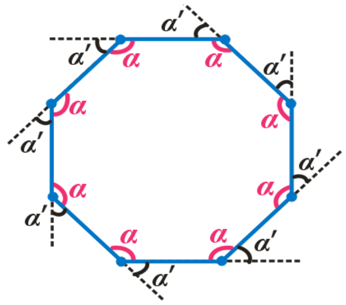
6-BOB
MUNTAZAM KO‘PBURCHAKLAR

1-§. Beshburchaklar. Oltiburchaklar. Muntazam ko‘pburchaklar

Agar qavariq ko‘pburchakning barcha tomonlari uzunliklari teng bo‘lsa, bunday ko‘pburchakka ***muntazam qavariq ko‘pburchak*** deyiladi (6.1-rasm).

Muntazam ko‘pburchakning barcha uchidagi ichki burchaklari ham tashqi burchaklari ham o‘zaro teng bo‘ladi.

Muntazam qavariq ko‘pburchakning har bir uchidagi ichki burchagi quyidagi formuladan aniqlanadi (n – uchlari soni):



6.1-rasm

$$\boxed{\alpha = \frac{n-2}{n}\pi.}$$

Muntazam qavariq ko‘pburchakning har bir uchidagi tashqi burchagi quyidagi formuladan aniqlanadi:

$$\boxed{\alpha' = \frac{2\pi}{n}.}$$

Quyidagi jadvalda ba’zi muntazam ko‘pburchaklar uchun ichki burchaklar yig‘indisi, ichki burchak hamda tashqi burchaklarning qiymatlari keltirilgan.

n	3	4	5	6	8	9	10	12	15	18	24
$\sum\alpha_i$	180^0	360^0	540^0	720^0	1080^0	1260^0	1440^0	1800^0	2340^0	2880^0	3960^0
α	60^0	90^0	108^0	120^0	135^0	140^0	144^0	150^0	156^0	160^0	165^0
α'	120	90^0	72^0	60^0	45^0	40^0	36^0	30^0	24^0	20^0	15^0

	0										

Jadvaldan ko‘rinib turibdiki, muntazam ko‘pburchak tomonlar soni ko‘payib borgan sari $(n \to \infty)$ ichki burchak 180^0 ga, tashqi burchak esa 0^0 ga intilib borar ekan.

1-teorema. Tekislikdan olingan ixtiyoriy uchtasi bir to‘g‘ri chiziqda yotmaydigan *n* ta nuqtani tutashtiruvchi kesmalar soni

$$\boxed{k = \frac{n(n-1)}{2}.}$$

ga teng.

Isbot. Bunda 1-nuqtani qolgan nuqtalar bilan tutashtirilganda $n-1$ ta, 2-nuqtani qolgan nuqtalar bilan tutashtirilganda $n-2$ ta, 3-nuqtani qolgan nuqtalar bilan tutashtirilganda $n-3$ ta va hokazo kesmalar hosil bo‘ladi. Bundagi kesmalar soni 1 dan $n-1$ gacha bo‘lgan arifmetik progressiyaning hadlari yig‘indisidir, ya’ni $S_n = \frac{a_1 + a_n}{2} n$ dan foydalanamiz. Bu yerda $a_1 = 0, a_n = n-1,\ d = 1,\ n = n-1,$ $S_n = k$ ekanini e’tiborga olsak, $k = \frac{0+n-1}{2} n = \frac{n(n-1)}{2}$ natija kelib chiqadi.

Mustaqil ishlash uchun masalalar

6.1. Har bir burchagi 135° bo‘lgan qavariq ko‘pburchakning nechta tomoni bor? javob: 8

6.2. Muntazam ko‘pburchakning tashqi burchagi 36° ga teng. Uning nechta tomoni bor? javob:10

6.3. Qavariq yettiburchakning har bir uchidan bittadan olingan tashqi burchaklari yig‘indisini toping. javob:360°

6.4. Agar qavariq ko‘pburchakning har bir ichki burchagi: l) 108°; 2) 120°; 3) 150°; 4)160°; 5) 90°; b) 135° ga teng bo‘lsa, ko‘pburchakning nechta tomoni bor?

6.5. Muntazam *n* burchakning bitta ichki burchagi va unga mos tashqi burchagi necha gradusga teng?

6.6. Agar qavariq ko‘pburchak ichki burchaklarining yig‘indisi tashqi burchaklari yig‘indisidan 4 marta katta bo‘lsa, uning tomonlari nechta?

6.7. Qavariq o‘nikkiburchak ichki burchaklarining yig‘indisi qavariq oltiburchak ichki burchaklarining yig‘indisidan necha marta katta?

6.8. Ichki burchaklari yig‘indisi uning har bir uchidan bittadan olingan tashqi burchaklari yig‘indisidan 6 marta katta bo‘lgan ko‘pburchakning tomoni nechta?

6.9. Muntazam ko‘pburchakning uchidagi ichki va bitta tashqi burchagi ayirmasi 120° ga teng bo‘lsa, uning tomoni nechta bo‘ladi?

6.10. Qavariq ko‘pburchak ichki burchaklari va bitta tashqi burchagining yig‘indisi: 1)1830°; 2)2340° 3) 1360° 4)1150° 5) 1715° ga teng bo‘lsa, ko‘pburchakning nechta tomoni bor?

6.11. Ichki burchaklar yig‘indisining tashqi burchaklari yig‘indisiga nisbati 15:4 bo‘ladigan qavariq ko‘pburchak mavjudmi?

6.12. Agar qavariq ko‘pburchakning diagonallari soni: l) 90 ta; 2) 35 ta; 3) 4850 ta; 4) 275 ta; 5) 405 ta; uning nechta tomoni bor?

6.13. Diagonallari 14 ta bo‘lgan qavariq ko‘pburchakning nechta tomoni bor?

6.14. Qavariq ko‘pburchak diagonallari soni uning tomonlari sonidan l) 2 marta ko‘p; 2) 2,5 marta ko‘p; 3) 3 marta ko‘p; 4) 7 marta ko‘p; 5) 11 marta ko‘p bo‘lsa, ko‘pburchakning tomonlari sonini toping.

6.15. Qavariq ko‘pburchak diagonallari soni uning tomonlari soni *n* dan: l) 2 ta ortiq; 2) 5 ta ortiq; 3) 11 ta ortiq; 4) 90 ta ortiq; 5)1000 ta ortiq; b) 75 ta ortiq; 7) 150 ta ortiq; 8) 250 ta ortiq bo‘lishi mumkinmi?

6.16. Diagonallari soni 80 tadan kam emas va 100 tadan ko‘p emasligi ma’lum. Tomonlar soni n nechtaga teng bo‘lishi mumkin?

6.17. Qavariq beshburchak burchaklaridan ikkitasi to‘g‘ri, qolganlari o‘zaro $2:3:3\frac{4}{7}$ nisbatda. Beshburchakning katta burchagini toping.

6.18. Qavariq beshburchak burchaklaridan ikkitasi to‘g‘ri, qolganlari o‘zaro $2:3:3\frac{1}{5}$ nisbatda. Beshburchakning katta burchagini toping.

6.19. Qavariq beshburchak burchaklaridan ikkitasi to‘g‘ri, qolganlari o‘zaro 2:3:4 nisbatda. Beshburchakning katta burchagini toping.

6.20. ABCDE qavariq beshburchakda *AB, BC, AE* va *DE* tomonlar o‘zaro teng. *CD* tomon *AC* va *AD* diagonallarga teng. $\angle BAE=120°$ bo‘lsa, $\angle BCD$ ni toping.

6.21. *ABCDE* qavariq beshburchakdagi *ABC*, *BCD*, *CDE*, *DEA*, *EAB* uchburchaklarning har birining yuzi 1 ga teng. *ABCDE* qavariq beshburchakning yuzini toping.

6.22. Qavariq yettiburchakning burchaklari kattaliklari 2:3:4:5:6:7:9 kabi nisbatda. Yettiburchakning katta burchagi bilan kichik burchagi orasidagi farqini toping.

6.23. Qavariq *p* burchakning 5 ta burchagining har biri 140° ga teng, qolgan burchaklari esa o‘tkir burchaklardir. Shu ko‘pburchaklarning tomonlari sonini toping.

6.24. Qavariq to‘rtburchakning burchaklaridan birining gradus o‘lchovi qolgan uchta burchaklari gradus o‘lchovlari yig‘indisining 80% ini tashkil qiladi. To‘rtburchakning shu burchagini toping.

6.25. *ABCD* to‘rtburchakning *AC* diagonali uni ikkita teng uchburchakka ajratadi. Agar $\angle ABC=\angle ADC=40°$ bo‘lsa, *ABCD* to‘rtburchak qavariq bo‘ladimi?

6.26. *ABCD* qavariq to‘rtburchakning perimetri 21 metrga teng. *AB* tomoni uzunligi eng ko‘pi bilan qanday natural songa teng bo‘lishi mumkin?

6.27. Muntazam oltiburchak ichidagi ixtiyoriy nuqtadan uning tomonlari yotgan to‘g‘ri chiziqlargacha bo‘lgan masofalar yig‘indisi 9 ga teng. Shu oltiburchakning perimetrini toping.

6.28. Muntazam oltiburchak ichidagi ixtiyoriy nuqtadan uning tomonlari yotgan to‘g‘ri chiziqlargacha bo‘lgan masofalar yig‘indisi 9 ga teng bo‘lsa, shu oltiburchakning yuzini toping

6.29. Muntazam oltiburchak tomonining uzunligi 1 ga teng. Shu oltiburchak tomonlarining o‘rtalari ketma-ket tutashtirildi, so‘ngra hosil bo‘lgan oltiburchak tomonlarining o‘rtari yana ketma-ket tutashtirildi va h.k. Hosil bo‘lgan barcha oltiburchaklar yuzlarining yig‘indisini toping.

6.30. Muntazam oltiburchakning tomoni $4\sqrt{6}$ ga teng. Shu ko‘pburchakka tengdosh bo‘lgan teng tomonli uchburchakning tomonini toping.

6.31. Agar muntazam oltiburchakning yuzi $54\sqrt{3}$ sm^2 bo‘lsa, uning tomoni uzunligi topilsin.(sm)

6.32. *ABCDEF* muntazam oltiburchakning yuzi 144 ga teng. *ABC* uchburchakning yuzini toping.

6.33. Ikkita o‘xshash ko‘pburchak perimetrlarining nisbati 2:3 kabi. Katta ko‘pburchakning yuzi 27 bo‘lsa, kichik ko‘pburchakning yuzini toping.

6.34. Ikkita o‘xshash ko‘pburchak yuzlarining nisbati 9:4 ga teng. Kichik ko‘pburchakning perimetri 4 sm. Katta ko‘pburchakning perimetrini toping.

6.35. Ko‘pburchakning tomoni 5 ga teng. Yuzasi berilgan ko‘pburchakning yuzasidan 4 marta katta hamda unga o‘xshash bo‘lgan ko‘pburchakning mos tomonini toping.

6.36. O‘xshash ko‘pburchaklarning yuzlari mos ravishda 121 va 225 sm^2 ga teng. Agar ko‘pburchaklardan ikkinchisining perimetri birinchisinikidan 16 sm katta bo‘lsa, ularning perimetri topilsin.(sm)

6.37. Muntazam beshburchakning bir uchidan o‘tkazilgan ikki diagonali orasidagi burchakni toping.

2-§. Muntazam ko‘pburchakka ichki va tashqi chizilgan aylanalar. Muntazam ko‘pburchak perimetri va yuzasi

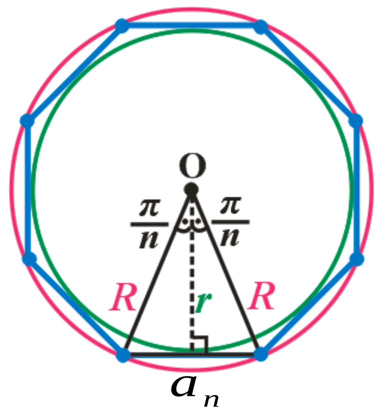
1-ta’rif. Ko‘pburchak tomonlariga urinuvchi aylanaga ko‘pburchakka ***ichki chizilgan aylana*** deyiladi. Ko‘pburchak uchlariga urinuvchi aylanaga ko‘pburchakka ***tashqi chizilgan aylana*** deyiladi.

2-teorema. Muntazam ko‘pburchakka ham ichki, ham tashqi aylana chizish mumkin. Tomoni a_n bo‘lgan muntazam ko‘pburchakka ichki va tashqi chizilgan aylana radiusi

$$\boxed{r=\frac{a_n}{2\operatorname{tg}\frac{\pi}{n}},\quad R=\frac{a_n}{2\sin\frac{\pi}{n}}.}$$

ga teng.

Isbot. Bunda ko‘pburchak markazini uning tomoni bilan tutashtirilganda yon tomonlari R ga, asosi a_n ga teng bo‘lgan teng yonli uchburchak hosil bo‘ladi (6.2-rasm). Bu teng yonli uchburchakning uchidagi burchagi $\frac{2\pi}{n}$ ga teng bo‘lib, uchidan asosiga tushirilgan balandlik bu burchakni $\frac{\pi}{n}$ ga teng bo‘lgan ikkita to‘g‘ri burchakli uchburchaklarga ajratadi.



6.2-rasm

Bu balandlik ko‘pburchakka ichki chizilgan aylana radiusidir, ya’ni $h_a=r$. To‘g‘ri burchakli uchburchak tangensi va sinusidan foydalanib so‘ralgan kattaliklarni osongina aniqlash mumkin. Unga ko‘ra so‘ralgan kattaliklar

$$\begin{cases}\operatorname{tg}\frac{\pi}{n}=\frac{a_n/2}{r}=\frac{a_n}{2r}\\ \sin\frac{\pi}{n}=\frac{a_n/2}{R}=\frac{a_n}{2R}\end{cases},\Rightarrow\begin{cases}r=\frac{a_n}{2\operatorname{tg}\frac{\pi}{n}}\\ R=\frac{a_n}{2\sin\frac{\pi}{n}}\end{cases}$$ bo‘ladi.

Muntazam ko‘pburchak tomonini ko‘pburchakka ichki va tashqi chizilgan aylana radiuslari orqali ifodalash mumkin:

$$\boxed{a_n = 2r\,\mathrm{tg}\frac{\pi}{n}, \quad a_n = 2R\sin\frac{\pi}{n}.}$$

Muntazam ko‘pburchakka ichki va tashqi chizilgan aylana radiuslari orasidagi bog‘lanish quyidagicha bo‘ladi:

$$\boxed{r = R\cos\frac{\pi}{n}.}$$

3-teorema. Muntazam qavariq ko‘pburchak yuzini R, r, a_n kattaliklar orqali ifodalash quyidagicha bo‘ladi:

$$\boxed{S_R = \frac{n}{2}\cdot\sin\frac{2\pi}{n}\cdot R^2, \quad S_r = n\cdot\mathrm{tg}\frac{\pi}{n}\cdot r^2, \quad S_a = \frac{n}{4}\cdot\mathrm{ctg}\frac{\pi}{n}\cdot a_n^{\,2}.}$$

Isbot. 6.2-rasmdagi teng yonli uchburchak yuzasi ikki tomon va ular orasidagi burchakka ko‘ra $S_0 = \frac{1}{2}\cdot\sin\frac{2\pi}{n}\cdot R^2$ ga teng bo‘ladi. Ko‘pburchak yuzasi bundan n marta katta, ya’ni $S = nS_0 = \frac{n}{2}\cdot\sin\frac{2\pi}{n}\cdot R^2$ bo‘ladi. Ko‘pburchak yuzini r orqali ifodalash uchun $R = r/\cos\frac{\pi}{n}$ formuladan foydalanib

$$S = \frac{n}{2}\cdot\sin\frac{2\pi}{n}\cdot\left(\frac{r}{\cos\frac{\pi}{n}}\right)^2 = \frac{n}{2}\cdot 2\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}\cdot\frac{1}{\cos^2\frac{\pi}{n}} = S_r = n\cdot\mathrm{tg}\frac{\pi}{n}\cdot r^2$$ formulani

keltirib chiqaramiz. Bunga esa $r = \frac{a_n}{2\,\mathrm{tg}\frac{\pi}{n}}$ formulani qo‘yish natijasida

$$S_a = n\cdot\mathrm{tg}\frac{\pi}{n}\cdot\left(\frac{a_n}{2\,\mathrm{tg}\frac{\pi}{n}}\right)^2 = \frac{n}{4}\cdot\mathrm{ctg}\frac{\pi}{n}\cdot a_n^{\,2}$$ formulani olamiz.

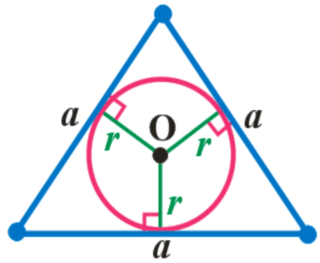
Endi bu formulaning xususiy hollardagi holatini ko‘rib chiqamiz.

$n=3$ bo‘lganda.

4-teorema. Muntazam uchburchakka ichki chizilgan aylana radiusi

$$\boxed{r=\frac{a_3}{2\sqrt{3}}=\frac{R}{2}=\frac{h}{3}.}$$

ga teng.



6.3-rasm

Isbot. Formuladan

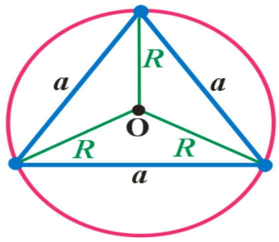
$$r=\frac{a_n}{2\operatorname{tg}\frac{\pi}{n}}=\frac{a_3}{2\operatorname{tg}\frac{\pi}{3}}=\frac{a_3}{2\cdot\sqrt{3}}=\frac{a_3}{2\sqrt{3}},$$

$r=\frac{a_3}{2\sqrt{3}}=\frac{R}{2}=\frac{h}{3}$ ekanligi kelib chiqadi(6.3-rasm).

5-teorema. Muntazam uchburchakka tashqi chizilgan aylana radiusi

$$\boxed{R=\frac{a_3}{\sqrt{3}}=2r=\frac{2}{3}h.}$$

ga teng.



6.4-rasm

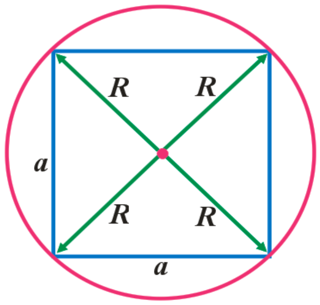
Isbot. Formuladan

$$R=\frac{a_n}{2\sin\frac{\pi}{n}}=\frac{a_3}{2\sin\frac{\pi}{3}}=\frac{a_3}{2\cdot\frac{\sqrt{3}}{2}}=\frac{a_3}{\sqrt{3}},$$

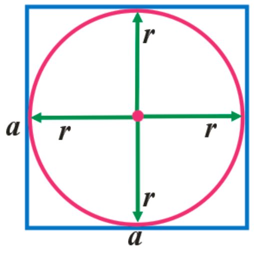
$R=\frac{a_3}{\sqrt{3}}=2r=\frac{2}{3}h$ ekanligi kelib chiqadi(6.4-rasm).

n=4 bo‘lganda. Kvadrat uchlariga urinadigan aylanaga kvadratga ***tashqi chizilgan aylana*** deyiladi. Kvadratga tashqi chizilgan aylananing diametri kvadrat diagonaliga teng (6.5-rasm):

$$\boxed{R=\frac{d}{2}=\frac{a}{\sqrt{2}}.}$$



6.5-rasm



6.6-rasm

Kvadrat tomonlariga urinadigan aylanaga kvadratga ***ichki chizilgan aylana*** deyiladi. Kvadratga ichki chizilgan aylananing diametri kvadrat tomoniga teng (6.6-rasm):

$$\boxed{r=\frac{a}{2}.}$$

6-teorema. Kvadratga tashqi va ichki chizilgan aylana radiuslari o‘zaro quyidagicha bog‘lanadi:

$$\boxed{R=\sqrt{2}\,r}$$

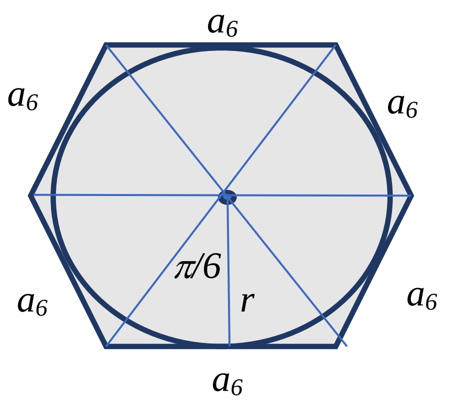
Isbot. Kvadratga tashqi va ichki chizilgan aylana radiuslaridan foydalanib aniqlaymiz. Unga ko‘ra

$$\begin{cases} r=\dfrac{a_4}{2\operatorname{tg}\dfrac{\pi}{4}}=\dfrac{a_4}{2}, \\ R=\dfrac{a_4}{2\sin\dfrac{\pi}{4}}=\dfrac{a_4}{2\dfrac{\sqrt{2}}{2}}=\dfrac{a_4}{\sqrt{2}} \end{cases} \Rightarrow \frac{r}{R}=\frac{\dfrac{a_4}{2}}{\dfrac{a_4}{\sqrt{2}}}=\frac{\sqrt{2}}{2}\Rightarrow R=\sqrt{2}r$$

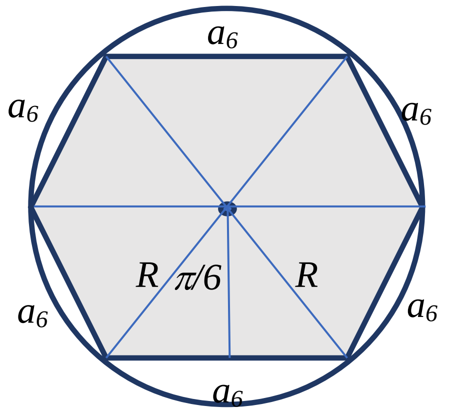
natija kelib chiqadi.

7-teorema. $n=6$ bo‘lganda. Muntazam oltiburchakka ichki chizilgan aylana radiusi quyidagicha bo‘ladi (6.7-rasm):

$$\boxed{r=\frac{\sqrt{3}a_6}{2}.}$$



6.7-rasm



6.8-rasm

Isbot. Formuladan $r = \dfrac{a_n}{2\operatorname{tg}\dfrac{\pi}{n}} = \dfrac{a_6}{2\operatorname{tg}\dfrac{\pi}{6}} = \dfrac{a_6}{2\cdot\dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}a_6}{2}$

ekanligi kelib chiqadi.

8-teorema. Muntazam oltiburchakka tashqi chizilgan aylana radiusi quyidagicha bo‘ladi (6.8-rasm):

$$\boxed{R = a_6.}$$

Isbot. Formuladan $R = \dfrac{a_n}{2\sin\dfrac{\pi}{n}} = \dfrac{a_6}{2\sin\dfrac{\pi}{6}} = \dfrac{a_6}{2\cdot\dfrac{1}{2}} = a_6$, ekanligi kelib chiqadi.

Muntazam ko’pburchakning perimetri $P_n = n\cdot a_n$ bo’ladi.

Quyidagi jadvalda masalalar yechishda ko‘p duch keladigan ba’zi muntazam ko‘pburchaklar uchun R, r, P_n, S_R, S_r, S_a kattaliklarning qiymatlari keltirilgan.

N №	R	r	P_n	S_R	S_r	S_a
3	$\frac{a_3}{\sqrt{3}}$	$\frac{a_3}{2\sqrt{3}}$	$3a_3$	$\frac{3\sqrt{3}}{4}R^2$	$3\sqrt{3}r^2$	$\frac{\sqrt{3}}{4}a_3^2$
4	$\frac{a_4}{\sqrt{2}}$	$\frac{a_4}{2}$	$4a_4$	$2R^2$	$4r^2$	a_4^2
6	a_6	$\frac{\sqrt{3}}{2}a_6$	$6a_6$	$\frac{3\sqrt{3}}{2}R^2$	$2\sqrt{3}r^2$	$\frac{3\sqrt{3}}{4}a_6^2$

3-§. Uchburchakka ichki va tashqi chizilgan aylanalar

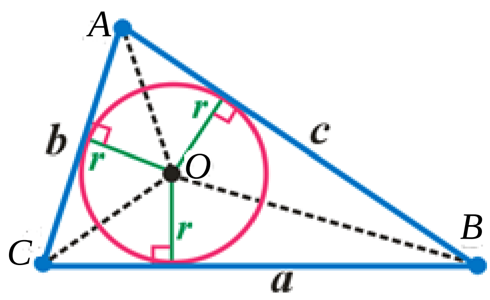
Uchburchakning barcha tomonlariga urinuvchi aylanaga uchburchakka ***ichki chizilgan aylana*** deyiladi. Uchburchak har qanday bo‘lmasin, unga ichki aylana chizish mumkin. Uchburchakka ichki chizilgan aylana markazi uchburchak uchlaridan chiquvchi bissektrisalarning kesishish chizig‘ida yotadi. Uchburchakka ichki chizilgan aylana markazi – O nuqta uchburchakning barcha tomonlaridan bir xil r masofada yotadi (6.9-rasm). Bu masofa esa ichki chizilgan aylana radiusidir.

9-teorema. Uchburchakka ichki chizilgan aylana radiusi

$$\boxed{r = \frac{2S}{a+b+c}}$$

ga teng.

Isbot. Uchta bissektrisa uchrashadigan O nuqta uchburchakka ichki chizilgan aylana markazi hamdir. Bissektrisalar O nuqtada uchrashganda berilgan ΔABC uchburchakni uchta $\Delta AOB, \Delta BOC, \Delta COA$ uchburchakka ajratadi (6.9-rasm).



6.9-rasm

Ichki chizilgan aylana radiusi r bu uchta uchburchakka O nuqtadan tushirilgan balandlik vazifasini ham o‘taydi. Bu uchburchaklar yuzalari

$$S_{AOB} = \frac{1}{2}AB \cdot r = \frac{1}{2}c\,r,\; S_{BOC} = \frac{1}{2}BC \cdot r = \frac{1}{2}a\,r,\; S_{COA} = \frac{1}{2}CA \cdot r = \frac{1}{2}b\,r$$

bo‘ladi. Bu uchta uchburchak yuzasi yig‘indisi berilgan ΔABC uchburchak yuzasini beradi, ya’ni $S_{ABC} = S_{AOB} + S_{BOC} + S_{COA}$ bo‘ladi. Shunga ko‘ra uchburchak yuzasi $S=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=\frac{1}{2}(a+b+c)r=\frac{1}{2}pr$ bo‘ladi. Bundan ichki chizilgan aylana radiusi $r=\frac{2S}{p}=\frac{2S}{a+b+c}$ hosil bo‘ladi.

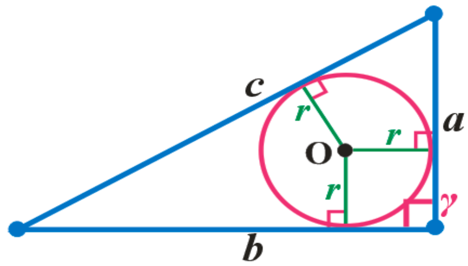
Yuqoridagi formuladan foydalanib, xususiy hollarni qarab chiqamiz.

10-teorema. To‘g‘ri burchakli uchburchakka ichki chizilgan aylana radiusi

$$\boxed{r=\frac{a+b-c}{2}}$$

ga teng.

Isbot. Isbotlash uchun ichki chizilgan aylana radiusini topish formulasi va Pifagor teoremasidan foydalanamiz (6.10-rasm). Shunga ko‘ra



6.10-rasm

$$r=\frac{2S}{a+b+c}=\frac{2\cdot\frac{1}{2}ab}{a+b+c}\cdot\frac{a+b-c}{a+b-c}=\frac{ab(a+b-c)}{(a+b)^2-c^2}=$$

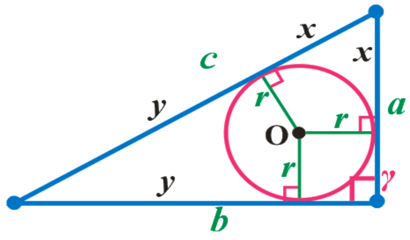
$$=\frac{ab(a+b-c)}{a^2+2ab+b^2-c^2}=\frac{ab(a+b-c)}{2ab}=\frac{a+b-c}{2}$$

bo‘ladi.

11-teorema. Agar to‘g‘ri burchakli uchburchakka ichki chizilgan aylana gipotenuzani urinish nuqtasida x va y kesmalarga ajratsa, u holda uchburchakning katetlari a va b, yuzasi S hamda ichki chizilgan aylana radiusi r quyidagi formulalar orqali aniqlanadi:

$$\boxed{\begin{gathered}a=\frac{\sqrt{(x+y)^2+4xy}+x-y}{2},\quad b=\frac{\sqrt{(x+y)^2+4xy}+y-x}{2},\\ r=\frac{\sqrt{(x+y)^2+4xy}-x-y}{2},\quad S=xy.\end{gathered}}$$

Isbot. Pifagor teoremasidan foydalanamiz. Unga ko‘ra(6.11-rasm)



6.11-rasm

$$(x+y)^2=(x+r)^2+(y+r)^2,\Rightarrow$$

$$x^2+2xy+y^2=x^2+2xr+r^2+y^2+2yr+r^2,\Rightarrow$$

$$r^2+(x+y)r-xy=0,\Rightarrow\ D=(x+y)^2+4xy,\ \Rightarrow$$

$$r=\frac{\sqrt{(x+y)^2+4xy}-x-y}{2}$$

bo‘ladi.

Uchburchak katetlari

$$a=r+x=\frac{\sqrt{(x+y)^2+4xy}+x-y}{2}\ \text{va}\ b=r+y=\frac{\sqrt{(x+y)^2+4xy}+y-x}{2};$$

yuzasi esa

$$S=\frac{1}{2}ab=\frac{1}{2}\frac{\sqrt{(x+y)^2+4xy}+x-y}{2}\cdot\frac{\sqrt{(x+y)^2+4xy}-(x-y)}{2}=\frac{1}{8}\left[(x+y)^2+4xy-(x-y)^2\right]=$$

$$=\frac{1}{8}\left[x^2+6xy+y^2-x^2+2xy-y^2\right]=xy$$

bo‘ladi.

12-teorema. Agar to‘g‘ri burchakli uchburchak gipotenuzasiga tushirilgan bissektrisa gipotenuzani x va y kesmalarga ajratsa, u holda uchburchakning katetlari a va b, yuzasi S hamda ichki chizilgan aylana radiusi r quyidagi formulalar orqali aniqlanadi:

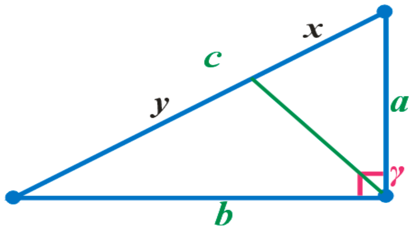
$$\boxed{a=\frac{x^2+xy}{\sqrt{x^2+y^2}},\ b=\frac{y^2+xy}{\sqrt{x^2+y^2}},\ r=\frac{x+y}{2\sqrt{x^2+y^2}}\left(x+y-\sqrt{x^2+y^2}\right),\ S=\frac{xy(x+y)^2}{2(x^2+y^2)}.}$$

Isbot. Bunda α burchak tangensi va sinusidan foydalanamiz (6.12-rasm). Unga ko‘ra

$$\operatorname{tg}\alpha=\frac{a}{b}=\frac{x}{y},\ \sin\alpha=\frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}}=\frac{x}{\sqrt{x^2+y^2}},$$

$$a=(x+y)\sin\alpha=(x+y)\frac{x}{\sqrt{x^2+y^2}}=\frac{x^2+xy}{\sqrt{x^2+y^2}},$$

$$b=\frac{a}{\operatorname{tg}\alpha}=a\frac{y}{x}=\frac{x^2+xy}{\sqrt{x^2+y^2}}\cdot\frac{y}{x}=\frac{y^2+xy}{\sqrt{x^2+y^2}}$$ bo‘ladi.

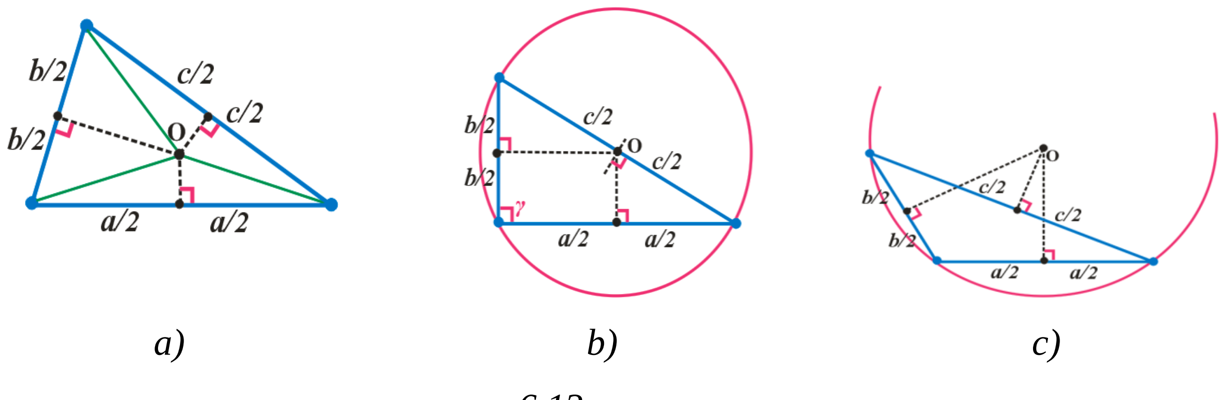


6.12-rasm

Uchburchak yuzasi esa $S=\frac{ab}{2}=\frac{1}{2}\frac{x^2+xy}{\sqrt{x^2+y^2}}\cdot\frac{y^2+xy}{\sqrt{x^2+y^2}}=\frac{xy(x+y)^2}{2(x^2+y^2)}$ bo‘ladi. Ichki chizilgan aylana radiusi to‘g‘ri burchakli uchburchak uchun $r=\frac{a+b-c}{2}$ ekanini e’tiborga olsak, $r=\frac{x+y}{2\sqrt{x^2+y^2}}\left(x+y-\sqrt{x^2+y^2}\right)$ natija kelib chiqadi.

Uchburchakning uchlariga urinuvchi aylanaga uchburchakka ***tashqi chizilgan aylana*** deyiladi. Uchburchak har qanday bo‘lmasin, unga tashqi aylana chizish mumkin.

Uchburchakka tashqi chizilgan aylana markazi tomonlar o‘rtalaridan perpendikular holda chiqarilgan o‘rta perpendikularlarning kesishish nuqtasida yotadi. Bu nuqta uchburchak uchlaridan bir xil R masofada joylashgan. Uchburchakka tashqi chizilgan aylana markazi o‘tkir burchakli uchburchakda uchburchak ichida, to‘g‘ri burchakli uchburchakda gipotenuza o‘rtasida, o‘tmas burchakli uchburchakda esa uchburchak tashqarisida yotadi (6.13-rasm).

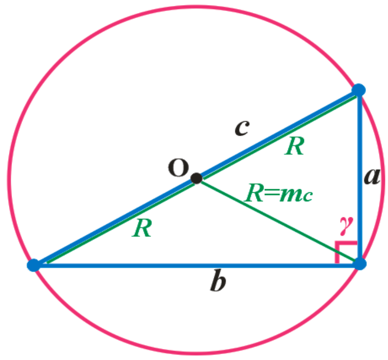


6.13-rasm

13-teorema. Uchburchakka tashqi chizilgan aylana radiusi

$$\boxed{R = \frac{abc}{4S}.}$$

ga teng.



6.14-rasm

Isbot. Sinuslar teoremasidan $\frac{c}{\sin\gamma} = 2R$ ekanini bilamiz. Shundan $R = \frac{1}{2}\frac{c}{\sin\gamma} = \frac{abc}{2bc\sin\gamma} = \frac{abc}{4S}$ hosil bo‘ladi.

14-teorema. To‘g‘ri burchakli uchburchakka tashqi chizilgan aylana radiusi gipotenuzaning yarmiga yoki gipotenuzaga tushirilgan mediana uzunligiga teng (6.14-rasm):

$$\boxed{R = \frac{c}{2} = m_c.}$$

Isbot. Formuladan $R = \frac{abc}{4S} = \frac{abc}{4 \cdot \frac{1}{2}ab} = \frac{c}{2}$ ekanligi kelib chiqadi.

Mustaqil ishlash uchun masalalar

6.38. To‘g‘ri burchakli uchburchakning bitta o‘tkir burchagi 40° ga teng. Bu uchburchakning tashqi aylana markazidan to‘g‘ri burchak uchiga o‘tkazilgan to‘g‘ri chiziq va gipotenuzasi orasidagi burchakni toping. javob: 80°

6.39. Asosi 12 va balandligi 8 bo‘lgan teng yonli uchburchakka ichki chizilgan aylanaga asosga parallel urinma o‘tkazilgan. Urinmaning uchburchak tomonlari bilan qisilgan kesimi uzunligini toping. javob:3

6.40. To‘g‘ri burchakli uchburchakka ichki va tashqi chizilgan aylana radiuslari mos ravishda 4 sm va 10 sm bo‘lsa, to‘g‘ri burchakli uchburchak yuzini toping. javob:96

6.41. Teng yonli to‘g‘ri burchakli uchburchakka tashqi chizilgan aylana radiusi $\sqrt{2}+1$ ga teng bo‘lsa, bu uchburchakka ichki chizilgan aylananing radiusini toping. javob:1

6.42. To‘g‘ri burchakli uchburchakka tashqi chizilgan aylananing radiusi $\sqrt{3}$ ga teng. Bu aylananing markazidan katetlarigacha bo‘lgan masofalarning nisbati $\sqrt{3}$ ga teng bo‘lsa, katta katetning uzunligini toping. javob: 3

6.43. To‘g‘ri burchakli uchburchak *ABC* da $\angle B$=30°, $\angle C$=90°, *O* tashqi chizilgan aylananing markazi. Agar *AO*=12 bo‘lsa, tashqi chizilgan aylananing radiusi uzunligini toping. javob: 6

6.44. Bir burchagi 120° li uchburchakka tashqi chizilgan aylananing radiusi $6\sqrt{3}$ ga teng bo‘lsa, bu uchburchakning asosining uzunligini toping. javob: 18

6.45. Teng yonli to‘g‘ri burchakli uchburchakka ichki chizilgan aylananing radiusi uzunligi $\sqrt{2}-1$ ga teng bo‘lsa, bu uchburchakning balandligi uzunligini toping. javob: 1

6.46. Teng yonli uchburchakning yon tomoni uzunligi 5 ga va asosidagi burchagi 30° ga teng bo‘lsa, tashqi chizilgan aylananing diametrini toping. javob: 10

6.47. O‘tkir burchagi 30° ga teng bo‘lgan to‘g‘ri burchakli uchburchakka tashqi chizilgan aylananing radiusi $3-\sqrt{3}$ ga teng bo‘lsa, bu uchburchakning perimetrini toping. javob: 6

6.48. O‘tkir burchagi 30° ga teng bo‘lgan to‘g‘ri burchakli uchburchakning kichik kateti va gipotenuzasining yig‘indisi 3 ga teng. Bu uchburchakka tashqi chizilgan aylananing radiusini toping. javob: 1

6.49. To‘g‘ri burchakli uchburchakning tashqi aylana markazi va to‘g‘ri burchagi uchini tutashtiruvchi kesma va katet orasidagi burchak 52° ga teng bo‘lsa, bu uchburchakning kichik o‘tkir burchagini toping. javob: 38°

6.50. To‘g‘ri burchakli teng yonli uchburchakning gipotenuzasi uzunligi $\frac{\sqrt{2}+1}{4}$ ga teng. Bu uchburchakning bir katetiga urinuvchi va bu katet qarshisidagi o‘tkir burchak uchidan o‘tuvchi aylana markazi gipotenuzada bo‘lsa, bu aylananing radiusini toping. javob: 0,25

6.51. O‘tkir burchagi 60° ga teng, to‘g‘ri burchakli uchburchakning katta katetiga urinuvchi va bu katet qarshisidagi burchak uchidan o‘tuvchi aylananing markazi gipotenuzada, hamda bu aylana radiusi uzunligi $\frac{3-\sqrt{3}}{5}$ ga teng bo‘lsa, bu uchburchakning perimetrini toping. javob: 1,8

6.52. To‘g‘ri burchakli uchburchakka ichki chizilgan aylananing radiusi 3 ga va bu uchburchakning kichik kateti uzunligi 10 ga teng bo‘lsa, tashqi chizilgan aylana radiusi uzunligini toping. javob: 7,25

6.53. To‘g‘ri burchakli uchburchakning katta katetiga urinuvchi va bu katet qarshisidagi o‘tkir burchak uchidan o‘tuvchi aylana markazi gipotenuzada. Bu uchburchakning katetlari uzunliklari 3 va $2\sqrt{10}$ ga teng bo‘lsa, bu aylana radiusining uzunligini toping. javob: 2,1

6.54. To‘g‘ri burchakli uchburchakka ichki chizilgan aylananing radiusi 2 ga va bu uchburchakning bir kateti uzunligi 14 ga teng bo‘lsa, bu uchburchakka tashqi chizilgan aylananing radiusini uzunligini toping. javob: 14,8

6.55. To‘g‘ri burchakli uchburchakka ichki chizilgan aylana gipotenuzaga urinish nuqtasida uni uzunliklari 3 va 10 ga teng kesmalarga ajratadi. Katta katetning uzunligini toping. javob: 12

6.56. To‘g‘ri burchakli uchburchakning gipotenuzasidagi nuqta, katetlardan teng uzoqlikda joylashgan, hamda gipotenuzani 6 va 8 uzunliklardagi kesmalarga ajratadi. Katta katetning uzunligini toping. javob: 11,2

6.57. To‘g‘ri burchakli uchburchakka ichki chizilgan aylananing radiusi 3 ga va bu uchburchakning bir kateti uzunligi 8 ga teng bo‘lsa, bu uchburchak gipotenuzasining uzunligini toping. javob: 17

6.58. To‘g‘ri burchakli uchburchakka ichki chizilgan aylananing radiusi 4 ga va bu uchburchakning bir kateti uzunligi 9 ga teng bo‘lsa, bu uchburchakning ikkinchi katetining uzunligini toping. javob: 40

6.59. To‘g‘ri burchakli uchburchakka ichki chizilgan aylana gipotenuzaga urinish nuqtasida uni uzunliklari 5 va 12 ga teng kesmalarga ajratadi. Kichik katetning uzunligini toping. javob:8

6.60. ABC uchburchakda $\angle BAC$=60° va tashqi chizilgan aylananing markazi O bo‘lsa hamda bu aylananing radiusi $\sqrt{3}$ ga teng bo‘lsa, OBC uchburchakning yuzini toping. javob: 0,75

6.61. ABC uchburchakda $\angle BAC$=45°, $\angle ABC$=30° va tashqi chizilgan aylananing markazi O bo‘lsa, hamda bu aylananing radiusi $\sqrt{2-\sqrt{3}}$ ga teng bo‘lsa, $AOBC$ to‘rtburchakning yuzini toping. javob: 0,25

6.62. ABC uchburchakda $\angle BAC$=60° va tashqi chizilgan aylananing markazidan BC tomongacha bo‘lgan masofa 1,3 ga teng bo‘lsa, bu tashqi chizilgan aylananing radiusi uzunligini toping. javob: 2,6

6.63. ABC uchburchakda $\angle BAC$=30°, $\angle ABC$=45° va tashqi chizilgan aylananing markazi O bo‘lsa, hamda bu aylananing radiusi $3-\sqrt{2}$ ga teng bo‘lsa, $AOBC$ to‘rtburchakning perimetrini toping. javob:7

6.64. Uchburchak ABC da BC tomon uzunligi tashqi chizilgan aylananing radiusi uzunligiga teng bo‘lsa, $\angle BAC$ ni toping. javob: 30°

6.65. ABC da $\angle ACB$=120° ga va tashqi chizilgan aylananing radiusi uzunligi $\sqrt{75}$ ga teng bo‘lsa, AB tomon uzunligini toping. javob: 15

6.66. Uchburchak ABC da $BC = 2\sqrt{2}$ va $\angle BAC$=45° ga teng bo‘lsa, bu uchburchakka tashqi chizilgan aylananing radiusi uzunligini toping. javob: 2

6.67. ABC uchburchakda $\angle BAC$=15°, $\angle ABC$=45° va tashqi chizilgan aylananing markazi O bo‘lsa, $\angle AOB$ ning kosinus qiymatini toping. javob:-0.5

6.68. *ABC* uchburchakda $\angle BAC$=45° va tashqi chizilgan aylananing markazi *O* nuqtada. Agar uchburchak *OBC* ning yuzi 18 ga teng bo'lsa, bu tashqi chizilgan aylana radiusining uzunligini toping. javob:6

6.69. *ABC* uchburchakda $\angle ABC$=45° va tashqi chizilgan aylananing radiusi $\sqrt{8}$ ga teng bo'lsa, *AC* tomonning uzunligini toping. javob: 4

6.70. *ABC* uchburchakda $\angle ABC$=60° va $AC = \frac{6}{\sqrt{\pi}}$ ga teng. Agar uchburchakning perimetri $\frac{15}{\sqrt{\pi}}$ ga teng bo'lsa, uchburchakka ichki chizilgan aylananing yuzini toping. javob: 0,75

6.71. To'g'ri burchakli uchburchakning gipotenuzasiga tushirilgan medianasining uzunligi $\frac{5}{\sqrt{\pi}}$ va uchburchakning perimetri $\frac{24}{\sqrt{\pi}}$ ga teng bo'lsa, uchburchakka ichki chizilgan aylana radiusining uzunligini toping. javob: 4

6.72. Uchburchak *ABC* ning perimetri 9 ga teng. Bu uchburchakka ichki chizilgan aylananing radiusi $\sqrt{3}$ ga teng. Agar *AC*=3,5 bo'lsa, ichki chizilgan aylananing markazidan *B* uchigacha bo'lgan masofani toping. javob: 2

6.73. To'g'ri burchakli uchburchakning to'g'ri burchagi uchidan ichki chizilgan aylana markazigacha bo'lgan masofa $\sqrt{2}$ ga teng. Tashqi chizilgan aylananing radiusi uzunligi esa 2,5 ga teng bo'lsa, uchburchakning perimetrini toping. javob: 12

6.74. Uchburchak *ABC* da $AC = 6\sqrt{3}$ va $\angle ABC$=60° ga teng. Agar uchburchakning perimetri $14\sqrt{3}$ ga teng bo'lsa, *B* uchidan ichki chizilgan aylana markazigacha bo'lgan masofani toping. javob:2

6.75. Uchburchak *ABC* da *AC*=13 va $\angle ABC$=120° ga teng. Agar ichki chizilgan aylananing radiusi 3 ga teng bo'lsa, bu uchburchakning perimetrini toping. javob: 28

6.76. To‘g‘ri burchakli uchburchakda ichki chizilgan aylana gipotenuzani urinish nuqtasida 2 va 3 ga teng kesmalarga ajratadi. Ichki chizilgan aylananing radiusi uzunligini toping. javob: 1

6.77. To‘g‘ri burchakli uchburchakning perimetri 24 ga va tashqi chizilgan aylananing radiusi 5 ga teng bo‘lsa, ichki chizilgan aylananing radiusi uzunligini toping. javob:2

6.78. To‘g‘ri burchakli uchburchakning ichki chizilgan aylana radiusi 3,5 ga va perimetri 36 ga teng. Tashqi chizilgan aylananing radiusini toping. javob: 7,25

6.79. Uchburchak *ABC* da *AB=BC* va *AC*=10 berilgan. *AB* tomon o‘rtasi *D* nuqtadan *AB* tomonga o‘tkazilgan perpendikular *BC* tomonni *E* nuqtada kesadi. Agar uchburchak *ABC* ning perimetri 40 ga teng bo‘lsa, *AEC* ning perimetrini toping. javob: 25

6.80. Uchburchakning tomonlari 2; 3,5 va 4 ga teng. Katta tomonda olingan nuqtadan qolgan ikkala tomongacha bo‘lgan masofalar teng. Bu nuqta orqali katta tomon qanday uzunlikdagi kesmalarga bo‘linadi? javob: 2,8; 1,6

6.81. Teng yonli uchburchakda balandlik uzunligi 32 ga teng. Yon tomoni va asosi 2:1 nisbatda bo‘lsa bu uchburchakka ichki chizilgan aylana radiusini toping. javob: 6,4

6.82. Uchburchakning tomonlari uzunliklari 5; 6 va 10 ga teng. Uchburchakning kichik tomoniga mediana va bissektrisa o‘tkazilgan, bu tomondagi mediana va bissektrisa asoslari orasidagi masofani toping. javob: 0,625

6.83. Uchburchakning tomonlari uzunliklari 8; 12 va 12,5 ga teng. Ikkita kichik tomonga urinuvchi aylana markazi katta tomonda. Bu aylana markazi katta tomondani qanday uzunliklardagi kesmalarga ajratadi? javob: 5; 7,5

6.84. To‘g‘ri burchakli uchburchakning 3 va 4,25 uzunliklardagi tomonlariga katta tomonga yopishgan burchaklarining bissektrisalari o‘tkazilgan, bu bissektrisalarning tomonlarni kesib o‘tgan nuqtalari orasidagi masofani toping. javob: 1,75

6.85. Teng yonli uchburchakda asosiga tushirilgan balandlikni ichki aylana markazi 5:3 nisbatda bo‘ladi. Yon tomoni 8,5 ga teng bo‘lsa, asosning uzunligini toping. javob: 10,2

6.86. To‘g‘ri burchakli uchburchakning katetlari 18 va 24 ga teng. Katta o‘tkir burchagi uchidan bissektrisa o‘tkazilgan, bu bissektrisaning katta katetdagi proyeksiyasi uzunligini toping. javob: 9

6.87. Uchburchak *ABC* ga ichki ravishda *BDEF* romb chizilgan, *D,E,F* nuqtalar mos ravishda *BC,AC* va *AB* tomonlarda yotadi. Agar *BC*=7, *AC*=5 va *AB*=3 bo‘lsa, *EC* ni toping. javob: 3,5

6.88. Teng yonli uchburchakda ichki chizilgan aylana radiusi asosga tushirilgan balandlikning $\frac{5}{11}$ qismiga teng. Agar uchburchakning asosi uzunligi 11 ga teng bo‘lsa, bu uchburchakning yon tomoni uzunligini toping. javob: 6,6

6.89. Uchburchak *ABC* da *BAC* va *BCA* burchaklari mos ravishda 60° va 45° ga teng. Bu uchburchakka tashqi chizilgan aylana radiusi $\sqrt{3-\sqrt{3}}$ ga teng bo‘lsa, bu uchburchakning yuzini toping. javob: 1,5

6.90. Uchburchak *ABC* da *BAC* va *ABC* burchaklar qiymati 30° va 75° ga teng. Bu uchburchakka tashqi chizilgan aylana radiusi $3\left(\sqrt{6}-\sqrt{2}\right)$ ga teng bo‘lsa, *AB* tomon uzunligini toping. javob:6

6.91. Teng yonli uchburchakka tashqi chizilgan aylana radiusi $4\sqrt{3}$ ga teng. Asosining o‘rtasi va yon tomoni o‘rtasini tutashtiruvchi kesma uzunligi tashqi chizilgan aylana radiusidan $\sqrt{2}$ marta kichik bo‘lsa, bu teng yonli uchburchakning yuzini toping. javob: 48

6.92. Uchburchak *ABC* da *BAC* va *B* burchaklar qiymati 30° va 75° ga teng. Agar $BC=3\left(\sqrt{6}+\sqrt{2}\right)$ ga teng bo‘lsa, bu uchburchakka tashqi chizilgan aylana markazidan *AC* tomongacha bo‘lgan masofani toping. javob:3

6.93. Uchburchak *ABC* da *BAC* va *BCA* burchaklar qiymati 45° va 60° ga teng. $BC = \sqrt[4]{12}$ ga teng va *O* uchburchak *ABC* ga tashqi chizilgan aylana markazi bo‘lsa, *BOA* uchburchakning yuzini toping. javob: 0,75

6.94. Uchburchak *ABC* da *BAC* va *BCA* burchaklar qiymati 30° va 75° ga teng. Agar uchburchakka tashqi chizilgan aylana radiusi $2-\sqrt{3}$ ga teng bo‘lsa, *BC* tomonga tushirilgan balandlik uzunligini toping. javob: 0,5

6.95. Uchburchak *ABC* da *BAC* va *BCA* burchaklar qiymati 45° va 60° ga teng. Tashqi chizilgan aylana markazidan *BC* tomongacha bo‘lgan masofa $\sqrt{6}$ ga teng bo‘lsa, *AB* tomon uzunligini toping. javob: 6

6.96. Uchburchak *ABC* da *BAC* va *BCA* burchaklar qiymati 45° va 60° ga teng. Tashqi chizilgan aylana markazidan *BC* tomongacha bo‘lgan masofa $3-\sqrt{3}$ ga teng bo‘lsa, *BC* tomonga tushirilgan balandlik uzunligini toping. javob:3

6.97. Uchburchak *ABC* da *BAC* va *BCA* burchaklar qiymati 45° va 60° ga teng. Tashqi chizilgan aylana radiusi $\sqrt{8}-\sqrt{2}$ teng bo‘lsa, *AC* tomonning uzunligini toping. javob:2

6.98. Uchburchak *ABC* da *BAC* va *BCA* burchaklar qiymati 45° va 60° ga teng. *AC* tomonning uzunligi $\frac{3+\sqrt{3}}{\sqrt{6}}$ ga teng bo‘lsa, tashqi chizilgan aylana markazini toping. javob: 1

6.99. Teng yonli uchburchakning yon tomoni 6 ga va asosi uzunligi 8 ga teng. Bu uchburchakka ichki chizilgan aylanaga asosga parallel qilib o‘tkazilgan urinmaning uchburchak tomonlari ichidagi kesmasi uzunligini toping. javob: 1,6

6.100.Teng yonli uchburchakning yon tomoni 17 ga va asosi uzunligi 16 ga teng. Bu uchburchakka ichki chizilgan aylanaga asosga tushirilgan balandlikga parallel qilib o‘tkazilgan urinmaning chiziqning uchburchak tomonlari ichidagi kesmasi uzunligini toping. javob:6

6.101.Rombning tomoni uzunligi 10 ga teng, katta diagonali uzunligi esa 16 ga teng. Rombga ichki chizilgan aylanaga kichik diagonaliga parallel qilib o‘tkazilgan urinmaning romb ichidagi kesmasi uzunligini toping. javob: 4,8

6.102. Radiusi $\sqrt{3}$ ga teng aylanaga tashqi chizilgan teng yonli trapetsiyaning katta asosi uzunligi 6 ga teng. Trapetsiyaning yon tomonlari davom ettirilgandan hosil qilingan teng yonli uchburchakning yon tomoni uzunligini toping. javob:6

6.103. Asosining uzunligi 15,5 ga teng bo‘lgan, teng yonli uchburchakka ichki chizilgan aylanaga asosga parallel qilib o‘tkazilgan urinmaning uchburchak tomonlari ichidagi kesmasi uzunligi 10,5 ga teng bo‘lsa, bu uchburchakning yon tomoni uzunligini toping. javob: 40,3

6.104. Teng yonli uchburchakning asosi uzunligi va asosga tushirilgan balandlik uzunligi 12 va 8 ga teng. Bu uchburchakka ichki chizilgan aylana asosga parallel qilib o‘tkazilgan urinma orqali teng yonli uchburchak va trapetsiyaga ajratilgan. Bu trapetsiyaning yuzini toping. javob: 45

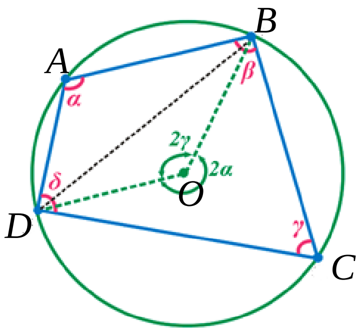
4-§. To‘rtburchakka ichki va tashqi chizilgan aylanalar

To‘rtburchak uchlariga urinadigan aylanaga to‘rtburchakka ***tashqi chizilgan aylana*** deyiladi. Umumiy holda berilgan hamma to‘rtburchaklarga ham tashqi aylana chizib bo‘lavermaydi. Buning uchun ma’lum bir shartlar bajarilish kerak.

15-teorema. Agar to‘rtburchakning qarama-qarshi burchaklari yig‘indisi 180^0 ga teng bo‘lsa, bunday to‘rtburchakka tashqi aylana chizish mumkin

$$\boxed{\begin{cases} \alpha+\gamma=180^0, \\ \beta+\delta=180^0. \end{cases}}$$

Isbot. BD vatarga kichik yoyning ixtiyoriy nuqtasida turib qaralganda bir xil α burchak ostida ko‘rinadi, shuningdek, A nuqtadan qaralganda ham. Bunga mos markaziy burchak 2α ga teng (6.15-rasm):. BD vatarga katta yoyning ixtiyoriy nuqtasida turib qaralganda bir xil γ burchak ostida ko‘rinadi, shuningdek, C nuqtadan qaralganda ham. Bunga mos markaziy burchak 2γ ga teng.



6.15-rasm

Bir aylana 2π ga tengligidan $2\alpha + 2\gamma = 2\pi$ bo‘ladi. Bundan esa qarama-qarshi burchaklar yig‘indisi $\alpha + \gamma = \pi$ ga tengligi kelib chiqadi.

Xuddi shuningdek, AC vatarga katta va kichik yoylardan qaralganda ko‘rinma burchaklardan $\beta + \delta = \pi$ ekanligi kelib chiqadi.

16-teorema. To‘rtburchakka tashqi chizilgan aylana radiusi

$$\boxed{d_1^2 = \frac{ab\left(c^2+d^2\right)+cd\left(a^2+b^2\right)}{ab+cd}, \qquad R = \frac{abx}{\sqrt{(2ab)^2-\left(a^2+b^2-x^2\right)^2}}.}$$

ga teng.

Isbot. Diagonal $d_1 = x$ deb belgilash kiritamiz (6.16-rasm):. Kosinuslar qonuni va $\alpha + \gamma = \pi$ ga ko‘ra

$$\begin{cases} x^2 = a^2 + b^2 - 2ab\cos\alpha \\ x^2 = c^2 + d^2 - 2cd\cos(\pi - \alpha) = c^2 + d^2 + 2cd\cos\alpha \end{cases}$$

bo‘ladi. Bundan

$$a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 + 2cd\cos\alpha \quad \Rightarrow \quad \cos\alpha = \frac{a^2+b^2-\left(c^2+d^2\right)}{2(ab+cd)}$$

kelib chiqadi. Bundan

$$x^2 = a^2 + b^2 - 2ab\frac{a^2 + b^2 - \left(c^2 + d^2\right)}{2\left(ab + cd\right)} =$$

$$= \frac{ab\left(a^2 + b^2\right) + cd\left(a^2 + b^2\right) - ab\left(a^2 + b^2\right) + ab\left(c^2 + d^2\right)}{ab + cd} =$$

$$= \frac{ab\left(c^2 + d^2\right) + cd\left(a^2 + b^2\right)}{ab + cd}$$

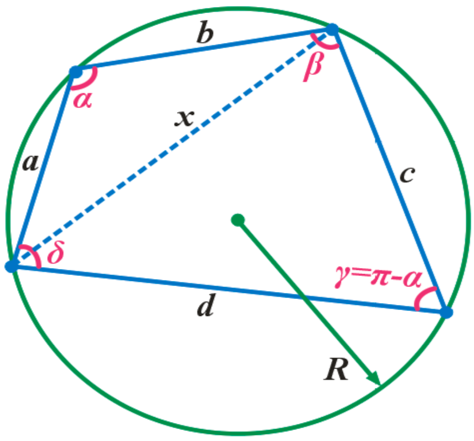
ifoda hosil bo‘ladi. $abcd$ to‘rtburchakka tashqi chizilgan aylana bir vaqtda abx hamda cdx uchburchaklarga ham tashqi chizilgan aylanadir. Shuning uchun tashqi chizilgan aylana radiusi

$$R = \frac{abx}{4S} = \frac{abx}{\sqrt{(2ab)^2 - \left(a^2 + b^2 - x^2\right)^2}}$$

yoki

$$R = \frac{cdx}{4S} = \frac{cdx}{\sqrt{(2cd)^2 - \left(c^2 + d^2 - x^2\right)^2}}$$

bo‘ladi.



6.16-rasm

Umumiy holda berilgan hamma to‘rtburchaklarga ham tashqi aylana chizib bo‘lmaydi. Buning uchun ma’lum bir shartlar bajarilish kerak.

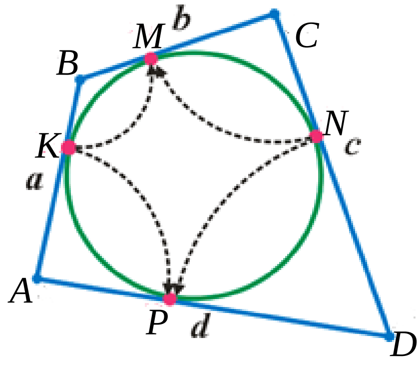
17-teorema. Agar to‘rtburchakning qarama-qarshi tomonlari yig‘indisi o‘zaro teng bo‘lsa, bunday to‘rtburchakka ichki aylana chizish mumkin.

$$\boxed{a + c = b + d.}$$

Isbot. Aylanadan tashqaridagi nuqtadan aylanaga 2 ta urinma o‘tkazish mumkin va urinish nuqtasigacha masofalar o‘zaro teng bo‘ladi. Shunga ko‘ra $AK = AP, BK = BM, CM = CN, DN = DP$ bo‘ladi (6.17-rasm):. Bundan esa

$$a + c = AB + CD = (AK + BK) + (CN + DN) =$$
$$= (AP + BM) + (CM + DP) =$$
$$= (AP + DP) + (BM + CM) = AD + BC = d + b$$

kelib chiqadi.

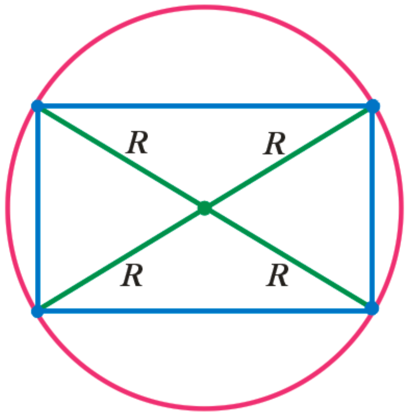


6.17-rasm

Faqat tomonlar uzunliklari ma’lum bo‘lganda to‘rtburchakka ichki chizilgan aylana radiusini aniqlab bo‘lmaydi. Bunda burchaklarni o‘zgartirib turli to‘rtburchaklar hosil qilish mumkin. Bu to‘rtburchaklarga esa turli aylanalar ichki chizish mumkin. To‘rtburchakka ichki chizilgan aylana radiusini aniqlash uchun to‘rtala tomondan tashqari yana bitta burchak berilgan bo‘lishi kerak.

Agar parallelogrammning qo‘shni tomonlar va qo‘shni burchaklar o‘zaro teng bo‘lmasa $(a \neq b, \alpha \neq \beta)$, bu parallelogrammga umumiy holda berilgan parallelogramm deyiladi. Umumiy holda berilgan parallelogrammga tashqi aylana ham, ichki aylana ham chizib bo‘lmaydi.

To‘g‘ri to‘rtburchak uchlariga urinuvchi aylanaga unga tashqi chizilgan aylana deyiladi. To‘g‘ri to‘rtburchak diagonali unga tashqi chizilgan aylana diametri bilan mos tushadi. Boshqacha aytganda, tashqi chizilgan aylana radiusi uning diagonali yarmiga teng bo‘ladi (6.18-rasm):



6.18-rasm

$$\boxed{R = \frac{d}{2} = \frac{1}{2}\sqrt{a^2 + b^2}.}$$

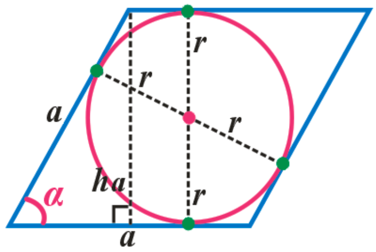
Rasmdan ko‘rinib turibdiki, to‘g‘ri to‘rtburchak diagonali uni ikkita to‘g‘ri burchakli uchburchakka ajratadi. Shundan ham ko‘rish mumkinki, to‘g‘ri burchakli uchburchakka tashqi chizilgan aylana markazi har doim gipotenuza markazida yotar ekan.

Umumiy holda berilgan to‘g‘ri to‘rtburchakka $(a \neq b)$ ichki aylana chizib bo‘lmaydi.

Rombga tashqi aylana chizib bo‘lmaydi.

Romb tomonlariga urinadigan aylanaga ***ichki chizilgan aylana*** deyiladi.

18-teorema. Rombga ichki chizilgan aylana radiusi romb balandligining yarmiga teng (6.19-rasm):



6.19-rasm

$$\boxed{r = \frac{h}{2} = \frac{1}{2} a \sin \alpha .}$$

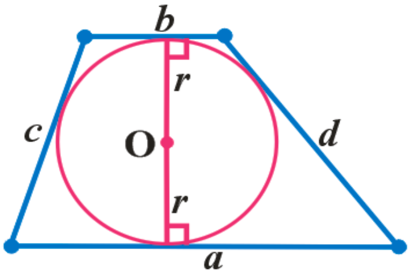
Isbot. Rombning balandligi unga ichki chizilgan aylana diametri bilan mos tushadi. Shuning uchun $2r = h$ yoki $r = \frac{h}{2} = \frac{1}{2} a \sin \alpha$ bo‘ladi.

Trapetsiya tomonlariga urinuvchi aylanaga trapetsiyaga ***ichki chizilgan aylana*** deyiladi. Hamma trapetsiyalarga ham ichki aylana chizib bo‘lmaydi.

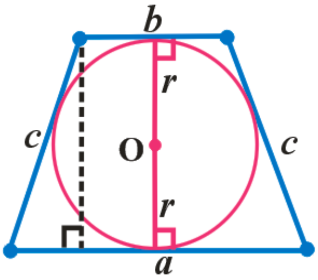
Trapetsiyaga ichki aylana chizish uchun trapetsiya asoslarining yig‘indisi yon tomonlar yig‘indisiga teng bo‘lishi kerak (6.20-rasm):

$$\boxed{a + b = c + d .}$$

Bu yerda: a, b – trapetsiya asoslari, c, d – trapetsiya yon tomonlari.



6.20-rasm



6.21-rasm

Trapetsiyaga ichki chizilgan aylana trapetsiya balandligining yarmiga teng:

$$\boxed{r = \frac{h}{2} .}$$

$$\boxed{\begin{cases} \cos\alpha = \dfrac{(a-b)^2 + c^2 - d^2}{2(a-b)c} \\ \cos\beta = \dfrac{(a-b)^2 + d^2 - c^2}{2(a-b)d} \end{cases}, \qquad \begin{cases} h = c\cdot\sin\alpha = c\cdot\sqrt{1-\cos^2\alpha} \\ h = d\cdot\sin\beta = d\cdot\sqrt{1-\cos^2\beta} \end{cases}.}$$

19-teorema. Teng yonli trapetsiyaga ichki aylana chizish uchun trapetsiya yon tomoni uning o‘rta chizig‘iga teng bo‘lishi kerak:

$$\boxed{c = \ell = \frac{a+b}{2}.}$$

Isbot. Bunda trapetsiya asoslar yig‘indisi yon tomonlar yig‘indisiga teng ekanligidan $a+b=c+c, \ \Rightarrow \ a+b=2c, \ \Rightarrow \ c=\dfrac{a+b}{2}=\ell$ natija kelib chiqadi.

20-teorema. Asoslari a va b bo‘lgan teng yonli trapetsiyaga ichki chizilgan aylana radiusi va balandlik

$$\boxed{r = \frac{1}{2}\sqrt{ab}, \ h = \sqrt{ab}.}$$

ga teng.

Isbot. Bunda yon tomonning katta asosdagi proyeksiyasi $x=\dfrac{a-b}{2}$, yon tomon esa $c=\dfrac{a+b}{2}$ ga teng bo‘ladi (6.21-rasm). Pifagor teoremasiga ko‘ra trapetsiya balandligi

$$h^2 = c^2 - x^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = \frac{a^2+2ab+b^2-a^2+2ab-b^2}{4} = \frac{4ab}{4} = ab,$$

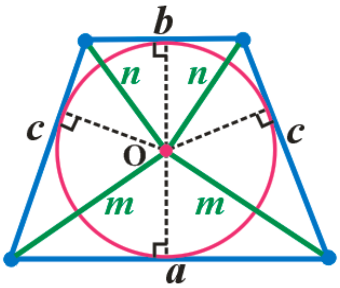
$h=\sqrt{ab}$ ga teng bo‘ladi. Ichki chizilgan aylana radiusi esa $r=\dfrac{h}{2}=\dfrac{1}{2}\sqrt{ab}$ bo‘ladi.

21-teorema. Teng yonli trapetsiyaga ichki chizilgan aylana markazidan trapetsiya uchlarigacha bo‘lgan masofalar m va n quyidagicha bo‘ladi (6.22-a rasm):

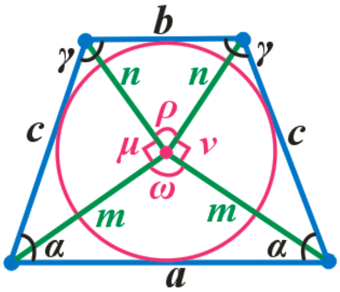
$$\boxed{m=\frac{1}{2}\sqrt{a^2+ab}\,,\quad n=\frac{1}{2}\sqrt{b^2+ab}\,.}$$

Isbot. Bunda Pifagor teoremasidan foydalanamiz. Unga ko‘ra

$$\begin{cases} m^2=r^2+\dfrac{a^2}{4}=\dfrac{1}{4}ab+\dfrac{a^2}{4}=\dfrac{a^2+ab}{4} \\ n^2=r^2+\dfrac{b^2}{4}=\dfrac{1}{4}ab+\dfrac{b^2}{4}=\dfrac{b^2+ab}{4} \end{cases}, \Rightarrow \begin{cases} m=\dfrac{1}{2}\sqrt{a^2+ab} \\ n=\dfrac{1}{2}\sqrt{b^2+ab} \end{cases}$$ bo‘ladi.



a)



b)

6.22-rasm

22-teorema. Teng yonli trapetsiyaga ichki chizilgan aylana markazidan trapetsiya uchlarigacha bo‘lgan masofalar m va n orasidagi burchaklar quyidagicha bo‘ladi (6.22-*b* rasm):

$$\boxed{\mu=\nu=90^0,\quad \cos\rho=\frac{a-b}{a+b},\quad \cos\omega=\frac{b-a}{b+a}=-\cos\rho,\quad \begin{cases}\omega=\gamma\\ \rho=\alpha\end{cases}.}$$

Isbot. Agar $m^2+n^2=c^2$ shart bajarilsa, u holda $\mu=\nu=90^0$ bo‘ladi. Tekshirib ko‘rilganda

$$m^2+n^2=c^2,\Rightarrow\ \frac{a^2+ab}{4}+\frac{b^2+ab}{4}=\left(\frac{a+b}{2}\right)^2\Bigg|\cdot 4,\Rightarrow\ a^2+2ab+b^2=(a+b)^2$$

haqiqatan ham to‘g‘ri tenglik hosil bo‘ldi. Demak, $\mu=\nu=90^0$ ekan. m va n orasidagi burchakni kosinuslar teoremasidan foydalanib aniqlasak,

$$\cos\rho=\frac{n^2+n^2-b^2}{2nn}=\frac{2n^2-b^2}{2n^2}=\frac{\dfrac{b^2+ab}{2}-b^2}{\dfrac{b^2+ab}{2}}=\frac{ab-b^2}{ab+b^2}=\frac{a-b}{a+b}$$ bo‘ladi. Bundan $a>b$

bo‘lgani uchun burchakning o‘tkir $(\rho<90^0)$ ekanligi kelib chiqadi. m va m

orasidagi burchakni kosinuslar teoremasidan foydalanib aniqlasak,

$$\cos\omega = \frac{m^2+m^2-a^2}{2mm} = \frac{2m^2-b^2}{2m^2} = \frac{\frac{a^2+ab}{2}-a^2}{\frac{a^2+ab}{2}} = \frac{ab-a^2}{ab+a^2} = \frac{b-a}{b+a} = -\cos\rho$$ bo‘ladi.

Bundan $a>b$ bo‘lgani uchun burchakning o‘tmas ekanligi kelib chiqadi. Endi trapetsiyaning katta asosidagi burchak kosinusini hisoblasak, u $\cos\alpha = \frac{x}{c} = \frac{\frac{a-b}{2}}{\frac{a+b}{2}} = \frac{a-b}{a+b} = \cos\rho$ bo‘ladi. Bundan $\alpha=\rho$ ekanligi kelib chiqadi.

Shuningdek, $\cos\gamma = -\cos\alpha = \frac{b-a}{b+a} = \cos\omega$ dan $\gamma=\omega$ ekanligi kelib chiqadi.

23-teorema. Teng yonli trapetsiyaga ichki chizilgan aylana markazidan trapetsiya uchlarigacha bo‘lgan masofalar m va n ma’lum bo‘lsa, bu trapetsiya yon tomoni, o‘rta chizig‘i, trapetsiyaga ichki chizilgan aylana radiusi hamda trapetsiyaning yuzi

$$\boxed{c=\ell=\sqrt{n^2+m^2}, \quad r=\frac{nm}{\sqrt{n^2+m^2}}, \quad S=2nm}$$

ga teng.

Isbot. m va n orasidagi burchak $\mu=\nu=90^0$ ekanligidan m va n lar to‘g‘ri burchak katetlari, *c* yon tomon esa gipotenuza ekanligi kelib chiqadi(6.22-a rasm). Demak gipotenuza $c=\sqrt{n^2+m^2}=\ell$ bo‘ladi. To‘g‘ri burchak uchidan gipotenuzaga tushirilgan balandlik katetlar ko‘paytmasining gipotenuzaga nisbatiga teng bo‘lishini bilamiz. Shunga ko‘ra $r=\frac{mn}{c}=\frac{mn}{\sqrt{m^2+n^2}}$ ga teng bo‘ladi. Trapetsiya yuzasi esa $S=\ell\cdot h=c\cdot 2r=\sqrt{n^2+m^2}\cdot 2\cdot\frac{mn}{\sqrt{m^2+n^2}}=2mn$ ekanligi kelib chiqadi.

Har qanday teng yonli trapetsiyaga tashqi aylana chizish mumkin. Demak, trapetsiyaga tashqi aylana chizish uchun u teng yonli bo‘lishi kerak ekan.

Trapetsiyaga tashqi chizilgan aylana radiusini aniqlash uchun, avvalo, aylana markazidan katta asosgacha bo‘lgan x masofani aniqlash lozimdir (6.23-rasm). Bu masofa quyidagicha bo‘ladi:

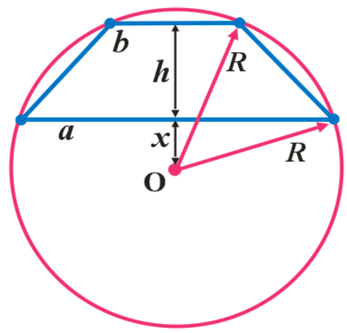
$$\boxed{x=\frac{a^2-b^2}{8h}-\frac{h}{2}.}$$

Isbot. Pifagor teoremasidan foydalanamiz. Unga ko‘ra

$$\begin{cases} R^2=\left(\dfrac{a}{2}\right)^2+x^2 & (1) \\ R^2=\left(\dfrac{b}{2}\right)^2+(h+x)^2 & (2) \end{cases},\ (1)=(2),\ \Rightarrow$$

$$\frac{a^2}{4}+x^2=\frac{b^2}{4}+h^2+2hx+x^2,\Rightarrow$$

$$\frac{a^2-b^2}{4}-h^2=2hx,\ \Rightarrow\ x=\frac{a^2-b^2}{8h}-\frac{h}{2}$$



formula kelib chiqadi.

6.23-rasm

24-teorema. Trapetsiyaga tashqi chizilgan aylana radiusi

$$\boxed{R=\sqrt{\left(\frac{a}{2}\right)^2+x^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a^2-b^2}{8h}-\frac{h}{2}\right)^2}}$$

ga teng.

Isbot. Pifagor teoremasidan foydalanamiz. Unga ko‘ra trapetsiyaga tashqi chizilgan aylana radiusi

$$R^2=\left(\frac{a}{2}\right)^2+x^2,\Rightarrow\ \sqrt{\left(\frac{a}{2}\right)^2+x^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a^2-b^2}{8h}-\frac{h}{2}\right)^2}$$ bo‘ladi.

$x=0$ shart bajarilganda trapetsiyaga tashqi chizilgan aylana radiusi katta asosning o‘rtasida yotadi:

$$\boxed{h=\frac{\sqrt{a^2-b^2}}{2}.}$$

Isbot. Buni yuqorida topilgan x masofa uchun $x=0$ shartdan keltirib chiqarish mumkin.

Asos uzunliklari va trapetsiya balandligining qiymatlariga qarab trapetsiyaga tashqi chizilgan aylana markazi asoslardan bir tarafda yoki asoslar orasida bo‘lishi mumkin. Buning uchun quyidagi shartlar bajarilishi kerak bo‘ladi:

$h<\frac{\sqrt{a^2-b^2}}{2}$ da asoslar bir tarafda,

$h>\frac{\sqrt{a^2-b^2}}{2}$ da asoslar turli tarafda.

Isbot Bularni yuqorida topilgan x masofa uchun $x>0$ yoki $x<0$ shartdan keltirib chiqarish mumkin.

Mustaqil ishlash uchun masalalar

6.105. Perimetri 100 ga teng teng yonli trapetsiyaga aylana ichki chizilgan. Bu aylananing yon tomonlariga uringan nuqtalari orasidagi masofa 16 ga teng bo‘lsa, bu aylana radiusini toping. javob: 10

6.106. *ABCD* to’g’ri burchakli trapetsiyada *AD*=16 ga teng va $CD=8\sqrt{3}$ Aylana *A, B, C* nuqtalardan o‘tib *AD* tomoni *N* nuqtada kesib o‘tadi. *ANB* burchak qiymati 60° ga teng. *BN* ning uzunligini toping. javob: 8

6.107. *ABCD* to’g’ri burchakli trapetsiyada *AB*=4 va *CD*=12. *A, B, C* uchlardan o‘tuvchi aylana *AD* ga urinadi. *AC* diagonal uzunligini toping. javob: $4\sqrt{3}$

6.108. *ABCD* trapetsiya asoslari *BC*=26, *AD*=39 va yon tomonlari *AB*=5 va *CD*=12. *A* va *B* uchdan o‘tuvchi va *CD* tomonga urinuvchi aylana radiusini toping. javob: 12,5

6.109. Tashqi aylana chizish mumkin bo‘lgan *ABCD* to‘rt burchakda *AB=BC, K* nuqta diagonllar kesishgan nuqtasi. *BK*=3 va *KD*=6 bo‘lsa, *AB* ning uzunligini toping. javob: $3\sqrt{3}$

6.110. *ABCD* parallelogrammda, *D* burchakning bissektrisasi *AB* tomonini *E* nuqtada kesadi. *ADE* uchburchakka ichki chizilgan aylana *AE* tomonga *P* nuqtada, *AD* tomonga esa *Q* nuqtada urinadi. *AD*=6 va *PQ*=3 bo‘lsa, *BAD* burchakning kattaligini toping. javob: 60°

6.111. *ABC* uchburchakning *A* va *C* uchlarida *AB* va *BC* tomonlarga urinuvchi aylananing uchburchak ichidagi yoyida olingan *M* nuqtadan, *AB* va *BC* tomonlargacha bo‘lgan masofalar 6 va 24 ga teng bo‘lsa, *M* nuqtadan *AC* tomongacha bo‘lgan masofani toping. javob: 12

6.112. Uchburchak *KLM* da *KN* va *LP* bissektrisalar *Q* nuqtada kesishadi. *PN*=1 va *M* nuqta *N, P, Q* dan o‘tuvchi aylanada yotadi. *NPQ* uchburchakning perimetrini toping. javob: $\frac{2\sqrt{3}}{3}+1$

6.113. Teng yonli uchburchakning balandliklari kesishgan nuqtasi, bu uchburchakga ichki chizilgan aylanada yotadi. Uchburchakning asosidagi burchagi kosinusini toping. javob: $\frac{2}{3}$

6.114. Uchburchakning tashqi va ichki chizilgan aylana markazlari uchburchakning bir tomoniga simmetrik ravishda joylashgan. Agar ichki chizilgan aylana radiusi $1+\sqrt{5}$ ga teng bo‘lsa, tashqi chizilgan aylana radiusini toping. javob: $3+\sqrt{5}$

6.115. *ABCD* trapetsiyaning *AD* asosini diametr qilib chizilgan aylana trapetsiyaning qolgan barcha tomonlariga urinadi. *AB* va *DC* tomonlar davom ettirilib *M* nuqtada kesishadi. *AM*=6 va *DM*=10 bo‘lsa, bu trapetsiyaning kichik asosi uzunligini toping. javob:4

6.116. Rombning diagonallari uzunliklari 12 va 16 ga teng. Rombga ichki chizilgan aylananing romb tomoniga uringan nuqtasidan kichik diagonalgacha bo‘lgan masofani toping. javob: 2,88

6.117. Yon tomoni uzunligi 2,5 va asosi uzunligi 3 ga teng bo‘lgan teng yonli uchburchakning yon tomoniga urinuvchi yarim aylananing markazi uchburchakning asosida yotsa, bu aylananing yon tomonlarga uringan nuqtalari orasidagi masofani toping. javob: 1,92

6.118. Rombning diagonallari uzunliklari $\sqrt{3}+1$ va $\sqrt{3}-1$ ga teng. Bu rombga ichki chizilgan aylananing romb tomonlariga urinish nuqtalari ketma-ket tutashtirilishidan hosil bo‘lgan to‘rt burchakning yuzini toping. javob: 0,125

6.119. Aylanaga tashqi chizilgan teng yonli trapetsiyaning yon tomoni uzunligini toping, bunda trapetsiyaning asosidagi burchagi $\frac{\pi}{3}$ ga va trapetsiyaning yuzasi $288\sqrt{3}$ ga teng. javob: 24

6.120. Teng yonli trapetsiya aylanaga tashqi chizilgan, aylananing yon tomoniga urinish nuqtasida yon tomoni 12 va 48 ga teng kesmalarga bo‘ladi. Bu trapetsiyaning yuzini toping. javob: 2880

6.121. Aylanaga tashqi chizilgan trapetsiya yuzi 144,5 ga teng. Trapetsiya asosidagi burchagi $\frac{\pi}{6}$ ga teng bo‘lsa, aylana radiusini toping. javob: 4,25

6.122. Teng yonli trapetsiya aylanaga tashqi chizilgan. Bu trapetsiya asosidagi burchagi $\frac{\pi}{3}$ ga va trapetsiya yuzasi $128\sqrt{3}$ ga teng bo‘lsa, bu trapetsiyaning yon tomoni uzunligini toping. javob: 16

6.123. Teng yonli trapetsiya radiusi uzunligi $2\sqrt{3}$ ga teng bo‘lgan aylanaga tashqi chizilgan, trapetsiyaning balandligi kichik asosidan ikki marta uzun bo‘lsa, bu trapetsiyaning yuzini toping. javob: 60

6.124. Teng yonli trapetsiyaga aylana ichki chizilgan, trapetsiya katta asosi aylana markazidan 120° burchak ostida ko‘rinadi. Agar trapetsiyaning yuzasi $50\sqrt{3}$ ga teng bo‘lsa, trapetsiyaning yon tomoni uzunligini toping. javob: 10

6.125. Teng yonli trapetsiyaga ichki aylana chizilgan. Trapetsiyaning balandligi yon tomondan ikki marta kichik va trapetsiya yuzi 162 ga teng bo‘lsa, ichki chizilgan aylana radiusini toping. javob: 4,5

6.126. Aylanaga tashqi chizilgan teng yonli trapetsiyaning yuzasi $98\sqrt{3}$ ga teng. Agar trapetsiyaning kichik asosidagi burchagi 120° ga teng bo‘lsa, trapetsiyaning o‘rta chizig‘i uzunligini toping. javob: 14

6.127. Aylanaga tashqi chizilgan trapetsiyaning yuzasi 321,5 ga teng. Trapetsiya asosidagi burchak qiymati 30° ga teng bo‘lsa, trapetsiyaning o‘rta chizig‘i uzunligini toping. javob: 25

6.128. Aylanaga tashqi chizilgan teng yonli trapetsiya yuzi 242 va aylana markazidan katta asosi 150° burchak ostida ko‘rinadi. Bu trapetsiya balandligi uzunligini toping. javob: 11

5-§. Muntazam ko‘pburchakka doir nostandart masalalar

1-masala. Tomoni *a* bo‘lgan muntazam *n* burchakli ko‘pburchak tomonlari o‘rtalariga va o‘zaro urinuvchi *n* ta ichki chizilgan aylana radiusi

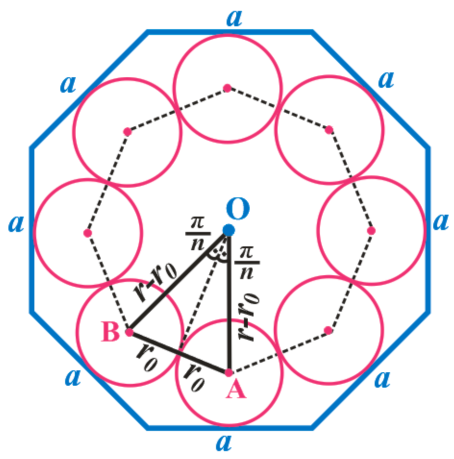
$$\boxed{r_0=\frac{\sin\frac{\pi}{n}}{1+\sin\frac{\pi}{n}}\cdot r=\frac{\cos\frac{\pi}{n}}{2\left(1+\sin\frac{\pi}{n}\right)}\cdot a.}$$

ga teng.

Isbot. Ichki chizilgan aylanalar markazlari tutashtirib chiqilsa, yana berilgan ko‘pburchakka o‘xshash kichikroq ko‘pburchak hosil bo‘ladi (6.24-rasm). Bu ko‘pburchak tomoni ichki chizilgan aylana diametriga teng bo‘lishi rasmdan ham ko‘rinib turibdi, ya’ni $AB=2r_0$. Berilgan ko‘pburchak markazini ikkita urinuvchi qo‘shni aylanalar markazlari bilan tutashtirilganda teng yonli uchburchak hosil bo‘ladi.

Bu teng yonli uchburchak asosi $AB=2r_0$ ga, yon tomoni esa $OA=OB=r-r_0$ ga teng bo‘ladi. Bu yerda r_0 – topishimiz kerak bo‘lgan ichki chizilgan aylana radiusi, r – berilgan tomoni a bo‘lgan muntazam n burchakli muntazam ko‘pburchakka ichki chizilgan aylana radiusi bo‘lib, uning qiymati $r=\dfrac{a}{2\operatorname{tg}\frac{\pi}{n}}$ ga teng bo‘ladi.

Bu teng yonli uchburchakning uchidagi burchagi $\frac{2\pi}{n}$ ga teng bo‘lib, uchidan asosiga tushirilgan balandlik bu burchakni $\frac{\pi}{n}$ ga teng bo‘lgan ikkita to‘g‘ri burchakli uchburchaklarga ajratadi. Bu to‘g‘ri burchakli uchburchak sinusidan foydalanib, so‘ralgan kattalikni osongina aniqlash mumkin. Unga ko‘ra so‘ralgan kattalik



6.24-rasm

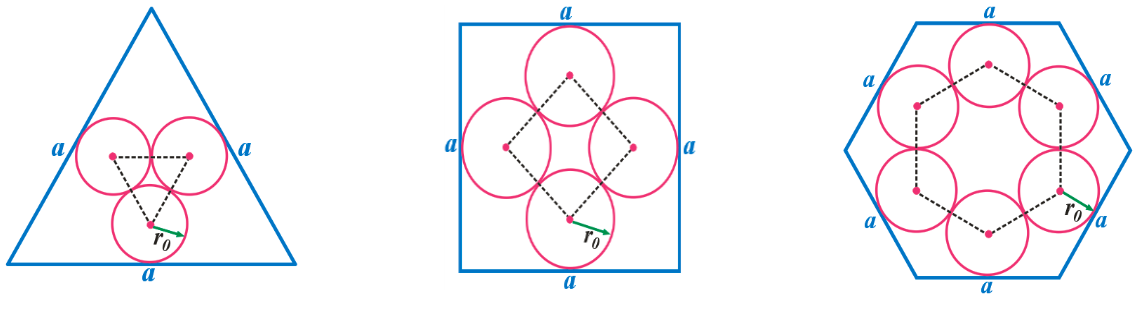
$$\sin\frac{\pi}{n}=\frac{r_0}{r-r_0},\Rightarrow\ \sin\frac{\pi}{n}\cdot r-\sin\frac{\pi}{n}\cdot r_0=r_0,\Rightarrow\ \sin\frac{\pi}{n}\cdot r=\left(1+\sin\frac{\pi}{n}\right)r_0,\Rightarrow$$

$$r_0=\frac{\sin\frac{\pi}{n}}{1+\sin\frac{\pi}{n}}\cdot r=\frac{\sin\frac{\pi}{n}}{1+\sin\frac{\pi}{n}}\cdot\frac{a}{2\operatorname{tg}\frac{\pi}{n}}=\frac{\cos\frac{\pi}{n}}{2\left(1+\sin\frac{\pi}{n}\right)}\cdot a$$

ekanligi kelib chiqadi.

Xususiy holda ba’zi ko‘pburchaklar uchun radiuslar quyida keltirilgan (6.25-rasm):

$$\boxed{r_0=\frac{2-\sqrt{3}}{2}a;}\qquad \boxed{r_0=\frac{\sqrt{2}-1}{2}a;}\qquad \boxed{r_0=\frac{\sqrt{3}}{6}a.}$$



6.25-rasm

2-masala. Tomoni *a* bo‘lgan muntazam *n* burchakli ko‘pburchak tomonlari o‘rtalariga va o‘zaro urinuvchi *n* ta tashqi chizilgan aylana radiusi

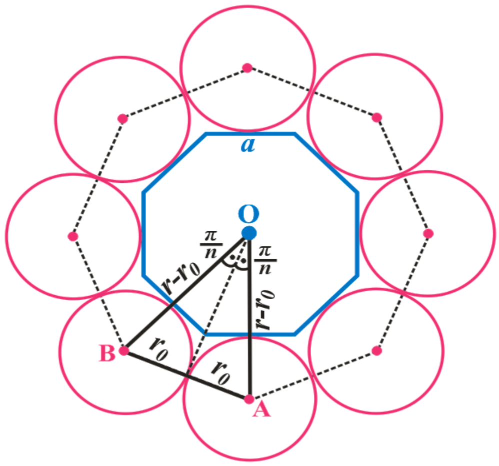
$$\boxed{r_0 = \frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}\cdot r = \frac{\cos\frac{\pi}{n}}{2\left(1-\sin\frac{\pi}{n}\right)}\cdot a.}$$

ga teng.

Isbot. Ichki chizilgan aylanalar markazlari tutashtirib chiqilsa, yana berilgan ko‘pburchakka o‘xshash kattaroq ko‘pburchak hosil bo‘ladi. Bu ko‘pburchak tomoni ichki chizilgan aylana diametriga teng bo‘lishi 6.26-rasmdan ko‘rinib turibdi, ya’ni $AB = 2r_0$.

Berilgan ko‘pburchak markazini ikkita urinuvchi qo‘shni aylanalar markazlari bilan tutashtirilganda teng yonli uchburchak hosil bo‘ladi. Bu teng yonli uchburchak asosi $AB = 2r_0$ ga, yon tomoni esa $OA = OB = r + r_0$ ga teng bo‘ladi. Bu yerda: r_0 – topishimiz kerak bo‘lgan ichki chizilgan aylana radiusi, r – berilgan tomoni a bo‘lgan muntazam n burchakli muntazam ko‘pburchakka ichki chizilgan aylana radiusi bo‘lib, uning qiymati $r = \dfrac{a}{2\operatorname{tg}\frac{\pi}{n}}$ ga teng bo‘ladi.

Bu teng yonli uchburchakning uchidagi burchagi $\frac{2\pi}{n}$ ga teng bo‘lib, uchidan asosiga tushirilgan balandlik bu burchakni $\frac{\pi}{n}$ ga teng bo‘lgan ikkita to‘g‘ri burchakli uchburchaklarga ajratadi. Bu to‘g‘ri burchakli uchburchak sinusidan foydalanib, so‘ralgan kattalikni osongina aniqlash mumkin. Unga ko‘ra so‘ralgan kattalik



6.26-rasm

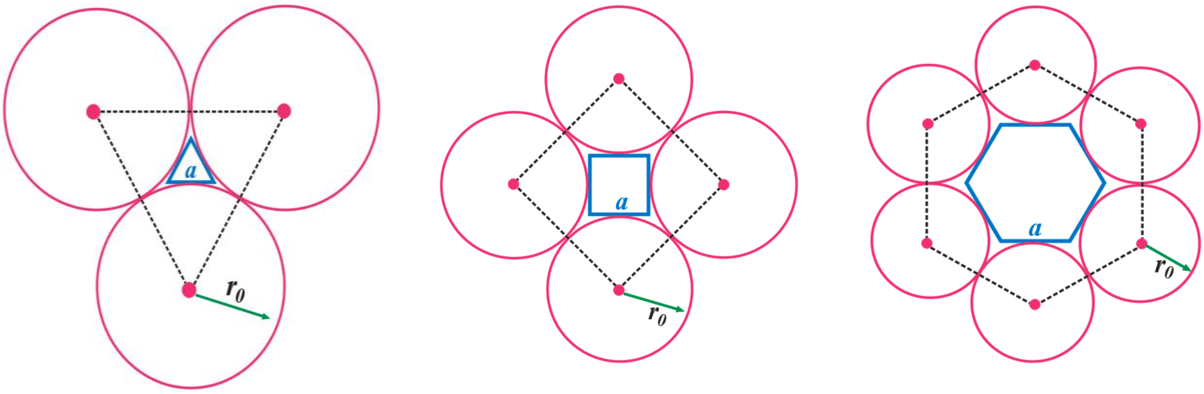
$$\sin\frac{\pi}{n}=\frac{r_0}{r+r_0},\ \Rightarrow\ \sin\frac{\pi}{n}\cdot r+\sin\frac{\pi}{n}\cdot r_0=r_0,\Rightarrow\ \sin\frac{\pi}{n}\cdot r=\left(1-\sin\frac{\pi}{n}\right)r_0,\ \Rightarrow$$

$$r_0=\frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}\cdot r=\frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}\cdot\frac{a}{2\operatorname{tg}\frac{\pi}{n}}=\frac{\cos\frac{\pi}{n}}{2\left(1-\sin\frac{\pi}{n}\right)}\cdot a$$

ekanligi kelib chiqadi.

Xususiy holda ba’zi ko‘pburchaklar uchun radiuslar quyida keltirilgan (6.27-rasm):

$$\boxed{r_0=\frac{2+\sqrt{3}}{2}a;}\qquad \boxed{r_0=\frac{\sqrt{2}+1}{2}a;}\qquad \boxed{r_0=\frac{\sqrt{3}}{2}a.}$$



6.27-rasm

3-masala. Tomoni a bo‘lgan muntazam n burchakli ko‘pburchakning burchak tomonlariga va o‘zaro urinuvchi n ta ichki chizilgan aylana radiusi

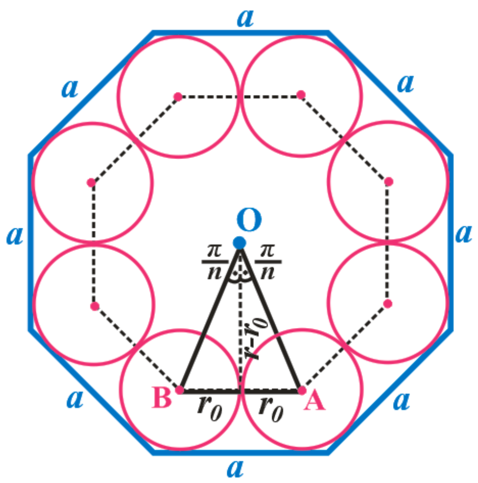
$$\boxed{r_0=\frac{\operatorname{tg}\frac{\pi}{n}}{1+\operatorname{tg}\frac{\pi}{n}}\cdot r=\frac{1}{2\left(1+\operatorname{tg}\frac{\pi}{n}\right)}\cdot a.}$$

ga teng.

Isbot. Ichki chizilgan aylanalar markazlari tutashtirib chiqilsa, yana berilgan ko‘pburchakka o‘xshash kichikroq ko‘pburchak hosil bo‘ladi (6.28-rasm).

Bu ko‘pburchak tomoni ichki chizilgan aylana diametriga teng bo‘lishi rasmdan ham ko‘rinib turibdi, ya’ni $AB=2r_0$. Berilgan ko‘pburchak markazini ikkita urinuvchi qo‘shni aylanalar markazlari bilan tutashtirilganda teng yonli uchburchak hosil bo‘ladi. Bu teng yonli uchburchak asosi $AB = 2r_0$ ga, yon tomoni esa $OA = OB$ ga teng bo‘ladi.

Bu yerda: r_0 – topishimiz kerak bo‘lgan ichki chizilgan aylana radiusi, r – berilgan tomoni a bo‘lgan muntazam n burchakli muntazam ko‘pburchakka ichki chizilgan aylana radiusi bo‘lib, uning qiymati $R = \dfrac{a}{2\sin\dfrac{\pi}{n}}$ ga teng bo‘ladi.



6.28-rasm

Bu teng yonli uchburchakning uchidagi burchagi $\dfrac{2\pi}{n}$ ga teng bo‘lib, uchidan asosiga tushirilgan balandlik bu burchakni $\dfrac{\pi}{n}$ ga teng bo‘lgan ikkita to‘g‘ri burchakli uchburchaklarga ajratadi. Bu to‘g‘ri burchakli uchburchak tangensidan foydalanib, so‘ralgan kattalikni osongina aniqlash mumkin. Unga ko‘ra so‘ralgan kattalik

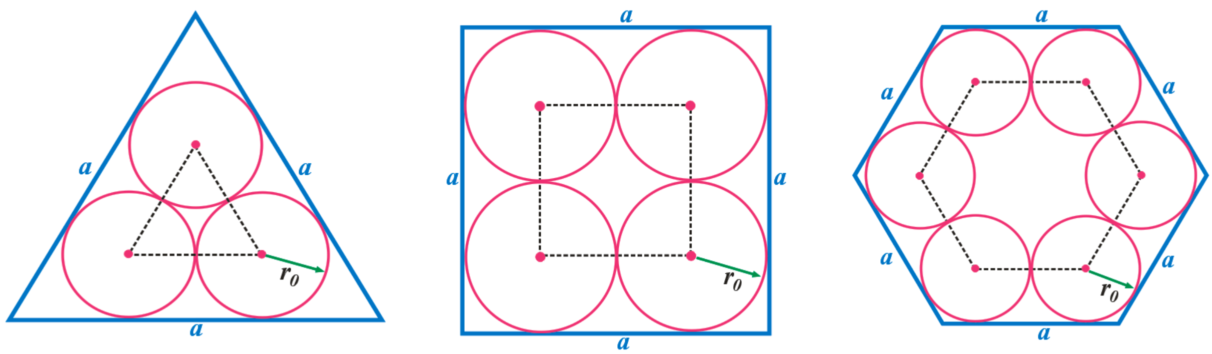
$$\operatorname{tg}\frac{\pi}{n} = \frac{r_0}{r - r_0}, \Rightarrow \operatorname{tg}\frac{\pi}{n}\cdot r - \operatorname{tg}\frac{\pi}{n}\cdot r_0 = r_0, \Rightarrow \operatorname{tg}\frac{\pi}{n}\cdot R = \left(1 + \operatorname{tg}\frac{\pi}{n}\right) r_0, \Rightarrow$$

$$\Rightarrow r_0 = \frac{\operatorname{tg}\dfrac{\pi}{n}}{1+\operatorname{tg}\dfrac{\pi}{n}}\cdot r = \frac{\operatorname{tg}\dfrac{\pi}{n}}{1+\operatorname{tg}\dfrac{\pi}{n}}\cdot\frac{a}{2\operatorname{tg}\dfrac{\pi}{n}} = \frac{1}{2\left(1+\operatorname{tg}\dfrac{\pi}{n}\right)}\cdot a$$

ekanligi kelib chiqadi.

Xususiy holda ba’zi ko‘pburchaklar uchun radiuslar quyida keltirilgan (6.29-rasm):

$$\boxed{r_0 = \frac{\sqrt{3}-1}{4}a;} \qquad \boxed{r_0 = \frac{1}{4}a;} \qquad \boxed{r_0 = \frac{3-\sqrt{3}}{4}a.}$$



6.29-rasm

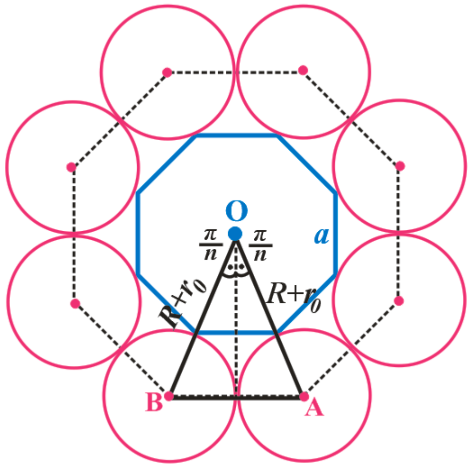
4-masala. Tomoni a bo‘lgan muntazam n burchakli ko‘pburchak burchaklariga va o‘zaro urinuvchi n ta tashqi chizilgan aylana radiusi

$$\boxed{r_0=\frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}\cdot R=\frac{1}{2\left(1-\sin\frac{\pi}{n}\right)}\cdot a.}$$

ga teng.

Isbot. Ichki chizilgan aylanalar markazlari tutashtirib chiqilsa, yana berilgan ko‘pburchakka o‘xshash kichikroq ko‘pburchak hosil bo‘ladi (6.30-rasm). Bu ko‘pburchak tomoni ichki chizilgan aylana diametriga teng bo‘lishi rasmdan ham ko‘rinib turibdi, ya’ni $AB=2r_0$. Berilgan ko‘pburchak markazini ikkita urinuvchi qo‘shni aylanalar markazlari bilan tutashtirilganda teng yonli uchburchak hosil bo‘ladi. Bu teng yonli uchburchak asosi $AB=2r_0$ ga, yon tomoni esa $OA=OB=R+r_0$ ga teng bo‘ladi. Bu yerda: r_0 – topishimiz kerak bo‘lgan ichki chizilgan aylana radiusi, R – berilgan tomoni a bo‘lgan muntazam n burchakli muntazam ko‘pburchakka tashqi chizilgan aylana radiusi bo‘lib, uning qiymati $R=\frac{a}{2\sin\frac{\pi}{n}}$ ga teng bo‘ladi.

Bu teng yonli uchburchakning uchidagi burchagi $\frac{2\pi}{n}$ ga teng bo‘lib, uchidan asosiga tushirilgan balandlik bu burchakni $\frac{\pi}{n}$ ga teng bo‘lgan ikkita to‘g‘ri burchakli uchburchaklarga ajratadi. Bu to‘g‘ri burchakli uchburchak sinusidan foydalanib, so‘ralgan kattalikni osongina aniqlash mumkin. Unga ko‘ra so‘ralgan kattalik



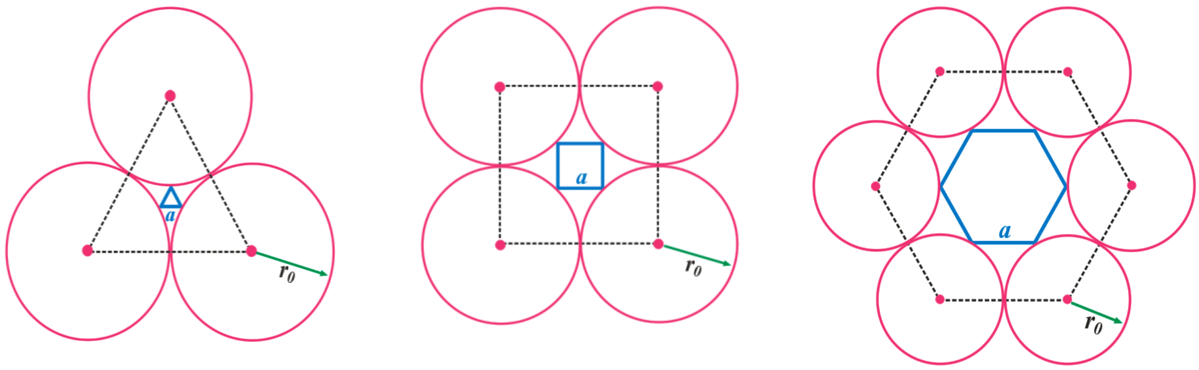
6.30-rasm

$$\sin\frac{\pi}{n} = \frac{r_0}{R+r_0}, \Rightarrow \sin\frac{\pi}{n}\cdot R + \sin\frac{\pi}{n}\cdot r_0 = r_0, \Rightarrow \sin\frac{\pi}{n}\cdot R = \left(1-\sin\frac{\pi}{n}\right)r_0, \Rightarrow$$

$$\Rightarrow r_0 = \frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}\cdot R = \frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}\cdot\frac{a}{2\sin\frac{\pi}{n}} = \frac{1}{2\left(1-\sin\frac{\pi}{n}\right)}\cdot a$$ ekanligi kelib chiqadi.

Xususiy holda ba’zi ko‘pburchaklar uchun radiuslar quyida keltirilgan (6.31-rasm):

$\boxed{r_0 = \left(2+\sqrt{3}\right)a;}$ $\boxed{r_0 = \frac{2+\sqrt{2}}{2}a;}$ $\boxed{r_0 = a.}$



6.31-rasm

5-masala. R radiusli aylanaga bir xil n ta aylana ichki chizilgan bo‘lsa, bu aylanalarning radiusi

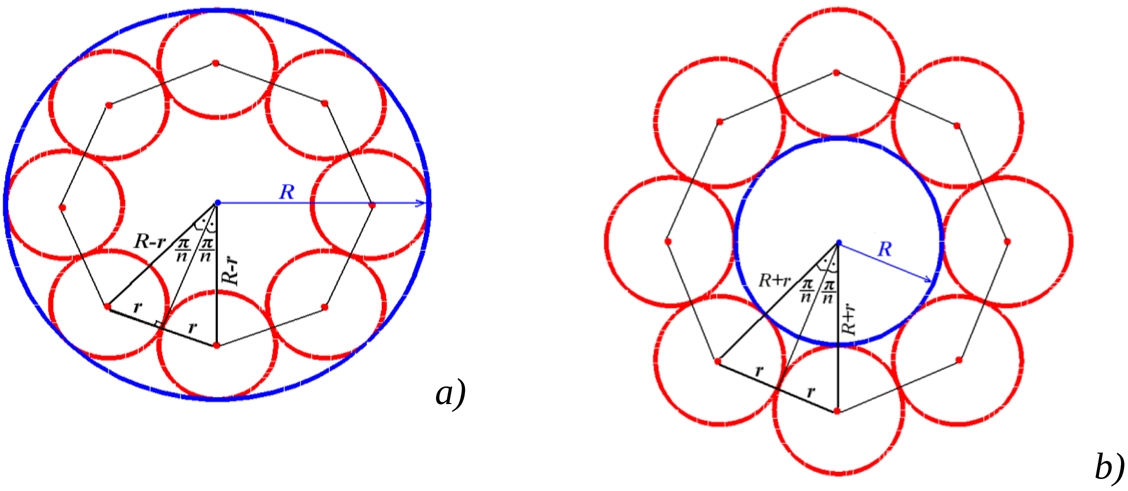
$$\boxed{r=\frac{\sin\frac{\pi}{n}}{1+\sin\frac{\pi}{n}}R.}$$

ga teng.

Isbot. Aylana markazlarini tutashtirib chiqilganda n burchakli muntazam ko‘pburchak hosil bo‘ladi. Berilgan aylana va unga ichki chizilgan ikkita qo‘shni aylana markazi tutashtirilganda asosi $2r$, yon tomoni $R-r$ bo‘lgan teng yonli uchburchak hosil bo‘ladi (6.32-*a* rasm). Bu teng yonli uchburchak uchidagi burchak $\frac{2\pi}{n}$ ga teng bo‘lib, bu uchdan asosga balandlik tushirilganda uchidagi burchak ikkita $\frac{\pi}{n}$ burchakka ajraladi. Bu burchak sinusidan foydalanib, so‘ralgan kattalikni aniqlaymiz. Unga ko‘ra

$$\sin\frac{\pi}{n}=\frac{r}{R-r},\ \Rightarrow\ R\sin\frac{\pi}{n}-r\sin\frac{\pi}{n}=r,\ \Rightarrow\ R\sin\frac{\pi}{n}=\left(1+\sin\frac{\pi}{n}\right)r,\ \Rightarrow\ r=\frac{\sin\frac{\pi}{n}}{1+\sin\frac{\pi}{n}}R$$

bo‘ladi.



6.32-rasm

6-masala. R radiusli aylanaga bir xil n ta aylana tashqi chizilgan bo‘lsa, bu aylanalarning radiusi

$$\boxed{r=\frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}R.}$$

ga teng.

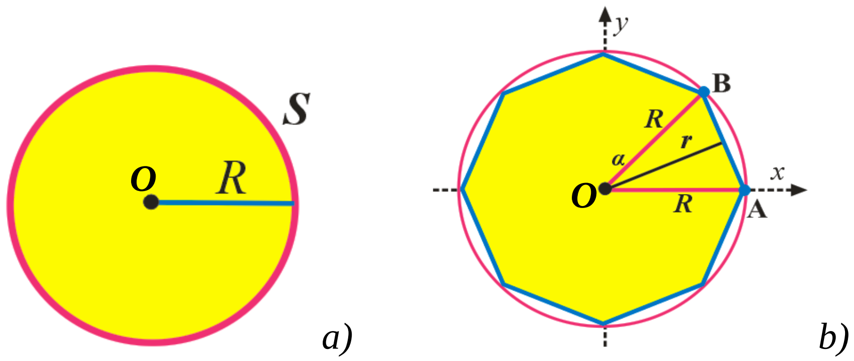
Isbot. Aylana markazlarini tutashtirib chiqilganda n burchakli muntazam ko‘pburchak hosil bo‘ladi. Berilgan aylana va unga ichki chizilgan ikkita qo‘shni aylana markazi tutashtirilganda asosi $2r$, yon tomoni $R+r$ bo‘lgan teng yonli uchburchak hosil bo‘ladi (6.32-*b* rasm). Bu teng yonli uchburchak uchidagi burchak $\frac{2\pi}{n}$ ga teng bo‘lib, bu uchdan asosga balandlik tushirilganda uchidagi burchak ikkita $\frac{\pi}{n}$ burchakka ajraladi. Bu burchak sinusidan foydalanib, so‘ralgan kattalikni aniqlaymiz. Unga ko‘ra

$$\sin\frac{\pi}{n}=\frac{r}{R+r},\ \Rightarrow\ R\sin\frac{\pi}{n}+r\sin\frac{\pi}{n}=r,\ \Rightarrow\ R\sin\frac{\pi}{n}=\left(1-\sin\frac{\pi}{n}\right)r,\ \Rightarrow\ r=\frac{\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}R$$

bo‘ladi.

6-§ Doira yuzasi. Simson to‘g‘ri chizig‘i. Ptolemey teoremasi

25-teorema. Doira yuzini $\boxed{S=\pi R^2}$ ga teng.



6.33-rasm

Isbot. Formulani ikki xil usulda isbot qilish mumkin:

Bu usulda 6.33-*b* rasmdan foydalanamiz. Bunda ΔOAB teng yonli uchburchak uchidagi burchak $\alpha=\frac{2\pi}{n}$ ga teng bo‘ladi. ΔOAB teng yonli uchburchakning yuzasi ko‘pburchakka tashqi chizilgan aylana radiusi orqali $S_{OAB}=\frac{1}{2}OA\cdot AB\cdot\sin\frac{2\pi}{n}=\frac{1}{2}R^2\cdot\sin\frac{2\pi}{n}$ ga teng bo‘ladi. Ko‘pburchak yuzasi ΔOAB yuzasidan n tasining yuziga teng, ya’ni $S=n\cdot S_{OAB}=\frac{n}{2}R^2\sin\frac{2\pi}{n}$ bo‘ladi. Ko‘pburchak tomonlari cheksiz ko‘p bo‘lganda ko‘pburchak va doira yuzalari orasidagi farq yo‘qolib boradi. Bunda ko‘pburchak yuzasi taxminan doira yuzasiga teng bo‘ladi. Demak,

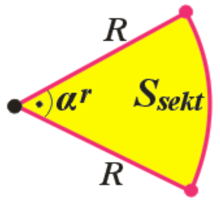
$$\lim_{n\to\infty}S_{ko'pburch}=S_{doira}\ \text{yoki}\ S_{doira}=\lim_{n\to\infty}\left(\frac{n}{2}R^2\sin\frac{2\pi}{n}\right)=\pi R^2\lim_{n\to\infty}\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$$

bo‘ladi. Endi biz $\frac{2\pi}{n}=\alpha$ deb belgilash kiritsak hamda 1-ajoyib limit $\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right)=1$ ekanini e’tiborga olsak, u holda doira yuzasi $S=\pi R^2\lim_{n\to\infty}\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)=\pi R^2\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right)=\pi R^2\cdot 1=\pi R^2$ natija kelib chiqadi.

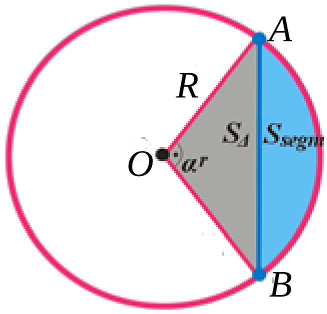
Sektor yuzasi markaziy burchakka to‘g‘ri proporsional bo‘lib, u radian yoki gradusda ushbu ko‘rinishda ifodalanadi (6.34-a rasm):

$$\boxed{S_{sektor}=\frac{1}{2}\alpha^r R^2\quad \text{yoki}\quad S_{sektor}=\frac{\pi}{360^0}\alpha^0 R^2.}$$

Bu yerda: α^r – burchakning radiandagi qiymati, α^0 – burchakning gradusdagi qiymati.



a)



b)

6.34-rasm

Doiradan vatar ajratgan kesimga doira segmenti deyiladi. Doira sektori yuzi segment va unga mos uchburchak yuzalari yig‘indisiga teng bo‘ladi (6.34-b rasm).

$$\boxed{S_{sertor} = S_{segment} + S_\Delta .}$$

26-teorema. Segment yuzi

$$\boxed{S_{segment} = S_{sertor} - S_\Delta = \frac{1}{2}R^2\left(\alpha^r - \sin\alpha^r\right).}$$

ga teng.

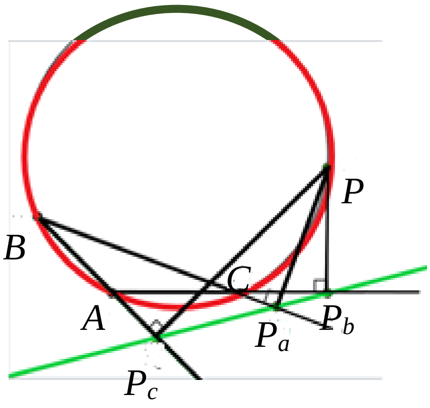
Isbot. Formulani isbotlash uchun 5.18-b rasmdan foydalanamiz. Bunda $S_{sektor} = \frac{1}{2}\alpha^r R^2$ hamda $S_\Delta = \frac{1}{2}R^2 \sin\alpha^r$ yuzalarni ayirganda segment yuzasi hosil bo‘ladi. Natijada,

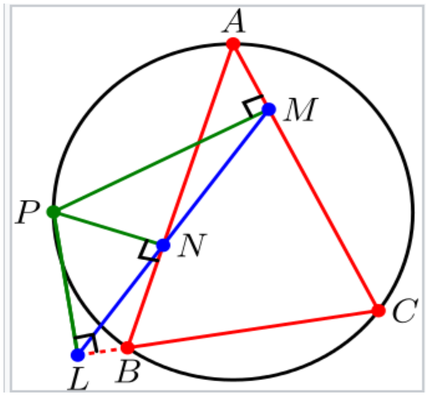
$$S_{segment} = S_{sertor} - S_\Delta = \frac{1}{2}R^2 = \frac{1}{2}R^2\alpha^r - \frac{1}{2}R^2\sin\alpha^r = \frac{1}{2}R^2\left(\alpha^r - \sin\alpha^r\right)$$

formulaga ega bo‘lamiz.

Aylanadagi P nuqtadan aylanaga ichki chizilgan uchburchakning tomonlariga yoki davomlariga tushirilgan perpendikularning asoslaridan o’tuvchi to’g’ri chiziq ***Simsonning to‘g‘ri chizig’i*** deyiladi. Bu yerda indeks perpendikular tushirilayotgan tomonni bildiradi (6.35-rasm).

27-teorema. (Simson teoremasi). *ABC* uchburchagining tomonlariga yoki ularning davomlariga aylanada yotgan ixtiyoriy *P* nuqtasidan tushirilgan perpendikular asoslari bitta to‘g‘ri chiziqda yotadi. Bu chiziq Simson chizig‘i deb nomlanadi



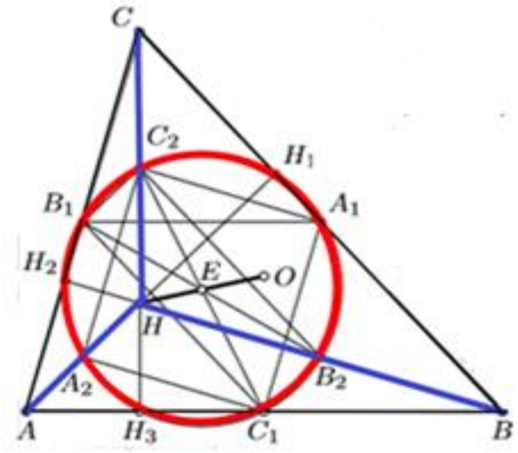


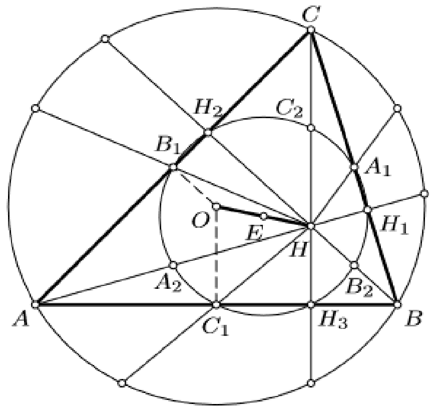
6.35-rasm

Qarama-qarshi faraz ham to‘g‘ri bo‘ladi: agar *P* nuqtadan *ABC* uchburchagining tomonlariga yoki ularning davomlariga tushirilgan perpendikular asoslari bitta to‘g‘ri chiziqda yotsa, u holda *P* nuqta uchburchakka ichki chizilgan aylanada yotadi.

Bizga *ABC* uchburchak berilgan bo‘lib, uning tomonlari o‘rtalari A_1,B_1,C_1 nuqtalar bo‘lsin. $A_1B_1C_1$ uchburchakka *ABC* uchburchakning ***o‘rta uchburchagi*** deb ataymiz $A_1B_1C_1$ uchburchakka tashqi γ aylanaga *ABC* uchburchakning ***Eyler aylanasi*** deb ataladi.

Uchburchak tomonlarining o‘rtalaridan, balandliklarining asoslaridan va ortomarkaz bilan uchburchak uchlarini tutashtiruvchi kesmalarning o‘rtalaridan iborat bo‘lgan to‘qqizta nuqta bir aylanada yotadi. Bu ***aylana to‘qq’iz nuqta*** aylanasi yoki ***Eyler aylanasi*** deyiladi.





6.37-rasm

6.36-rasm

Bu yerda:

H_1, H_2, H_3 - uchburchak balandliklarining asoslari;

A_1, B_1, C_1 - uchburchak tomonlarining ortalari;

A_2, B_2, C_2 - Uchburchak uchlari bilan ortomarkazni tutashtiruvchi kesmaning o‘rtasi;

C_1C_2 - diametr;

O – uchburchakka tashqi chizilgan aylana markazi.

Eyler aylanasining quyidagi hossalarini keltiramiz:

1. Uchburchakka tashqi chizilgan aylananing markazi, uning tomonlari o‘rta perpendikularlari kesishgan nuqtada bo‘ladi;

2. γ Eyler aylanasining radiusi *ABC* uchburchakka tashqi chizilgan aylana radiusining yarmiga teng;

3. *ABC* va $A_1B_1C_1$ uchburchaklarning medianalari kesishgan nuqtalari (sentroidlari) ustma-ust tushadi, *ABC* va $A_1B_1C_1$ uchburchaklar o‘zlarining umumiy sentroidlariga nisbatan (markazi sentroidlar yotgan nuqtada) koeffitsiyenti 2 bo‘lgan gomotetik uchburchaklardir;

4. Istalgan *ABC* uchburchakda uning ortosentri *H*, tashqi chizilgan aylana markazi *O*, sentroidi *E* bir to‘g‘ri chiziqda yotadi (Eyler to‘g‘ri chizig‘i) va $2OE=OH$ tenglik o‘rinli bo‘ladi;

5. *ABC* uchburchak uchun Eyler aylanasi markazi *OH* kesmaning or`tasi bilan ustma-ust tushadi.

28-teorema. γ - *ABC* uchburchak uchun Eyler aylanasiga uchburchak tomonlari o‘rtalaridan tashqari yana 6 ta nuqta: *ABC* uchburchak balandliklari asoslari, uning uchlarini ortomarkazi bilan tutashtiruvchi kesmalarning o‘rtalari tegishli bo‘ladi (6.37-rasm).

Uchburchakkka ichki va tashqi aylanalarning hossalari hammamizga ma’lum. Uchburchak bilan bo‘g‘liq yana uchta aylana borki ular adabiyotlarda juda kam yoritilgan. Uchburchakning bir tomoni va qolgan ikki tomonining davomiga urinuvchi aylana ***sirtdan ichki chizilgan aylana*** deb aytiladi.

Bizga ma’lumki berilgan ABC uchburchakning yuzi quyidagi formulalar bilan hisoblanadi

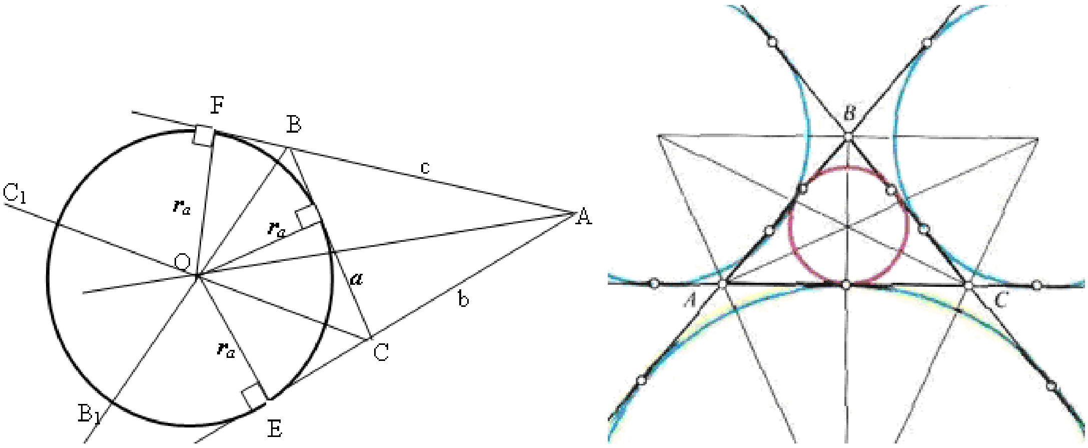
$$S = S_{ABC} = \frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2} = pr \qquad (1)$$

bu yerda h_a, h_b, h_c - mos ravishda a, b, c tomonlariga tushirilgan balandliklar, r - ABC uchburchakka ichki chizildan aylana radiusi, *p* –yarim perimetr va $p = \frac{a+b+c}{2}$.

Bundan quyidagilarni aniqlaymiz

$$h_a = \frac{2S}{a},\; h_b = \frac{2S}{b},\; h_c = \frac{2S}{c},\; r = \frac{S}{p}. \qquad (2)$$

r_a- uchburchakning *BC* tomoni va *AB*, *AC* tomonlarining davomiga urinuvchi ***sirtdan ichki chizilgan aylana***ning radiusi, r_b- uchburchakning *AC* tomoni va *AB*, *BC* tomonlarining davomiga urinuvchi ***sirtdan ichki chizilgan aylana***ning radiusi, r_c- uchburchakning *AB* tomoni va *AC*, *BC* tomonlarining davomiga urinuvchi ***sirtdan ichki chizilgan aylana***ning radiusi bo‘lsin.



a) *b)*

6.38-rasm

Bizga ma’lumki ABC uchburchakka tashqarida ichki chizilgan aylana markazi O nuqta C va B burchaklarning tashqi bissiktrisalari kesishish nuqtasi, ya’ni CC_1 va BB_1 to‘g‘ri chiziqlarning kesishish nuqtasidir. Chunki aylanaga o’tkazilgan urinmaning hossasiga ko’ra $BF = BN$, $BO = ON = r_a$ va $\angle F = \angle N = 90^0$ bundan uchburchak tengligining alomatiga ko’ra $\Delta BOF = \Delta BON$. Bu uchburchaklar tengligidan $\angle OBF = \angle OBN$ tengligi kelib chiqadi. Demak BO to’g’ri chiziq B burchakning bissektrisasi ekan. Shu kabi CO to’g’ri chiziq C burchak bissiktrisasi ekanligini ko’rsatish mumkin. A va O nuqtalarni tutashtirib (1) va (2) larga kora uchburchaklarning yuzalarini aniqlaymiz

$$S_{AOB} = \frac{cr_a}{2},\ S_{AOC} = \frac{br_a}{2} \text{ va } S_{CBO} = \frac{ar_a}{2}. \qquad (3)$$

(3) dan foydalanib ABC uchburchakning yuzi uchun

$S = S_{ABC} = S_{AOB} + S_{AOC} - S_{OCB}$ ya’ni $S = \frac{cr_a}{2} + \frac{br_a}{2} - \frac{ar_a}{2} = \frac{r_a}{2}\left(c + b - a\right)$ tenglikni olamiz. Bu tenglikdan r_a radiusni topsak

$$r_a = \frac{2S}{b + c - a} = \frac{2S}{2\left(\frac{a+b+c}{2}\right) - 2a} = \frac{S}{p - a}$$

Formulani olamiz, bu yerda p –yarim perimetr. Shunga o‘xshash usul bilan qolgan ikkita radiusni ham topamiz va demak

$$r_a = \frac{S}{p - a},\ r_b = \frac{S}{p - b},\ r_c = \frac{S}{p - c}. \qquad (4)$$

Umuman olganda $r_a \neq r_b \neq r_c$. Agar $r_a = r_b = r_c$ bo’lsa, undan $a = b = c$ akanligi kelib chiqadi va $r_a < r_b < r_c$ bo‘lsa, $a < b < c$ bo ‘ladi.

Ichki, tashqi va tashqarida ichki chizilgan aylanalarning radiuslari orasida o‘zaro qanday bog‘liqlik mavjud ekanligini aniqlashga harakat qilib ko‘ramiz.

Bizga tashqarida ichki chizilgan aylanalarning radiuzlari kattaliklari r_a, r_b, r_c berilgan bo‘lsin, (4) formuladan quyidagilarni hisoblaymiz

$$\frac{1}{r_a}=\frac{p-a}{S},\ \frac{1}{r_b}=\frac{p-b}{S},\ \frac{1}{r_c}=\frac{p-c}{S}. \qquad (5)$$

(5) tengliklarni mos ravishda ikkitadan qo‘shib,

$$\frac{1}{r_a}+\frac{1}{r_b}=\frac{2p-a-b}{S}=\frac{c}{S},\ \frac{1}{r_a}+\frac{1}{r_c}=\frac{b}{S},\ \frac{1}{r_b}+\frac{1}{r_c}=\frac{a}{S} \qquad (6)$$

ga ega bo ‘lamiz.

(6) ga kora uchburchak tomonlarining nisbati kelib chiqadi

$$a:b:c=\left(\frac{1}{r_b}+\frac{1}{r_c}\right):\left(\frac{1}{r_a}+\frac{1}{r_c}\right):\left(\frac{1}{r_a}+\frac{1}{r_b}\right),$$

$$\frac{a}{c}=\frac{\frac{1}{r_b}+\frac{1}{r_c}}{\frac{1}{r_a}+\frac{1}{r_b}}=\frac{\frac{r_c+r_b}{r_cr_b}}{\frac{r_a+r_b}{r_ar_b}}=\frac{r_ar_c+r_ar_b}{r_cr_a+r_cr_b},\ \frac{b}{c}=\frac{\frac{1}{r_a}+\frac{1}{r_c}}{\frac{1}{r_a}+\frac{1}{r_b}}=\frac{\frac{r_c+r_a}{r_cr_a}}{\frac{r_a+r_b}{r_ar_b}}=\frac{r_br_c+r_ar_b}{r_cr_a+r_cr_b},$$

va bundan quyidagi tengliklarni olamiz

$$a=\left(\frac{r_ar_c+r_ar_b}{r_ar_c+r_br_c}\right)c,\ b=\left(\frac{r_ar_b+r_br_c}{r_ar_c+r_br_c}\right)c. \qquad (7)$$

(5) tengliklarni hadma had qo‘shib

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{p-a}{S}+\frac{p-b}{S}+\frac{p-c}{S}=\frac{3p-2p}{S}=\frac{p}{S}=\frac{1}{r}$$

tenglikni hosil qilamiz va

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{a}{ah_a}+\frac{b}{bh_b}+\frac{c}{ch_c}=\frac{a}{2S}+\frac{b}{2S}+\frac{c}{2S}=\frac{2p}{2S}=\frac{p}{S}=\frac{1}{r}$$

ekanligidan foydalanib, quyidagiga ega bo‘lamiz

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c},$$

$$\frac{h_ah_b+h_ah_c+h_bh_c}{h_ah_bh_c}=\frac{r_ar_b+r_ar_c+r_br_c}{r_ar_br_c},$$

$$\frac{r_ar_br_c}{h_ah_bh_c}=\frac{r_ar_b+r_ar_c+r_br_c}{h_ah_b+h_ah_c+h_bh_c}.$$

(4) formuladan tashqarida ichki chisilgan aylana radiuslarini juft – jufti bilan ko‘paytirib qo‘shamiz va Geron formulasiga ko’ra

$$r_a r_b + r_a r_c + r_b r_c = S^2\left(\frac{1}{(p-a)(p-b)} + \frac{1}{(p-a)(p-c)} + \frac{1}{(p-c)(p-b)}\right) =$$

$$= S^2\frac{p-c+p-b+p-a}{(p-a)(p-b)(p-c)} = S^2\frac{3p-a-b-c}{(p-a)(p-b)(p-c)} = \frac{S^2 p}{(p-a)(p-b)(p-c)} =$$

$$= \frac{S^2 p^2}{p(p-a)(p-b)(p-c)} = \frac{S^2 p^2}{S^2} = p^2$$

tenglikni olamiz.

Bundan uchburchakning yarim perimetri uchun formula kelib chiqadi

$$p = \sqrt{r_a r_b + r_a r_c + r_b r_c}\ . \tag{8}$$

(7) va (8) lardan uchburchak tomonlarini topamiz

$$a+b+c = 2\sqrt{r_a r_b + r_a r_c + r_b r_c}\ ,$$

$$\left(\frac{r_a r_c + r_a r_b}{r_a r_c + r_b r_c}\right)c + \left(\frac{r_b r_c + r_a r_b}{r_a r_c + r_b r_c}\right)c + c = 2\sqrt{r_a r_b + r_a r_c + r_b r_c}\ ,$$

$$\left(\frac{r_a r_c + r_a r_b}{r_a r_c + r_b r_c} + \frac{r_b r_c + r_a r_b}{r_a r_c + r_b r_c} + 1\right)c = \frac{2(r_a r_b + r_a r_c + r_b r_c)}{r_a r_c + r_b r_c}c = 2\sqrt{r_a r_b + r_a r_c + r_b r_c}\ ,$$

$$c = \frac{(r_a r_c + r_b r_c)\sqrt{r_a r_b + r_a r_c + r_b r_c}}{r_a r_b + r_a r_c + r_b r_c} = \frac{(r_a + r_b)r_c}{\sqrt{r_a r_b + r_a r_c + r_b r_c}}\ .$$

Huddi shu kabi a va b tomonlar uchun quyidagi formulalarni olish mumkin, shunday qilib

$$a = \frac{(r_c + r_b)r_a}{\sqrt{r_a r_b + r_a r_c + r_b r_c}}\ , b = \frac{(r_a + r_c)r_b}{\sqrt{r_a r_b + r_a r_c + r_b r_c}}\ ,\ c = \frac{(r_a + r_b)r_c}{\sqrt{r_a r_b + r_a r_c + r_b r_c}}\ . \tag{9}$$

(4) da va $r = \dfrac{S}{p}$ formulaga kora quyidagi tenglikni

$$r r_a r_b r_c = \frac{S^4}{p(p-a)(p-b)(p-c)}$$

hosil qilamiz, bundan Geron formulasiga ko’ra berilgan uchburchak yuzi uchun quyidagi tenglikga ega bo‘lamiz

$$S = \sqrt{r r_a r_b r_c}\ . \tag{10}$$

Uchburchakning uchta tashqarida ichki chizilgan aylana radiuslari berilgan bo‘lsa, (9) formula orqali uchburchakning tomonlarini bir qiymatli aniqlash mumkinligini ko‘rsatish mumkin. (9) formuladan ixtiyoriy r_a, r_b, r_c uchun $a+b>c$, $a+c>b$, $b+c>a$ uchburchak tengsizligi o‘rinli ekanligi ham kelib chiqadi.

(6) va (9) formulalardan

$$\frac{1}{r_a}+\frac{1}{r_b}=\frac{c}{S},\quad \frac{r_a+r_b}{r_a r_b}=\frac{1}{S}\frac{r_c\left(r_a+r_b\right)}{\sqrt{r_a r_b+r_a r_c+r_b r_c}}=\frac{c}{S}$$

tengliklarni olamiz. Bundan esa

$$S=\frac{r_a r_b r_c}{\sqrt{r_a r_b+r_a r_c+r_b r_c}} \qquad (11)$$

va tashqi chizilgan aylana radiusining formulasidan

$$R=\frac{abc}{4S}=\frac{\frac{(r_c+r_b)r_a}{\sqrt{r_a r_b+r_a r_c+r_b r_c}}\frac{(r_a+r_c)r_b}{\sqrt{r_a r_b+r_a r_c+r_b r_c}}\frac{(r_b+r_a)r_c}{\sqrt{r_a r_b+r_a r_c+r_b r_c}}}{4\frac{r_a r_b r_c}{\sqrt{r_a r_b+r_a r_c+r_b r_c}}},$$

$$R=\frac{(r_a+r_b)(r_a+r_c)(r_b+r_c)}{4(r_a r_b+r_a r_c+r_b r_c)} \qquad (12)$$

ekanligi kelib chiqadi.

Ichki chizilgan aylana radiusi formulasi, (8) va (11) lardan

$$r=\frac{S}{p}=\frac{\frac{r_a r_b r_c}{\sqrt{r_a r_b+r_a r_c+r_b r_c}}}{\sqrt{r_a r_b+r_a r_c+r_b r_c}}=\frac{r_a r_b r_c}{r_a r_b+r_a r_c+r_b r_c} \qquad (13)$$

formula hosil bo‘ladi.

Natijada, (12) va (13) ga ko’ra quyidagiga ega bo’lamiz

$$\frac{R}{2r}=\frac{(r_a+r_b)(r_a+r_c)(r_b+r_c)}{8r_a r_b r_c}. \qquad (14)$$

Koshi tengsizligia ko‘ra, $r_a+r_b\geq 2\sqrt{r_ar_b}$,$r_a+r_c\geq 2\sqrt{r_ar_c}$ va $r_b+r_c\geq 2\sqrt{r_br_c}$ bo ‘lgani uchun (14) ga ko’ra $\frac{R}{2r}=\frac{(r_a+r_b)(r_a+r_c)(r_b+r_c)}{8r_ar_br_c}\geq\frac{2\sqrt{r_ar_b}\,2\sqrt{r_ar_c}\,2\sqrt{r_br_c}}{8r_ar_br_c}=1$,

ya’ni $R\geq 2r$.

Agar $r_a=r_b=r_c$ bo ‘lsa, $a=b=c$ bo‘ladi va muntazam uchburchakda $R=2r$ tenglik o‘rinli.

Uchburchakning tashqarida ichki chizilgan aylanalarining radiuslari ko‘paymasini topamiz, (3) va Geron formulasiga ko’ra

$$r_ar_br_c=\frac{S^3}{(p-a)(p-b)(p-c)}=\frac{pS^3}{p(p-a)(p-b)(p-c)}=\frac{pS^3}{S^2}=pS\,.$$

Endi uchburchakning tashqarida ichki chizilgan aylanalarining radiuslari yigindisini topamiz

$$r_a+r_b+r_c-r=\frac{S}{p-a}+\frac{S}{p-b}+\frac{S}{p-c}-\frac{S}{p}=\frac{S}{p(p-a)(p-b)(p-c)}$$

$$\big(p(p-c)(p-b)+p(p-a)(p-c)+p(p-a)(p-b)+(p-a)(p-b)(p-c)\big)=$$

$$=\frac{S}{S^2}\big(p(p-c)(p-a+p-b)+(p-a)(p-b)(p-p+c)\big)=$$

$$=\frac{pc(p-c)+c(p-a)(p-b)}{S}=\frac{c\left(p^2-pc+p^2-pa-pb+ab\right)}{S}=$$

$$=\frac{c\left(2p^2-p(a+b+c)+ab\right)}{S}=\frac{abc}{S}=4R.$$

Natijada $r_a+r_b+r_c=4R+r$ yig‘indining formulasifa ega bo‘lamiz.

Berilgan uchburchakning tashqarida ichki chizilgan aylanalarining radiuslari r_a,r_b,r_c orqali uch o‘zgaruvchili elementar simmetrik ko‘phadga ega bo‘ldik

$$\begin{cases} r_a+r_b+r_c=4R+r, \\ r_ar_b+r_ar_c+r_br_c=p^2, \\ r_ar_br_c=pS. \end{cases} \tag{15}$$

Shunday qilib, agar uchta to‘g‘ri chiziq uchburchak tashkil etsa, har qaysi to‘g‘ri chiziqqa urinuvchi to‘rtta aylana mavjud ekan (6.38-rasm). Nemis matematigi K.Feyerbax planimetriyaning eng ajoyib va chiroyli teoremalaridan birini isbotlagan.

29-teorema. (*Feyerbax teoremasi*). To‘qqiz nuqtali aylana uchburchakka ichki chizilgan aylana va sirtdan ichki chizilgan uchta aylanaga urinadi.

E’tirof etish lozimki, aslida Eyler aylanasi nafaqat ichki va sirtdan ichki chizilgan aylanalarga, balki bu uchburchak bilan bog‘liq bo‘lgan yana 64 ta aylanaga urinar ekan.

30-teorema. (***Ptolemey teoremasi***) Aylanaga ichki chizilgan to‘rtburchak diagonallari ko‘paytmasi to‘rtburchak qarama-qarshi tomonlari ko‘paytmasi yig‘indisiga teng.

Isbot. Ichki chizilgan $ABCD$ to‘rtburchakda AC va BD diagonallar uchun

$$AC\cdot BD = AB\cdot CD + AD\cdot BC$$

ekanini isbotlaymiz.

Aylanaga ichki chizilgan to‘rtburchak qarama-qarshi burchaklari yig‘indisi 180° ga teng bo‘lgani uchun $\cos\angle D = -\cos\angle B$ bo‘ladi.

ABC va ACD uchburchaklar uchun kosinuslar teoremasidan:

$$AC^2 = AB^2 + BC^2 - 2AB\cdot BC\cdot\cos\angle B,\quad AC^2 = CD^2 + AD^2 - 2CD\cdot AD\cdot\cos\angle D.$$

Bu tengliklarning birinchisini $CD\cdot AD$ ga, ikkinchisini $AB\cdot BC$ ga ko‘paytirib qo‘shamiz:

$$AC^2\cdot AB\cdot BC + AC^2\cdot CD\cdot AD = AB^2\cdot CD\cdot AD + BC^2\cdot CD\cdot AD -$$
$$-2AB\cdot BC\cdot CD\cdot AD\cdot\cos\angle B + CD^2\cdot AB\cdot BC + AD^2\cdot AB\cdot BC + 2AB\cdot BC\cdot CD\cdot AD\cdot\cos\angle B.$$

Tenglikni soddalashtirib quyidagiga kelamiz:

$$AC^2(AB\cdot BC + CD\cdot AD) = AB\cdot CD(AB\cdot AD + BC\cdot CD) + BC\cdot AD(BC\cdot CD + AD\cdot AB)\Rightarrow$$

$$AC^2 = \frac{(AB\cdot CD + BC\cdot AD)(BC\cdot CD + AD\cdot AB)}{(AB\cdot BC + CD\cdot AD)}.$$

Shunga o‘xshash, ABD va BCD uchburchaklardan

$$BD^2 = \frac{(AB\cdot CD + BC\cdot AD)(AB\cdot BC + AD\cdot CD)}{(AB\cdot AD + BC\cdot CD)}.$$

Hosil bo‘lgan ikki tenglikni hadma-had ko‘paytirib, har ikki tomonni kvadrat ildiz chiqaramiz: $AC\cdot BD = AB\cdot CD + AD\cdot BC$.

Mustaqil ishlash uchun masalalar

6.129. Qavariq teng tomonli *ABCDEF* oltiburchakning *A, C, E* burchaklari to‘g‘ri burchaklar. Agar uning tomoni uzunligi $3\sqrt{3-\sqrt{3}}$ ga teng bo‘lsa, oltiburchak yuzini toping. javob: 27

6.130. Qavariq teng tomonli *ABCDEF* oltiburchakning *B* va *F* burchaklari to‘g‘ri burchaklar. Agar uning tomoni uzunligi $\frac{11}{\sqrt{4+\sqrt{3}+\sqrt{7}}}$ ga va *CE*=*AB* bo‘lsa, oltiburchak yuzini toping. javob: 30,25

6.131. Qavariq teng tomonli *ABCDE* beshburchakning *B* va *E* burchaklari to‘g‘ri burchaklar. Agar uning tomoni uzunligi $5\sqrt{4-\sqrt{7}}$ ga teng bo‘lsa, beshburchak yuzini toping. javob: 56,25

6.132. Qavariq teng tomonli *ABCDEF* oltiburchakning *F* burchagi to‘g‘ri burchak. Agar *ACE* uchburchak teng tomonli bo‘lib, tomoni uzunligi $\sqrt{\frac{3-\sqrt{3}}{2}}$ ga teng bo‘lsa, oltiburchak yuzini toping. javob: 0,75 448.

6.133. Qavariq teng tomonli *ABCDEF* oltiburchakning *A* va *E* uzunligi burchaklari to‘g‘ri burchaklar hamda $\angle C$=60°. Agar uning tomoni $\frac{1}{\sqrt{\sqrt{7}+\sqrt{3}+4}}$ ga teng bo‘lsa, oltiburchak yuzini toping. javob: 0,25

6.134. Qavariq teng tomonli *ABCDEF* oltiburchakning *B* va *F* burchaklari to‘g‘ri burchaklar hamda $\angle D$=120°. Agar uning tomoni uzunligi $\frac{1}{\sqrt{\sqrt{7}+\sqrt{3}+4}}$ ga teng bo‘lsa, *ACDE* to‘rt burchakning yuzini toping. javob: 75

6.135. Qavariq teng tomonli *ABCDEF* oltiburchakning $\angle A$, $\angle C$, $\angle E$ lari 150° ga teng. Agar uning tomoni uzunligi $5\sqrt{3-\sqrt{3}}$ ga teng bo‘lsa, oltiburchak yuzini toping. javob: 75

6.136. Qavariq teng tomonli *ABCDE* beshburchakning burchaklari uchun *C*=*D* va $\angle A$=90° ma’lum. Agar uning tomoni uzunligi $\sqrt[4]{56\sqrt{2}-77}$ ga teng bo‘lsa, *BCDE* to‘rt burchakning yuzini toping. javob: 1,75

6.137. Qavariq teng tomonli *ABCDE* beshburchakning *A* burchagi to‘g‘ri burchakli va *BC* tomoni *BE* diagonaliga perpendikular. Agar bu besh burchakning tomoni uzunligi $7\sqrt{2\sqrt{2}-\sqrt{3}}$ ga teng bo‘lsa, *BCDE* to‘rt burchakning yuzini toping. javob: 61,25

6.138. Qavariq teng tomonli *ABCDE* beshburchakning *B* burchagi to‘g‘ri burchakli va *CD* tomoni *CA* diagonaliga perpendikular. Agar diagonal $DF=\sqrt{2\sqrt{2}-\sqrt{3}}$ ga teng bo‘lsa, *ACDF* to‘rt burchakning yuzini toping. javob: 1,25