_1, ..., z_n) \in C^{(n+2)}(Q)$, $Q = S \times R, \ S \subset R^{n+1}$–ochiq to‘plam;

b) $F^0_{y^{(n)}y^{(n)}}(x, y, z_1, ..., z_n) \geq 0 \ \ (\leq 0), \ \forall (x, y, z_1, ..., z_n) \in Q;$

v) $y^0(x) \in C^{(2n)}[x_0, x_1]$ joiz statsionar funksiya;

g) kuchaytirilgan Lejandr sharti bajarilsin:

$$F^0_{y^{(n)}y^{(n)}}(x) > 0 \ \ (< 0), \ \forall x \in [x_0, x_1];$$

d) kuchaytirilgan Yakobi sharti bajarilsin: $(x_0, x_1]$ oraliqda *$y^0(x)$* chiziq bo‘ylab x_0 nuqtaga qo‘shma nuqta mavjud emas. U holda *$y^0(x)$* – (11), (12) masalada kuchli lokal minimal (maksimal) bo‘ladi.

3. Bir necha o'zgaruvchili funksiyalarga bog'liq bo'lgan funksionalning ekstremumi

Biz ikki o'zgaruvchili $z = z(x, y)$ funksiyaga bog'liq bo'lgan, quyidagi,

$$J[z(x, y)] = \iint_D F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})dxdy \qquad (18)$$

funksionalni qaraymiz, bunda F - o'z argumentlarining uch marta differensiallanuvchi funksiyasi ($F \in C^{(3)}(D)$).

O'zining ikkinchi tartibli xususiy hosilalari bilan birgalikda D sohada uzluksiz bo'lib, D sohaning ∂D chegarasida berilgan qiymatlarni qabul qiluvchi funksiyalar ichida, (18) funksionalga ekstremum qiymat beruvchi $z = z(x, y)$ funksiyani topish talab qilinadi.

Agar $z = z(x, y)$ sirtda (18) funksional ekstremumga erishsa, ekstremumning zaruriy sharti $\delta J[z(x, y)] = 0$ (funksional birinchi variatsiyasining nolga teng bo'lishi), sirt quyidagi

$$F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\} = 0 \qquad (19)$$

Eyler – Ostrogradskiy tenglamasini qanoatlantirishini beradi, bunda

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y},$$

$$\frac{\partial}{\partial x}\{F_p\} = F_{px} + F_{pz}\frac{\partial z}{\partial x} + F_{pp}\frac{\partial p}{\partial x} + F_{pq}\frac{\partial q}{\partial x}, \qquad (20)$$

$$\frac{\partial}{\partial y}\{F_q\} = F_{qy} + F_{qz}\frac{\partial z}{\partial y} + F_{qp}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y} \qquad (21)$$

(19) – ikkinchi tartibli xususiy hosilali differensial tenglamadan iborat. Endi agar funksional,

$$J[z(x_1, x_2, ..., x_n)] = \iint_D ... \int F(x_1, x_2, ..., x_n, z, p_1, p_2, ..., p_n)dx_1 dx_2 \cdots dx_n \qquad (22)$$

ko‘rinishda bo‘lsa, bunda $p_k = \frac{\partial z}{\partial x_k}$ $(k = 1,2,...,n),$ ekstremumning zaruriy sharti,

$$F_z - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\{F_{p_i}\} = 0 .$$

Eyler – Ostrogradskiy tenglamasi ko‘rinishini oladi.

Bu tenglamaning yechimi bo‘lgan $z(x_1, x_2,..., x_n)$ funksiya n o‘lchovli D sohaning ∂D chegarasida berilgan chegaraviy shartlarni qanoatlantirishi kerak.

II. Nazariy savollar.

1. Variatsion hisob asosiy masalasining umumlashmasi. Funksional bir necha funksiyalarga bog‘liq bo‘lgan masala.
2. Eyler tenglamalari sistemasi.
3. Variatsion hisob asosiy masalasining umumlashmasi. Funksionalda yuqori tartibli hosilalar qatnashgan masala.
4. Eyler – Puasson tenglamasi.
5. Bir necha o‘zgaruvchili funksiyalarga bog‘liq funksionallar.
6. Eyler - Ostrogradskiy tenglamasi.

III. Amaliy topshiriqlar.

1 – masala. Quyidagi variatsion masala uchun Eyler tenglamalari sistemasini tuzing.

1. $J(y) = \int_0^1 \left(y_1'^2 + y_2'^2 - 2y_1 y_2\right)dx \to \min,$

$y_1(0) = 0, y_2(0) = 0, y_1(1) = -1, y_2(1) = 1, y_1 y_2 \in C^{(1)}[0;1]$

2. $J(y) = \int_1^2 \left(y_1'^2 + y_2'^2 + 3y_1' y_2' + 4y_1^2\right)dx \to \min,$

$y_1(1) = 1, y_2(1) = 1, y_1(2) = 0, y_2(1) = -1, y_1 y_2 \in C^{(1)}[1;2]$

3. $J(y)=\int\limits_0^2\left(y_1'^2+y_2'^2-2y_1'x^3+y_2e^{2x}\right)dx\to\min,$

$y_1(0)=0, y_2(0)=1, y_1(2)=1, y_2(2)=0, y_1y_2\in C^{(1)}[0;2]$

4. $J(y)=\int\limits_0^1\left(y_1'^2+y_2'^2-4y_1y_2\right)dx\to\min,$

$y_1(0)=0, y_2(0)=0, y_1(1)=-1, y_2(1)=1, y_1y_2\in C^{(1)}[0;1]$

5. $J(y)=\int\limits_1^2\left(y_1'^2+y_2'^2-y_1'y_2'+2y_1^2\right)dx\to\min,$

$y_1(1)=1, y_2(1)=1, y_1(2)=0, y_2(1)=-1, y_1y_2\in C^{(1)}[1;2]$

6. $J(y)=\int\limits_0^2\left(y_1'^2+y_2'^2+y_1'x^3-2y_2e^{3x}\right)dx\to\min,$

$y_1(0)=0, y_2(0)=1, y_1(2)=1, y_2(2)=0, y_1y_2\in C^{(1)}[0;2]$

7. $J(y)=\int\limits_0^1\left(y_1'^2+y_2'^2+16y_1y_2\right)dx\to\min,$

$y_1(0)=0, y_2(0)=0, y_1(1)=-1, y_2(1)=1, y_1y_2\in C^{(1)}[0;1]$

8. $J(y)=\int\limits_1^2\left(y_1'^2+y_2'^2-3y_1'y_2'+y_1^2\right)dx\to\min,$

$y_1(1)=1, y_2(1)=1, y_1(2)=0, y_2(1)=-1, y_1y_2\in C^{(1)}[1;2]$

9. $J(y)=\int\limits_0^2\left(y_1'^2+y_2'^2-3y_1'x^3-y_2e^{2x}\right)dx\to\min,\ y_1(0)=0,$

$y_2(0)=1, y_1(2)=1, y_2(2)=0, y_1y_2\in C^{(1)}[0;2]$

10. $J(y)=\int\limits_0^1\left(y_1'^2+y_2'^2-y_1y_2\right)dx\to\min,$

$y_1(0)=0, y_2(0)=0, y_1(1)=-1, y_2(1)=1, y_1y_2\in C^{(1)}[0;1]$

11. $J(y)=\int\limits_1^2\left(y_1'^2+y_2'^2+y_1'y_2'-3y_1^2\right)dx\to\min,$

$y_1(1)=1, y_2(1)=1, y_1(2)=0, y_2(1)=-1, y_1y_2\in C^{(1)}[1;2]$

12. $J(y)=\int\limits_0^2\left(y_1'^2+y_2'^2+4y_1'x^3-2y_2e^{3x}\right)dx\to\min,$

$y_1(0)=0, y_2(0)=1, y_1(2)=1, y_2(2)=0, y_1 y_2 \in C^{(1)}[0;2]$

13. $J(y)=\int_0^1 \left(y_1'^2+y_2'^2+8y_1y_2\right)dx \to \min,$

14. $J(y)=\int_1^2 \left(y_1'^2+y_2'^2+4y_1'y_2'-5y_1^2\right)dx \to \min,$

$y_1(0)=0, y_2(0)=0, y_1(1)=-1, y_2(1)=1, y_1 y_2 \in C^{(1)}[0;1]$

$y_1(1)=1, y_2(1)=1, y_1(2)=0, y_2(1)=-1, y_1 y_2 \in C^{(1)}[1;2]$

15. $J(y)=\int_0^2 \left(y_1'^2+y_2'^2-5y_1'x^3+4y_2e^{2x}\right)dx \to \min,$

$y_1(0)=0, y_2(0)=1, y_1(2)=1, y_2(2)=0, y_1 y_2 \in C^{(1)}[0;2]$

16. $J(y)=\int_0^1 \left(y_1'^2+y_2'^2-2y_1y_2+y_1\right)dx \to \min,$

$y_1(0)=0, y_2(0)=0, y_1(1)=-1, y_2(1)=1, y_1 y_2 \in C^{(1)}[0;1]$

17. $J(y)=\int_1^2 \left(y_1'^2+y_2'^2-y_1'y_2'-4y_1^2\right)dx \to \min,$

$y_1(1)=1, y_2(1)=1, y_1(2)=0, y_2(1)=-1, y_1 y_2 \in C^{(1)}[1;2]$

18. $J(y)=\int_0^2 \left(y_1'^2+y_2'^2-6y_1'x^3-y_2e^{3x}\right)dx \to \min,$

$y_1(0)=0, y_2(0)=1, y_1(2)=1, y_2(2)=0, y_1 y_2 \in C^{(1)}[0;2]$

19. $J(y)=\int_0^1 \left(y_1'^2+y_2'^2-y_1y_2+2y_2\right)dx \to \min,$

$y_1(0)=0, y_2(0)=0, y_1(1)=-1, y_2(1)=1, y_1 y_2 \in C^{(1)}[0;1]$

20. $J(y)=\int_1^2 \left(y_1'^2+y_2'^2-4y_1'y_2'-6y_1^2\right)dx \to \min,$

$y_1(1)=1, y_2(1)=1, y_1(2)=0, y_2(1)=-1, y_1 y_2 \in C^{(1)}[1;2]$

21. $J(y_1,y_2)=\int_0^1 \left(y_1'^2+y_2'^2-3y_1'y_2'+4y_1^{\,2}\right)dx \to \min,$

$y_1(0)=1,\quad y_2(0)=1,\quad y_1(1)=0,\quad y_2(1)=-1,\quad y_1, y_2 \in C^{(1)}[0:1].$

masala uchun Eyler tenglamalari sistemasini tuzing.

Yechilishi: Bu masala funktsionali uchun Eyler tenglamalari sistemasi

$$\begin{cases} F_{y_1} - \dfrac{d}{dx}F_{y'_1} = 0 \\ F_{y_2} - \dfrac{d}{dx}F_{y_2'} = 0 \end{cases}$$

koʻrinishda boʻlishi kerak. Integral ostidagi funksiya $F = y_1'^2 + y_2'^2 - 3y_1'y_2' + 4y_1^2$. Bu funksiyadan y_1, y_1' va x lar boʻyicha hosilalar olib, quyidagilarga ega boʻlamiz:

$$F_{y_1} = 8y_1, \ F_{y_1'} = 2y_1' - 3y_2', \ \frac{d}{dx}F_{y_1'} = 2y_1'' - 3y_2''.$$

Bundan y_1 boʻyicha Eyler tenglamasi $8y_1 - 2y_1'' - 3y_2'' = 0$ koʻrinishga keladi.

Endi integral ostidagi funksiyadan y_2, y'_2 va x lar boʻyicha hosilalar olib,

$$F_{y_2} = 0, \ F_{y_2'} = 2y_2' - 3y_1', \ \frac{d}{dx}F_{y_2'} = 2y_2'' - 3y_1''$$

larga ega boʻlamiz. Bundan y_2 boʻyicha Eyler tenglamasi $-2y_2'' + 3y_1'' = 0$ yoki $2y_2'' - 3y_1'' = 0$ koʻrinishda boʻladi. Natijada, Eyler tenglamalari sistemasi,

$$\begin{cases} 2y_1'' - 8y_1 - 3y_2'' = 0 \\ 2y_2'' - 3y_1'' = 0 \end{cases}$$

koʻrinishda boʻladi.

<u>2 – masala.</u> Quyidagi variatsion masala uchun Eyler – Puasson tenglamasini tuzing.

1. $J(y) = \int_0^1 \left(y''^2 + y'^2 + 3y'x^3\right)dx \to \min, y\ (0) = 0, y'(0) = 0, y\ (1) = 1, y'(1) = -1, y \in C^{(2)}[0;1]$

2. $J(y) = \int_0^{\pi} \left(y''^2 - 3yy' + y^2\right)dx \to \min, y\ (0) = 0, y'(0) = 0, y\ (\pi) = 1, y'(\pi) = -1, y \in C^{(2)}[0;\pi]$

3. $J(y)=\int_1^2\left(y''^2+y'^2-2y^2\right)dx\to\min,\ y\ (1)=1, y'(1)=-1, y\ (2)=0, y'(2)=0, y\in C^{(2)}[1;2]$

4. $J(y)=\int_0^1\left(y''^2+4y'^2-y'x^3\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (1)=1, y'(1)=-1, y\in C^{(2)}[0;1]$

5. $J(y)=\int_0^{\pi}\left(y''^2-yy'+4y^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (\pi)=1, y'(\pi)=-1, y\in C^{(2)}[0;\pi]$

6. $J(y)=\int_1^2\left(y''^2+4y'^2-y^2\right)dx\to\min,\ y\ (1)=1, y'(1)=-1, y\ (2)=0, y'(2)=0, y\in C^{(2)}[1;2]$

7. $J(y)=\int_0^1\left(y''^2-y'^2+2y'x^3\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (1)=1, y'(1)=-1, y\in C^{(2)}[0;1]$

8. $J(y)=\int_0^{\pi}\left(y''^2+2yy'-y^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (\pi)=1, y'(\pi)=-1, y\in C^{(2)}[0;\pi]$

9. $J(y)=\int_1^2\left(y''^2+6y'^2+2y^2\right)dx\to\min,\ y\ (1)=1, y'(1)=-1, y\ (2)=0, y'(2)=0, y\in C^{(2)}[1;2]$

10. $J(y)=\int_0^1\left(y''^2-4y'^2-2y'x^3\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (1)=1, y'(1)=-1, y\in C^{(2)}[0;1]$

11. $J(y)=\int_0^{\pi}\left(y''^2+yy'+9y^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (\pi)=1, y'(\pi)=-1, y\in C^{(2)}[0;\pi]$

12. $J(y)=\int_1^2\left(y''^2+5y'^2+y^2\right)dx\to\min,\ y\ (1)=1, y'(1)=-1, y\ (2)=0, y'(2)=0, y\in C^{(2)}[1;2]$

13. $J(y)=\int_0^1\left(y''^2+y'^2-4yx^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (1)=1, y'(1)=-1, y\in C^{(2)}[0;1]$

14. $J(y)=\int_0^{\pi}\left(y''^2-6yy'-y^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (\pi)=1, y'(\pi)=-1, y\in C^{(2)}[0;\pi]$

15. $J(y)=\int_1^2\left(y''^2+2y'^2-4y^2\right)dx\to\min,$

$y\ (1)=1, y'(1)=-1, y\ (2)=0, y'(2)=0, y\in C^{(2)}[1;2]$

16. $J(y)=\int_0^1\left(y''^2-y'^2+3yx^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (1)=1, y'(1)=-1, y\in C^{(2)}[0;1]$

17. $J(y)=\int_0^{\pi}\left(y''^2+4yy'-y^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (\pi)=1, y'(\pi)=-1, y\in C^{(2)}[0;\pi]$

18. $J(y)=\int\limits_1^2\left(y''^2+4y'^2+3y^2\right)dx\to\min,\ y\ (1)=1, y'(1)=-1, y\ (2)=0, y'(2)=0, y\in C^{(2)}[1;2]$

19. $J(y)=\int\limits_0^1\left(y''^2+4y'^2-3y'x^3\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (1)=1, y'(1)=-1, y\in C^{(2)}[0;1]$

20. $J(y)=\int\limits_0^\pi\left(y''^2-5yy'-4y^2\right)dx\to\min,\ y\ (0)=0, y'(0)=0, y\ (\pi)=1, y'(\pi)=-1, y\in C^{(2)}[0;\pi]$

21. Ushbu

$$J(y)=\int\limits_0^1\left(y''^2+y'^2-2y'x^3\right)dx\to\min,$$

$$y_0(0)=0,\ \ y_0'(0)=0,\ \ y_0(1)=1,\ \ y_0'(1)=-1,\ \ y\in C^{(2)}[0:1],$$

variatsion masala uchun Eyler – Puasson tenglamasini tuzing.

Yechilishi: Eyler – Puasson tenglamasi

$$F_y-\frac{d}{dx}F_{y'}+\frac{d^2}{dx^2}F^0_{y''}+...+(-1)\frac{d^n}{dx^n}F^0_{y(n)}=0$$

tenglamadan iborat. Integral ostidagi funksiya $F=y''^2+y'^2-2y'x^3$. Bu funksiyadan y bo‘yicha xususiy hosila olamiz: $F_y=0$. Endi integral ostidagi funksiyadan y' bo‘yicha xususiy hosila olib, $F_{y'}=2y'-2x^3$ ifodaga ega bo‘lamiz. $F_{y'}=2y'-2x^3$ funksiyadan x bo‘yicha to‘la hosila olib, $\frac{d}{dx}F_{y'}=2y''-6x^2$ ifodaga ega bo‘lamiz. Endi integral ostidagi funksiyadan y'' bo‘yicha xususiy hosila olib, $F_{y''}=2y''$ ifodaga ega bo‘lamiz. $F_{y''}=2y''$ funksiyadan x bo‘yicha hosilalar olib, $\frac{d}{dx}F_{y''}=2y'''$ va $\frac{d^2}{dx^2}F_{y''}=2y^{IV}$ ifodalarga ega bo‘lamiz. Bu olingan hosilalarni yuqoridagi tenglamaga keltirib qo‘yib,

$$-2y''+6x^2+2y^{IV}=0$$

yoki

$$y^{IV}-y''+3x^2=0$$

Eyler – Puasson tenglamasiga ega bo‘lamiz.

<u>**3 – masala.**</u> Quyidagi funksional uchun Eyler - Ostrogradskiy tenglamasini tuzing:

1. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial z}{\partial x}\right)^4+\left(\frac{\partial z}{\partial y}\right)^4+2zf(x,y)\right]dxdy$

2. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial z}{\partial x}\right)^4+\left(\frac{\partial z}{\partial y}\right)^4+4zf(x,y)\right]dxdy$

3. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial z}{\partial x}\right)^4+\left(\frac{\partial z}{\partial y}\right)^4+6zf(x,y)\right]dxdy$

4. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial z}{\partial x}\right)^4+\left(\frac{\partial z}{\partial y}\right)^4+8zf(x,y)\right]dxdy$

5. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial z}{\partial x}\right)^4+\left(\frac{\partial z}{\partial y}\right)^4+5zf(x,y)\right]dxdy$

6. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial z}{\partial x}\right)^4+\left(\frac{\partial z}{\partial y}\right)^4+3zf(x,y)\right]dxdy$

7. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial y^2}\right)^2+2\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2-2zf(x,y)\right]dxdy$

8. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial y^2}\right)^2+2\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2-4zf(x,y)\right]dxdy$

9. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial y^2}\right)^2+2\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2-6zf(x,y)\right]dxdy$

10. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial y^2}\right)^2+2\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2-8zf(x,y)\right]dxdy$

11. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial y^2}\right)^2+2\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2-zf(x,y)\right]dxdy$

12. $J[z(x,y)]=\iint\limits_D\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial y^2}\right)^2+2\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2-7zf(x,y)\right]dxdy$

13. $J[u(x,y,z)] = \iiint\limits_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dxdydz$

14. $J[u(x,y,z)] = \iiint\limits_D \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + 2\left(\frac{\partial^2 u}{\partial z \partial y}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial z}\right)^2 - 2uf(x,y,z) \right] dxdydx$

15-20. $J[z(x,y)] = \iint\limits_D \left[a_1(x,y)\left(\frac{\partial z}{\partial x}\right)^2 + a_2(x,y)\left(\frac{\partial z}{\partial y}\right)^2 - b(x,y)z^2 + 2zf(x,y) \right] dxdy$ **.**

15. $a_1(x,y) = x^2 + y^2, \quad a_2(x,y) = x + y, \quad b(x,y) = (x+y)^2$

16. $a_1(x,y) = x^2 + 2y^2, \quad a_2(x,y) = 2x + y, \quad b(x,y) = (x-y)^2$

17. $a_1(x,y) = 3x^2 + y^2, \quad a_2(x,y) = x - y, \quad b(x,y) = (2x+y)^2$

18. $a_1(x,y) = x^2 - 4y^2, \quad a_2(x,y) = x - 3y, \quad b(x,y) = (x+3y)^2$

19. $a_1(x,y) = 2x^2 - y^2, \quad a_2(x,y) = 3x + 2y, \quad b(x,y) = (2x-3y)^2$

20. $a_1(x,y) = 4x^2 - y^2, \quad a_2(x,y) = 5x + y^2, \quad b(x,y) = (x+y)^3$

21. $J[z(x,y)] = \iint\limits_D \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 \right] dxdy$ funksional uchun Eyler - Ostrogradskiy tenglamasini tuzing.

Yechilishi: Funksionalning ifodasidagi funksiya:

$$F(x,y,z,p,q) = p^2 - q^2$$

ko‘rinishda bo‘lganligidan, Eyler - Ostrogradskiy tenglamasi,

$$-\frac{\partial}{\partial x}(2p) - \frac{\partial}{\partial y}(-2q) = 0$$

yoki

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

ko‘rinishni oladi.

4§. Optimal boshqaruv masalalari (Pontryaginning maksimum prinsipi. Terminal boshqaruv masalasi. Chiziqli boshqaruv sistemalari).

1. Optimal boshqaruv masalasining qo‘yilishi.

Avvalo optimal boshqaruv amaliy masalalaridan birini keltiramiz: v_0 boshlang‘ich tezlikka ega bo‘lgan birlik massali material nuqtani modul bo‘yicha birdan oshmaydigan kuch ta’sirida gorizontal to‘g‘ri chiziq bo‘ylab A nuqtadan B nuqtaga shunday ko‘chirish talab qilinadiki, bunda material nuqta B nuqtaga v_1 tezlik bilan eng qisqa vaqtda yetib kelsin.

Qo‘yilgan masala optimal boshqaruvning tezkor masalasidan iborat. Uning matematik modelini tuzamiz.

Ox o‘qda $A(\alpha)$ va $B(\beta)$ nuqtalarni olaylik. Material nuqta $t=t_0$ boshlang‘ich vaqtda A nuqtada, $t=t_1(t_1>t_0)$ vaqtda esa, B nuqtada bo‘lsin.

$T=t_1-t_0$ - material nuqtaning ko‘chish vaqtidan iborat.

$x=x(t)$-material nuqtaning t vaqtda bosib o‘tgan yo‘li, $u=u(t)$-material nuqtaga t vaqt momentida ta’sir etayotgan kuch miqdori bo‘lsin.

U vaqtda $\dot{x}=\dfrac{dx}{dt}=v$ - material nuqtaning tezligi, $\ddot{x}=\dfrac{d^2x}{dt^2}=a$ - material nuqtaning tezlanishi bo‘ladi.

Nyutonning ikkinchi qonuniga ko‘ra $ma=u$ tenglik o‘rinli, bu yerda m–material nuqtaning massasi. $m=1, a=\ddot{x}$ ekanligini hisobga olsak,

$$\ddot{x}=u \qquad (1)$$

tenglamaga ega bo‘lamiz. Masalaning qo‘yilishiga ko‘ra,

$$\left.\begin{aligned} x(t_0)=\alpha, \quad \dot{x}(t_0)=v_0 \\ x(t_1)=\beta, \quad \dot{x}(t_1)=v_1 \end{aligned}\right\} \qquad (2)$$

shartlar kelib chiqadi. Bundan tashqari, $u(t)$ - kuchga

$$|u(t)| \le 1, \; t \in [t_0, t_1]$$

cheklashlar qo‘yiladi. $u(t)$ - boshqarish funksiyasi (qisqacha, boshqarish) deyiladi. Odatda u bo‘lakli-uzluksiz funksiyalar sinfidan, deb qaraladi. Bunday funksiyalar joiz boshqarishlar sinfini tashkil etadi.

Shunday qilib, qo‘yilgan masalaning matematik modeli quyidagicha:

shunday $|u^*(t)| \le 1, \; t \in [t_0, t_1^*]$ joiz boshqarishni topish talab qilinadiki, (1) tenglamaning unga mos keluvchi $x^*(t)$ yechimi (2) shartlarni qanoatlantirsin va bunda ko‘chish vaqti $T = t_1^* - t_0$ minimal bo‘lsin.

$x_1 = x, \; x_2 = \dot{x}$ o‘zgaruvchilarni kiritib, bu masalani,

$$\left.\begin{array}{l} T(u) = t_1 - t_0 \to \min, \\ \dot{x}_1 = x_2, \dot{x}_2 = u, \\ x_1(t_0) = \alpha, x_1(t_1) = \beta, \\ x_2(t_0) = v_0, x_2(t_1) = v_1, \\ |u| \le 1 \end{array}\right\} \qquad (3)$$

ko‘rinishda yozish mumkin.

(3) masala, geometrik tilda, $\{x_1, x_2\}$ tekislikda shunday $x^*(t) = \{x_1^*(t), x_2^*(t)\}$ trayektoriyani qurishni bildiradiki, u eng qisqa $T^* = t_1^* - t_0$ vaqtda $A = \{\alpha, v_0\}$ nuqtadan $B = \{\beta, v_1\}$ nuqtaga ko‘chib o‘tadi.

2. Pontryaginning maksimum prinsipi.

Maksimum prinsipi optimal boshqaruv masalalarida optimallikning asosiy zaruriy sharti hisoblanadi. Bu natija XX asrning 50-yillari ikkinchi yarmida akademik L.S.Pontryagin boshchiligidagi sovet matematiklari tomonidan olingan.

Quyidagi:

$$J(u,x)=\int_{t_0}^{t_1} f_0(x(t),u(t),t)dt+g_0(x^0,x(t_1))\to \inf \tag{4}$$

$$\left.\begin{array}{l} \dot{x}(t)=f(x(t),u(t),t), t_0\le t\le t_1 \\ x(t_0)=x^0, g_i(x(t_1))\le 0,\ i=1,...,k, \\ g_i(x(t_1))=0, i=k+1,...,s \\ u=u(t)\in V, \end{array}\right\} \tag{5}$$

optimal boshqaruv masalasini qaraymiz. Bu masalada t_0, t_1 vaqt momentlari belgilangan (oʻzgarmas), x^0 - berilgan boshlangʻich nuqta.

Faraz qilamizki, $f(x,u,t)=(f_1(x,u,t),....,f_n(x,u,t))$ vektor-funksiyaning $f_i(x,u,t)$ komponentalari va $f_0(x,u,t),\ g_0(x)$ funksiyalar x boʻyicha uzluksiz xususiy hosilalarga ega boʻlsin.

Maksimum prinsipini bayon qilish uchun Gamilton-Pontryagin funksiyasi deb ataluvchi,

$$\begin{aligned} H(x,u,t,\psi,a_0)&=-a_0 f_0(x,u,t)+\psi_1 f_1(x,u,t)+....+\psi_n f_n(x,u,t)= \\ &=-a_0 f_0(x,u,t)+\psi^T f(x,u,t) \end{aligned} \tag{6}$$

funksiyani qaraymiz, bu yerda $\psi=(\psi_1,...,\psi_n),\ a_0=const$.

$u=u(t)$ - joiz boshqaruv, $x(t)=x(t,u,x^0)$ - unga mos joiz trayektoriya, $[t_0,t_1]$ oraliqda aniqlangan boʻlsin. $(u(t),x(t)),\ t_0\le t\le t_1$ juftlikka mos boʻlgan, $\psi=\psi(t)=(\psi_1(t),...,\psi_n(t))$ oʻzgaruvchilarga nisbatan,

$$\psi_i(t) = \frac{\partial H(x,u,t,\psi(t),a_0)}{\partial x_1}\Bigg|_{u=u(t)}^{x=x(t)} =$$
$$= a_0 f_{0x_i}(x(t),u(t),t) - \sum_{j=1}^{n} \psi_j(t) f_{jx_i}(x(t),u(t),t),(t_0 \le t \le t_1) \quad (7)$$

sistemani qaraymiz. Unga *qo'shma sistema* deyiladi. (7) qo'shma sistemani vektorli shakldа

$$\dot{\psi}(t) = -H_x(x(t),u(t),t,\psi(t),a_0), \; t_0 \le t \le t_1 \quad (8)$$

kabi yozish mumkin, bu yerda $H_x = (H_{x_1},...,H_{x_n})$

Agar (5) sistema *x,u* ga nisbatan chiziqli, ya'ni

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t),(t_0 \le t \le t_1)$$

ko'rinishda bo'lsa,

$$H(x,u,t,\psi,a_0) = -a_0 f_0(x,u,t) + \psi'(A(t) + B(t)u + f(t))$$

va (8) qo'shma sistema

$$\dot{\psi}(t) = a_0 f_{0x}(x(t),u(t),t) - A'(t)\psi(t),(t_0 \le t \le t_1)$$

kabi bo'ladi, bu yerda '- transponirlash belgisi.

(7) qo'shma sistema –chiziqli differensial tenglamalar sistemasidan iborat bo'lib, u $\psi(t_0) = \psi_0$ boshlang'ich shartni qanoatlantiruvchi yagona yechimga ega.

<u>1-teorema.</u> Agar $(x(t),u(t),t_0,t_1)$ - (12) (13) masalaning yechimi bo'lsa, shunday $a_0 a_1,...,a_s$ sonlar va $\psi(t) = (\psi_1(t),...,\psi_n(t)),t_0 \le t \le t_1$ vektor-funksiya mavjud bo'ladiki, ular:

1) $a = (a_0, a_1,..., a_n) \ne 0, \; a_0 \ge 0,..., a_k \ge 0;$ (9)

2) $\psi(t)$ funksiya *(x(t),u(t))* ga mos keluvchi (7) qo'shma sistemaning yechimidan iborat;

3) $u(t)$ optimal boshqarishning barcha $t\in[t_0,t_1]$ uzluksizlik nuqtalarida $H(x(t),u,t,\psi(t),a_0)$ funksiya, $u=(u_1,\ldots,u_m)$ oʻzgaruvchi boʻyicha, V toʻplamda aniq yuqori chegarasiga $u=u(t)$ boʻlganda erishadi, ya'ni

$$\sup_{u\in V} H(x(t),u,t,\psi(t),a_0) = H(x(t),u(t),t,\psi(t),a_0),(t_0 \le t \le t_1) \qquad (10)$$

shartlarni qanoatlantiradi:

4) $$\psi(t_1) = -\sum_{j=0}^{s} a_j g_{jx_i}(x(t_1),t_0,t_1) \qquad (11)$$

$$a_j g_j(x(t_1),t_0,t_1) = 0, \quad j = 1,2,\ldots,k \qquad (12)$$

4) shartlarga transversallik shartlari deyiladi.

3. Maksimum prinsipining chegaraviy masalasi.

Maksimum prinsipidan amaliyotda qanday foydalanish mumkinligini koʻrib oʻtamiz.

$H(x,u,t,\psi,a_0)$ funksiyani $u=(u_0 u_1,\ldots,u_n),$ oʻzgaruvchining funksiyasi deb qaraymiz va har bir belgilangan (x,t,ψ,a_0) da

$$H(x,u,t,\psi,a_0) \to \sup, u \in V \qquad (13)$$

maksimallashtirish masalasini yechamiz.

$$u = u(x,t,\psi,a_0) \in V \qquad (14)$$

shu masalaning yechimi boʻlsin, ya'ni

$$H(x,u(x,t,\psi,a_0),t,\psi,a_0) = \sup_{u\in V} H(x,u,t,\psi,a_0) \qquad (15)$$

tenglik bajarilsin. Agar optimal boshqarish masalasi yechimga ega boʻlsa, maksimum shartiga koʻra (14) funksiya aniqlangan boʻladi. Koʻp hollarda (14) funksiyani oshkor koʻrinishda yozish mumkin boʻladi. Masalan, agar

$$f_j(x,u,t)=f_0^0(x,t)+\sum_{i=1}^{m} f_j^0(x,t)u_i,\quad j=1,2,...,n$$
$$V=\{\mathrm{u}=(\mathrm{u}_1,...,\mathrm{u}_\mathrm{m})\in \mathrm{R}^\mathrm{m},\alpha_\mathrm{i}\le u_i\le \beta_i,\qquad i=1,2,...,m\}$$

($\alpha_\mathrm{i},\beta_i$ -berilgan sonlar) bo‘lsa,

$$H(x,u,t,\psi,a_0)=-a_0 f_0^0(x,t)+\sum_{j=1}^{n}\psi_j f_j^0(x,t)+\sum_{i=1}^{m}\varphi_i(x,t,\psi,a_0)u_i$$

bo‘ladi, bu yerda

$$\varphi_i(x,t,\psi,a_0)=-a_0 f_0^0(x,t)+\sum_{j=1}^{n}\psi_j f_j^0(x,t)$$

U vaqtda (14) masala yechimi $u(x,t,\psi,a_0)$ ning koordinatalari

$$u=u_i(x,t,\psi,a_0)=\begin{cases}\beta_i,\varphi_i(x,t,\psi,a_0)>0\\ \alpha_i,\ \varphi_i(x,t,\psi,a_0)<0,\quad i=1,...,m\end{cases}$$

ko‘rinishda bo‘lishi ravshan. Xususiy holda, agar $\alpha_i=-1,\beta_i=+1$ bo‘lsa,

$$u_i=sign\varphi_i(x,t,\psi,a_0),i=1,2,...,m$$

bo‘ladi.

Agar V to‘plam

$$V=\left\{u\in R^m:|u|=(\sum_{i=1}^{m}u_i^2)^{\frac{1}{2}}\le r\right\}$$

ko‘rinishda bo‘lsa, (14) funksiyani oshkor shaklda,

$$u(x,t,\psi,a_0)=\frac{\varphi(x,t,\psi,a_0)}{\|\varphi_i(x,t,\psi,a_0)\|}r$$

kabi yozish mumkin, bu yerda $\varphi=(\varphi_1,...,\varphi_n)$,

Faraz qilaylik, bizga (14) funksiya ma’lum bo‘lsin. U vaqtda t,ψ o‘zgaruvchilarga nisbatan quyidagi 2n ta differensial tenglamalar sistemasini qaraymiz:

$$\left.\begin{array}{l}\dot{x}=f(x,u(x,t,\psi,a_0),t)\\ \dot{\psi}=-H(x,u(x,t,\psi,a_0),t,\psi,a_0),t_0\le t\le t_1\end{array}\right\}\qquad(16)$$

Differensial tenglamalar kursidan yaxshi ma’lumki, (16) tenglamalar sistemasining umumiy yechimi 2n ta ixtiyoriy parametrlarga (masalan, $x(t_0)=(x_1(t_0),...,x_n(t_0)),\ \psi(t_0)=(\psi_1(t_0),...,\psi_n(t_0)),$ boshlang‘ich shartlarga) bog‘liq bo‘ladi.

Bundan tashqari, maksimum prinsipidagi $a_0 a_1,...,a_s$ parametrlar ham noma’lum, ularni aniqlash uchun yana s+1 ta shart kerak bo‘ladi. Shunday qilib, noma’lum 2n+s+1 ta parametrlarni aniqlash uchun 2n+s+1 ta shart zarur. Ularni maksimum prinsipidan, masalan, 1-teoremadagi (10), (11) shartlar hamda

$$g_j(x(t_j))=0,\quad j=k+1,...,s \qquad (17)$$

shartlardan olamiz. Bu shartlar jami 2n+s ta tenglamalarni beradi. Yetishmayotgan yana bitta tenglamani olish uchun $H(x,u,t,\psi,a_0)$ funksiyaning $\psi_1,\psi_2,...,\psi_n,a_0$ o‘zgaruvchilarga nisbatan chiziqli va bir jinsli ekanligini, ya’ni $H(x,u,t,a\psi,aa_0)=aH(x,u,t,\psi,a_0),\forall a\in R^1$ ekanligini hisobga olamiz. U vaqtda (15) shartdan

$$u(x,t,a\psi,aa_0)=u(x,t,\psi,a_0),\ \forall a>0 \qquad (18)$$

ekanligi kelib chiqadi. Demak, maksimum prinsipida $a_1,a_2,...,a_s,\ \psi_1,...,\psi_n$ o‘zgaruvchilar musbat ko‘paytuvchi aniqligida topiladi. Demak,

$$\|a\|^2=\sum_{i=0}^{S}a_i^2=1 \qquad (19)$$

deb olish mumkin. Agar $a_0>0$ ekanligi ma’lum bo‘lsa, (19) shart o‘rniga $a_0=1$ deb olish ham mumkin. (10), (11), (17), (19) tenglamalar sistemasini yechganda,

$$a_0\ge 0,a_1\ge 0,...,a_n\ge 0,\quad g_i(x(t_1),t_0,t_1)\le 0,\quad i=1,...,k \quad (20)$$

shartlarning bajarilishi hisobga olinadi. Shunday qilib, maksimum prinsipi asosida, (15) maksimum shartidan, (16) tenglamalar sistemasi va (10), (11), (17), (19), (20) shartlardan iborat, maxsus chyegaraviy masalaga ega bo'ldik. Bu masalaga *maksimum prinsipining chegaraviy masalasi* deyiladi.

Agar *x(t)*, $\psi(t), a_1, a_2, ..., a_s$ lar - maksimum prinsipining chegaraviy masalasi yechimidan iborat bo'lsa, ularni (14) ga qo'yib,

$$u(t) = u(x(t), t, \psi(t), a_0) \qquad t_0 \leq t \leq t_1 \qquad (21)$$

funksiyani hosil qilamiz. Agar bu funksiya $[t_0,t_1]$ oraliqda bo'lakli-uzluksiz bo'lsa, u optimalikka shubhali boshqarish bo'ladi. Agar optimal boshqaruv masalasining yechimi mavjud va maksimum prinsipining chegaraviy masalasi yagona yechimga ega bo'lsa, (21) bo'lakli-uzluksiz funksiya optimal boshqarishdan iborat bo'ladi.

4. Terminal boshqaruv masalasi. Maksimum prinsipi.

Boshqarish obyekti

$$\dot{x} = f(x,u,t),\ \ t \in [t_0, t_1] \qquad (22)$$

vektorli differensial tenglama bilan berilgan boʻlsin, bu yerda $x = (x_1,...,x_n),\ u = (u_1,...,u_m),\ f = (f_1,..., f_n),\ f_i(x,u,t)$ funksiyalarni $f_{ixj}(x,u,t)$ xususiy hosilalari bilan birga uzluksiz deb hisoblaymiz. Joiz boshqarishlar $[t_0, t_1]$ oraliqda aniqlangan boʻlakli - uzluksiz va $V \subset R^m$ toʻplamdan qiymatlar qabul qiluvchi *u=u(t)* *m*– vektor funksiyalardan iborat. Har bir *u=u(t)* joiz boshqarishga mos (22) tenglamaning $x = x(t)$ joiz trayektoriyasi

$$x(t_0) = x^0 \qquad (23)$$

shartni qanoatlantiradi. Qaralyotgan obyektni boshqarish,

$$J(u) = \varphi(x(t_1)) \qquad (24)$$

terminal kriteriy orqali sifat jihatidan baholanadi, bu yerda $\varphi(x) - R^n$ da uzluksiz differensiallanuvchi funksiya. Shunday $u^*(t)$ boshqarishni topish kerakki, $J(u^*) = \inf\limits_{u \in U} J(u)$ boʻlsin, bu yerda *U*– barcha joiz boshqarishlar toʻplami. Shunday qilib, quyidagi

$$\left.\begin{aligned} &J(u) = \varphi(x(t_1)) \to \inf \\ &\dot{x} = f(x,u,t),\ \ t \in [t_0, t_1] \\ &x(t_0) = x^0,\ \ u = u(t) \in V \end{aligned}\right\} \qquad (25)$$

terminal boshqaruv masalasini qaraymiz. Bu masalada trayektoriyalarning chap uchi mahkamlangan ((23) shartga q.), oʻng uchi esa erkin $(x(t_1) \in R^n)$.

2-teorema. Agar $u^*(t),\ t\in[t_0,t_1]$ – optimal boshqarish, $x^*(t),\ t\in[t_0,t_1]$ optimal trayektoriya bo‘lsa,

$$H\left(x^*(t),\psi^*(t),u^*(t),t\right)=\max_{u\in V} H\left(x^*(t),\psi^*(t),u,t\right),\quad t\in[t_0,t_1] \qquad (26)$$

maksimum sharti bajariladi, bu yerda

$$H(x,\psi,u,t)=\psi' f(x,u,t)=\sum_{j=1}^{n}\psi_j f_j(x,u,t),$$

$\psi'(t),\ t\in[t_0,t_1]$ funksiya

$$\dot{\psi}=\frac{\partial H\left(x^*(t),\psi,u^*(t),t\right)}{\partial x} \qquad (27)$$

$$\psi(t_1)=\frac{\partial\varphi\left(x^*(t_1)\right)}{\partial x} \qquad (28)$$

qo‘shma sistemaning yechimidir.

3. Chiziqli terminal boshqaruv masalasi.

Chiziqli boshqaruv sistemasi uchun chiziqli terminal kriteriyli quyidagi masalani qaraymiz:

$$\left.\begin{aligned} &J(u)=c'x(t_1)\to\min,\\ &\dot{x}=A(t)x+b(t,u),\ \ t\in[t_0,t_1]\\ &x(t_0)=x^0,\ \ u=u(t)\in V,\ \ t\in[t_0,t_1]\end{aligned}\right\} \qquad (29)$$

bu yerda

$x\in R^n,\ u\in R^m,\ V\subset R^n,\ A(t)-n\times n-$matrisa-funksiya, $b(t,u)=(b_1(t,u),...,b_n(t,u))$, $c\in R^n,\ x^0\in R^n$.

A(t) matrisaning elementlarini $[t_0,t_1]$ da uzluksiz, $b_i(t,u),\ i=\overline{1,n}$ funksiyalarni $[t_0,t_1]\times V$ da uzluksiz deb faraz qilamiz.

(29) masala uchun quyidagi teorema o‘rinlidir.

3-teorema. $u=u(t),\ \ t\in[t_0,t_1]$ bo‘lakli-uzluksiz funksiyaning (29) masalada optimal boshqarish bo‘lishi uchun,

$$\max_{v\in V}\psi'(t)b(t,v)=\psi'(t)b(t,u(t)),\ \ t\in[t_0,t_1] \qquad (30)$$

shartning bajarilishi zarur va yetarlidir, bu yerda $\psi(t), t\in[t_0,t_1]$ funksiya

$$\dot{\psi}=-A'(t)\psi,\ \ \psi(t_1)=-c \qquad (31)$$

qo‘shma sistemaning yechimidan iborat.

6. **Chiziqli tezkor masala. Optimallikning zaruriy va yetarli shartlari.**

$$\dot{x}=A(t)x+B(t)u,\quad x\in R^n,\quad u\in R^m \qquad (32)$$

sistema uchun ikki nuqtali tezkor masalani qaraymiz: shunday $u^*(t)\in U$ boshqarishni topish talab qilinadiki, unga mos $x^*(t)$ trayektoriya , t_0, t_1 vaqt momentlarida berilgan x^0, x^1 nuqtalardan o‘tsin, ya’ni

$$x^*(t_0)=x^0, x^*(t_1^*)=x^1 \qquad (x^0\neq x^1)$$

shartlar bajarilsin va t_1-t_0 o‘tish vaqti minimal bo‘lsin. Bunday $u^*(t)$ ga optimal boshqarish, $x^*(t)$ ga optimal trayektoriya, t_1^* ga optimal vaqt momenti (tezkor momenti) deyiladi, $(u^*(t), x^*(t), t_1^*)$ esa, masalaning yechimidir. Qaralayotgan masalani, qisqacha ,

$$\left.\begin{array}{l} t_1-t_0\to\min \\ \dot{x}=A(t)x+B(t)u,\ \ t\in[t_0,t_1] \\ x(t_0)=x^0, x(t_1)=x^1, \\ u(t)\in U \end{array}\right\} \qquad (33)$$

ko‘rinishda belgilaymiz.

Quyidagi funksiyani kiritamiz:

$$\rho(t)=\min_{\|c\|=1}\left[c'F(t_1,t)x^0+\int_{t_0}^{t}\max_{u\in V}c'F(t,\tau)B(\tau)u\,d\tau-c'x\right],\ \ t\geq t_0$$

Bu funksiya barcha $t_0 > t_1$ nuqtalarda uzluksizdir.

Agar ixtiyoriy $c \in R^n, c \neq 0$ vektor va ixtiyoriy $[t_0, t_1]$ kesma uchun $c'F(t_1,t)B(t)$ funksiya V toʻplamning har bir *k*- oʻlchovli yoqiga $[t_0, t_1]$ oraliqning chekli sondan koʻp boʻlmagan nuqtalari yoki kesmalarida ortogonal boʻlsa, (32) sistema regularlik shartini qanoatlantiradi, deyiladi. Quyidagi tasdiq oʻrinli:

1-lemma [2]. Agar (32) sistema regulyarlik shartini qanoatlantirsa, $Q(t_1)$- yopiq, chegaralangan, qavariq toʻplam boʻladi.

Chiziqli boshqaruv sistemalarini oʻrganishda, ekstremal prinsip deb ataluvchi, quyidagi tasdiqdan keng foydalaniladi.

2-lemma. Faraz qilaylik, (32) sistema regularlik shartini qanoatlantirsin. U vaqtda har bir $c \in R^n, c \neq 0$ vektor va $t_1 \geq t_0$ son uchun *$Q(t_1)$* toʻplamning shunday $\bar{x}$ chegaraviy nuqtasi mavjud boʻladiki, unda

$$c'\bar{x} = \max_{x \in Q(t_1)} c'x \qquad (34)$$

munosabat bajariladi. (34) tenglikning oʻrinli boʻlishi uchun, shunday $\bar{u}(t) \in U$ boshqarish topilib,

$$\bar{x} = F(t_1, t_0)x^0 + \int_{t_0}^{t_1} F(t_1, t)B(t)u(t)dt, \qquad (35)$$

$$c'F(t_1, t)B(t)\bar{u}(t) = \max_{v \in V} c'F(t_1, t)B(t)v, \ t \in [t_0, t_1] \qquad (36)$$

tengliklarning bajarilishi zarur va yetarlidir.

4-teorema. Faraz qilaylik, (32) sistema regularlik shartini qanoatlantirsin va $(u^*(t), x^*(t), t_1^*)$ - (33) masalaning yechimi boʻlsin. U vaqtda:

1) t_1^* optimal vaqt momenti

$$\min_{\|c\|=1} \left[c'F(t, t_0)x^0 + \int_{t_0}^{t} \max_{u \in V} c'F(t, \tau)B(\tau)ud\tau - c'x \right] = 0 \qquad (37)$$

tenglamaning minimal ildiziga teng;

2) $u^*(t)$ optimal boshqarish

$$c^{*'} F(t_1^*, t_0)B(t)u^*(t) = \max_{u\in V} c'F(t_1^*, t_0)B(t)u, \ t_0 \le t \le t_1^* \qquad (38)$$

maksimum shartini qanoatlantiradi, bu yerda $c^* \in R^n, \ c^* \neq 0$ vektor $t = t_1^*$ bo‘lganda (37) ning chap tomonidagi ifodaning ixtiyoriy minimum nuqtasidir;

3) $x^*(t)$ optimal trayektoriya

$$\dot{x}^* = A(t)x^* + B(t)u^*(t), \ x^*(t_0) = x^0 \qquad (39)$$

munosabatlarni qanoatlantiradi.

(32) chiziqli sistema uchun normallik sharti har bir $c\neq 0$ va $t_1>t_0$ da (36) maksimum shartidan $u(t)$ bo‘lakli-uzluksiz funksiyaning $[t_0,t_1]$ dagi barcha uzluksizlik nuqtalarida bir qiymatli aniqlanishini ifodalaydi.

Normallik sharti, regularlik shartidan kuchliroq talabdir, chunki normallik shartiga ko‘ra, ixtiyoriy $c\neq 0$ va $t_1>t_0$ uchun $c'F(t_1,t)B(t)$ funksiya V to‘plamning yoqlariga $[t_0,t_1]$ oraliqning faqat alohida olingan nuqtalaridagina ortogonal bo‘lishi mumkin, ya’ni $[t_0,t_1]$ ning qism intervallarida ortogonallik qaralmaydi.

5-teorema. Faraz qilaylik, (39) sistema normallik shartini qanoatlantirsin. Agar $u^*(t)$ boshqarish, $x^*(t)$ trayektoriya va t_1^* vaqt momenti 4-teoremadagi 1)-3) shartlarni qanoatlantirsa, $(u^*(t), x^*(t), t_1^*)$ - (33) tezkor masalaning yechimi bo‘ladi.

Optimallikning zaruriy va yetarli shartlarini ifodalovchi 4-5-teoremalarga qo‘shimcha qilib shuni aytish mumkinki, normallik sharti bajarilganda optimal boshqarish (va demak, optimal trayektoriya ham) yagona bo‘ladi. Bu tasdiq (38) maksimum shartidan osongina kelib chiqadi.

7. Tezkor masala uchun Pontryaginning maksimum prinsipi.

Optimal boshqaruv nazariyasida optimallikning zaruriy sharti Pontryaginning maksimum prinsipi ko'rinishida ifodalanadi. Quyida 4-teoremada keltirilgan zaruriy shartlarni maksimum prinsipi shaklida yozish mumkinligini ko'rsatamiz.

Quyidagi

$$\psi^*(t) = F(t_1^*, t)' c^* \qquad (40)$$

funksiyani kiritamiz va (38) shartni

$$\psi^{*'}(t)B(t)u^*(t) = \max_{v \in V} \psi^{*'}(t)B(t)u, \; t_0 \le t \le t_1 \qquad (41)$$

ko'rinishda yozamiz. (40) funksiya

$$\dot{\psi} = -A'(t)\psi, \psi(t_1^*) = c^* \qquad (42)$$

qo'shma sistemaning yechimidir. Agar

$$H(x, \psi, u, t) = \psi'[A(t)x + B(t)u]$$

Gamilton-Pontryagin funksiyasidan foydalansak, $x^*(t)$ ning (39) ni, $u^*(t)$ ning (38) ni va $\psi^*(t)$ ning esa, (42) ni qanoatlantirishini,

$$\dot{x}^*(t) = H_\psi(x^*(t), \psi^*(t), u^*(t), t) \qquad (43)$$

$$\dot{\psi}^*(t) = -H_x(x^*(t), \psi^*(t), u^*(t), t) \qquad (44)$$

$$H(x^*(t), \psi^*(t), u^*(t), t) = \max_{v \in V} H(x^*(t), \psi^*(t), u, t), \quad t \in [t_0, t_1^*] \qquad (45)$$

ko'rinishda yozish mumkin. Agar bu sistemani yana bitta

$$H(x^*(t_1^*), \psi^*(t_1^*), u^*(t_1^*), t_1^*) \ge 0 \qquad (46)$$

munosabat bilan to'ldirsak, tezkor masala uchun quyidagi maksimum prinsipiga ega bo'lamiz.

3-teorema (maksimum prinsipi). Faraz qilaylik, (32) sistema uchun regularlik sharti bajarilsin. Agar $(u^*(t), x^*(t), t_1^*)$ - (33) masalaning yechimi bo‘lsa, shunday trivial (aynan nol) bo‘lmagan $\psi^*(t)$ vektor - funksiya topiladiki, (43)-(46) shartlar bajariladi.

II. Nazariy savollar.

1. Sodda mexanik harakatni boshqarish. Optimal boshqaruv tezkor masalasining qo‘yilishi va uning matematik modeli.
2. Optimal boshqaruv masalasining umumiy qo‘yilishi.
3. Pontryaginning maksimum prinsipi.
4. Gamilton- Pontryagin funksiyasi.
5. Maksimum prinsipining chegaraviy masalasi.
6. Terminal boshqaruv masalasi.
7. Funksional orttirmasi formulasi.
8. «Ignasimon» variatsiya.
9. Ekstremal boshqarishhlar.
10. Chiziqli terminal boshqaruv masalasi.
11. Optimallikning zaruriy va yetarli shartlari.
12. Chiziqli boshqaruv sistemasi. Ekstremal prinsip.
13. Regular sistemalar. Normal sistemalar.
14. Chiziqli tezkor masala. Optimallikning zaruriy va yetarli shartlari.
15. Chiziqli tezkor masala uchun Pontryaginning maksimum prinsipi.
16. Chiziqli statsionar tezkor masala.

III. Amaliy topshiriqlar

Terminal boshqaruv masalasida Gamilton - Pontryagin funksiyasi, qo‘shma sistema, optimallik shartini qanoatlantiruvchi boshqarishning ko‘rinishini toping.

1. $J(u) = x_1(1) + x_2(1) \to \min,\ \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1^2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,1]$.

2. $J(u) = x_1^2(1) + x_2(1) \to \min,\ \dot{x}_1 = 2x_1,\ \dot{x}_2 = x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,1]$.

3. $J(u) = 2x_1(1) - x_2^2(1) \to \min,\ \dot{x}_1 = x_1 - x_2,\ \dot{x}_2 = x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,1]$.

4. $J(u) = x_1^2(1) + x_2^2(1) \to \min,\ \dot{x}_1 = x_1^2 - 2u,\ \dot{x}_2 = x_1 + 2x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,1]$.

5. $J(u) = x_1^2(1) + x_1(1)x_2(1) \to \min,\ \dot{x}_1 = x_1 + x_2,\ \dot{x}_2 = x_1 - x_2 - u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,1]$.

6. $J(u) = 3x_1(2) - x_2(2) \to \min,\ \dot{x}_1 = -x_2 - u,\ \dot{x}_2 = x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,2]$.

7. $J(u) = 2x_1^2(2) + 3x_2(2) \to \min,\ \dot{x}_1 = x_1^2 + x_2,\ \dot{x}_2 = -x_1 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,2]$.

8. $J(u) = x_1(2) + 2x_2^2(2) \to \min,\ \dot{x}_1 = -x_1 + x_2^2,\ \dot{x}_2 = x_2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,2]$.

9. $J(u) = -2x_1^2(2) + x_2^2(2) \to \min,\ \dot{x}_1 = x_2 - 3u,\ \dot{x}_2 = -x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,2]$.

10. $J(u) = 3x_1(2)x_2(2) - x_2^2(2) \to \min,\ \dot{x}_1 = 2x_1 - x_2,\ \dot{x}_2 = x_2^2 - 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,2]$.

11. $J(u) = x_1(3) + 2x_2(3) \to \min,\ \dot{x}_1 = x_1 + 2u,\ \dot{x}_2 = -x_1 + x_2^2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,3]$.

12. $J(u) = 2x_1^2(3) - x_2(3) \to \min,\ \dot{x}_1 = 2x_1 + x_2,\ \dot{x}_2 = x_1^2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,3]$.

13. $J(u)=x_1^2(3)+4x_2^2(3)\to\min,\ \dot{x}_1=x_1^2-2x_2,\ \dot{x}_2=x_1+x_2-u,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 3,\ t\in T=[0,3]$.

14. $J(u)=x_1^2(3)+3x_2^2(3)\to\min,\ \dot{x}_1=2x_1-x_2^2,\ \dot{x}_2=3x_2+u,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 4,\ t\in T=[0,3]$.

15. $J(u)=5x_1(3)x_2(3)\to\min,\ \dot{x}_1=x_1-x_2+3u,\ \dot{x}_2=-x_1^2,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 5,\ t\in T=[0,3]$.

16. $J(u)=4x_1(4)-x_2(4)\to\min,\ \dot{x}_1=-x_1^2+2u,\ \dot{x}_2=x_1+3x_2,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 1,\ t\in T=[0,4]$.

17. $J(u)=2x_1^2(4)+2x_2(4)\to\min,\ \dot{x}_1=x_2^2-u,\ \dot{x}_2=-x_1+x_2,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 2,\ t\in T=[0,4]$.

18. $J(u)=3x_1(4)-x_2^2(4)\to\min,\ \dot{x}_1=x_1+2x_2,\ \dot{x}_2=x_2^2+3u,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 3,\ t\in T=[0,4]$.

19. $J(u)=x_1^2(4)+4x_2^2(4)\to\min,\ \dot{x}_1=x_1^2-x_2,\ \dot{x}_2=-x_2^2+2u,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 4,\ t\in T=[0,4]$.

20. $J(u)=x_1^2(4)+x_1(4)x_2(4)-x_2^2(4)\to\min,\ \dot{x}_1=x_2-2u,\ \dot{x}_2=x_1^2+3x_2,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 5,\ t\in T=[0,4]$.

21. $J(u)=x_1(1)-5x_2(1)\to\min,\ \dot{x}_1=x_1+x_2,\ \dot{x}_2=-x_1^3+u,$
$x_1(0)=x_2(0)=0,\ |u(t)|\le 1,\ t\in T=[0,1]$

Yechilishi:

$$H(x,\psi,u)=\psi'\,f(x,u,t)=\sum_{i=1}^{n}\psi_i f_i(x,u,t)$$

$$n=2,\quad f_1(x,u,t)=x_1+x_2,\quad f_2(x,u,t)=-x_1^3+u$$

boʻlgani uchun,

Gamilton – Pontryagin funksiyasi $H(x,\psi,u)=\psi_1(x_1+x_2)+\psi_2(x_1^3+u)$ boʻladi.

U holda qoʻshma sistema $\dot{\psi}_1=-\dfrac{\partial H}{\partial x_1}=-\psi_1+3x_1^2\psi_2,\quad \dot{\psi}_2=-\dfrac{\partial H}{\partial x_2}=-\psi_1$

koʻrinishda boʻladi. $\varphi(x(1))=x_1(1)-5x_2(1)$ boʻlgani uchun qoʻshma sistema uchun boshlangʻich shartlar

$\psi_1(1) = -\dfrac{\partial\varphi(x(1))}{\partial x_1} = -1, \quad \psi_2(1) = -\dfrac{\partial\varphi(x(1))}{\partial x_2} = 5$ ko‘rinishga ega. Maksimum shartini yozamiz:

$$H(x^0, \psi^0, u^0, t) = \max_{|u|\le 1} H(x^0, \psi^0, u, t)$$
$$\psi_1{}^0(x_1{}^0 + x_2{}^0) + \psi_2{}^0(-x_1{}^{0^3} + u^0) = \max_{|u|\le 1}\left(\psi_1{}^0(x_1{}^0 + x_2{}^0) + \psi_2{}^0(-x_1{}^{0^3} + u)\right)$$

Bu yerdan $\psi_2^0 u^0 = \max_{|u|\le 1} \psi_2^0 u$. Demak, optimal boshqarish

$$u^0 = sign\psi_2^0 = \begin{cases} 1, & агар\,\psi_2^0 > 0 \\ -1, & агар\,\psi_2^0 < 0 \end{cases}$$

ko‘rinishga ega.

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VARIATSION HISOB VA OPTIMALLASHTIRISH USULLARI

fanidan

AMALIY MASHG‘ULOTLAR

(uslubiy qo‘llanma)

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