

Mavzu; Parametrga bog'liq xosmas integrallar

1⁰. Parametrga bog'liq xosmas integral tushunchasi. Faraz qilaylik, $f(x, y)$ funksiya

$$M = \{(x, y) \in R^2 : x \in [a, +\infty), y \in E \subset R\}$$

to'plamda berilgan bo'lsin. Bu funksiya har bir tayin $y \in E$ da x o'zgaruvchining funktsiyasi sifatida $[a, +\infty)$ da integrallanuvchi, ya'ni

$$\int_a^{+\infty} f(x, y) dx$$

xosmas integral yaqinlashuvchi. Ravshanki, integralning qiymati y o'zgaruvchiga bog'liq bo'ladi:

$$F(y) = \int_a^{+\infty} f(x, y) dx. \quad (1)$$

Masalan, $y > 1$ bo'lganda

$$\int_1^{+\infty} \frac{dx}{x^y} = \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^y} = \lim_{t \rightarrow \infty} \frac{1}{1-y} (t^{1-y} - 1) = \frac{1}{y-1}$$

bo'ladi. Demak, bu holda

$$F(y) = \frac{1}{y-1}$$

bo'ladi.

(1) integral parametrga bog'liq chegarasi cheksiz xosmas integral, y esa parametr deyiladi.

Xuddi shunga o'xshash

$$F_1(y) = \int_{-\infty}^a f(x, y) dx, \quad F_2(y) = \int_{-\infty}^{+\infty} f(x, y) dx$$

parametrga bog'liq xosmas integrallar tushunchalari kiritiladi.

Aytaylik, $f(x, y)$ funksiya

$$M = \{(x, y) \in R^2 : x \in [a, b), y \in E \subset R\}$$

to'plamda berilgan bo'lsin. Bu funktsiya har bir tayin $y \in E$ da x o'zgaruvchining funktsiyasi sifatida qaralganda uning uchun b maxsus nuqta bo'lib, u $[a, b)$ da integrallanuvchi, ya'ni

$$\int_a^b f(x, y) dx$$

xosmas integral yaqinlashuvchi bo'lsin. Ravshanki, bu holda ham integralning qiymati y o'zgaruvchiga bog'liq bo'ladi:

$$\Phi(y) = \int_a^b f(x, y) dx . \quad (2)$$

Masalan, $0 < y < 1$ bo'lganda

$$\int_1^2 \frac{dx}{(2-x)^y} = \lim_{t \rightarrow 2-0} \int_1^t (2-x)^{-y} dx = \lim_{t \rightarrow 2-0} \frac{1}{y-1} [(2-t)^{1-y} - 1] = \frac{1}{1-y}$$

bo'ladi. Demak, bu holda

$$\Phi(y) = \frac{1}{1-y}$$

bo'ladi.

(2) integral parametrga bog'liq, chegaralanmagan funktsiyaning xosmas integrali, y esa parametr deyiladi.

Umumiy holda, parametrga bog'liq, chegaralanmagan funktsiyaning chegarasi cheksiz integrali tushunchasi ham yuqoridagidek kiritiladi.

Parametrga bog'liq xosmas integrallarning funktsional xossalari (limiti, uzluksizligi, differentsiallanishi integrallanishi)ni

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral uchun keltirish bilan kifoyalanamiz.

2^o. Integralning tekis yaqinlashishi. Aytaylik, $f(x, y)$ funktsiya

$$M = \{(x, y) \in R^2 : x \in [a, +\infty), y \in E \subset R\}$$

to'plamda berilgan bo'lib, har bir tayin $y \in E$ da

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

xosmas integral yaqinlashuvchi bo'lsin. Ta'rifga binoan

$$F(y) = \int_a^{+\infty} f(x, y) dx = \lim_{t \rightarrow +\infty} \int_a^t f(x, y) dx \quad (a < t < \infty)$$

bo'ladi.

Natijada berilgan $f(x, y)$ funktsiya yordamida

$$G(y, t) = \int_a^t f(x, y) dx, \quad (a < t < \infty)$$

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

funktsiyalar yuzaga keladi va

$$\lim_{t \rightarrow +\infty} G(y, t) = F(y) \quad (y \in E)$$

munosabat bajariladi.

Demak, $G(y, t)$ funktsiya $t \rightarrow +\infty$ da limit funktsiya $F(y)$ ga ega bo'ladi.

1-ta'rif. Agar $t \rightarrow +\infty$ da $G(y, t)$ funktsiya limit funktsiya $F(y)$ ga E to'plamda tekis yaqinlashsa,

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral E to'plamda tekis yaqinlashuvchi deyiladi.

Integralning E to'plamda tekis yaqinlashuvchiligini quyidagicha anglash lozim:

1) har bir tayin $y \in E$ da $\int_a^{+\infty} f(x, y) dx$ xosmas integral yaqinlashuvchi;

2) $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $\forall t > \delta$ va $\forall y \in E$ uchun

$$\left| \int_t^{+\infty} f(x, y) dx \right| < \varepsilon$$

tengsizligi bajariladi.

1-misol. Ushbu

$$\int_0^{+\infty} e^{-x} \cos xy dx$$

xosmas integralning $(-\infty, +\infty)$ da tekis yaqinlashuvchi ekani ko'rsatilsin.

◀ Har bir tayin $y \in (-\infty, +\infty)$ da qaralayotgan xosmas integralning yaqinlashuvchi ekanligi ravshan.

$\forall \varepsilon > 0$ ga ko'ra $\delta = \ln \frac{2}{\varepsilon}$ deyilsa, unda $\forall t > \delta$ va $\forall y \in (-\infty, +\infty)$ uchun

$$\left| \int_t^{+\infty} e^{-x} \cos xy dx \right| \leq \int_t^{+\infty} e^{-x} dx = e^{-t} \leq e^{-\delta} = e^{-\ln \frac{2}{\varepsilon}} = \frac{\varepsilon}{2} < \varepsilon$$

bo'ladi. Demak, berilgan integral $(-\infty, +\infty)$ da tekis yaqinlashuvchi. ▶

2-ta'rif. Agar $t \rightarrow +\infty$ da $G(y, t)$ funktsiya limit funktsiya $F(y)$ ga E to'plamda tekis yaqinlashmasa,

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral E to'plamda tekis yaqinlashmaydi deyiladi.

Integralning E to'plamda yaqinlashuvchi, ammo uning shu to'plamda tekis yaqinlashmaydi degani quyidagini anglatadi:

1) har bir tayin $y \in E$ da $\int_a^{+\infty} f(x, y) dx$ xosmas integral yaqinlashuvchi;

2) $\forall \delta > 0$ olinganda ham, shunday $\varepsilon_0 > 0$, $y_0 \in E$ va $t_1 > \delta$ bo'lgan $t_1 \in [a, +\infty)$ topiladiki,

$$\left| \int_{t_1}^{+\infty} f(x, y_1) dx \right| \geq \varepsilon_0$$

bo'ladi.

2-misol. Ushbu

$$\int_0^{+\infty} ye^{-xy} dx$$

xosmas integralning $(0, +\infty)$ da tekis yaqinlashmasligi ko'rsatilsin.

◀ Ravshanki,

$$\int_0^{+\infty} ye^{-xy} dx = \lim_{t \rightarrow +\infty} \int_0^t ye^{-xy} dx = \lim_{t \rightarrow +\infty} (1 - e^{-ty}) = 1.$$

Demak, berilgan xosmas integral yaqinlashuvchi. Aytay-lik, $y \in E = (0, +\infty)$ bo'lsin. Ixtiyoriy musbat δ sonni olaylik. Agar $\varepsilon_0 = \frac{1}{3}$, $t_0 > \delta$ va $y_0 = \frac{1}{t_0}$ deb olsak, u holda

$$\left| \int_{t_0}^{+\infty} y_0 e^{-xy_0} dx \right| = e^{-t_0 y_0} = e^{-1} > \frac{1}{3} = \varepsilon_0$$

bo'ladi. Bu esa $\int_0^{+\infty} ye^{-xy} dx$ integral $E = (0, +\infty)$ da tekis yaqinlashmasligini bildiradi. ▶

Yuqoridagi

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

parametr ga bog'liq xosmas integralning parametr y bo'yicha E to'plamda tekis yaqinlashishini quyidagicha ham ta'rif-lasa bo'ladi.

3-ta'rif. Agar

$$\lim_{t \rightarrow +\infty} \sup_{y \in E} \left| F(y) - \int_a^t f(x, y) dx \right| = \lim_{t \rightarrow +\infty} \sup_{y \in E} \left| \int_t^{+\infty} f(x, y) dx \right| = 0$$

($a < t < +\infty$) bo'lsa,

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

xosmas integral E to'plamda tekis yaqinlashuvchi deyiladi.

3-misol. Ushbu

$$F(y) = \int_1^{+\infty} \frac{dx}{x^y}$$

xosmas integralning $E = [2, +\infty)$ to'plamda tekis yaqinlashuvchi ekani ko'rsatilsin.

◀ Ravshanki, $1 < t < +\infty$ uchun

$$0 \leq \sup_{y \in [2, +\infty)} \left| \int_t^{+\infty} \frac{dx}{x^y} \right| = \sup_{y \in [2, +\infty)} \frac{1}{(y-1)t^{y-1}} \leq \frac{1}{t}$$

bo'lib,

$$\lim_{t \rightarrow +\infty} \sup_{y \in [2, +\infty)} \left| \int_t^{+\infty} \frac{dx}{x^y} \right| = 0$$

bo'ladi. Demak, berilgan xosmas integral $E = [2, +\infty)$ to'plamda tekis yaqinlashuvchi. ►

Endi integralning tekis yaqinlashishini ifodalovchi teoremani keltiramiz.

1-teorema. Ushbu

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integralning E to'plamda tekis yaqinlashuvchi bo'lishi uchun $\forall \varepsilon > 0$ olinganda ham y ga bog'liq bo'lmagan shunday $\delta = \delta(\varepsilon) > 0$ topilib, $t' > \delta, t'' > \delta$ tengsizliklarni qanoatlantiruvchi $\forall t', t''$ va $\forall y \in E$ da

$$\left| \int_{t'}^{t''} f(x, y) dx \right| < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

Bu teoremaning isboti ravshan.

3⁰. Parametrga bog'liq xosmas integrallarning parametr bo'yicha tekis yaqinlashish alomatlari.

2-teorema (Veyershtrass alomati). Aytaylik, $f(x, y)$ funksiya

$$M = \{(x, y) \in R^2 : x \in [a, +\infty), y \in E \subset R\}$$

to'plamda berilgan va har bir tayin $y \in E$ da $f(x, y)$ funksiya $[a, +\infty)$ da integrallanuvchi bo'lsin.

Agar $[a, +\infty)$ da aniqlangan shunday $\varphi(x)$ funksiya topilsaki,

1) $\forall x \in [a, +\infty), \forall y \in E$ uchun $|f(x, y)| \leq \varphi(x)$ bo'lsa,

2) ushbu $\int_a^{+\infty} \varphi(x) dx$ xosmas integral yaqinlashuvchi bo'lsa, u holda

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral E to'plamda tekis yaqinlashuvchi bo'ladi.

◀ Modomiki, $\int_a^{+\infty} \varphi(x) dx$ yaqinlashuvchi ekan, unda $\forall \varepsilon > 0$ olinganda ham,

shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $t' > \delta, t'' > \delta$ bo'lganda

$$\left| \int_{t'}^{t''} \varphi(x) dx \right| < \varepsilon$$

tengsizlik bajariladi.

Ayni paytda,

$$\left| \int_{t'}^{t''} f(x, y) dx \right| \leq \int_{t'}^{t''} |f(x, y)| dx \leq \int_{t'}^{t''} \varphi(x) dx \quad (t' < t'')$$

bo'lganligi sababli

$$\left| \int_{t'}^{t''} f(x, y) dx \right| < \varepsilon$$

bo'ladi. Yuqorida keltirilgan 1-teoremaga muvofiq

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral E to'plamda tekis yaqinlashuvchi bo'ladi. ▶

4-misol. Ushbu

$$\int_0^{+\infty} \frac{\cos xy}{1+x^2} dx \quad (y \in E = (-\infty, +\infty))$$

integralning tekis yaqinlashuvchi ekani ko'rsatilsin.

◀ Ravshanki, $\forall x \in [0, +\infty)$ va $\forall y \in (-\infty, +\infty)$ uchun

$$|f(x, y)| = \left| \frac{\cos xy}{1+x^2} \right| \leq \frac{1}{1+x^2}$$

bo'ladi. Ayni paytda,

$$\int_0^{+\infty} \frac{1}{1+x^2} dx$$

xosmas integral yaqinlashuvchi bo'lganligi sababli Veyersh-trass alomatiga ko'ra berilgan integral $E = (-\infty, +\infty)$ da tekis yaqinlashuvchi bo'ladi. ►

Integrallarning tekis yaqinlashishini aniqlashda ko'p foydalaniladigan Abel hamda Dirixle alomatlarini isbot-siz keltiramiz.

3-teorema (Abel alomati). $f(x, y)$ va $g(x, y)$ funktsiyalar

$$M = \{(x, y) \in R^2 : x \in [a, +\infty), y \in E \subset R\}$$

to'plamda berilgan bo'lib, quyidagi shartlar bajarilsin:

- 1) har bir tayin $y \in E$ da $g(x, y)$ funktsiya $[a, +\infty)$ da monoton bo'lsin;
- 2) $\forall (x, y) \in M$ uchun $|g(x, y)| \leq c$, ($c = \text{const}$) bo'lsin;
- 3) ushbu

$$\int_a^{+\infty} f(x, y) dx$$

integral E to'plamda tekis yaqinlashuvchi bo'lsin.

U holda

$$\int_a^{+\infty} f(x, y) g(x, y) dx$$

integral E to'plamda tekis yaqinlashuvchi bo'ladi.

4-teorema (Dirixle alomati). $f(x, y)$ va $g(x, y)$ funktsiyalar M to'plamda berilgan bo'lib, quyidagi shartlar bajarilsin:

- 1) $\forall t \geq a$ hamda $\forall t \in E$ da

$$\left| \int_a^t f(x, y) dx \right| \leq c \quad (c = \text{const})$$

tengsizlik bajarilsin;

2) har bir tayin $y \in E$ da $g(x, y)$ funktsiya limit funktsiya $\varphi(x) = 0$ ga tekis yaqinlashsin.

U holda

$$\int_a^{+\infty} f(x, y) g(x, y) dx$$

integral E to'plamda tekis yaqinlashuvchi bo'ladi.

5-misol. Ushbu

$$\int_0^{+\infty} \frac{\sin xy}{x} dx \quad (y \in E = [1, 2])$$

integral tekis yaqinlashuvchilikka tekshirilsin.

◀ Berilgan integralda

$$f(x, y) = \sin xy, \quad g(x, y) = \frac{1}{x}$$

deyilsa, unda

1) $\forall t > 0, \forall y \in [1, 2]$ uchun

$$\left| \int_0^t f(x, y) dx \right| = \left| \int_0^t \sin xy dx \right| = \left| \frac{1 - \cos ty}{y} \right| \leq 2,$$

2) $x \rightarrow +\infty$ da $g(x, y) = \frac{1}{x}$ funktsiya $E = [1, 2]$ da nolga tekis yaqinlashuvchi.

Dirixle alomatiga ko'ra berilgan integral $E = [1, 2]$ da tekis yaqinlashuvchi bo'ladi. ▶

Mashqlar

1. Ushbu

$$\int_0^{+\infty} \frac{y \cos xy^2 dx}{y + x^y}$$

integralning $E = [2, 10]$ to'plamda tekis yaqinlashishi isbotlansin.

2. Ushbu

$$\int_0^{+\infty} \frac{dx}{1 + x^y} \quad (y > 1)$$

integral tekis yaqinlashishga tekshirilsin.

3. Aytaylik, $f(x)$ funktsiya R da uzluksiz bo'lib, $\forall x \in R$ da $f(x) \geq 0$ bo'lsin. Ushbu

$$\int_0^{+\infty} f(y-x) dx, \quad \int_{-\infty}^0 f(y-x) dx$$

integrallarning y parametr bo'yicha ixtiyoriy chekli $[a, b] \subset R$ segmentda tekis yaqinlashuvchi bo'lishi isbotlansin.

Parametrga bog‘liq xosmas integrallarning funktsional xossalari

Ushbu ma’ruzada parametrga bog‘liq xosmas integral

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

ning limiti, uzluksizligi, differentsiallanishi hamda integrallanishi masalalarini bayon etamiz.

1^o. $F(y)$ funktsiyaning limiti. Aytaylik, $f(x, y)$ funktsiya

$$M = \{(x, y) \in R^2 : x \in [a, +\infty), y \in E \subset R\}$$

to‘plamda berilgan, $y_0 \in R$ esa E to‘plamning limit nuqtasi bo‘lsin.

1-teorema. $f(x, y)$ funktsiya qo‘yidagi shartlarni bajar-sin:

1) har bir tayin $y \in E$ da $f(x, y)$ funktsiya x o‘zgaruvchining funksiyasi sifatida $[a, +\infty)$ da uzluksiz;

2) $y \rightarrow y_0$ da $f(x, y)$ funktsiya ixtiyoriy $[a, t]$ da ($a < t < \infty$) limit funktsiya $\varphi(x)$ ga tekis yaqinlashsin;

3) ushbu $F(y) = \int_a^{+\infty} f(x, y) dx$ integral E to‘plamda tekis yaqinlashuvchi

bo‘lsin. U holda $y \rightarrow y_0$ da $F(y)$ funktsiya limitga ega va

$$\lim_{y \rightarrow y_0} F(y) = \int_a^{+\infty} \varphi(x) dx$$

bo‘ladi.

◀Teoremaning 1- va 2- shartlarining bajarilishidan $\varphi(x)$ funktsiyaning $[a, \infty)$ da uzluksiz bo‘lishini topamiz. Binobarin, $\varphi(x)$ ixtiyoriy $[a, t]$ da ($a < t < \infty$) integrallanuvchi bo‘ladi.

Modomiki,

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral E to'plamda tekis yaqinlashuvchi ekan, unda 77-ma'ruzadagi 1-teoremaga ko'ra

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, t' > \delta, t'' > \delta, \forall t', t'', \forall y \in E:$$

$$\left| \int_{t'}^{t''} f(x, y) dx \right| < \varepsilon$$

bo'ladi.

Keyingi tengsizlikda, $y \rightarrow y_0$ da limitga o'tsak, u holda

$$\left| \int_{t'}^{t''} \varphi(x) dx \right| \leq \varepsilon$$

tengsizlik hosil bo'ladi. Bundan $\varphi(x)$ funktsiyaning $[a, \infty)$ da integrallanuvchiligi kelib chiqadi.

Ushbu

$$\left| \int_a^{+\infty} f(x, y) dx - \int_a^{+\infty} \varphi(x) dx \right|$$

ayirmani qaraymiz. Uning uchun quyidagi tengsizlik bajariladi:

$$\left| \int_a^{+\infty} f(x, y) dx - \int_a^{+\infty} \varphi(x) dx \right| \leq \int_a^t |f(x, y) - \varphi(x)| dx + \left| \int_t^{+\infty} f(x, y) dx \right| + \left| \int_t^{+\infty} \varphi(x) dx \right|. \quad (a < t < \infty)$$

(1)

Bu tengsizlikning o'ng tomonidagi qo'shiluvchilarni baholaymiz. $\int_a^{+\infty} f(x, y) dx$

integral E to'plamda tekis yaqinlashuvchi bo'lganligi sababli,

$$\forall \varepsilon > 0, \exists \delta_1 = \delta_1(\varepsilon) > 0, \forall t > \delta_1, \forall y \in E$$

$$\left| \int_t^{+\infty} f(x, y) dx \right| < \frac{\varepsilon}{3} \quad (2)$$

bo'ladi.

$$\int_a^{+\infty} \varphi(x) dx$$

integral yaqinlashuvchi bo'lganligi sababli

$$\forall \varepsilon > 0, \exists \delta_2 = \delta_2(\varepsilon) > 0, \forall t > \delta_2 :$$

$$\left| \int_t^{+\infty} \varphi(x) dx \right| < \frac{\varepsilon}{3} \quad (3)$$

bo'lad.

Ravshanki, $\forall t > \delta_0 : (\delta_0 = \max(\delta_1, \delta_2))$ da (2) va (3) tengsizliklar bir yo'la bajariladi. Funktsiya $y \rightarrow y_0$ da $f(x, y)$ $[a, t]$ da ($t > \delta_0$) limit funktsiya $\varphi(x)$ ga tekis yaqinlashuvchi bo'lganligi sababli

$$\forall \varepsilon > 0, \exists \delta' = \delta'(\varepsilon) > 0, |y - y_0| < \delta' \quad \forall y \in E, \forall x \in [a, t] \quad (a < t < \infty)$$

$$|f(x, y) - \varphi(x)| < \frac{\varepsilon}{3(t - a)} \quad (4)$$

bo'lad.

(1), (2), (3) va (4) munosabatlardan

$$\left| \int_a^{+\infty} f(x, y) dx - \int_a^{+\infty} \varphi(x) dx \right| < \varepsilon$$

bo'lishi kelib chiqadi. Demak

$$\lim_{y \rightarrow y_0} F(y) = \lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \varphi(x) dx. \blacktriangleright$$

Keyingi tenglikni quyidagicha ham yozish mumkin

$$\lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \left[\lim_{y \rightarrow y_0} f(x, y) \right] dx.$$

1- misol. Ushbu

$$\lim_{y \rightarrow +0} \int_a^{+\infty} e^{-xy} \frac{\sin x}{x} dx = \int_a^{+\infty} \frac{\sin x}{x} dx$$

tenglik isbotlansin.

◀ Agar $\varphi(x) = \frac{\sin x}{x}$ funktsiyaning $x = 0$ nuqtadagi qiymati-ni $\varphi(0) = 1$ deb

olinsa, unda

$$f(x, y) = e^{-xy} \frac{\sin x}{x}$$

funktsiya $\{(x, y) \in R^2 : x \in [0, +\infty), y \in [0, +\infty)\}$ to'plamda uzluksiz bo'ladi.

Ravshanki, har bir tayin $y \in [0, +\infty)$ da $f(x, y)$ funktsiya x o'zgaruvchining funktsiyasi sifatida $[0, +\infty)$ da uzluksiz bo'lib, $y \rightarrow +\infty$ da bu funktsiya ixtiyoriy $[0, t]$ da $(0 < t < +\infty)$ $\varphi(x) = \frac{\sin x}{x}$ funktsiyaga tekis yaqinlashadi.

Endi,

$$\int_0^{+\infty} e^{-xy} \frac{\sin x}{x} dx$$

xosmas integralni parametr y bo'yicha $[0, +\infty)$ da tekis yaqinlashuvchi bo'lishini ko'rsatamiz.

Agar 77-ma'ruzada keltirilgan Abel alomatida $f(x, y)$ funktsiya sifatida $\frac{\sin x}{x}$, $g(x, y)$ funktsiya sifatida e^{-xy} funktsiyalar olinsa, ular uchun Abel alomatining barcha shartlarining o'rinli bo'lishini ko'rsatish qiyin emas. Demak, alomatga ko'ra

$$\int_0^{+\infty} e^{-xy} \frac{\sin x}{x} dx$$

integral tekis yaqinlashuvchi.

Yuqorida keltirilgan 1-teoremaga binoan

$$\lim_{y \rightarrow +0} \int_0^{+\infty} e^{-xy} \frac{\sin x}{x} dx = \int_0^{+\infty} \left(\lim_{y \rightarrow +0} e^{-xy} \frac{\sin x}{x} \right) dx$$

bo'lib, undan

$$\lim_{y \rightarrow +0} \int_0^{+\infty} e^{-xy} \frac{\sin x}{x} dx = \int_0^{+\infty} \frac{\sin x}{x} dx$$

bo'lishi kelib chiqadi. ►

2⁰. $F(y)$ funktsiyaning uzluksizligi. Aytaylik, $f(x, y)$ funktsiya

$$M_0 = \{(x, y) \in R^2 : x \in [a, +\infty), y \in [c, d]\}$$

to'plamda berilgan bo'lsin.

2-teorema. Agar $f(x, y)$ funktsiya M_0 to'plamda uzluksiz bo'lib,

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral $[c, d]$ da tekis yaqinlashuvchi bo'lsa, u holda $F(y)$ funktsiya $[c, d]$ da uzluksiz bo'ladi.

◀ Ixtiyoriy $y_0 \in [c, d]$, $y_0 + \Delta y \in [c, d]$ nuqtalarni olib, $F(y)$ funksiyaning orttirmasini topamiz:

$$\Delta F(y_0) = F(y_0 + \Delta y) - F(y_0) = \int_a^{+\infty} [f(x, y_0 + \Delta y) - f(x, y_0)] dx.$$

Shartga ko'ra $\int_a^{+\infty} f(x, y) dx$ integral $[c, d]$ da tekis yaqinlashuvchi. Unda

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \forall t_0 > \delta, \forall y \in [c, d]:$$

$$\left| \int_{t_0}^{+\infty} f(x, y) dx \right| < \frac{\varepsilon}{3} \quad (5)$$

bo'ladi. Ravshanki, $f(x, y)$ funktsiya

$$M_{t_0} = \{(x, y) \in R^2 : x \in [a, t_0], y \in [c, d]\} \quad (a < t_0 < +\infty)$$

to'plamda tekis uzluksiz bo'ladi. Unda

$$\forall \varepsilon > 0, \exists \delta_1 = \delta_1(\varepsilon) > 0, \Delta y < \delta_1(\varepsilon)$$

$$|f(x, y_0 + \Delta y) - f(x, y_0)| < \frac{\varepsilon}{3(t_0 - a)} \quad (6)$$

bo'ladi. Agar $\delta_0 = \max\{\delta, \delta_1\}$ deyilsa uning uchun (5) va (6) tengsizliklar bir yo'la bajariladi. (5) va (6) munosabatlarni e'tiborga olib topamiz:

$$\begin{aligned} |\Delta F(y_0)| &= \left| \int_a^{+\infty} [f(x, y_0 + \Delta y) - f(x, y_0)] dx \right| \leq \\ &\leq \int_a^{t_0} |f(x, y_0 + \Delta y) - f(x, y_0)| dx + \left| \int_{t_0}^{+\infty} f(x, y_0 + \Delta y) dx \right| + \left| \int_{t_0}^{+\infty} f(x, y_0) dx \right| < \varepsilon. \end{aligned}$$

Demak,

$$\lim_{\Delta y \rightarrow 0} \Delta F(y_0) = 0.$$

Bu esa $F(y)$ funktsiyaning $[c, d]$ oraliqda uzluksizligini bildiradi. ►

2-misol. Ushbu

$$F(y) = \int_0^{+\infty} e^{-(x-y)^2} dx$$

integral parametr y ning uzluksiz funktsiyasi bo'lishi ko'rsatilsin.

◄ Berilgan integralda

$$x - y = t$$

almashtirish bajaramiz. Unda

$$F(y) = \int_{-y}^{+\infty} e^{-t^2} dt = \int_{-y}^0 e^{-t^2} dt + \int_0^{+\infty} e^{-t^2} dt = \int_0^y e^{-t^2} dt + \int_0^{+\infty} e^{-t^2} dt$$

bo'lib, bu yig'indining har bir qo'shiluvchisi y ning uzluksiz funktsiyasi bo'lgani uchun berilgan integral parametr y ning uzluksiz funktsiyasi bo'ladi. ►

3^o. $F(y)$ funktsiyani differentsiallash. Faraz qilaylik $f(x, y)$ funktsiya M_0 to'plamda berilgan bo'lsin.

3-teorema. $f(x, y)$ funktsiya quyidagi shartlarni qanoatlantirsin:

- 1) $f(x, y)$ funktsiya M_0 to'plamda uzluksiz;
- 2) $f'_y(x, y)$ xususiy hosila mavjud va u M_0 to'plamda uzluksiz;
- 3) Har bir tayin $y \in [c, d]$ da

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral yaqinlashuvchi;

4) Ushbu

$$\int_a^{+\infty} f'_y(x, y) dx$$

integral $[c, d]$ da tekis yaqinlashuvchi.

U holda $F(y)$ funktsiya $[c, d]$ da $F'(y)$ hosilaga ega va

$$F'(y) = \int_a^{+\infty} f'_y(x, y) dx$$

bo‘ladi.

◀ $y_0 \in [c, d]$, $y_0 + \Delta y \in [c, d]$ nuqtalarni olib, topamiz:

$$\frac{F(y_0 + \Delta y) - F(y_0)}{\Delta y} = \int_a^{+\infty} \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y} dx.$$

Lagranj teoremasiga ko‘ra

$$\frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y} = f'_y(x, y_0 + \theta \Delta y), \quad (0 < \theta < 1),$$

$$\frac{F(y_0 + \Delta y) - F(y_0)}{\Delta y} = \int_a^{+\infty} f'_y(x, y_0 + \theta \Delta y) dx$$

bo‘ladi. Demak,

$$\lim_{\Delta y \rightarrow 0} \frac{F(y_0 + \Delta y) - F(y_0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \int_a^{+\infty} f'_y(x, y_0 + \theta \Delta y) dx. \quad (7)$$

Shartga ko‘ra $f'_y(x, y)$ funktsiya M_0 to‘plamda uzluksiz. Kantor teoremasiga binoan u

$$M_t = \{(x, y) \in R^2 : x \in [a, t] \quad y \in [c, d]\} \quad (a < t < \infty)$$

to‘plamda tekis uzluksiz bo‘ladi. Unda

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, |\Delta y| < \delta(\varepsilon) \quad \forall x \in [a, t]$$

$$|f'_y(x, y_0 + \theta \Delta y) - f'_y(x, y_0)| < \varepsilon$$

bo‘ladi. Demak $\Delta y \rightarrow 0$ da $f'_y(x, y_0 + \theta \Delta y)$ funktsiya $f'_y(x, y_0)$ ga tekis yaqinlashadi. Shartga ko‘ra

$$\int_a^{+\infty} f'_y(x, y_0) dx$$

integral tekis yaqinlashuvchi. Unda

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, t' > \delta; t'' > \delta; \quad \forall t', t'', \quad \forall y \in [c, d]:$$

$$\left| \int_{t'}^{t''} f'_y(x, y) dx \right| < \varepsilon$$

jumladan,

$$\left| \int_{t'}^{t''} f'_y(x, y_0 + \theta \Delta y) dx \right| < \varepsilon$$

bo‘ladi. Keyingi tengsizlikning bajarilishidan esa

$$\int_a^{+\infty} f'_y(x, y_0 + \theta \Delta y) dx$$

integralning tekis yaqinlashuviligi kelib chiqadi. Ushbu ma’ruzada keltirilgan 1-teoremani (7) tenglikning o‘ng tomoniga qo‘llab, topamiz.

$$\lim_{\Delta y \rightarrow 0} \int_a^{+\infty} f'_y(x, y_0 + \theta \Delta y) dx = \int_a^{+\infty} [\lim_{\Delta y \rightarrow 0} f'_y(x, y_0 + \theta \Delta y)] dx = \int_a^{+\infty} f'_y(x, y_0) dx. \quad (8)$$

(7) va (8) munosabatlardan

$$F'(y_0) = \int_a^{+\infty} f'_y(x, y_0) dx \quad (9)$$

bo‘lishi kelib chiqadi.

(9) munosabatni quyidagicha ham yozish mumkin:

$$\frac{d}{dy} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \frac{\partial f(x, y)}{\partial y} dx.$$

Bu differentsiallash amalini integral ostiga o‘tkazish qoidasini ifodalaydi. ►

4⁰. $F(y)$ funktsiyani integrallash. Aytaylik, $f(x, y)$ funktsiya

$$M_0 = \{(x, y) \in R^2 : x \in [a, +\infty), y \in [c, d]\}$$

to‘plamda berilgan bo‘lsin.

4-teorema. Agar $f(x, y)$ funktsiya M_0 to‘plamda uzluksiz va

$F(y) = \int_a^{+\infty} f(x, y) dx$ integral $[c, d]$ da tekis yaqinlashuvchi bo‘lsa, u holda $F(y)$

funktsiya $[c, d]$ da integrallanuvchi va

$$\int_c^d F(y) dy = \int_c^d \left[\int_a^{+\infty} f(x, y) dx \right] dy = \int_a^{+\infty} \left[\int_c^d f(x, y) dy \right] dx$$

bo‘ladi.

◄Ravshanki, $F(y)$ funktsiya $[c, d]$ da uzluksiz bo‘ladi. Binobarin, u $[c, d]$ da integrallanuvchi.

Shartga ko‘ra

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integral $[c, d]$ da tekis yaqinlashuvchi. Unda

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \forall t > \delta, \forall y \in [c, d]:$$

$$\left| \int_t^{+\infty} f(x, y) dx \right| < \varepsilon$$

bo'lad i Shu munosabatdagi t ni olib topamiz:

$$\int_a^{+\infty} f(x, y) dx = \int_a^t f(x, y) dx + \int_t^{+\infty} f(x, y) dx.$$

Natijada

$$\begin{aligned} \int_c^d \left[\int_a^{+\infty} f(x, y) dx \right] dy &= \int_c^d \left[\int_a^t f(x, y) dx \right] dy + \int_c^d \left[\int_t^{+\infty} f(x, y) dx \right] dy = \\ &= \int_a^t \left[\int_c^d f(x, y) dy \right] dx + \int_c^d \left[\int_t^{+\infty} f(x, y) dx \right] dy \end{aligned}$$

bo'lad i.

Agar

$$\left| \int_c^d F(y) dy - \int_a^t \left[\int_c^d f(x, y) dy \right] dx \right| \leq \int_c^d \left| \int_t^{+\infty} f(x, y) dx \right| dy < \varepsilon(d - c)$$

bo'lishini e'tiborga olsak, unda

$$\int_c^d F(y) dy = \lim_{t \rightarrow \infty} \int_a^t \left[\int_c^d f(x, y) dy \right] dx = \int_a^\infty \left[\int_c^d f(x, y) dy \right] dx$$

bo'lib,

$$\int_c^d \left[\int_a^{+\infty} f(x, y) dx \right] dy = \int_a^{+\infty} \left[\int_c^d f(x, y) dy \right] dx$$

ekanligi kelib chiqadi. ►

Mashqlar

1. Agar $f(x)$ funktsiya $(0, \infty)$ da integrallanuvchi bo'lib,

$$F(y) = \int_0^\infty e^{-yx} f(x) dx$$

bo'lsa,

$$\lim_{y \rightarrow +0} F(y) = \lim_{y \rightarrow +0} \int_0^{\infty} e^{-yx} f(x) dx = \int_0^{\infty} f(x) dx$$

bo'lishi isbotlansin.

2. Ushbu

$$F(y) = \int_0^{\infty} e^{-y^2 x} y dx \quad (-\infty < y < +\infty)$$

funktsiyani uzluksizlikka tekshirilsin.

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