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MATEMATIKA 1

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S O‘ Z   B O SHI

Ushbu o‘quv qo‘llanma O‘zR Oliy va o‘rta maxsus ta’lim vazirligi tomonidan tasdiqlangan “Oliy matematika,” fanining o‘quv rejasiga to‘la mos keladi va bu o‘quv qo‘llanma bakalavriatning quyidagi ta’lim yo‘nalishi talabalariga mo‘ljallangan:

5330500 – Kompyuter injiniringi (“Kompyuter injiniringi”, “AT-servis”, “Multimedia texnologiyalari”);

5330300 – Axborot xavfsizligi;

5330600 – Dasturiy injiniring;

5350100 – Telekommunikatsiya texnologiyalari (“Telekommunikatsiyalar”, “Teleradioeshittirish”, Mobil tizimlari);

5350200 –Televizion texnologiyalar (“Audiovizual texnologiyalar”, “Telestudiya tizimlari va ilovalari”);

5350300 – Axborot-kommunikatsiya texnologiyalari sohasida iqtisodiyot va menejment;

5350400 – Axborot-kommunikatsiya texnologiyalari sohasida kasb ta’limi;

5350500 –Pochta aloqasi texnologiyasi;

5350600 –Axborotlashtirish va kutubxonashunoslik.

O‘quv qo‘llanma ikki jilddan iborat bo‘lib, uning birinchi jildida chiziqli algebra, tekslikda va fazoda analitik geometriya, vektorlar algebrasi elementlari, matematik analizga kirish, bir o‘zgaruvchili funksiyalarning differensial hisobi, funksiyalarni hosilalar yordamida tekshirish, bir o‘zgaruvchili funksiyalarning integral hisobi kiritilgan. Har bir paragrafda dastlab qisqacha nazariy ma’lumotlar keltirilib, keyin esa turli tipdagi misol va masalalarning batafsil yechilish usullar ko‘rsatilib, kerakli uslubiy ko‘rsatmalar berilgan. Har bir bo‘lim uchun mustaqil yechish uchun yetarli miqdorda misol va masalalar hamda test savollari berilgan. Undan tashqari har bir bo‘limda berilayotgan nazariy bilimlarni amaliyot bilan bog‘lovchi masalalar yechib ko‘rsatilgan va mustaqil bajarish uchun topshiriqlar berilgan.

Kitob hajmini ixchamlashtirish maqsadida unda quyidagi belgilashlar kiritilgan:

►- masala va misollar yechilishining boshlanishi;

◄- masala va misollar yechilishining tugallanishi.

Mazkur qo‘llanma yaratishda mualliflar o‘zlarining Toshkent axborot texnologiyalari universitetining talabalariga ko‘p yillar mobaynida o‘qilgan ma’ruzalar va talabalar bilan o‘tkazilgan amaliy mashg‘ulotlarni asos qilib olingan, shuningdek mavjud adabiyotlardan ham foydalanilgan. O‘quv qo‘llanma kamchiliklardan holi emas, albatta. Qo‘llanmadagi kamchiliklarni bartaraf etishga va uning sifatini yaxshilashga qaratilgan fikr va mulohazalarini bildirganlarga mualliflar avvaldan o‘z minnatdorchliklarini bildiradilar.

MUNDARIJA

I BOB. CHIZIQLI ALGEBRA ELEMENTLARI

1.1. Determinantlar va ularning xossalari. Determinantlarni hisoblash usullari

Ikkinchi tartibli determinant deb

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad (1.1)$$

tenglik bilan aniqlanadigan songa aytiladi. Qisqacha, Δ deb belgilanadi. Bu yerda a_{11}, a_{12}, a_{21}, a_{22} -determinantning ***elementlari*** deyiladi.

a_{11}, a_{12} va a_{21}, a_{22} mos ravishda determinantning 1- va 2-satrlari, a_{11}, a_{21} va a_{12}, a_{22} mos ravishda determinantning 1- va 2-ustunlari deyiladi. Ya'ni

$$a_{ij} : \begin{cases} i - \text{satr tartibi} \\ j - \text{ustun tartibi.} \end{cases}$$

Determinantning ixtiyoriy satri yoki ustuni determinantning ***qatori*** deb ataladi. a_{11}, a_{22} -elementlar joylashgan diagonal ***bosh diagonal*** deyiladi. a_{21}, a_{12} -elementlar joylashgan diagonal ***yordamchi diagonal*** deyiladi.

1.1-misol

Hisoblang: $\begin{vmatrix} 3 & 2 \\ -4 & 5 \end{vmatrix}$

► (1.1) formulani qo'llaymiz:

$$\begin{vmatrix} 3 & 2 \\ -4 & 5 \end{vmatrix} = 3 \cdot 5 - 2 \cdot (-4) = 15 + 8 = 23. ◄$$

Eslatma. Determinantning elementlari funksiyalar bo'lishi ham mumkin, shuning uchun determinantning qiymati, umuman olganda, funksiyadir.

1.2-misol

Hisoblang: $\begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix}$.

► $\begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix} = \cos^2 x - \sin^2 x = \cos 2x$. ◄

Uchinchi tartibli determinant deb

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \qquad (1.2)$$

tenglik bilan aniqlanadigan songa aytiladi. Ko‘pincha, determinant tartibiga mos ravishda Δ_3 deb ham belgilanadi.

1.3-misol

Hisoblang: $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 2 & -3 & 5 \end{vmatrix}$.

► (1.2) formulani qo‘llaymiz:

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 2 & -3 & 5 \end{vmatrix} = 2\cdot1\cdot5 + (-1)\cdot4\cdot2 + 3\cdot1\cdot(-3) - 3\cdot1\cdot2 - (-1)\cdot1\cdot5 - 2\cdot4\cdot(-3) =$$

$$= 10 - 8 - 9 - 6 + 5 + 24 = 16. \blacktriangleleft$$

Determinantning a_{ij} elementining M_{ij} ***minori*** deb, uning $i-$ satri va $j-$ ustunini o‘chirishdan hosil bo‘lgan determinantga aytiladi.

Masalan, uchunchi tartibli determinant uchun

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Determinantning a_{ij} elementining A_{ij} ***algebraik to‘ldiruvchisi*** deb,

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$$

tenglik bilan aniqlanadigan songa aytiladi.

Masalan, uchunchi tartibli determinant uchun

$$A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}.$$

1.4-misol

Quyidagi $\begin{vmatrix} 1 & 1 & 4 \\ -1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix}$ determinantning M_{23} minorini hisoblang

► Determinantning $2-$ satri va $3-$ ustunini o‘chiramiz:

$$\begin{vmatrix} 1 & 1 & 4 \\ -1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} = 1\cdot 2 - 1\cdot(-3) = 2+3 = 5.$$ Demak, $M_{23} = 5$. ◄

1.5-misol

Quyidagi $\begin{vmatrix} 2 & 1 & -4 \\ 1 & 3 & 5 \\ 3 & 2 & -1 \end{vmatrix}$ determinantning A_{32} va A_{13} algebraik to‘ldiruvchilarini hisoblang.

► $A_{32} = (-1)^{3+2} M_{32}$, ya’ni $A_{32} = -M_{32}$ bo‘lgani uchun, determinantning $3-$ satri va $2-$ ustunini o‘chiramiz:

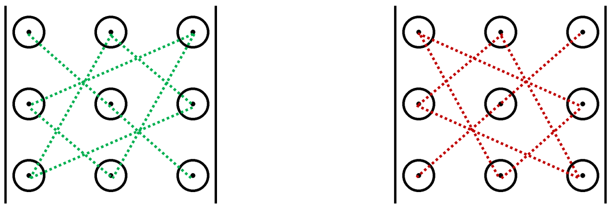
$$A_{32} = -\begin{vmatrix} 2 & 1 & -4 \\ 1 & 3 & 5 \\ 3 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = -\left(2\cdot 5 - (-4)\cdot 1\right) = -14 \cdot$$

$A_{13} = (-1)^{1+3} M_{32}$ yoki $A_{13} = M_{13}$ bo‘lgani uchun, determinantning $1-$satri va $3-$ ustunini o‘chirib hisoblaymiz.

$$A_{13} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = 1\cdot 2 - 3\cdot 3 = -7.$$

Demak, $A_{32} = -14$, $A_{13} = -7$. ◄

Determinant hisoblashning (1.2) formulasini eslab qolish uchun quyidagi sxemani keltiramiz:



Hisoblashning bu qoidasi ***uchburchak usuli*** deyiladi. Qulaylik uchun deteminantning birinchi va ikkinchi ustunini quyidagicha parallel ko‘chirib, bosh diagonal va yordamchi diagonalga parallel chiziqlar bo‘yicha ko‘paytmalar tuzamiz (***Sarrius usuli***):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} =$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}\ , \qquad (1.3)$$

bunda bosh diagonal bo‘yicha hosil qilinga qo‘shuluvchilar musbat ishora bilan, yordamchi diagonal bo‘yicha hosil qilingan qo‘shiluvchilar manfiy ishora bilan olinadi.

1.6-misol

Quyidagi $\begin{vmatrix} 1 & 1 & 4 \\ -1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix}$ determinantni hisoblang

► Determinantning birinchi va ikkinchi ustunini parallel ko‘chirib yozib Sarryus usulida hisoblaymiz:

$$\begin{vmatrix} 1 & 1 & 4 \\ -1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix}\begin{matrix} 1 & 1 \\ -1 & 2 \\ -3 & 2 \end{matrix}=1\cdot2\cdot5+1\cdot3\cdot(-3)+4\cdot(-1)\cdot2-4\cdot2\cdot(-3)-1\cdot3\cdot2-1\cdot(-1)\cdot5=$$

$$=10-9-8+24-6+5=16.◄$$

(1.2) formulani algebraik to‘lduruvchilar yordamida quyidagicha ifodalaymiz:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13} \qquad (1.4)$$

1.7-misol

Quyidagi $\begin{vmatrix} 2 & 5 & -4 \\ 1 & -3 & 5 \\ 3 & 2 & -1 \end{vmatrix}$ determinantni hisoblang

► (1.4) formulani qo‘llaymiz, buning uchun avval A_{11}, A_{12} va A_{13} larni hisoblaymiz:

$$A_{11}=\begin{vmatrix} -3 & 5 \\ 2 & -1 \end{vmatrix}=3-10=-7,\ A_{12}=-\begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix}=-(-1-15)=16,$$

$$A_{13}=\begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix}=2-(-9)=11.$$

$$\begin{vmatrix} 2 & 5 & -4 \\ 1 & -3 & 5 \\ 3 & 2 & -1 \end{vmatrix}=2\cdot(-7)+5\cdot16-4\cdot11=-14+80-44=22.◄$$

Determinantning xossalari:

1. *Determinantning barcha satrlarini mos ustunlari bilan almashtirish natijasida qiymati o'z garmaydi*

2. *Determinantning biror qatoridagi barcha elementlari nolga teng bo'lsa, uning qiymati nolga teng bo'ladi.*

3. *Determinantning ikkita parrallel qatorining o'rinlarini o'zaro almashtirish natijasida determinant qiymatining ishorasi qarama-qarshisiga o'zgaradi.*

4. *Determinantning ikkita parrallel qatori bir xil bo'lsa, uning qiymati nolga teng bo'ladi.*

5. *Agar determinantning biror qatori bir xil ko'paytuvchiga ega bo'lsa, bu ko'paytuvchini determinant belgisidan tashqariga chiqarish mumkin. Demak, determinantni biror songa ko'paytirish uchun uning biror qatori elementlarini shu songa ko'paytirish kifoya.*

6. *Determinantning ikkita parrallel qatori mos pavishda proporsional bo'lsa, uning qiymati nolga teng bo'ladi.*

7. *Determinantning qiymati uning biror qatori elementlarini mos algebraik to'ldiruvchilariga ko'paytirilib qo'shilganiga teng.*

8. *Agar determinantning biror qator elementlari yig'indilardan iborat bo'lsa, u holda bu determinant ikki determinant yig'indisiga teng bo'ladi, bunda birinchi determinantda shu qator birinchi qo'shuluvchilardan, ikkinchisida esa ikkinchi qo'shuluvchilardan tashkil topgan bo'ladi.*

Masalan,

$$\begin{vmatrix} a_{11}+b_1 & a_{12} & a_{13} \\ a_{21}+b_2 & a_{22} & a_{23} \\ a_{31}+b_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{33} \\ b_3 & a_{32} & a_{33} \end{vmatrix}.$$

9. *Agar determinantning biror qatori elementlarini ixtiyoriy songa ko'paytirib, parallel qatori elementlariga mos ravishda qo'shilsa, determinant qiymati o'zgarmaydi.*

10. *Determinantning biror qatori elementlarini parallel qator mos elementlarining algebraik to'ldiruvchilariga ko'paytmalari yig'indisi nolga teng.*

Masalan, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$.

9-xossa yordamida, (1.4) formuladan ko'ra umumiyroq bo'lgan, determinantni ***biror qatori bo'yicha yoyib hisoblash usuli*** hosil bo'ladi. Masalan, uchunchi tartibli determinant uchun

$$\Delta_3 = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}, \qquad (1.5)$$

$$\Delta_3 = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}. \qquad (1.6)$$

Bu yerda (1.5) va (1.6) formulalar mos ravishda determinantning ixtiyoriy $i-$ ***satri*** va $j-$ ***ustuni bo'yicha yoyilmasi*** deyiladi.

1.8-misol

Quyidagi $\begin{vmatrix} 4 & -2 & 0 \\ 3 & 5 & 6 \\ -3 & 4 & 0 \end{vmatrix}$ determinantni biror qatori bo'yicha yoyib hisoblang

► Determinantni eng ko'p nol element qatnashgan qatorini aniqlaymiz. Bu yerda uchunchi ustunda eng ko'p nol element bo'lgani uchun, (1.6) formulani qo'llaymiz: $\Delta_3 = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} = 6A_{23}$, chunki $a_{13} = a_{33} = 0$.

$$\begin{vmatrix} 4 & -2 & 0 \\ 3 & 5 & -6 \\ -3 & 4 & 0 \end{vmatrix} = -6 \cdot (-1)^{2+3} \begin{vmatrix} 4 & -2 \\ -3 & 4 \end{vmatrix} = 6 \cdot (16-6) = 60. \blacktriangleleft$$

Quyida biz *n*-tartibli determinantning ko'rinishini keltiramiz:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Quyida biz, asosan, yuqori tartibli determinantlarni hisoblashda keng qo'llanadigan ikkita hisoblash usulini keltiramiz.

1. Yuqori tartibli determinantni, asosiy xossalaridan foydalanib, biror qatorining bitta elementidan boshqa barcha elementlarini nolga aylantirilib, so'ng 9-xossa yordamida *tartibini pasaytirib hisoblash* mumkin.

1.9-misol

Quyidagi $\Delta = \begin{vmatrix} 2 & -4 & 1 & 5 \\ 1 & -3 & 2 & 5 \\ 2 & 2 & 0 & -3 \\ 3 & -1 & 1 & 2 \end{vmatrix}$ determinantni hisoblang.

► Determinantning birinchi satrini -2 va -1 ga ko'paytirib, mos ravishda ikkinchi va to'rtinchi satriga qo'shamiz:

$$\Delta = \begin{vmatrix} 2 & -4 & 1 & 5 \\ -3 & 5 & 0 & -5 \\ 2 & 2 & 0 & -3 \\ 1 & 3 & 0 & -3 \end{vmatrix}$$

$3-$ustun bo‘yicha yoyib(9-xossa), ya’ni $\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} + a_{43}A_{43}$ va $a_{23} = a_{33} = a_{43} = 0$ ekanini e’tiborga olib, tartibini pasaytiramiz. Hosil bo‘lgan uchunchi tartibli determinantni esa uchburchak usulida yechamiz.

$$\Delta = 1\cdot(-1)^{1+3}\begin{vmatrix} -3 & 5 & -5 \\ 2 & 2 & -3 \\ 1 & 3 & -3 \end{vmatrix} = 18-15-30+10+30-27 = -14 \,.$$ ◄

2. Bosh diagonalidan yuqorisidagi yoki pastidagi barcha elementlari nollardan iborat bo‘lgan determinant ***uchburchak shaklidagi determinant*** deyiladi. Bunday determinantning qiymati bosh diagonali elementlari ko‘paytmasiga teng. Har qanday determinantni uchburchak shakliga keltirib hisoblash mumkin.

1.10-misol

Quyidagi $\Delta = \begin{vmatrix} 2 & -4 & 1 & 5 \\ 1 & -3 & 2 & 5 \\ 2 & 2 & 0 & -3 \\ 3 & -1 & 1 & 2 \end{vmatrix}$ determinantni uchburchak shakliga keltirib hisoblang.

► Determinantning a_{11} elementini 1 ga aylantirish uchun birinchi va ikkinchi satrlarining o‘rinlarini almashtiramiz. Hosil bo‘lgan birinchi satrni -2, -2 va -3 ga ko‘paytirib, mos ravishda ikkinchi , uchunchi va to‘rtinchi satrlarga qo‘shamiz.

$$\Delta = -\begin{vmatrix} 1 & -3 & 2 & 5 \\ 2 & -4 & 1 & 5 \\ 2 & 2 & 0 & -3 \\ 3 & -1 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 2 & 5 \\ 0 & 2 & -3 & -5 \\ 0 & 8 & -4 & -13 \\ 0 & 8 & -5 & -13 \end{vmatrix}$$

Uchinchi satrini -1 ga ko‘paytirib to‘rtinchi satrlarga qo‘shamiz:

$$\Delta = -\begin{vmatrix} 1 & -3 & 2 & 5 \\ 0 & 2 & -3 & -5 \\ 0 & 8 & -4 & -13 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

Ikkinchi satrini -4 ga ko‘paytirib uchinchi satriga qo‘shamiz. So‘ng uchinchi va to‘rtinchi satrlar o‘rinlarini almashtiramiz:

$$\Delta=-\begin{vmatrix}1&-3&2&5\\0&2&-3&-5\\0&0&8&7\\0&0&-1&0\end{vmatrix}=\begin{vmatrix}1&-3&2&5\\0&2&-3&-5\\0&0&-1&0\\0&0&8&7\end{vmatrix}$$

Uchinchi satrini 8 ga ko‘paytirib to‘rtinchi satriga qo‘shamiz :

$$\Delta=\begin{vmatrix}1&-3&2&5\\0&2&-3&-5\\0&0&-1&0\\0&0&0&7\end{vmatrix}=1\cdot 2\cdot(-1)\cdot 7=-14\,.$$ ◄

1-Auditoriya topshiriqlari

1. Berilgan ikkinchi tartibli deteminantlarni hisoblang.

a) $\begin{vmatrix}5&3\\7&-4\end{vmatrix}$; b) $\begin{vmatrix}4&-7\\-2&-3\end{vmatrix}$; d) $\begin{vmatrix}tgx&-1\\1&tgx\end{vmatrix}$.

2. Tenglamani yeching.

a) $\begin{vmatrix}x+3&2\\7&x-2\end{vmatrix}=0$; b) $\begin{vmatrix}\sin 2x&-\cos 2x\\\sin 3x&\cos 3x\end{vmatrix}=0$

3. Berilgan uchinchi tartibli deteminantlarni hisoblang.

a) $\begin{vmatrix}1&-2&4\\-3&5&5\\2&-1&3\end{vmatrix}$; b) $\begin{vmatrix}10&-2&4\\-15&3&6\\20&-1&5\end{vmatrix}$; d) $\begin{vmatrix}1&2&5\\5&-3&7\\4&6&5\end{vmatrix}$.

4. Berilgan uchinchi tartibli deteminantlarni satr yoki ustun bo‘yicha yoyib hisoblang.

a) $\begin{vmatrix}5&0&6\\4&0&5\\2&4&3\end{vmatrix}$; b) $\begin{vmatrix}2&2&-1\\7&0&3\\3&4&0\end{vmatrix}$, d) $\begin{vmatrix}1&2&3\\4&5&6\\7&8&0\end{vmatrix}$.

5. Berilgan deteminantlarni uchburchak shakliga keltirib hisoblang.

a) $\begin{vmatrix}2&-3&2&4\\-3&2&2&5\\1&5&-3&0\\0&-1&1&2\end{vmatrix}$; b) $\begin{vmatrix}4&0&-3&5\\3&2&-2&1\\1&3&1&0\\5&6&2&-1\end{vmatrix}$.

6. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ deteminantlarni $a-b$, $a-c$ va $b-c$ larga bo‘linishini isbotlang.

7. $\begin{vmatrix} 1 & 6 & 9 \\ 2 & 3 & 4 \\ 3 & 2 & 5 \end{vmatrix}$ deteminantni hisoblamasdan, 13 ga bo‘linishini isbotlang.

1-Mustaqil yechish uchun testlar

1. To‘g‘ri tengliklarni aniqlang

1) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} d & c \\ b & a \end{vmatrix}$, 2) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$, 3) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ a & b \end{vmatrix}$, 4) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} b & a \\ d & c \end{vmatrix}$.

A) 1), 3); B) 1), 2); D) 2), 3); E) 3), 4).

2. $\begin{vmatrix} 2 & 1 & 2 \\ -2 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ determinantning a_{21} elementining M_{21} minorini toping:

A) 4 B) -4 D) 2 E) -2.

3. $\begin{vmatrix} 2 & 1 & 2 \\ -2 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$ determinantning a_{21} elementining A_{21} algebraik to‘ldiruvchisiini toping:

A) 4 B) -4 D) 2 E) -2.

4. $\begin{vmatrix} 0 & 3 & 7 \\ 1 & -3 & 4 \\ 0 & 2 & 6 \end{vmatrix}$ deteminantni hisoblang

A) 4 B) -4 D) 2 E) -2.

5. Agar n-tartibli determinantning satrlarini teskari tartibda yozib chiqilsa qiymati qanday o‘zgaradi?

A) $(-1)^n$ ga ko‘payadi; B) $(-1)^{n-1}$ ga ko‘payadi; D) $(-1)^{\frac{n(n-1)}{2}}$ ga ko‘payadi; E) o‘zgarmaydi.

1.2. Matritsalar va ular ustida amallar. Teskari matritsa

Berilgan m ta satr va n ta ustundan iborat to‘g‘ri burchakli ushbu

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ yoki } A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad (1.7)$$

jadvalga $m \times n$ o‘lcovli ***matritsa*** deyiladi. Bu yerda a_{ij} $(i=1, 2,...,m;\ j=1, 2,...,n)$ - matritsaning elementlari deyiladi. Matritsalar lotin alifbosidagi bosh harflar bilan belgilanadi. Ba’zan, o‘lchamlarini ifodalash uchun $A_{m\times n}$ kabi belgilanadi.

Matritsalar qisqacha,

$$A=\left(a_{ij}\right)\ \ (i=\overline{1,m};\ j=\overline{1,n}) \text{ yoki } A=\left\|a_{ij}\right\|\ \ (i=\overline{1,m};\ j=\overline{1,n}) \qquad (1.8)$$

ko‘rinishda ham yoziladi.

Agar matritsada $i=1$ bo‘lsa, bunday

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$

matritsa ***satr matritsa*** deyiladi.

Agar matritsada $j=1$ bo‘lsa, bunday

$$A=\begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{n1} \end{bmatrix}$$

matritsa ***ustun matritsa*** deyiladi.

Matritsada $m=n$ bo‘lsa, ***kvadrat matritsa*** deyiladi:

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad (1.9)$$

Bosh diagonalidagi elementlari birlardan va qolgan elementlari nollardan iborat bo‘lgan kvadrat matritsa ***birlik matritsa*** deyiladi va E deb belgilanadi. Masalan,

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matritsaning barcha elementlari nollardan iborat bo‘lsa, ***nol matritsa*** deyiladi va Q deb belgilanadi.

Mos elementlari teng, ya’ni $a_{ij} = b_{ij}$ bo‘lgan bir xil o‘lchamli A va B matritsalar ***teng matritsalar*** deyiladi.

Matritsaning satrlarini mos ustunlariga almashtirishdan hosil bo‘lgan matritsa ***transponirlangan matritsa*** deyiladi va A^T kabi belgilanadi.

Masalan, $A = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$ matritsa berilgan bo‘lsa, A^T matritsani hisoblash uchun satrlarini mos ustunlariga almashtiramiz: $A^T = \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$.

Bir xil o‘lchovli A va B matritsalarni qo‘shish va ayirish mumkin.

Bir xil o‘lchovli ***A va B matritsalarni yig‘indisi(ayirmasi)*** deb shunday C matritsaga aytiladiki, uning elementlari A va B matritsalarning mos elementlari yig‘indisiga(ayirmasiga) teng. $C = A + B$ ($C = A - B$) kabi belgilanadi.

A matritsani λ songa ko‘paytmasi deb, barcha elementlarini λ songa ko‘paytirishdan hosil bo‘lgan B matritsaga aytiladi, $B = \lambda A$ kabi belgilanadi.

Matritsalarni qo‘shish va songa ko‘paytirish quyidagi xossalarga ega:

i. $A + B = B + A$

ii. $A + Q = A$

iii. $\lambda(A + B) = \lambda A + \lambda B$

iv. $(\lambda + \mu)A = \lambda A + \mu A$

1.11-misol

Quyida $A = \begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ matritsalar berilgan bo‘lsa $A + B$ va $2A - B$ matritsalarni hisoblang

► $A + B = \begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -2+3 & 5+(-1) \\ 3+2 & -1+4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$;

$$2A - B = 2 \cdot \begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot(-2) & 2\cdot 5 \\ 2\cdot 3 & 2\cdot(-1) \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} -4-3 & 10-(-1) \\ 6-2 & -2-3 \end{bmatrix} = \begin{bmatrix} -7 & 11 \\ 4 & -5 \end{bmatrix}.$$ ◄

1.12-misol

Quyida $A=\begin{bmatrix}2 & -1 & 3\\ 4 & 0 & 5\end{bmatrix}$ $B=\begin{bmatrix}-3 & 1\\ -5 & 4\\ 0 & 2\end{bmatrix}$ matritsalar berilgan. A^T+2B va $A-B^T$ matritsalarni hisoblang

$$\blacktriangleright\ A^T+B=\begin{bmatrix}2 & 4\\ -1 & 0\\ 3 & 5\end{bmatrix}+2\cdot\begin{bmatrix}-3 & 1\\ -5 & 4\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2+2\cdot(-3) & 4+2\cdot 1\\ -1+2(-5) & 0+2\cdot 4\\ 3+2\cdot 0 & 5+2\cdot 2\end{bmatrix}=\begin{bmatrix}-4 & 6\\ -11 & 8\\ 3 & 9\end{bmatrix};$$

$$A-B^T=\begin{bmatrix}2 & -1 & 3\\ 4 & 0 & 5\end{bmatrix}-\begin{bmatrix}-3 & -5 & 0\\ 1 & 4 & 2\end{bmatrix}=\begin{bmatrix}2-(-3) & -1-(-5) & 3-0\\ 4-1 & 0-4 & 5-2\end{bmatrix}=\begin{bmatrix}5 & 4 & 3\\ 3 & -4 & 3\end{bmatrix}.\blacktriangleleft$$

Berilgan $m\times k$ o‘lchovli A matritsani $k\times n$ o‘lchovli B matritsaga ***ko‘paytmasi*** deb, shunday $m\times n$ o‘lchovli C matritsaga aytiladiki, uning elementlari

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+...+a_{ik}b_{kj} \tag{1.10}$$

tenglik bilan aniqlanadi. $C=A\cdot B$ kabi belgilanadi.

Demak, birinchi matritsaning ustunlari soni ikkinchi matritsaning satrlari soniga teng bo‘lgan holdagini ularni ko‘paytirish mumkin. Umuman olganda, $A\cdot B$ ko‘paytma mavjud bo‘ganda $B\cdot A$ ko‘paytma mavjud bo‘lavermaydi. $B\cdot A$ ko‘paytma mavjud bo‘gan holda ham, umuman olganda, $AB\neq BA$.

Agar $A\cdot B=B\cdot A$ bo‘lsa, A va B matritsalar ***kommutativ*** matritsalar deyiladi.

1.13-misol

Quyida $A=\begin{bmatrix}3 & 1 & 2\\ -2 & 0 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ -5 & 3\\ 0 & 4\end{bmatrix}$ matritsalar berilgan. $A\cdot B$ va $B\cdot A$ ko‘paytmalarni hisoblang

► Bu yerda $A_{2\times 3}$ va $B_{3\times 2}$ bo‘lgani uchun AB matritsa 2×2 o‘lchovli bo‘ladi:

$$A\cdot B=\begin{bmatrix}3 & 1 & 2\\ -2 & 0 & 5\end{bmatrix}\cdot\begin{bmatrix}1 & 2\\ -5 & 3\\ 0 & 4\end{bmatrix}=\begin{bmatrix}3\cdot 1+1\cdot(-5)+2\cdot 0 & 3\cdot 2+1\cdot 3+2\cdot 4\\ -2\cdot 1+0\cdot(-5)+5\cdot 0 & -2\cdot 2+0\cdot 3+5\cdot 4\end{bmatrix}=$$

$$=\begin{bmatrix}-2 & 17\\ -2 & 16\end{bmatrix}.$$

$B\cdot A$ matritsa esa 3×3 o‘lchovli bo‘ladi:

$$B\cdot A=\begin{bmatrix}1&2\\-5&3\\0&4\end{bmatrix}\cdot\begin{bmatrix}3&1&2\\-2&0&5\end{bmatrix}=\begin{bmatrix}1\cdot3+2\cdot(-2)&1\cdot1+2\cdot0&1\cdot2+2\cdot5\\-5\cdot3+3\cdot(-2)&-5\cdot1+3\cdot0&-5\cdot2+3\cdot5\\0\cdot4+4\cdot(-2)&0\cdot1+4\cdot0&0\cdot2+4\cdot5\end{bmatrix}=$$

$$=\begin{bmatrix}-1&1&12\\-21&-5&5\\-8&0&20\end{bmatrix}. \quad A\cdot B\neq B\cdot A.$$ ◀

1.14-misol

Quyida $A=\begin{bmatrix}2&2\\1&2\end{bmatrix}$, $B=\begin{bmatrix}3&4\\2&3\end{bmatrix}$ matritsalar berilgan. $A\cdot B$ va $B\cdot A$ ko‘paytmalarni hisoblang

▶ $A\cdot B=\begin{bmatrix}2&2\\1&2\end{bmatrix}\cdot\begin{bmatrix}3&4\\2&3\end{bmatrix}=\begin{bmatrix}2\cdot3+2\cdot2&2\cdot4+2\cdot3\\1\cdot3+2\cdot2&1\cdot4+2\cdot3\end{bmatrix}=\begin{bmatrix}10&14\\7&10\end{bmatrix};$

$B\cdot A=\begin{bmatrix}3&4\\2&3\end{bmatrix}\cdot\begin{bmatrix}2&2\\1&2\end{bmatrix}=\begin{bmatrix}3\cdot2+4\cdot1&3\cdot2+4\cdot2\\2\cdot2+3\cdot1&2\cdot2+3\cdot2\end{bmatrix}=\begin{bmatrix}10&14\\7&10\end{bmatrix}.$

$A\cdot B=B\cdot A$, demak, A va B matritsalar kommutativlanadigan matritsalar.◀

Matritsalarni ko‘paytirish quyidagi xossalarga ega:

i. $(\lambda A)B=\lambda(AB)$

ii. $(A+B)C=AC+AB$

iii. $A(B+C)=AB+AC$

iv. $A(BC)=(AB)C$

Transponirlangan matritsa uchun esa quyidagi formulalar o‘rinli:

1. $\left(A^T\right)^T=A$
2. $(AB)^T=B^T\cdot A^T$

Agar A kvadrat matritsaning determinanti noldan farqli bo‘lsa, ya’ni $\det A\neq0$ bo‘lsa, A matritsa ***xosmas matritsa*** deyiladi.

Agar $\det A=0$ bo‘lsa, A matritsa ***xos matritsa*** deyiladi.

Agar $AA^{-1}=A^{-1}A=E$ tenglik o‘rinli bo‘lsa, A^{-1}matritsa Axosmas matrtsaning ***teskari matritsasi*** deyiladi. Bu yerda E matritsa A matritsa o‘lchovi bilan bir xil o‘lchovli birlik matritsadir.

Xosmas matritsa A uchun yagona A^{-1} teskari matritsa mavjud va quyidagi formula bilan hisoblanadi:

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \qquad (1.11)$$

Teskari matritsa quyidagi xossalarga ega:

1. $\det\left(A^{-1}\right) = \frac{1}{\det A}$
2. $(AB)^{-1} = B^{-1} \cdot A^{-1}$

1.15-misol

Quyidagi matritsalarning teskarilarini toping

a) $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, b) $A = \begin{bmatrix} 2 & -4 & 1 \\ 1 & -5 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

► a) $\det A = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = -10 \neq 0$ algebraik to‘ldiruvchilarni hisoblaymiz:

$$A_{11} = 4,\ A_{12} = -3,\ A_{21} = -2,\ A_{22} = -1$$

Natijada, (2.5) formulaga ko‘ra,

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 4 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -0,4 & 0,2 \\ 0,3 & 0,1 \end{bmatrix}$$

Tekshirish:

$$AA^{-1} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -0,4 & 0,2 \\ 0,3 & 0,1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E$$

a) Uchunchi tartibli determinantni hisoblaymiz, $\det A = -8$ va algebraik to‘ldiruvchilar: $A_{11} = -2$, $A_{12} = 2$, $A_{13} = 4$, $A_{21} = 3$, $A_{22} = 1$, $A_{23} = -2$, $A_{31} = -7$, $A_{32} = -5$, $A_{33} = -6$. U holda,

$$A^{-1} = -\frac{1}{8}\begin{pmatrix} -2 & 3 & -7 \\ 2 & 1 & -5 \\ 4 & -2 & -6 \end{pmatrix}$$ ◄

1.3. Matritsalar ustida amallarning tatbiqlari

1.3.1. Elektr tarmoqlariga tatbiqi

Eng sodda elektr zanjiri ikkita asosiy komponentadan, ***elektr manba*** (bu yerda E elektr potensial voltlarda (V) o‘lchanadi) va ***rezistordan*** (bu yerda R qarshilik Omlarda (Ω)o‘lchanadi). Biz (A) amperlarda o‘lchanadigan elektr tokini aniqlash bilan qiziqamiz.

Ba’zi hollarda ikki nuqta orasidagi elektr potensial ikki nuqta orasidagi ***kamayuvchi kuchlanish*** deb ataladi. Elektr toki va kamayuvchi kuchlanish musbat yoki manfiy bo‘lishi mumkin.

Elektr zanjirda elektr toki oqimi quyidagi uchta qoida orqali boshqariladi:

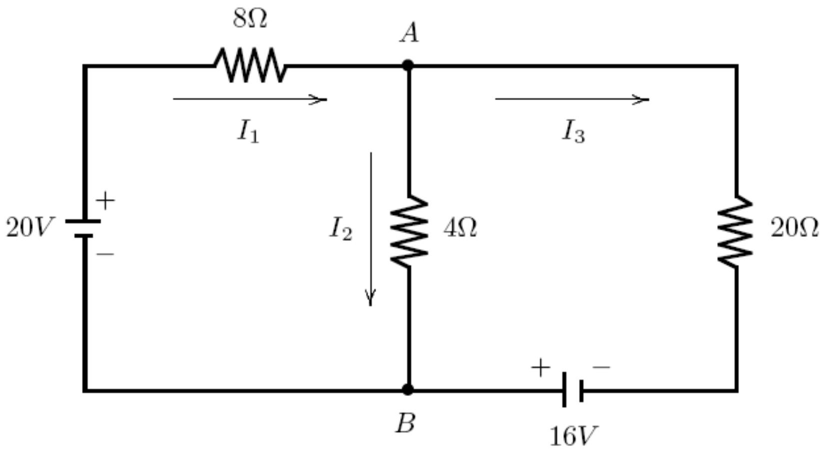
- Om qonuni: $E = IR$ formula bilan berilgan bo‘lsa ham o‘tayotgan I elektr toki bilan R qarshilik yordamida rezistor orqali E kuchlanish kamayadi;
- Toki kuchi qonuni: ichki ixtiyoriy nuqtadagi elektr toki oqimining yig‘indisi tashqi nuqtaning elektr toki oqimlarining yig‘indisiga teng;
- Kuchlanish qonuni: ixtiyoriy yopiq halqa bo‘ylab kamayuvchi kuchlanishlar yig‘indisi nolga teng.

Ixtiyoriy halqa bo‘ylab, soat mili bo‘yicha musbat yo‘nalishni tanlaymiz va soat miliga qarama-qarshi yo‘nalishni esa manfiy yo‘nalish deb olamiz.

- Rezistor bo‘ylab kamayuvchi kuchlanish agar elektr toki oqimlari halqaning musbat yo‘nalishida bo‘lsa, u holda musbat va agar elektr toki oqimlari halqaning manfiy yo‘nalishida bo‘lsa, u holda manfiy olinadi;
- Elektr manba bo‘ylab kamayuvchi kuchlanish agar halqaning musbat yo‘nalishi "+" dan "–" ga o‘tsa, u holda musbat va agar halqaning manfiy yo‘nalishi "-" dan "+" ga o‘tsa, u holda manfiy olinadi.

1.16-misol

Quyidagi diagrammada ko‘rsatilgan elektr zanjirni qaraylik:



Biz I_1, I_2 va I_3 elektr tokini aniqlashga harakat qilamiz. A nuqtaga elektr toki qonunini tatbiq qilsak, u holda $I_1 = I_2 + I_3$ ga ega bo‘lamiz. B nuqtaga ham elektr toki qonunini tatbiq qilsak, u holda yana o‘sha natijaga ega bo‘lamiz. Bundan quyidagi chiziqli tenglamaga ega bo‘lamiz:

$$I_1 - I_2 - I_3 = 0.$$

So‘ngra keling, dastlab chap tomondagi halqani qaraylik va soat mili bo‘yicha yo‘nalishni musbat yo‘nalish deb olaylik. Om qonuniga ko‘ra 8Ω resistor bo‘ylab $8I_1$ kamayuvchi kuchlanish bo‘lsa, 4Ω resistor bo‘ylab esa $4I_2$ kamayuvchi kuchlanish bo‘ladi. Boshqa tomondan esa, halqaning musbat yo‘nalishi "-" dan "+" ga o‘tganligi sababli elektr manba bo‘ylab manfiy $20V$ kamayuvchi kuchlanish o‘tadi. Bu halqaga kuchlanish qonunini tatbiq qilsak, u holda $8I_1 + 4I_2 - 20 = 0$ ni hosil qilamiz va quyidagi chiziqli tenglamaga ega bo‘lamiz:

$$8I_1 + 4I_2 = 20 \text{ yoki } 2I_1 + I_2 = 5.$$

So‘ngra keling, endi o‘ng tomondagi halqani qaraylik va soat mili bo‘yicha yo‘nalishni musbat yo‘nalish deb olaylik. Om qonuniga ko‘ra 20Ω resistor bo‘ylab $20I_3$ kamayuvchi kuchlanish bo‘lsa, 4Ω resistor bo‘ylab esa $-4I_2$ kamayuvchi kuchlanish bo‘ladi. Boshqa tomondan esa, halqaning musbat yo‘nalishi "-" dan "+" ga o‘tganligi sababli elektr manba bo‘ylab manfiy $16V$ kamayuvchi kuchlanish o‘tadi. Bu halqaga kuchlanish qonunini tatbiq qilsak, u holda $20I_3 - 4I_2 - 16 = 0$ ni hosil qilamiz va quyidagi chiziqli tenglamaga ega bo‘lamiz:

$$4I_2 - 20I_3 = -16 \text{ yoki } I_2 - 5I_3 = -4.$$

Bulardan quyidagi uchta chiziqli tenglamalar sistemasiga ega bo‘lamiz:

$$\begin{cases} I_1 - I_2 - I_3 = 0, \\ 2I_1 + I_2 \qquad = 5, \\ \quad I_2 - 5I_3 = -4. \end{cases} \qquad (1)$$

Bu sistemaning kengaytirilgan matrissasi

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 1 & 0 & 5 \\ 0 & 1 & -5 & -4 \end{array}\right).$$

Bu matrissani eshelon koʻrinishga keltirsak

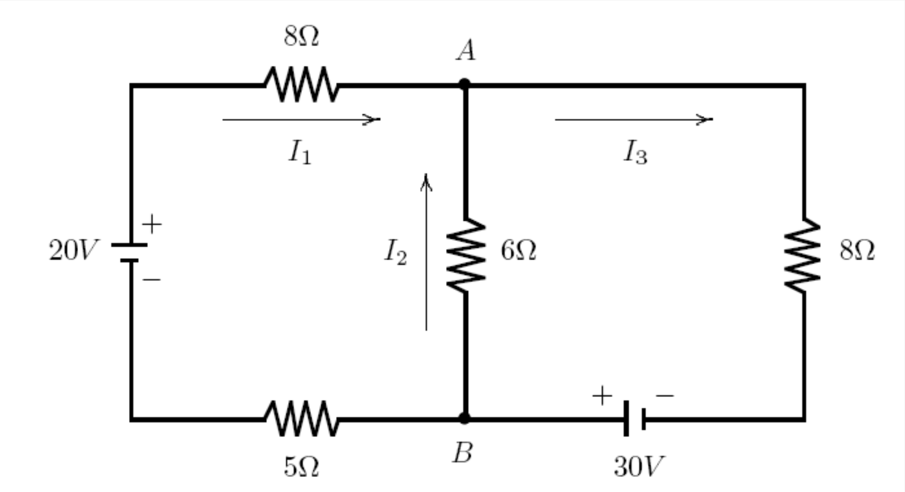
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Bundan $I_1 = 2$ va $I_2 = I_3 = 1$. Bu yerda izoh sifatida aytish kerakki, halqa tashqarisida qaramaymiz.

Faraz qilaylik, yana soat mili boʻyicha yoʻnalishni musbat yoʻnalish deb olaylik. Om qonuniga koʻra 8Ω rezistor boʻylab $8I_1$ kamayuvchi kuchlanish boʻlsa, 20Ω rezistor boʻylab esa $20I_3$ kamayuvchi kuchlanish boʻladi. Boshqa tomondan, elektr manba boʻylab har ikkalasi ham manfiy $20V$ va $16V$ kamayuvchi kuchlanish boʻladi. Bu halqaga kuchlanish qonunini qoʻllasak, u holda $8I_1 + 20I_3 - 36 = 0$ boʻladi. Ammo bu tenglama (1) dagi oxirgi ikkita tenglamaning kombinatsiyasini tashkil qiladi.

1.17-masala

Quyidagi diagrammada koʻrsatilgan elektr zanjirni qaraylik:



Biz I_1, I_2 va I_3 elektr tokini aniqlashga harakat qilamiz. A nuqtaga elektr toki qonunini tatbiq qilsak, u holda $I_1 + I_2 = I_3$ ga ega bo‘lamiz. B nuqtaga ham elektr toki qonunini tatbiq qilsak, u holda yana o‘sha natijaga ega bo‘lamiz. Bundan quyidagi chiziqli tenglamaga ega bo‘lamiz:

$$I_1 + I_2 - I_3 = 0.$$

So‘ngra keling, dastlab chap tomondagi halqani qaraylik va soat mili bo‘yicha yo‘nalishni musbat yo‘nalish deb olaylik. Om qonuniga ko‘ra 8Ω resistor bo‘ylab $8I_1$ kamayuvchi kuchlanish bo‘lsa, 6Ω resistor bo‘ylab esa $-6I_2$ kamayuvchi kuchlanish va 5Ω resistor bo‘ylab esa $5I_1$ kamayuvchi kuchlanish bo‘ladi. Boshqa tomondan esa, halqaning musbat yo‘nalishi "-" dan "+" ga o‘tganligi sababli elektr manba bo‘ylab manfiy $20V$ kamayuvchi kuchlanish o‘tadi. Bu halqaga kuchlanish qonunini tatbiq qilsak, u holda $8I_1 - 6I_2 + 5I_1 - 20 = 0$ ni hosil qilamiz va quyidagi chiziqli tenglamaga ega bo‘lamiz:

$$13I_1 - 6I_2 = 20.$$

So‘ngra keling, endi o‘ng tomondagi halqani qaraylik va soat mili bo‘yicha yo‘nalishni musbat yo‘nalish deb olaylik. Om qonuniga ko‘ra 8Ω resistor bo‘ylab $8I_1$ kamayuvchi kuchlanish bo‘lsa, o‘ngdagi 8Ω resistor bo‘ylab $8I_3$ kamayuvchi kuchlanish bo‘lsa, 5Ω resistor bo‘ylab esa $5I_1$ kamayuvchi kuchlanish bo‘ladi. Boshqa tomondan esa, halqaning musbat yo‘nalishi "-" dan "+" ga o‘tganligi sababli har bir holda elektr manba bo‘ylab manfiy $30V$ va $20V$ kamayuvchi kuchlanish o‘tadi. Bu halqaga kuchlanish qonunini tatbiq qilsak, u holda $8I_1 + 8I_3 + 5I_1 - 50 = 0$ ni hosil qilamiz va quyidagi chiziqli tenglamaga ega bo‘lamiz:

$$13I_1 + 8I_3 = 50.$$

Bulardan quyidagi uchta chiziqli tenglamalar sistemasiga ega bo‘lamiz:

$$\begin{cases} I_1 + I_2 - I_3 = 0, \\ 13I_1 - 6I_2 \qquad = 20, \\ 13I_1 \qquad + 8I_3 = 50. \end{cases} \qquad (2)$$

Bu sistemaning kengaytirilgan matrissasi

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 13 & -6 & 0 & 20 \\ 13 & 0 & 8 & 50 \end{array}\right).$$

Bu matrissani eshelon ko‘rinishga keltirsak

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Bundan $I_1 = 2$, $I_2 = 1$ va $I_3 = 3$. Bu yerda izoh sifatida aytish kerakki, halqa tashqarisida qaramaymiz.

Faraz qilaylik, yana soat mili bo‘yicha yo‘nalishni musbat yo‘nalish deb olaylik. Om qonuniga ko‘ra 8Ω rezistor bo‘ylab $8I_3$ kamayuvchi kuchlanish bo‘lsa, 6Ω rezistor bo‘ylab esa $6I_2$ kamayuvchi kuchlanish bo‘ladi. Boshqa tomondan, elektr manba bo‘ylab manfiy $30V$ kamayuvchi kuchlanish bo‘ladi. Bu halqaga kuchlanish qonunini qo‘llasak, u holda $8I_3 + 6I_2 - 30 = 0$ bo‘ladi. Ammo bu tenglama (2) dagi oxirgi ikkita tenglamaning kombinatsiyasini tashkil qiladi.

1.3.2. Iqtisodiyatga tatbiqi

Bu bo‘limda biz iqtisodchi Leontifga tegishli aniq sodda ayirboshlashlarni tasvirlaymiz. Har bir tarmoq uchun umumiy ishlab chiqarishni bilamiz, shuningdek, qanday tarmoqlar orasida almashish beriladi. Tarmoqning umumiy ishlab chiqarish qiymati mahsulotning narxi sifatida tan olingan.

Leontif bir xil xarajatni talab qiladigan har bir tarmoq yo‘nalishi uchun foyda bilan shunday tarmoqlarning umumiy ishlab chiqarishiga belgilangan bu narxlar o‘rtasida muvozanot mavjud ekanligini ko‘rsatgan edi.

1.18-masala

Iqtisod uchta A, B, C tarmoqdan tuzilgan bo‘lib, quyidagi jadvalda bir-biridan xarid qilish shartnomasi keltirilgan:

Har bir tarmoq ishlab chiqarishining miqdori

	A	B	C
A tarmoqdan xarid qilishi	0.2	0.6	0.1
B tarmoqdan xarid qilishi	0.4	0.1	0.5
C tarmoqdan xarid qilishi	0.4	0.3	0.4

A, B, C tarmoqlarning umumiy ishlab chiqarishining qiymatini mos ravishda p_A, p_B, p_C orqali belgilaylik. U holda har bir tarmoq uchun uning qiymatiga mos keluvchi xarajat uchun quyidagi sistemaga ega bo‘lamiz:

$$\begin{cases} 0.2p_A + 0.6p_B + 0.1p_C = p_A, \\ 0.4p_A + 0.1p_B + 0.5p_C = p_B, \\ 0.4p_A + 0.3p_B + 0.4p_C = p_C. \end{cases}$$

Bu sistemadan quyidagi bir jinsli chiziqli tenglamalar sistemasiga ega bo‘lamiz:

$$\begin{cases} 0.8p_A - 0.6p_B - 0.1p_C = 0, \\ 0.4p_A - 0.9p_B + 0.5p_C = 0, \\ 0.4p_A + 0.3p_B - 0.6p_C = 0. \end{cases}$$

Bu sistemaning kengaytirilgan matrissasi

$$\left(\begin{array}{ccc|c} 0.8 & -0.6 & -0.1 & 0 \\ 0.4 & -0.9 & 0.5 & 0 \\ 0.4 & 0.3 & -0.6 & 0 \end{array}\right) \text{ yoki } \left(\begin{array}{ccc|c} 8 & -6 & -1 & 0 \\ 4 & -9 & 5 & 0 \\ 4 & 3 & -6 & 0 \end{array}\right).$$

Bu matrissaning satrlari ustida elementar almashtirishni bajarsak, u holda

$$\left(\begin{array}{ccc|c} 16 & 0 & -13 & 0 \\ 0 & 12 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

matrissaga ega bo‘lamiz. Agar $p_C = t$ ni ozod o‘zgaruvchi deb qabul qilsak, u holda $(p_A, p_B, p_C) = t \cdot \left(\frac{13}{16}, \frac{11}{12}, 1\right)$ yechimni topamiz yoki agar $p_C = 48t$ ni ozod o‘zgaruvchi, bu yerda t haqiqiy parametr deb qabul qilsak, u holda $(p_A, p_B, p_C) = t \cdot (39, 44, 48)$ yechimni topamiz. Misol uchun, $t = 10^6$ deb tanlasak, u holda mos ravishda A, B, C uchta tarmoq uchun 39, 44 va 48 million narxlarga ko‘tarilib boradi.

1.3.3. Kimyoga tatbiqi

Kimyoviy tenglamalar reaktantlar va mahsulotlardan tuzilgan. Shunday teng kuchli tenglamalar masalasi quyidagi ikki qoidaga:

- Massaning saqlanishi: kimyoviy reaksiyalarning buzilishi yoki atomlarning yetishmasligi;
- Zaryadning saqlanishi: mahsulotlarning umumiy zaryadiga reaktantlarning umumiy zaryadi teng.

1.19-masala

Quyidagi kimyoviy tenglama orqali berilgan ammiakning oksidlanishidan azot oksidi va suvning paydo bo'lishini qaraylik:

$$(x_1)NH_3 + (x_2)O_2 \to (x_3)NO + (x_4)H_2O.$$

Bu yerda reaktantlar ammiak (NH_3) va kislorod (O_2), mahsulotlar esa azot oksidi (NO) va suv (H_2O). Bizning masala yuqoridagi tenglama teng kuchli bo'ladigan shunday eng kichik musbat butun x_1, x_2, x_3 va x_4 ning qiymatlarini topishdan iborat. Buni bajarishda, kimyoviy tenglamaning ikkala tomonida qatnashayotgan har bir turdagi atomning umumiy sonini tenglashtiramiz:

$$N \text{ atom:} \qquad x_1 = x_3,$$

$$H \text{ atom:} \qquad 3x_1 = 2x_4,$$

$$O \text{ atom:} \qquad 2x_2 = x_3 + x_4.$$

Berilganlardan to'rtta x_1, x_2, x_3 va x_4 o'zgaruvchili uchta bir jinsli chiziqli tenglamalar sistemasini ko'rish mumkin

$$\begin{cases} x_1 \qquad - x_3 \qquad = 0, \\ 3x_1 \qquad\qquad - 2x_4 = 0, \\ \qquad 2x_2 - x_3 - x_4 = 0. \end{cases}$$

Bu sistemaning kengaytirilgan matrissasi

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right)$$

Satr ustida elementar almashtirishni bajarsak, u holda

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right)$$

bo‘ladi. Agar $x_4 = t$ ozod o‘zgaruvchi kiritsak, u holda sistemaning umumiy yechimi $(x_1,...,x_4) = t\left(\frac{2}{3},\frac{5}{6},\frac{2}{3},1\right)$ ga ega bo‘lamiz. Agar $t = 6$ deb olsak, u holda kimyoviy tenglama teng kuchli bo‘ladigan eng kichik musbat butun yechim $(x_1,...,x_4) = (4,5,4,6)$ ni hosil qilamiz. Demak,

$$4NH_3 + 5O_2 \rightarrow 4NO + 6H_2O.$$

1.20-masala

Quyidagi kimyoviy tenglamani qaraylik:

$$(x_1)CO + (x_2)CO_2 + (x_3)H_2 \rightarrow (x_4)CH_4 + (x_5)H_2O.$$

Biz kimyoviy tenglamaning ikkala tomonida qatnashayotgan har bir turdagi atomning umumiy sonini tenglashtiramiz:

C atom: $\quad x_1 + x_2 = x_4,$

O atom: $\quad x_1 + 2x_2 = x_5,$

H atom: $\quad 2x_3 = 4x_4 + 2x_5.$

Berilganlardan beshta x_1, x_2, x_3, x_4 va x_5 o‘zgaruvchili uchta bir jinsli chiziqli tenglamalar sistemasini ko‘rish mumkin

$$\begin{cases} x_1 + x_2 - x_4 = 0, \\ x_1 + 2x_2 - x_5 = 0, \\ 2x_3 - 4x_4 - 2x_5 = 0. \end{cases}$$

Bu sistemaning kengaytirilgan matrissasi

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -4 & -2 & 0 \end{array}\right).$$

Eshalon ko‘rinishda satrni kamaytirsak

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \end{array}\right).$$

Agar biz ikkita $x_4 = s$ va $x_5 = t$ ozod o‘zgaruvchilarni kiritsak, u holda sistemaning umumiy yechimi $(x_1,...,x_5) = s(2,-1,2,1,0) + t(-1,1,1,0,1)$ bo‘ladi. Kimyoviy tenglama teng kuchli bo‘ladigan $s = 2$ va $t = 3$ larni tanlasak, u holda $(x_1,...,x_5) = (1,1,7,2,3)$ yechimga ega bo‘lamiz. Natijada

$$CO + CO_2 + 7H_2 \rightarrow 2CH_4 + 3H_2O;$$

Kimyoviy tenglama teng kuchli bo‘ladigan $s = 3$ va $t = 4$ larni tanlasak, u holda $(x_1,...,x_5) = (2,1,10,3,4)$ yechimga ega bo‘lamiz. Natijada

$$2CO + CO_2 + 10H_2 \rightarrow 3CH_4 + 4H_2O;$$

Kimyoviy tenglama teng kuchli bo‘ladigan $s = 3$ va $t = 5$ larni tanlasak, u holda $(x_1,...,x_5) = (1,2,11,3,5)$ yechimga ega bo‘lamiz. Natijada

$$CO + 2CO_2 + 11H_2 \rightarrow 3CH_4 + 5H_2O.$$

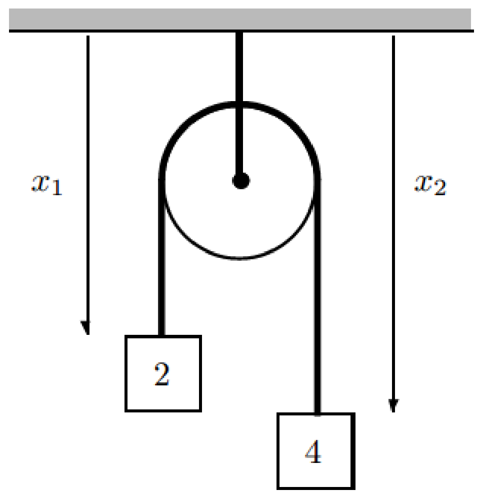
1.3.4. Mexanikaga tatbiqi

Bu bo‘limda biz quyidagi asosiy ikki jismga bog‘langan troslar va qo‘zg‘almas bloklar og‘irliklari sistemasi muommosini qaraymiz.

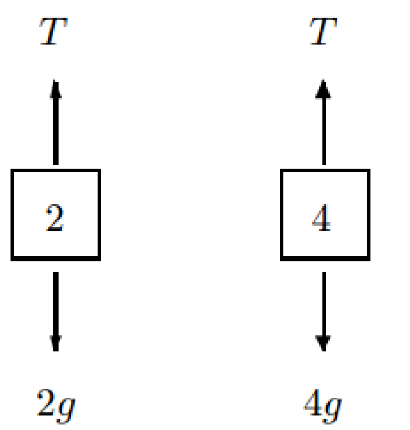
- Agar bir yoki bir nechta qo‘zg‘almas blok aylanasiga tros o‘tkazilgan bo‘lsa, u holda trosning oxirlaridagi ikkita taranglik kuchi bir xil bo‘ladi.
- Jismning harakatidagi Nyutonning ikkinchi qonuni: $F = m\ddot{x}$ formulaga egamiz, bu yerda F orqali kuchni, m bilan massani va $\ddot{x}$ bilan esa tezlanishni belgilaymiz.

1.21-misol

Quyidagi diagrammada ko‘rsatilganidek, shiftdan osib qo‘yilgan qo‘zg‘almas blok aylanasiga o‘tkazilgan trosning oxirlariga og‘irliklari 2 va 4 kg bo‘lgan ikki jism mahkamlangan.



Bizni har bir jismning tezlanishi va trosning taranglik kuchini topish masalasi qiziqtiradi. Bu yerda x_1 va x_2 kattaliklar bilan o‘lchanadigan masofalarni qulaylik uchun pastga yo‘nalgan va bu yo‘nalishni musbat yo‘nalish deb olamiz, shu sababli pastga yo‘nalgan ixtiyoriy tezlanish musbat bo‘ladi. Dastlab, har bir jismga Nyutonning ikkinchi qonunini tatbiq qilamiz. Quyidagi chizmada ko‘rsatilgan ikkita jismga nisbatan ta’sir kuchlarini umumlashtiraylik:



Bu yerda T orqali trosning taranglik kuchini va g orqali esa erkin tushish tezlanishni belgilaymiz. (Pastga yo‘nalgan) ikki jismga Nyutonning ikkinchi qonunini tatbiq qilsak, u holda quyidagi tenglamalar hosil bo‘ladi:

$$2\ddot{x}_1 = 2g - T \text{ va } 4\ddot{x}_2 = 4g - T .$$

Biz trosnning uzunligini $x_1 + x_2 = C$ ko‘rinishda deb qabul qilsak, u holda $\ddot{x}_1 + \ddot{x}_2 = 0$ bo‘ladi. Bu tenglamalarni umumlashtirsak, u holda $\ddot{x}_1, \ddot{x}_2, T$ uchta o‘zgaruvchiga nisbatan quyidagi chiziqli tenglamalar sistemasiga ega bo‘lamiz:

$$\begin{cases} 2\ddot{x}_1 \quad\quad + T = 2g, \\ \quad\quad 4\ddot{x}_2 + T = 4g, \\ \ddot{x}_1 + \ddot{x}_2 \quad\quad = 0. \end{cases}$$

Bu sistemaning kengaytirilgan matrissasi

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2g \\ 0 & 4 & 1 & 4g \\ 1 & 1 & 0 & 0 \end{array}\right)$$

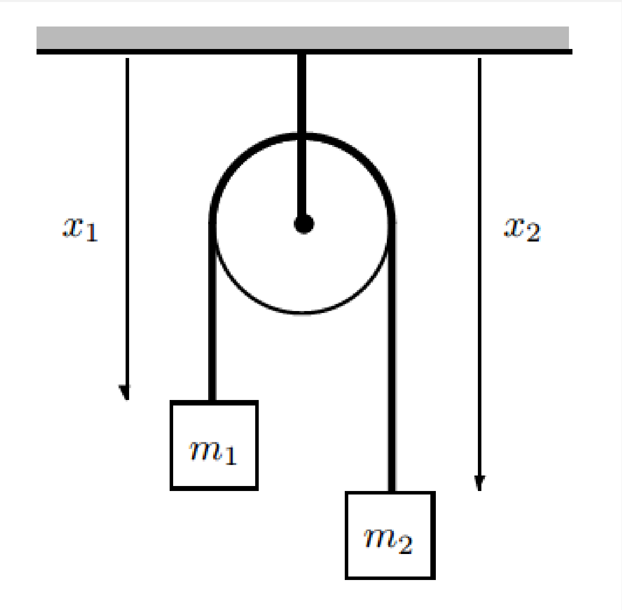
satrlari ustida elementar almashtirishlarni bajarsak, u holda quyidagi matrissaga ega bo‘lamiz:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 2g \\ 0 & 0 & 3 & 8g \end{array}\right).$$

Bundan esa sistemaning yechimi $(\ddot{x}_1, \ddot{x}_2, T) = \left(-\frac{1}{3}g, \frac{1}{3}g, \frac{8}{3}g\right)$ ga ega bo‘lamiz.

1.22-misol

Biz yuqorida keltirilgan masalani umumlashtiramiz. Quyidagi diagrammada ko‘rsatilganidek, shiftdan osib qo‘yilgan qo‘zg‘almas blok aylanasiga o‘tkazilgan trosning oxirlariga og‘irliklari m_1 va m_2 kg bo‘lgan ikki jism mahkamlangan.



Natijada $\ddot{x}_1, \ddot{x}_2, T$ uchta o‘zgaruvchiga nisbatan quyidagi chiziqli tenglamalar sistemasiga ega bo‘lamiz:

$$\begin{cases} m_1\ddot{x}_1 \qquad + T = m_1 g, \\ \qquad m_2\ddot{x}_2 + T = m_2 g, \\ \ddot{x}_1 + \ddot{x}_2 \qquad = 0. \end{cases}$$

Bu sistemaning kengaytirilgan matrissasi

$$\left(\begin{array}{ccc|c} m_1 & 0 & 1 & m_1 g \\ 0 & m_2 & 1 & m_2 g \\ 1 & 1 & 0 & 0 \end{array}\right)$$

satrlari ustida elementar almashtirishlarni bajarsak, u holda quyidagi matrissaga ega bo‘lamiz:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & m_1 m_2 & m_1 & m_1 m_2 g \\ 0 & 0 & m_1 + m_2 & 2m_1 m_2 g \end{array}\right).$$

Bu sistemaning yechimi

$$(\ddot{x}_1, \ddot{x}_2, T) = \left(\frac{m_1 - m_2}{m_1 + m_2} g, \frac{m_2 - m_1}{m_1 + m_2} g, \frac{2m_1 m_2}{m_1 + m_2} g \right).$$

Agar $m_1 = m_2$ bo‘lsa, u holda $\ddot{x}_1 = \ddot{x}_2 = 0$ bo‘ladi, shuning uchun bu jismlar qo‘zg‘almas. Boshqa tomondan, agar $m_2 > m_1$ bo‘lsa, u holda $\ddot{x}_2 > 0$ va $\ddot{x}_1 < 0$ bo‘ladi. U holda

$$T < \frac{2m_1 m_2}{m_1 + m_1} g = m_2 g \text{ va } T < \frac{2m_1 m_2}{m_2 + m_2} g = m_1 g .$$

Bundan $m_1 g < T < m_2 g$.

2-Mustaqil ishlash uchun misollar

1. Quyidagi chiziqli tenglamalar sistemasini qaraylik:

$$\begin{cases} 2x_1 + 5x_2 + 8x_3 = 2, \\ x_1 + 2x_2 + 3x_3 = 4, \\ 3x_1 + 4x_2 + 4x_3 = 1. \end{cases}$$

a) Bu sistema uchun kengaytirilgan matrissani yozing;

b) Kengaytirilgan matrissaning satrlari ustida elementar almashtirishlarni bajarib satrlarini eshelon ko‘rinishga keltiring;

c) b) qismdagi javobingizdan foydalanib, chiziqli tenglamalar sistemasini yeching.

2. Quyidagi chiziqli tenglamalar sistemasini qaraylik:

$$\begin{cases} 4x_1 + 5x_2 + 8x_3 = 0, \\ x_1 \qquad\quad + 3x_3 = 6, \\ 3x_1 + 4x_2 + 6x_3 = 9. \end{cases}$$

a) Bu sistema uchun kengaytirilgan matrissani yozing;

b) Kengaytirilgan matrissaning satrlari ustida elementar almashtirishlarni bajarib satrlarini eshelon ko‘rinishga keltiring;

c) b) qismdagi javobingizdan foydalanib, chiziqli tenglamalar sistemasini yeching.

3. Quyidagi chiziqli tenglamalar sistemasini qaraylik:

$$\begin{cases} x_1 - x_2 - 7x_3 + 7x_4 = 5, \\ -x_1 + x_2 + 8x_3 - 5x_4 = -7, \\ 3x_1 - 2x_2 - 17x_3 + 13x_4 = 14, \\ 2x_1 - x_2 - 11x_3 + 8x_4 = 7. \end{cases}$$

a) Bu sistema uchun kengaytirilgan matrissani yozing;

b) Kengaytirilgan matrissaning satrlari ustida elementar almashtirishlarni bajarib satrlarini eshelon ko‘rinishga keltiring;

c) b) qismdagi javobingizdan foydalanib, chiziqli tenglamalar sistemasini yeching.

4. Quyidagi chiziqli tenglamalar sistemasini yeching:

$$\begin{cases} x + 3y - 2z = 4, \\ 2x + 7y + 2z = 10. \end{cases}$$

5. Quyida berilgan har bir kengaytirilgan matrissani eshelon ko‘rinishdagi matrissaga yoki satrlarini eshelon ko‘rinishga keltiring va hosil qilingan

matrissadan foydalanib chiziqli tenglamalar sistemasining yechimini toping:

a) $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 3 & 2 & -1 & 3 & 6 \\ 4 & 3 & 1 & 4 & 11 \\ 2 & 1 & -3 & 2 & 1 \end{array}\right)$ b) $\left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 1 \\ 2 & -5 & -3 & 12 & 2 \\ 7 & 1 & 8 & 5 & 7 \end{array}\right)$

6. Quyida keltirilgan har bir matrissa ustida elementar almashtirishlarni bajarib satrlarini eshelon ko‘rinishga keltiring:

a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 11 & 21 & 31 & 41 & 51 \\ 2 & 12 & 22 & 32 & 42 & 52 \\ 3 & 13 & 23 & 33 & 43 & 53 \end{pmatrix}$

b) Beshta o‘zgaruvchili $x=(x_1,x_2,x_3,x_4,x_5)$ chiziqli tenglamalar sistemasini qaraylik va bu sistemani $Ax=b$ matrissa ko‘rinishda ifodalaylik, bu yerda x ustun matrissadagidek yoziladi. Faraz qilaylik, ushbu $(A|b)$ kengaytirilgan matrissa satrlar ustida elementar almashtirishlar yordamida satrlari echelon ko‘rinishga keltirilgan bo‘lsin:

$$\left(\begin{array}{ccccc|c} 1 & 3 & 2 & 0 & 6 & 4 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

a) Qaysi biri bazis o‘zgaruvchi va qaysinisi ozod o‘zgaruvchi?

b) Chiziqli tenglamalar sistemasining barcha yechimlarini aniqlang.

c) Beshta o‘zgaruvchili $x=(x_1,x_2,x_3,x_4,x_5)$ chiziqli tenglamalar sistemasini qaraylik va bu sistemani $Ax=b$ matrissa ko‘rinishda ifodalaylik, bu yerda x ustun matrissadagidek yoziladi. Faraz qilaylik, ushbu $(A|b)$ kengaytirilgan matrissa satrlar ustida elementar almashtirishlar yordamida satrlari echelon ko‘rinishga keltirilgan bo‘lsin:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & 5 \\ 0 & 1 & 3 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

a) Qaysi biri bazis o‘zgaruvchi va qaysinisi ozod o‘zgaruvchi?

b) Chiziqli tenglamalar sistemasining barcha yechimlarini aniqlang.

d) Quyidagi chiziqli tenglamalar sistemasini qaraylik:

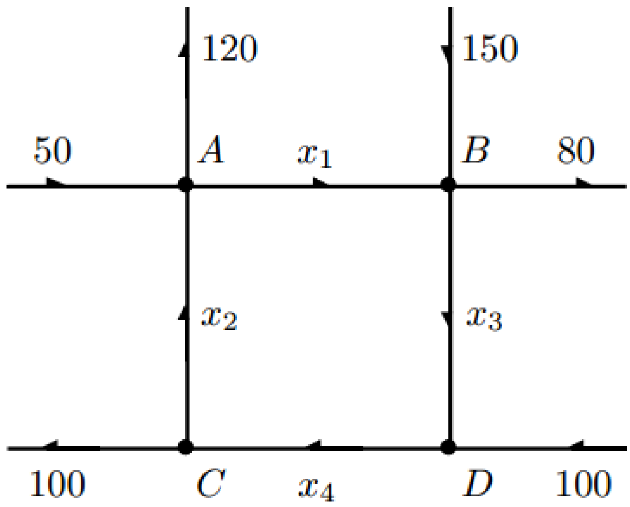
$$\begin{cases} x_1 + \lambda x_2 - x_3 = 1, \\ 2x_1 + x_2 + 2x_3 = 5\lambda + 1, \\ x_1 - x_2 + 3x_3 = 4\lambda + 2, \\ x_1 - 2\lambda x_2 + 7x_3 = 10\lambda - 1. \end{cases}$$

a) λ o‘zgaruvchiga bog‘langan kengaytirilgan matrissaning satrlarini eshelon ko‘rinishga keltiring. [Ko‘rsatma: Bir yoki ikki qadamdan keyin, ayniqsa λ yo nol yoki noldan farqli ekanligini bilmasligimiz sababli juda noqulay hisoblashlarga duch kelamiz. Tenglamalar sistemasini x_1, x_3, x_2 o‘zgaruvchilar ketma-ketligida yozadigan bo‘lsak, u holda kengaytirilgan matrissaning ikkinchi va uchunchi ustunlari almashtiriladi].

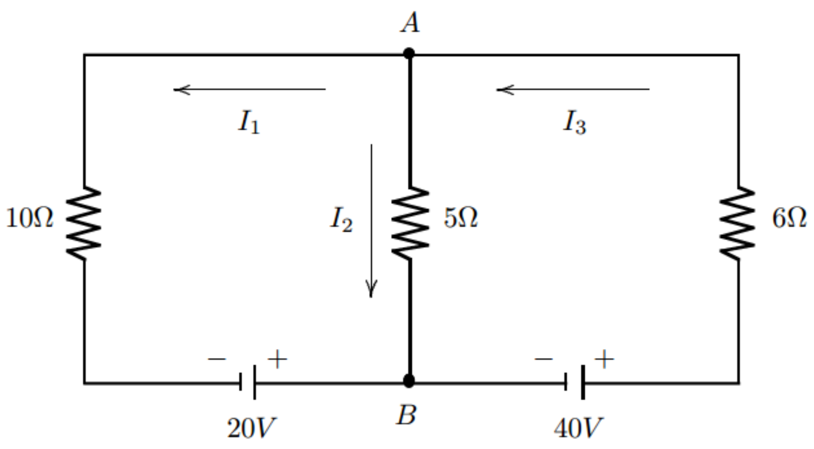
b) Sistema yechimga ega bo‘ladigan har qanaqa λ ning qiymatini toping.

c) Sistemani yeching.

d) Quyidagi sistemada ko‘chaning bir yo‘lidagi x_4 uchun minimum qiymatni toping.



13. Quyidagi diagrammada ko‘rsatilgan elektr zanjirni qaraylik.



I_1 , I_2 va I_3 elektr toklarini aniqlaylik. Buning uchun elektr zanjirdagi qonunlarga aloqador bo‘lgan barcha asoslarni diqqat bilan har bir qadamda izohlab berishimiz shart. Xususan, qaralayotgan har bir halqaning musbat yo‘nalishini aniq ko‘rsatishimiz kerak va halqada har bir rezistor va elektr manba bo‘ylab kuchlanish tushishini ta’minlovchi

1.3.5. Kompyuter grafikasiga tatbiqi

Bizga ma’lumki matritsalar va ular ustida amallar axborot xavfsizligi, iqtisodiyot nazariyasi, va boshqa ko‘pgina sohalarda keng qo‘llaniladi. Bundan tashqari matritsalarning o‘rni kompyuter grafikasida ham ahamiyatli. Bu maqolada biz matritsalarning aynan kompyuter grafikasiga tatbiqini qaraymiz.

Kompyuter grafikasi bu- kompyuter yordamida tasvirlar bilan ishlash hisoblanib, bunda obyektning joylashgan oʻrni, kattaligi, rangi va h.k kabi xususiyatlari ustida ish olib boriladi. Bunda esa matritsalar ustida amallar tasvirlarni parallel koʻchirish, burish, kattalashtirish yoki kichiklashtirish, 3 oʻlchamli obyektni 2 oʻlchamli obyektga oʻtkazish (proyeksiyalash) kabi harakatlarda qoʻllaniladi.

Tekislikda ixtiyoriy nuqtasi (x,y) boʻlgan Ω obyektni olaylik. Bu nuqta ustida quyidagi amallarni qaraymiz:

a) (x,y) nuqtani $(x',y')=(x+ax,y+by)$ nuqtaga parallel koʻchirish (Ω obyektni Ox oʻqi boʻylab a masofaga va Oy oʻqi boʻylab b masofaga parallel koʻchirish);

b) (x,y) nuqtani koordinata boshiga nisbatan φ burchakka burish (Ω obyekt holatini Ox oʻqiga nisbatan φ burchakka burish);

c) (x,y) nuqtani (kx,ky) nuqtaga oʻtkazish (Ω obyektni kattalashtirish) , bu yerda $k>0$ soni *kattalashtirish koeffisienti*.

Yuqoridagi amallar uchun quyidagi teoremani keltiramiz:

1.1-Teorema. *Berilgan Ω obyektning holati quyidagi matritsalar ustida amallar yordamida oʻzgaradi:*

a) *Parallel koʻchirish:* $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ax \\ by \end{bmatrix}$

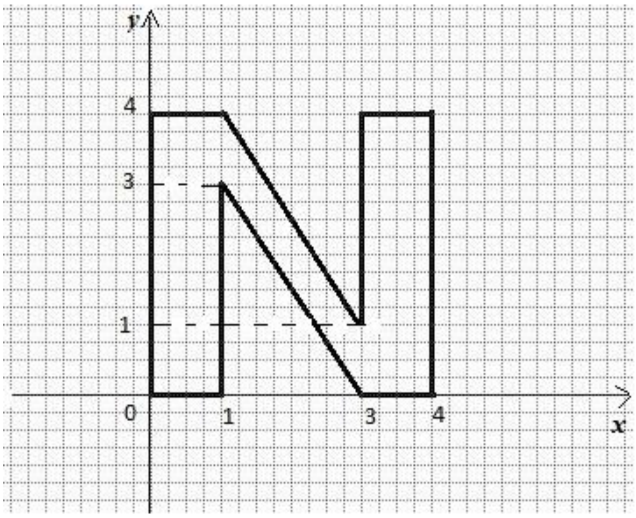
b) *φ burchakka burish:* $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

c) *$k>0$ marta kattalashtirish:* $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} kx & 0 \\ 0 & ky \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

Bu teoremaning isboti analitik geometriya va chiziqli algebra kurslaridan maʼlum boʻlgan tekislikning biror (x,y) nuqtasini parallel koʻchirish, koordinatalar boshiga nisbatan φ burchakka burish hamda matritsalar ustida amallar orqali kelib chiqadi.

1.23-misol

Quyidagi chizmada keltirilgan N harfini qaraylik (1.1-chizma):



1.1-chizma

Koordinatalar boshidan boshlab soat strelkasi bo‘ylab bu harfning 10 ta uchlari koordinatalari quyidagicha bo‘ladi:

$$(0,0),\ (0,4),\ (1,4),\ (3,1),\ (3,4),\ (4,4),\ (4,0),\ (3,0),\ (1,3),\ (1,0)$$

Bu uchlarning koordinatalarini matritsaning mos ustun elementlari sifadida qarab hosil bo‘lgan N matritsani boshqa matritsalarga ko‘paytiramiz.

$$N=\begin{bmatrix} 0 & 0 & 1 & 3 & 3 & 4 & 4 & 3 & 1 & 1 \\ 0 & 4 & 4 & 1 & 4 & 4 & 0 & 0 & 3 & 0 \end{bmatrix}$$

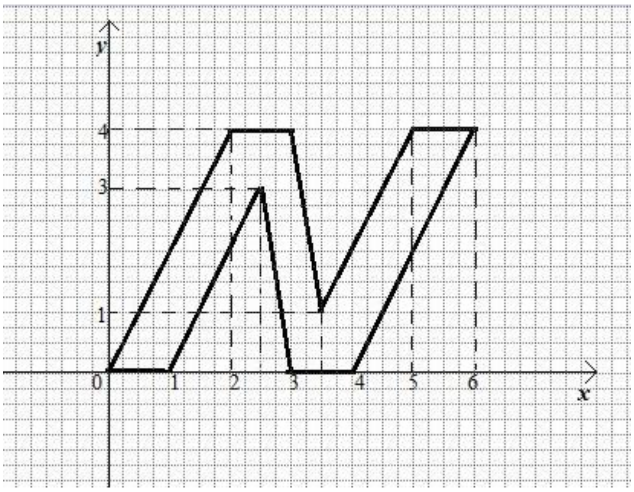
a) Bu matritsani

$$A=\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

matritsaga o‘ngdan ko‘paytiramiz:

$$A\cdot N=\begin{bmatrix}1 & \frac{1}{2}\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & 1 & 3 & 3 & 4 & 4 & 3 & 1 & 1\\ 0 & 4 & 4 & 1 & 4 & 4 & 0 & 0 & 3 & 0\end{bmatrix}=$$
$$=\begin{bmatrix}0 & 2 & 3 & 3{,}5 & 5 & 6 & 4 & 3 & 2{,}5 & 1\\ 0 & 4 & 4 & 1 & 4 & 4 & 0 & 0 & 3 & 0\end{bmatrix}$$

Natijada quyidagi N harfiga ega bo‘lamiz (1.2-chizma):



1.2-chizma

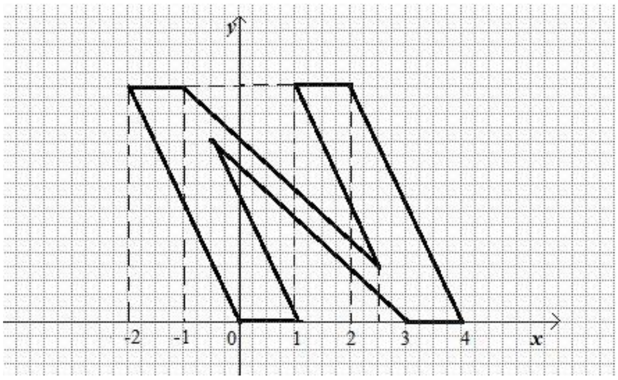
b) Xuddi shunday N matritsani yana

$$B=\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 1\end{bmatrix}$$

matritsaga ko‘paytiramiz va yangi ko‘rinishdagi N harfiga ega bo‘lamiz

(1.3-chizma):

$$B\cdot N=\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & 1 & 3 & 3 & 4 & 4 & 3 & 1 & 1\\ 0 & 4 & 4 & 1 & 4 & 4 & 0 & 0 & 3 & 0\end{bmatrix}=$$
$$=\begin{bmatrix}0 & -2 & -1 & 2{,}5 & 1 & 2 & 4 & 3 & -0{,}5 & 1\\ 0 & 4 & 4 & 1 & 4 & 4 & 0 & 0 & 3 & 0\end{bmatrix}$$



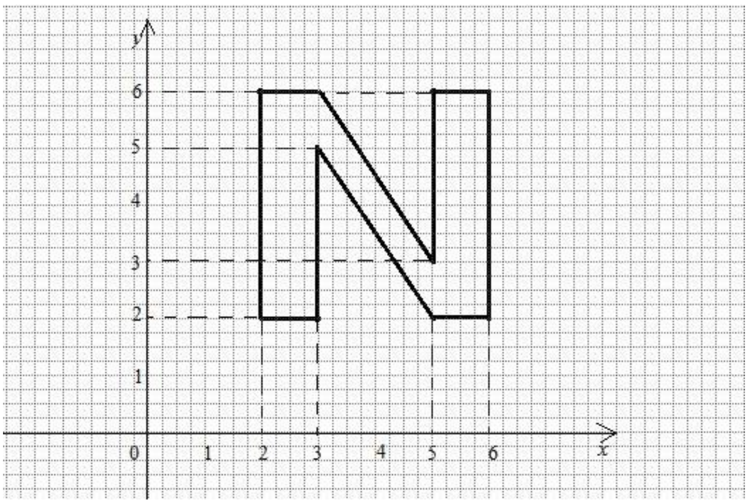
1.3-chizma

c) Endi yuqoridagi teoremaga ko‘ra quyidagi C matritsani tanlab olamiz va uni

N matritsaga qo‘shamiz hamda N harfini parallel ko‘chiramiz (1.4-chizma):

$$C=\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix},$$

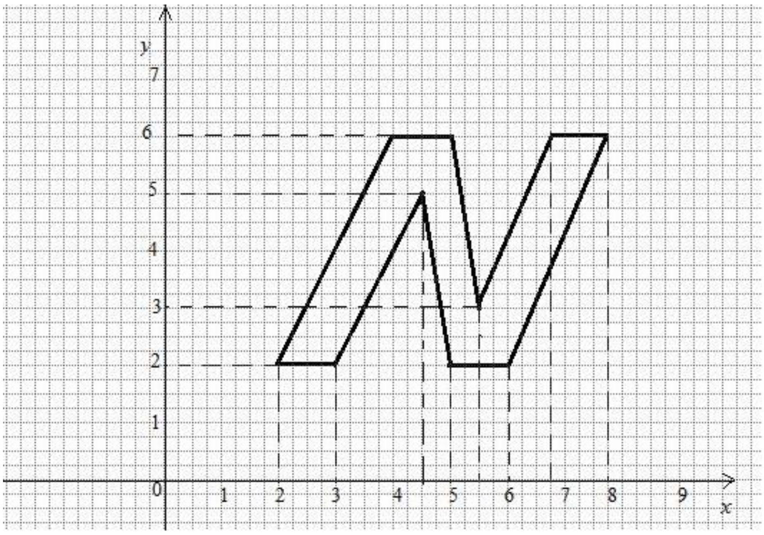
$$C+N=\begin{bmatrix} 2 & 2 & 3 & 5 & 5 & 6 & 6 & 5 & 3 & 3 \\ 2 & 6 & 6 & 3 & 6 & 6 & 2 & 2 & 5 & 2 \end{bmatrix}$$



1.4-chizma

d) Yuqoridagi hisoblashlardan foydalanib N harfining yana yangi ko‘rinishini hosil qilamiz (1.5-chizma):

$$C+A\cdot N=\begin{bmatrix} 2 & 4 & 5 & 5{,}5 & 7 & 8 & 6 & 5 & 4{,}5 & 3 \\ 2 & 6 & 6 & 3 & 6 & 6 & 2 & 2 & 5 & 2 \end{bmatrix}$$



1.5-chizma

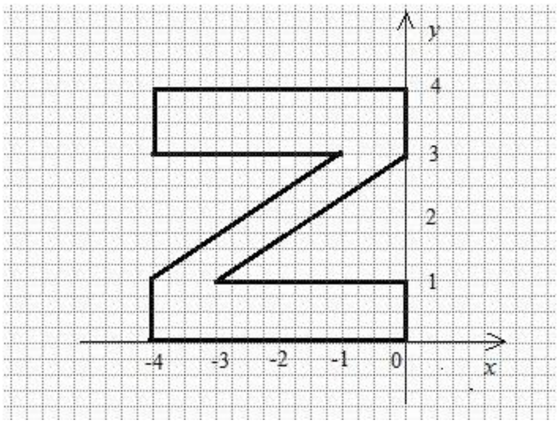
e) Endi 1-chizmadagi rasmni to‘g‘ri burchak ostida buramiz. Bunda yuqoridagi teoremaga ko‘ra $\varphi = 90^0$ uchun

$$K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

matritsaga ega bo‘lamiz.

$$K \cdot N = \begin{pmatrix} 0 & -4 & -4 & -1 & -4 & -4 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 3 & 4 & 4 & 3 & 1 & 1 \end{pmatrix}$$

Bunda N harfi Z harfiga o‘zgaradi (1.6-chizma).



1.6-chizma

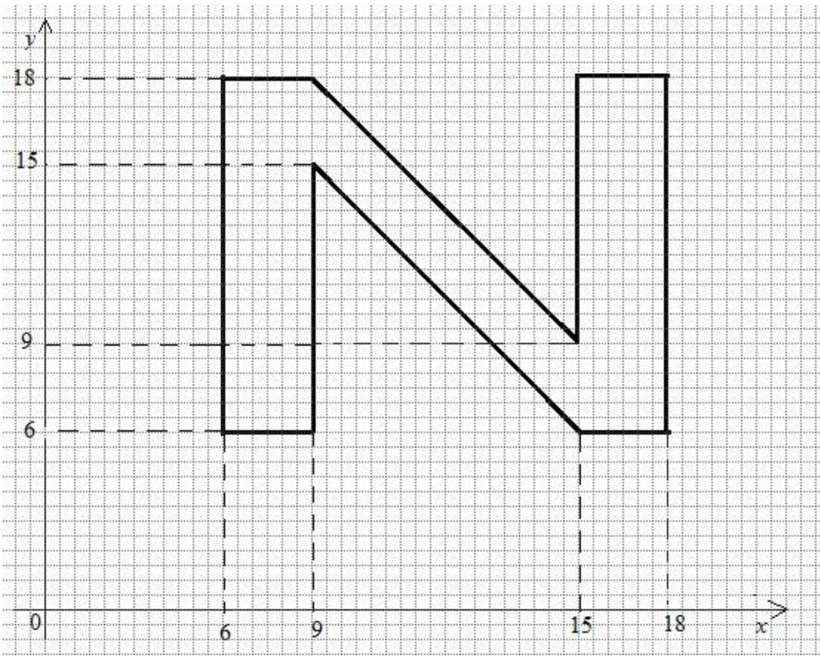
f) Yana teorema asosida N harfinining o‘lchamini 3 marta kattalashtiramiz, ya’ni $C + N$ matritsani

$$G=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix};$$

matritsaga ko‘paytiramiz (1.7-chizma):

$$G\cdot(C+N)=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}\cdot\begin{pmatrix}2 & 2 & 3 & 5 & 5 & 6 & 6 & 5 & 3 & 3\\ 2 & 6 & 6 & 3 & 6 & 6 & 2 & 2 & 5 & 2\end{pmatrix}$$

$$=\begin{pmatrix}6 & 6 & 9 & 15 & 15 & 18 & 18 & 15 & 9 & 9\\ 6 & 18 & 18 & 9 & 18 & 18 & 6 & 6 & 15 & 6\end{pmatrix}$$



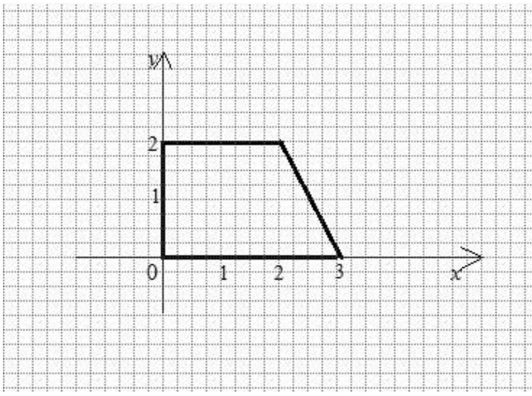
1.7-chizma

1.24-misol

Endi uchlari

$$(0;0),(0;2),(2;2),(3;0)$$

nuqtalarda bo‘lgan ko‘pburchakni qaraylik (1.8-chizma). bu koordinatalarning chizmadagi ko‘rinishi quyidagicha:



1.8-chizma

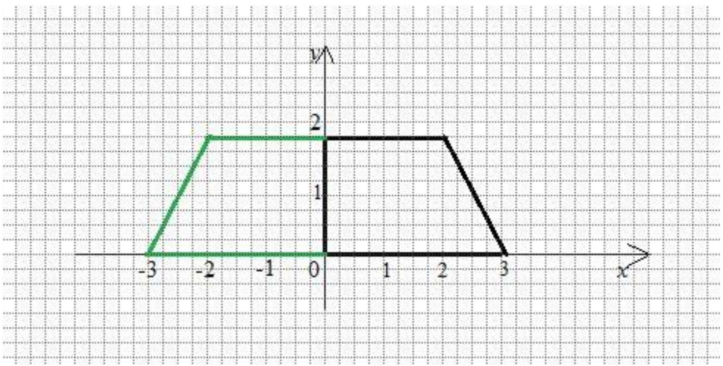
a) Endi bu koordinatalardan matritsa tuzib olamiz:

$$T = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

va uni y o'qi bo'ylab simmetrik ko'chirish uchun F matritsa tuzib olamiz:

$$F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T \cdot F = \begin{pmatrix} 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

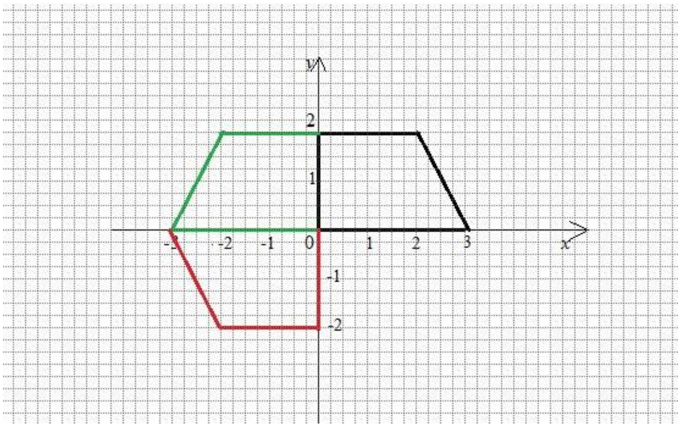


1.9-chizma

b) birinchi shaklni koordinata tekisligini 3 chi choragiga tushirish uchun L matritsa tuzib olamiz uni ko'rinishi quyidagicha bo'ladi:

$$L = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T \cdot L = \begin{pmatrix} 0 & 0 & -2 & -3 \\ 0 & -2 & -2 & 0 \end{pmatrix}$$

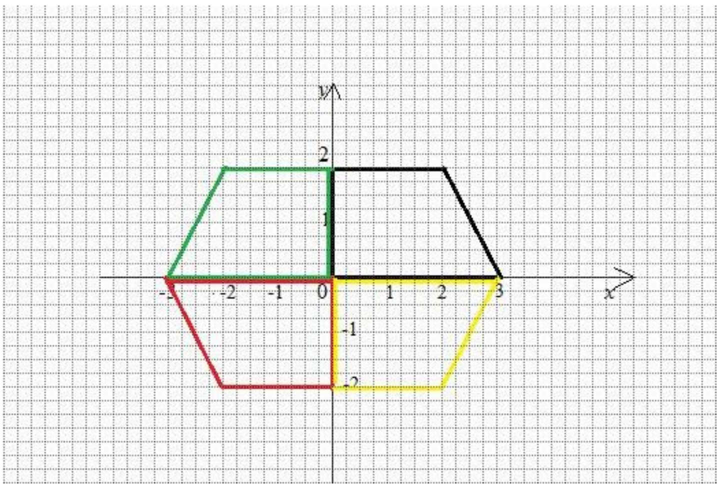


1.10-chizma

c) endi koordinata tekisligining 4 choragiga tushirish uchun yana bir matritsa tuzib olamiz:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T \cdot M = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & -2 & -2 & 0 \end{pmatrix}$$



1.11-chizma

1.3.6. Matritsaning o‘yinlar nazariyasiga tatbiqi

Ikkita o‘yinchidan iborat o‘yinni qaraylik. Odatiy tarzda R o‘yinchini mumkin bo‘lgan i=1,2,…,m ko‘chishlarga ega satr o‘yinchisi, C o‘yinchini esa j=1,2,…,n mumkin bo‘lgan ustun ko‘chishlar o‘yinchisi sifatida belgilaylik. Har bir $(i = 1,2,3, \dots \dots ., m)$ va $(j = 1,2,3,4 \dots \dots . ., n)$ lar uchun a_{ij} orqali agar R o‘yinchi I inchi satr bo‘yicha va C o‘yinchi j ustun bo‘yicha ko‘chganda C o‘yinchining R o‘yinchini tutib olishidagi o‘yin tugashini belgilaymiz. Bu sonlar o‘yin tugashi matritsasini hosil qiladi:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Matritsa elementlari musbat, manfiy yoki nol bo‘lishi mumkin. Faraz qilaylik har bir $i = 1,2,\dots,m$ lar uchun R o‘yinchi i satrga p_i ehtimol bilan va har bir $j = 1,2,\dots,n$ lar uchun C o‘yinchi j satrga q_j ehtimol bilan ko‘chsin. U holda ravshanki,

$$p_1 + p_2 + \dots + p_m = 1 \quad \text{va} \quad q_1 + q_2 + \dots . + q_n = 1$$

bo‘ladi. O‘yinchilar bir –biriga bog‘liqsiz ko‘chishlar hosil qiladi deb qabul qilamiz. U holda har bir $i = 1,2,\dots,m$ va $j = 1,2,\dots,n$ lar uchun $p_i q_j$ son R o‘yinchi i satr bo‘yicha va C o‘yinchi j ustun bo‘yicha ko‘chish ehtimolligini beradi. U holda quyidagi qo‘sh yig‘indi kutilgan C o‘yinchi R o‘yinchini tutishidagi o‘yin tugashini hosil ifodalaydi:

$$E_A(p,q) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

Quyidagi

$$p = (p_1, p_2, ..., p_m) \text{ va } \quad q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

matritsalar mos ravishda *R* va *C* o‘yinchilar strategiyalarini ifodalaydi.
Ravshanki, kutilgan o‘yin tugashi

$$E_A(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} p_i q_j = (p_1, p_2, ..., p_m)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = pAq$$

Endi biz quyidagi masalalarni qaraymiz: Faraz qilaylik A matritsa fiksirlangan bo‘lsin. R o‘yinchi kutilgan $E_A(p,q)$ o‘yin tugashini maksimumga erishtiradigan p strategiyani tanlashi mumkinmi? Shu o‘rinda C o‘yinchi kutilgan $E_A(p,q)$ o‘yin tugashini minimumga erishtiradigan q strategiyani tanlashi mumkinmi?

1.2-Teorema (O‘yin tugashining fundamental teoremasi). *R o‘yinchining har qanday p strategiyasi va C o‘yinchining har qanday q strategiyasi uchun shunday p^* va q^* strategiyalari mavjudki*

$$E_A(p^*,q) \geq E_A(p^*,q^*) \geq E_A(p,q^*)$$

bo‘ladi.

Eslatma. p^* strategiya R o‘yinchining optimal strategiyasi sifatida va q^* strategiya C o‘yinchining optimal strategiyasi sifatida ma’lum. $E_A(p^*,q^*)$ miqdor o‘yinning qiymati hisoblanadi. Optimal strategiylar yagona bo‘lishi zarur emas. Agar p^{**} va q^{**} lar boshqa optimal strategiyalar bo‘lsa u holda $E_A(p^*,q^*) = E_A(p^{**},q^{**})$ bo‘ladi.

Bu yerda o‘yin tugashi matritsasi A egar nuqtalarni o‘z ichiga oladi. Agar a_{ij} element A matritsaning satrlaridagi eng kichik va ustunlaridagi eng katta element bo‘lsa u egar nuqta deyiladi. Bu holda strategiyalar quyidagicha bo‘ladi:

$$p^* = (0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0) \qquad \text{va} \qquad q = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Bu yerda 1 lar i pozitsiyada p^* va j pozitsiyada q^* optimal strategiyalarning sodir bo‘lishlaridir, shuning uchun o‘yinning qiymati a_{ij} bo‘ladi.

1.25-misol

Sportchilar maktabi o‘qituvchisi eshkakchilar (R) va kriketchilar (C) orasidan 100 ta talaba tanlashni talab qildi. Talabalar o‘zlaricha tarkib tuzolmaydilar va eshkakchilar murabbiysi hamda kriketchilar murabbiysi tanlaydi. Maktab 3 ta eshkakchilar murabbiysi hamda 4 ta kriketchilar murabbiylarini yollashi mumkin. Har bir senariyda eshkakchilar kriketchilardan oldin tanlanishi quyidagicha, bu yerda $R1, R2$ va $R3$ lar mumkin bo‘lgan eshkakchilar murabbiylari va $C1, C2, C3$ va $C4$ lar kriketchilar murabbiylari belgilangan:

	C1	C2	C3	C4
R1	75	50	45	60
R2	20	60	30	55
R3	45	70	35	30

[misol uchun, agar R2 va C1 murabbiylar tanlangan bo‘lsa u holda 20 ta talaba eshkakchilardan va qolgan 80 ta talaba kriketchilardan tanlangan bo‘ladi.]

Dastlab biz har bir elementdan 50 ni ayiramiz va o‘yin tugashi matritsasini hosil qilamiz:

$$A = \begin{pmatrix} 25 & 0 & -5 & 10 \\ -30 & 10 & -20 & 5 \\ -5 & 20 & -15 & -20 \end{pmatrix}$$

[misol uchun, yuqori chap element agar har bir sport 50 ta talaba bilan boshlansa u holda 25 ta kriketchi talabalar eshkakchilarga yutqazadilr.] Birinchi satr va uchinchi ustunda joylashgan -5 soni egar nuqta bo‘ladi, shuning uchun eshkakchilar uchun optimal strategiya R1 murabbiydan foydalanish hamda kriketchilar uchun optimal strategiya C3 murabbiydan foydalanishdir.

Umuman, egar nuqtalar mavjud bo‘lmasligi mumkin, shuning uchun masala qat’iy aniqlanmagan. U holda optimal masalaning yechimi chiziqli dasturlashtirish usulllar yordamida topiladi. Quyidagi o‘yin tugashi matritsasi 2×2

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

matritsa bo‘lsa va egar nuqtalarni o‘z ichiga olmasa, u holda biz $p_2 = 1 - p_1$ va $q_2 = 1 - q_1$ deb yozishimiz mumkin. U holda

$$E_A(p,q) = a_{11}p_1q_1 + a_{12}p_1(1-q_1) + a_{21}q_1(1-p_1)q_1 + a_{22}(1-p_1)(1-q_1) =$$
$$= ((a_{11} - a_{12} - a_{21} + a_{22})p_1 - (a_{22} - a_{21}))q_1 + (a_{12} - a_{22})p_1 + a_{22}$$

Faraz qilaylik

$$p_1 = p_1^* = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

bo‘lsin. U holda q ga bog‘liqsiz holda

$$E_A(p^*,q)=\frac{(a_{12}-a_{22})(a_{22}-a_{21})}{a_{11}-a_{12}-a_{21}+a_{22}}+a_{22}=\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}-a_{12}-a_{21}+a_{22}}$$

bo‘ladi. Xuddi shunday agar

$$q_1=q_1^*=\frac{a_{22}-a_{12}}{a_{11}-a_{12}-a_{21}+a_{22}}$$

bo‘lsa, u holda p ga bog‘liqsiz holda

$$E_A(p,q^*)=\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}-a_{12}-a_{21}+a_{22}}$$

bo‘ladi. Demak, barcha p va q strategiyalar uchun

$$E_A(p^*,q)=E_A(p^*,q^*)=E_A(p,q^*)$$

bo‘ladi.

Ta’kidlash kerakki,

$$p^*=\left(\frac{a_{22}-a_{21}}{a_{11}-a_{12}-a_{21}+a_{22}}\quad \frac{a_{11}-a_{12}}{a_{11}-a_{12}-a_{21}+a_{22}}\right) \qquad (1)$$

va

$$q^*=\begin{pmatrix}\dfrac{a_{22}-a_{12}}{a_{11}-a_{12}-a_{21}+a_{22}}\\[2mm] \dfrac{a_{11}-a_{21}}{a_{11}-a_{12}-a_{21}+a_{22}}\end{pmatrix} \qquad (2)$$

Bundan,

$$E_A(p^*,q^*)=\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}-a_{12}-a_{21}+a_{22}}.$$

1.26-misol

Quyidagi o‘yin tugashi matritsasini qaraylik:

$$A=\begin{pmatrix}4 & -1 & -6 & 4\\ -6 & 2 & 0 & 8\\ -3 & -8 & 7 & -5\end{pmatrix}$$

a) Agar $p=(1/3\ \ 0\ \ 2/3)$ va $q=\begin{pmatrix}1/4\\1/4\\1/4\\1/4\end{pmatrix}$ strategiyalar bo‘lsa ***kutilgan yechim*** nimaga teng?

b) Faraz qilaylik R o‘yinchi $p=(1/3\ \ 0\ \ 2/3)$ strategiyani tanlagan bo‘lsin. C o‘yinchi qanday strategiyani tanlagan bo‘lishi mumkin.

c) Faraz qilaylik C o‘yinchi $q=\begin{pmatrix}1/4\\1/4\\1/4\\1/4\end{pmatrix}$ strategiyasini qabul qilsin. R o‘yinchi qanday strategiyani qabul qilishi mumkin.

1.27-misol Optimal strategiyalar yagona emasligiga sodda misol tuzing.

1.28-misol (1) va (2) matritsa elemetlari $[0,1]$ oraliqda o‘zgarishini ko‘rsating.

3-Auditoriya topshiriqlari

1. A va B matritsalar berilgan. $A+B$, $2A-B$ va $A+3B$ matritsalarni toping

a) $A=\begin{bmatrix}3 & 0 & -1\\ 5 & 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}-1 & 3 & 4\\ 0 & 2 & 5\end{bmatrix}$;

b) $A=\begin{bmatrix}2 & -4\\ 1 & -1\\ 5 & -3\end{bmatrix}$, $B=\begin{bmatrix}-3 & 2\\ -2 & 4\\ 0 & 5\end{bmatrix}$

2. A va B matritsalar berilgan. AB va BA matritsalarni toping

a) $A=\begin{bmatrix}1 & -2 & 4\\ -3 & 5 & 0\\ 2 & -1 & 3\end{bmatrix}$, $B=\begin{bmatrix}3 & 2 & 5\\ 4 & 0 & -1\\ 1 & 1 & -3\end{bmatrix}$;

b) $A=\begin{bmatrix}3 & 2 & -1\\ 7 & 4 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & -4\\ 1 & -1\\ 5 & -3\end{bmatrix}$;

d) $A=\begin{bmatrix}2 & -5 & 3 & 0\end{bmatrix}$, $B=\begin{bmatrix}4\\ 2\\ -3\\ 5\end{bmatrix}$;

e) $A=\begin{bmatrix}11 & 10\\ 2 & 9\end{bmatrix}$, $B=\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}$.

3. A, B va C matritsalar berilgan. $(AB)C=A(BC)$ ekanini tekshiring

$A=\begin{bmatrix}5 & 3\\ 2 & -1\\ 4 & 7\end{bmatrix}$, $B=\begin{bmatrix}4 & -3\\ -5 & 2\end{bmatrix}$, $C=\begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}$.

4. Berilgan A matritsaning A^{-1} teskari matritsasini toping

a) $A=\begin{bmatrix}3 & -1 & 3\\ 2 & -1 & 2\\ 2 & 1 & 3\end{bmatrix}$;

b) $A=\begin{bmatrix}2 & 4 & -1\\ 5 & 3 & 4\\ 3 & 7 & 0\end{bmatrix}$

3-Mustaqil yechish uchun testlar

1. $A=\begin{pmatrix}5&2\\1&4\end{pmatrix}$ va $B=\begin{pmatrix}-3&1\\5&-2\end{pmatrix}$ matritsa berilgan, $A+2B$ matritsani toping

$A)\begin{pmatrix}-1&4\\11&0\end{pmatrix}$, $B)\begin{pmatrix}-5&1\\17&-7\end{pmatrix}$, $C)\begin{pmatrix}-1&4\\8&0\end{pmatrix}$, $D)\begin{pmatrix}-11&4\\11&0\end{pmatrix}$

2. $K=\begin{pmatrix}1&-2&3\\-2&3&-4\end{pmatrix}$ matritsa berilgan bo‘lsa, $2K$ matritsani toping

$A)\begin{pmatrix}2&-4&6\\-2&3&-4\end{pmatrix}$, $B)\begin{pmatrix}1&-2&3\\-4&6&-8\end{pmatrix}$, $C)\begin{pmatrix}2&-4&6\\-4&6&-8\end{pmatrix}$, $D)$ A va B to'g'ri

3. $A=\begin{pmatrix}5&2\\1&4\end{pmatrix}$ va $B=\begin{pmatrix}-3&1\\5&-2\end{pmatrix}$ matritsalar berilgan, $A\cdot B$ matritsani toping

$A)\begin{pmatrix}-13&7\\25&-7\end{pmatrix}$, $B)\begin{pmatrix}-5&7\\8&10\end{pmatrix}$, $C)\begin{pmatrix}-5&1\\17&-7\end{pmatrix}$, $D)\begin{pmatrix}-10&2\\5&-2\end{pmatrix}$

4. Teskari matritsani topish formulasini ko‘rsating?

A) $A^{-1}=\dfrac{1}{\det A}\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}$, B) $A^{-1}=\dfrac{1}{\det A}\begin{pmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{pmatrix}$

C) $A^{-1}=\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}$, D) $A^{-1}=\begin{vmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{vmatrix}$

5. $A=\begin{pmatrix}3&2&0\\1&3&1\\5&3&0\end{pmatrix}$ bo‘lsa, A^{-1} teskari matritsani toping

A) $A^{-1}=-\begin{pmatrix}-3&5&-12\\0&0&1\\2&-3&7\end{pmatrix}$, B) $A^{-1}=\begin{pmatrix}-3&5&-12\\0&0&1\\2&-3&7\end{pmatrix}$

C) $A^{-1}=\begin{pmatrix}-3&0&2\\5&0&-3\\-12&1&7\end{pmatrix}$ D) $A^{-1}=\begin{pmatrix}-3&0&2\\-5&0&3\\-12&-1&7\end{pmatrix}$

1.4. Matritsa rangi. Chiziqli algebraik tenglamalar sistemasi. Kroneker-Kapelli teoremasi

1.4.1. Matritsaning rangi

To‘g‘ri burchakli (xususiy holda kvadrat) A matritsa berilgan bo‘lsin. Uning biror k ta satr va k ta ustunini ajratamiz, kesishmada turgan elementlardan k – tartibli determinanat hosil qilamiz. Bu determinant matritsaning ***k -tartibli minori*** deb ataladi.

Masalan, ushbu $A=\begin{bmatrix}1 & 1 & 7 & 1\\ 2 & 3 & 4 & -2\\ -1 & 6 & 2 & 5\end{bmatrix}$ matritsaning 2-tartibli minorlaridan biri $M_2=\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}=3-2=1$ bo‘ladi. 3-tartibli minorlaridan biri $M_3=\begin{vmatrix}1 & 1 & 1\\ 2 & 3 & -2\\ -1 & 6 & 5\end{vmatrix}=15+12+2+3-10+12=34$ bo‘ladi. Berilgan matritsaning 18 ta *2-tartibli*, 4 ta *3-tartibli* minori bor.

Matritsaning rangi deb, uning noldan farqli minorlarining eng yuqori tartibiga aytiladi, *rangA* yoki $r(A)$ kabi belgilanadi.

Matritsada elementar almashtirishlar deb, quyidagi almashtirishlarga aytiladi:

a) Nollardan iborat qatorlarni o‘chirish.

b) Ikkita parallel qatorlarni o‘rnini almashtirish.

c) Bir qatorning barcha elementlarini biror songa ko‘paytirib, boshqa qatorning mos elementlariga qo‘shish.

d) Qatorning barcha elementlarini noldan farqli bir xil songa ko‘paytirish

Bu almashtirishlar natijasida hosil bo‘lgan matritsa berilgan matritsaga ***ekvivalent matritsa*** deyiladi va $A\sim B$ kabi belgilanadi.

1.3-Teorema. *Matritsalar ustida elementar almashtirishlar natijasida uning rangi o‘z garmaydi.*

Matritsaning rangini 2 xil usulda topish mumkin.

1-usul. ***O‘rab turuvchi minorlar usuli***

Bu usulda birinchi noldan farqli k *-tartibli* minori topiladi. k –tartibli minorni o‘z ichiga oluvchi barcha $k+1$ tartibli minorlar ***o‘rab turuvchi minorlar*** deyiladi. k –tartibli minor noldan farqli bo‘lib, bu minorni o‘rab turuvchi barcha $k+1$ tartibli minorlar nolga teng bo‘lganda, matritsaning rangi shu noldan farqli minor tartibiga teng bo‘ladi. Bu usul hisoblash ishlarini ancha kamaytirish imkoniyatini beradi. Agar o‘rab turuvchi $k+1$ tartibli minorlardan birortasi nolga teng bo‘lmasa, ana shu minorni o‘rab turuvchi minorlarni tekshirilib, bu jaroyon davom ettiriladi.

2-usul. Zinasimon usul (yoki elementar almashtirishlar usuli)

Bu usulda elementar almashtirishlar yordamida matritsa uchburchakli matritsa
ko‘rinishiga keltiriladi. Natijada hosil bo‘lgan matritsaning noldan farqli satrlari soni matritsaning rangiga teng bo‘ladi.

1.29-misol

Berilgan $A=\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 4 & -2 & 0 \\ 1 & 2 & -7 & 5 & 6 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$ matritsa rangini ikki xil usulda aniqlang.

► *1-usul.* $M_2=\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix}=4+2=6\neq 0$. Bu minorni o‘rab turuvchi 3-tartibli minorlar soni 6 ta(umumiy holda, 3-tartibli minorlari 40 ta).

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 4 \\ 1 & 2 & -7 \end{vmatrix}=-28+8-4+4-8+28=0,\quad \begin{vmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ 1 & -7 & 5 \end{vmatrix}=20+2-14-4-14+10=0,$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 4 & 0 \\ 1 & -7 & 6 \end{vmatrix}=24-28-8+12=0,\quad \begin{vmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ 3 & 9 & -5 \end{vmatrix}=-20+6+18-12+18-10=0,$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 4 \\ 3 & 6 & 9 \end{vmatrix}=36+24-12+12-24-36=0,\quad \begin{vmatrix} 1 & -1 & 2 \\ 2 & 4 & 0 \\ 3 & 9 & -2 \end{vmatrix}=-8+36-24-4=0$$

Demak, berilgan matritsa uchun $rangA=2$ bo‘ladi.

2-usul. Quyidagicha elementar almashtirishlar olib boramiz: 1)1-satr elementlarini $-2,-1,-3$ larga ko‘paytirib, mos ravishda 2-, 3-, 4-satr elementlariga qo‘shamiz; 2) 2-satr elementlarini $1,-2$ larga ko‘paytirib, mos ravishda 3-, 4-satr elementlariga qo‘shamiz; 3) nollardan iborat satrlarini o‘chiramiz.

$$A=\begin{bmatrix}1 & 2 & -1 & 1 & 2\\ 2 & 4 & 4 & -2 & 0\\ 1 & 2 & -7 & 5 & 6\\ 3 & 6 & 9 & -5 & -2\end{bmatrix}\sim\begin{bmatrix}1 & 2 & -1 & 1 & 2\\ 0 & 0 & 6 & -4 & -4\\ 0 & 0 & -6 & 4 & 4\\ 0 & 0 & 12 & -8 & -8\end{bmatrix}\sim\begin{bmatrix}1 & 2 & -1 & 1 & 2\\ 0 & 0 & 6 & -4 & -4\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\sim$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 1 & 2\\ 0 & 0 & 6 & -4 & -4\end{bmatrix}.$$ Demak, $rangA=2$ ekan.◄

Tartibi matritsa rangiga teng bo‘lgan minor ***bazis minor*** deb ataladi. Kesishmasida bazis minor elementlari turgan satrlar va ustunlar ***bazis satrlar va ustunlar*** deyiladi. Matritsaning istalgan satri(ustuni) uning bazis satrlarining (ustunlarining) chiziqli kombinatsiyasidan iborat bo‘ladi. Bazis satrlar (ustunlar) chiziqli erkli satrlar(ustunlar) bo‘ladi.

1.4-Teorema. *Agar matritsaning rangi r ga teng bo‘lsa, u holda unda r ta chiziqli erkli satr topiladi, qolgan barcha satrlar esa bu r ta satrning chiziqli kombinatsiyasi bo‘ladi.*

Natija. *Matritsaning rangi undagi chiziqli erkli satrlar(ustunlar) soniga teng.*

1.4.2. Chiziqli algebraik tenglamalar sistemasi

Quyidagi umumiy ko‘rinishdagi n ta noma’lumli m ta tenglamalar sistemasini qaraymiz:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n = b_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = b_m \end{cases} \qquad (1.12)$$

Bu yerda, $x_1, x_2, ..., x_n$ – noma’lumlar, $a_{11}, a_{12}, ..., a_{mn}$ -koeffitsientlar, $b_1, b_2, ..., b_n$ - ozod hadlar.

(1.12) tenglamalar sistemasining ***yechimi*** deb, shunday n ta $\left(x_1^o, x_2^o, ..., x_n^o\right)$ sonlar to‘plamiga aytiladiki, bu sonlar (1.12) sistemaning barcha tenglamalarini to‘g‘ri tenglikka aylantiradi.

Agar (1.12) sistema yechimga ega bo‘lsa, u ***birgalikdagi sistema*** deyiladi. Agar bu yechim yagona bo‘lsa, sistema ***aniq sistema*** deyiladi. Agar (1.12) sistema cheksiz ko‘p yechimga ega bo‘lsa, u ***aniqmas*** sistema, agar tenglamalar sistemasi umuman yechimga ega bo‘lmasa, u ***birgalikda bo‘lmagan sistema*** deyiladi.

(1.12) tenglamalar sistemasi uchun quyidagi matritsalarni tuzamiz:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \qquad (1.13)$$

A matritsa (1.12) sistemaning ***asosiy matritsasi*** deyiladi. B matritsa ***kengaytirilgan matritsa*** deyiladi.

Bu matritsalarning ranglar $rangA \leq rangB$. munosabat bilan bog‘langan.

Agar A matritsaning rangi n noma’lumlar sonidan kichik bo‘lsa, u holda bu tenglamalar sistemasida $n-k$ ta o‘zgaruvchi chiziqli erkli bo‘lib, k ta o‘zgaruvchi chiziqli bog‘liq o‘zgaruvchilar bo‘ladi. Bu holda (4.1) tenglamalar sistemasida k ta tenglama qoldiriladi. Qolgan tenglamalar bu tenglamalarning chiziqli kombinatsiyasidan iborat bo‘ladi. Qoldirilgan tenglamalarda $n-k$ ta o‘zgaruvchini tenglamalarning o‘ng tomoniga o‘tkaziladi. Bu o‘zgaruvchilar chiziqli erkli o‘zgaruvchilar deyiladi. Tenglamalarni yechishda chiziqli erkli o‘zgaruvchilarga qiymatlar berilib, qolgan k ta o‘zgaruvchilarning ularga mos qiymatlari topiladi.

1.5-Teorema (*Kroneker – Kapelli teoremasi*). *Chiziqli algebraik tenglamalar sistemasi birgalikda bo‘lishi uchun uning asosiy matritsasi bilan kengaytirilgan matritsasining rangi teng bo‘lishi zarur va yetarli, ya’ni* $rangA = rangB$.

Shunday qilib: $rangA \neq rangB$ bo‘lsa, tenglamalar sistemasi birgalikda emas;

$rangA = rangB = r = n$ bo‘lsa, tenglamalar sistemasi yagona yechimga ega;

$rangA = rang\ B = r < n$ bo‘lsa, tenglamalar sistemasi cheksiz ko‘p yechimga ega.

1.4.3. Bir jinsli chiziqli tenglamalar sistemasi

Agar chiziqli algebraik tenglamalar sistemasining barcha ozod hadlari nolga teng bo‘lsa, bunday sistema ***bir jinsli chiziqli tenglamalar sistemasi*** deyiladi.Ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n = 0 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = 0 \end{cases} \qquad (1.14)$$

tenglamalar sistemasi bir jinsli tenglamalar sistemasi.

Bu yerda $b_1 = b_2 = ... = b_m = 0$ bo‘lib A va B matritsalar ranglari teng,

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & 0 \\ a_{21} & a_{22} & ... & a_{2n} & 0 \\ ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} & 0 \end{bmatrix}$$

ya’ni *r(A)=r(B).*

Kroneker-Kapelli teoremasiga ko‘ra, bir jinsli chiziqli tenglamalar sistemasi hamma vaqt birgalikda bo‘ladi. (1.14) tenglamalar sistemasi doim nollardan iborat ***trivial yechim*** deb ataladigan yechimga ega:

$$x_1 = x_2 = ... = x_n = 0 \qquad (1.15)$$

1.6-Teorema. *Bir jinsli chiziqli tenglamalar sistemasining determinanti nolga teng bo‘lganda, va faqat shu holdagina bu sistema noldanfarqli yechimlarga ega bo‘ladi.*

1.7-Teorema. *(1.14) tenglamalar sistemasi noldan farqli yechimga ega bo‘lishi uchun A matritsaning rangi noma’lumlar sonidan kichik, ya’ni* $r(A) < n$ *bo‘lishi zarur va yetarli.*

Bir jinsli chiziqli tenglamalar sistemasining yechimlarining har qanday chiziqli kombinatsiyasi yana shu sistemaning yechimi bo‘ladi.

$r(A) = k < n$ bo‘lsa, u holda (1.14) sistemaning fundamental yechimlar sistemasi $n-k$ ta yechimdan iborat bo‘ladi. Fundamental yechimlar sistemasini aniqlash uchun bazis noma'lumlarni aniqlaymiz. Ularni $x_1, x_2, ..., x_k$ deb belgilaymiz. Bu noma'lumlarni $x_{k+1}, x_{k+2}, ..., x_n$ chiziqli erkli noma'lumlar orqali ifodalab olinadi. Bu *n-k* ta noma'lumga ixtiyoriy qiymatlar berib, $x_1, x_2, ..., x_k$ o'zgaruvchilarning mos aniq qiymatlarini topamiz. Bu topilgan yechimlar (1.14) ning fundamental yechimlar sistemasi bo'ladi. Ko'pincha normallangan fundamental yechimlar sistemasi olinadi.

1.30-misol

Fundamental va umumiy yechimlar sistemasi topilsin.

$$\begin{cases} x_1 + 2x_2 + 4x_3 - 3x_4 = 0 \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 = 0 \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 + 8x_2 + 24x_3 + 19x_4 = 0 \end{cases}$$

► $A = \begin{bmatrix} 1 & 2 & 4 & -3 \\ 3 & 5 & 6 & -4 \\ 4 & 5 & -2 & 3 \\ 3 & 8 & 24 & 19 \end{bmatrix}$, $\quad M_2 = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 4 & 5 & -2 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & -3 \\ 3 & 5 & -4 \\ 4 & 5 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 3 & 8 & 24 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & -3 \\ 3 & 5 & -4 \\ 3 & 8 & 19 \end{vmatrix} = 0.$$

$$rangA = 2, \quad \begin{cases} x_1 + 2x_2 + 4x_3 - 3x_4 = 0 \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 = 0 \end{cases}, \quad \begin{cases} x_1 = 8x_3 - 7x_4 \\ x_2 = -6x_3 + 5x_4 \end{cases}.$$

$x_3 = 1,\ x_4 = 0$ va $x_3 = 0,\ x_4 = 1$ deb olib, $(8; -6; 1; 0)$ va $(-7; 5; 0; 1)$ fundamental yechimlarni hosil qilamiz. Umumiy yechim: $\{8a - 7b; -6a + 5b; a; b\}$ ◄

Natija. *Agar bir jinsli tenglamalar sistemasining tenglamalari soni noma'lumlar sonidan kichik bo'lsa, bu sistema nolmas yechimga ega bo'ladi va bu yechimlar cheksiz ko'p bo'ladi.*

Agar bir jinsli tenglamalar sistemasining rangi $r < n$ bo'lsa sistemadagi shu rangni tashkil qiluvchi minorlar turgan satrdagi tenglamalarni ajratamiz, ular qolgan $n - r$ dona tenglamalarning chiziqli kombinatsiyalardan iborat bo'ladi.

1.31-misol

Fundamental va umumiy yechimlar sistemasi topilsin

$$\begin{cases} 2x_1 - 4x_2 + 5x_3 + 3x_4 = 0 \\ x_1 - 6x_2 + 4x_3 + 2x_4 = 0 \\ 4x_1 - 8x_2 + 17x_3 + 11x_4 = 0 \end{cases}$$

► $A = \begin{bmatrix} 2 & -4 & 5 & 3 \\ 3 & -6 & 4 & 2 \\ 4 & -8 & 17 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 5 & 3 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 5 & 3 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$

$rang(A) = 2$.

$$\begin{cases} 2x_1 - 4x_2 + 5x_3 + 3x_4 = 0 \\ -7x_3 - 5x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = 2x_2 - 4{,}6x_3 \\ x_4 = -1{,}4x_3 \end{cases}$$

Javob: $\{2a - 4{,}6b;\ a;\ b; -1.4b\}$.◄

4-Auditoriya topshiriqlari

1. Berilgan A matritsaning rangini ta'rif yordamida toping.

$$\text{a) } A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -4 & 4 \\ 3 & 6 & -6 \end{bmatrix}; \quad \text{b) } A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 3 & 4 \\ 4 & 3 & -8 \end{bmatrix}; \quad \text{d) } A = \begin{bmatrix} -2 & 0 & -2 \\ 2 & -3 & 4 \\ -4 & 3 & -8 \end{bmatrix}.$$

2. Berilgan A matritsaning rangini o'rab turuvchi minorlar usulida yeching.

$$a)\quad A = \begin{bmatrix} 3 & -1 & 2 & 1 \\ 2 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad b)\quad A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$c)\quad A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 2 & -1 & 4 & 2 \\ 4 & 0 & 1 & 1 \\ 5 & -2 & 3 & 2 \end{bmatrix} \qquad d)\quad A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 1 & 4 & 2 \\ -1 & 4 & -3 & 1 \\ 2 & -2 & 5 & -2 \end{bmatrix}$$

3. Berilgan A matritsaning rangini elementar almashtirishlar usulida yeching.

$$a)\quad A = \begin{bmatrix} 1 & 3 & -2 & 0 & -1 & 4 \\ 2 & 1 & 3 & 2 & 5 & 3 \\ -1 & 4 & 1 & 3 & 0 & 3 \\ 5 & 2 & 4 & 1 & 9 & 7 \\ 4 & -1 & 3 & -2 & 7 & 3 \end{bmatrix} \qquad b)\quad A = \begin{bmatrix} 1 & 5 & 2 & 1 & -2 & 2 \\ -1 & 1 & 3 & 2 & -2 & 5 \\ 2 & 4 & -1 & -3 & 0 & -3 \\ 3 & 9 & 1 & -2 & -2 & -1 \\ 5 & 1 & -5 & -8 & 4 & -16 \end{bmatrix}$$

4. Sistema birgalikda bo'ladimi?

$$a)\ \begin{cases} x_1 - 2x_2 + 5x_3 = 0 \\ 2x_1 - 5x_2 - 2x_4 = -2 \\ 3x_1 + 2x_2 + 6x_3 = 16 \end{cases} \qquad b)\ \begin{cases} 2x_1 + 3x_2 - 7x_3 = 2 \\ 3x_1 - x_2 + x_4 = 9 \\ 4x_1 - 5x_2 + 9x_3 = 14 \end{cases}$$

5. Bir jinsli chiziqli algebraik tenglamalar sistemasini yeching.

$$a)\ \begin{cases} x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 - 2x_2 + 5x_4 = 0 \\ 2x_1 - 2x_2 + 3x_3 = 0 \end{cases} \qquad b)\ \begin{cases} 2x_1 - 4x_2 + 5x_3 = 0 \\ x_1 + 2x_2 - 3x_4 = 0 \\ 3x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

6. Fundamental va umumiy yechimlar sistemasi topilsin.

$$a)\ \begin{cases} 4x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ 2x_1 + 8x_2 + 3x_3 - 5x_4 = 0 \\ x_1 - 5x_2 - 2x_3 + 4x_4 = 0 \\ 3x_1 + 3x_2 + x_3 - x_4 = 0 \end{cases} \qquad b)\ \begin{cases} x_1 - 5x_2 + 2x_3 + 4x_4 = 0 \\ 2x_1 - 3x_2 - 3x_3 + x_4 = 0 \\ x_1 + 2x_2 - 5x_3 - 3x_4 = 0 \\ 3x_1 - 8x_2 - x_3 + 5x_4 = 0 \end{cases}$$

4-Mustaqil yechish uchun testlar

1. Matitsa rangini toping.

$$A = \begin{bmatrix} -1 & 3 & 2 & 4 \\ 1 & 5 & 6 & 4 \\ 3 & 0 & 3 & -3 \end{bmatrix}$$

$A)\ rang(A) = 3 \quad B)\ rang(A) = 2 \quad C)\ rang(A) = 1 \quad D)\ rang(A) = 4$

2. Matritsa rangi bu -

A) Matritsaning o‘lchami B) matritsaning determinanti.

C) noldan farqli eng katta minorining qiymati D) noldan farqli minorlarining eng katta tartibi

3. Matritsaning ko‘rsatilgan minorini o‘rab turuvchi minori noto‘g‘ri berilgan variantni aniqlang:

$$A = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 1 & 5 & 6 & 4 \\ 3 & 0 & 3 & -3 \\ -2 & 4 & 5 & -1 \end{bmatrix}; \quad M_2 = \begin{vmatrix} 1 & 6 \\ 3 & 3 \end{vmatrix}$$

$$A)\begin{vmatrix}1&3&2\\1&5&6\\3&0&3\end{vmatrix},\quad B)\begin{vmatrix}1&6&4\\3&3&-3\\-2&5&-1\end{vmatrix},\quad C)\begin{vmatrix}1&2&-4\\1&6&4\\3&3&-1\end{vmatrix},\quad D)\begin{vmatrix}1&5&6\\3&0&3\\-2&4&5\end{vmatrix}$$

4. O‘lchami 4×4 bo‘lgan matritsaning nechta 2-tartibli va nechta 3-tartibli minorlari mavjud?

A) 16 va 9; B) 25 va 16; C) 36 va 9; D) 36 va 16.

5. Agar n ta noma’lumli chiziqli algebraik tenglamalar sistemasi aniqmas sistema bo‘lsa. uning asosiy A va kengaytirilgan B matritsalari ranglari qanday bog‘langan bo‘ladi?

A) $rang(A)=rang(B)<n$ B) $rang(A)\neq rang(B)$

C) $rang(A)=rang(B)=n$ D) $rang(A)<n$

6. Agar 5 ta noma’lumli chiziqli bir jinsli algebraik tenglamalar sistemasi asosiy matritsasi rangi $r=3$ bo‘lsa, uning fundamental yechimlari soni nechta bo‘ladi?

A) 3ta; B) 2ta; D) 5ta; E) 4ta.

1.5. Chiziqli algebraik tenglamalar sistemasini yechish usullari

1.5.1. Chiziqli tenglamalar sistemasini yechishning Kramer usuli.

Determinantlarni chiziqli tenglamalar sistemasini yechishga tatbiqi bo‘lgan ***Kramer(determinant) usuli*** bilan tanishamiz. Aytaylik bizga n ta no‘malumli n ta chiziqli tenglamalar sistemasi berilgan bo‘lsin.

$$\begin{cases}a_{11}x_1+a_{12}x_2+......+a_{1n}x_n=b_1\\a_{21}x_1+a_{22}x_2+.....+a_{2n}x_n=b_2\\..\\a_{n1}x_1+a_{n2}x_2+.....+a_{nn}x_n=b_n\end{cases}\qquad(1.16)$$

Bu yerda $x_1,x_2,...,x_n$ -noma’lumlar, $a_{11},a_{12},...,a_{nn}$ -koeffitsientlar, $b_1,b_2,...,b_n$ - ozod sonlar.

1.8-Teorema. *Agar (1.16)-tenglamalar sistemasining asosiy determinanti* ($\Delta\neq0$) *noldan farqli bo‘lsa, u holda sistema yagona yechimga ega bo‘ladi va u quyidagi formulalardan topiladi .*

$$\Delta \neq 0\ ,\ \ x_1 = \frac{\Delta_{x_1}}{\Delta},\ \ x_2 = \frac{\Delta_{x_2}}{\Delta},\ \ldots,\ x_n = \frac{\Delta_{x_n}}{\Delta} \qquad (1.17)$$

Bu ***Kramer*** formulasidan iborat. Bu yerda $\Delta \neq 0$ ga bosh determinant, $\Delta_{x_1}, \Delta_{x_2}, \Delta_{x_3}, \ldots, \Delta_{x_n}$ larga yordamchi determinantlar deyiladi. Soddalik uchun uch noma'lumli, uchta chiziqli tenglamalar sistemasini qaraymiz.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases} \qquad (1.18)$$

uch noma'lumli uchta chiziqli tenglamalar sistemasini yechishda dastlab bosh(asosiy) determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1.19)$$

topiladi. $\Delta \neq 0$ bo'lsin. Undan so'ng yordamchi determinantlar hisoblanadi(bunda bosh determinantning ustun elementlari mos ravsihda ozod hadlar bilan almashtiriladi):

$$\Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},\quad \Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},\quad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \qquad (1.20)$$

Noma'lumlar quyidagi formulalar yordamida hisoblanadi:

$$x = \frac{\Delta_x}{\Delta},\qquad y = \frac{\Delta_y}{\Delta},\qquad z = \frac{\Delta_z}{\Delta} \qquad (1.21)$$

1.32-misol

Ushbu sistemani Kramer usulida yeching:

$$\begin{cases} x + 5y - z = 3, \\ 2x + 4y - 3z = 2, \\ 3x - y - 3z = -7 \end{cases}.$$

►Quyidagi determinantlarni tuzamiz va hisoblaymiz:

$$\Delta = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 4 & -3 \\ 3 & -1 & -3 \end{vmatrix} = -16; \qquad \Delta_x = \begin{vmatrix} 3 & 5 & -1 \\ 2 & 4 & -3 \\ -7 & -1 & -3 \end{vmatrix} = 64;$$

$$\Delta_y=\begin{vmatrix}1 & 3 & -1\\ 2 & 2 & -3\\ 3 & -7 & -3\end{vmatrix}=-16; \qquad \Delta_z=\begin{vmatrix}1 & 5 & 3\\ 2 & 4 & 2\\ 3 & -1 & -7\end{vmatrix}=32.$$

Bundan, $x=\dfrac{64}{-16}=-4,\quad y=\dfrac{-16}{-16}=1,\quad z=\dfrac{32}{-16}=-2.$ ◄

Agar bosh determinant nolga teng bo‘lsa, tenglamalar sistemasi yechimga ega bo‘lmaydi yoki cheksiz ko‘p yechimga ega bo‘ladi. Ya’ni

1) $\Delta=0$ bo‘lib, Δ_x, Δ_y, Δ_z lardan kamida bittasi nolga teng bo‘lmasa, (4.3) tengamalar sistemasi yechimga ega bo‘lmaydi,

2) $\Delta=0$ bo‘lib, $\Delta_x=0$, $\Delta_y=0$, $\Delta_z=0$ bo‘lsa, sistema cheksiz ko‘p yechimga ega bo‘ladi.

1.33-misol

Ushbu sistemani Kramer usulida yeching:

$$\begin{cases} x+2y-3z=7\\ 2x+y-2z=9\\ 3x-z=10\end{cases}$$

► Bosh determinantini hisoblaymiz:

$$\Delta=\begin{vmatrix}1 & 2 & -3\\ 2 & 1 & -2\\ 3 & 0 & -1\end{vmatrix}=-1-12+0+9-0+4=0.$$

Yordamchi determinantlarni hisoblaymiz:

$$\Delta_x=\begin{vmatrix}7 & 2 & -3\\ 9 & 1 & -2\\ 10 & 0 & -1\end{vmatrix}=-7-40+0+30-0+18=1.$$

$\Delta=0$ bo‘lib, $\Delta_x=1\neq 0$ bo‘lgani uchun berilgan tenglamalar sistemasi yechimga ega emas.◄

1.34-misol

Ushbu sistemani Kramer usulida yeching:

$$\begin{cases} x-2y+z=5\\ 2x-z=3\\ 3x+2y-3z=1\end{cases}$$

►Quyidagi determinantlarni tuzamiz va hisoblaymiz:

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ 3 & 2 & -3 \end{vmatrix} = 0 + 6 + 4 - 0 + 2 - 12 = 0,$$

$$\Delta_x = \begin{vmatrix} 5 & -2 & 1 \\ 3 & 0 & -1 \\ 1 & 2 & -3 \end{vmatrix} = 0 + 2 + 6 - 0 + 10 - 18 = 0,$$

$$\Delta_y = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 3 & -1 \\ 3 & 1 & -3 \end{vmatrix} = -9 - 15 + 2 - 9 + 1 + 30 = 0,$$

$$\Delta_z = \begin{vmatrix} 1 & -2 & 5 \\ 2 & 0 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 0 - 18 + 20 - 0 - 6 + 4 = 0.$$

$\Delta = 0$ bo‘lib, $\Delta_x = 0$, $\Delta_y = 0$, $\Delta_z = 0$ bo‘lgani uchun sistema cheksiz ko‘p yechimga ega bo‘ladi.

Bu holda 2 ta tenglamani qoldirib, erkli noma’lum, masalan, z ni tenlikning o‘ng tomoniga o‘tkazamiz.

$$\begin{cases} x - 2y + z = 5 \\ 2x \quad\quad - z = 3 \end{cases} \qquad \begin{cases} x - 2y = 5 - z \\ 2x \quad\quad = 3 + z \end{cases}$$

Hosil bo‘lgan ikki noma’lumli tenglamalar sistemasini yana Kramer usulida yechamiz.

$$\Delta = \begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} = 4,\ \Delta_x = \begin{vmatrix} 5 - z & -2 \\ 3 + z & 0 \end{vmatrix} = 6 + 2z,\ \Delta_y = \begin{vmatrix} 1 & 5 - z \\ 2 & 3 + z \end{vmatrix} = -7 + 3z.$$

Demak, tenglamaning yechimi: $\left\{ \frac{z+3}{2}; \frac{3z-7}{4}; z \right\}$. ◀

1.5.2. Chiziqli algebraik tenglamalar sistemasini yechishning matrisa usuli.

Aytaylik bizga n ta no‘malumli n ta chiziqli (1.18) tenglamalar sistemasi berilgan bo‘lsin. Ushbu belgilashlarni kiritamiz:

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}; \qquad X=\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}; \qquad B=\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} \qquad (1.22)$$

U holda (1.18) sistemani matrisalarni ko‘paytirish qoidasidan foydalanib, ushbu ekvivalent shaklda yozish mumkin:

$$\boxed{A\cdot X=B} \qquad (1.23)$$

Bu yerda A-noma’lumlar oldidagi koeffisientlardan tuzilgan matrisa, B-ozod hadlardan tuzilgan ustun matrisa, X -noma’lumlardan tuzilgan ustun matrisa

Agar A matrisa xosmas, ya’ni $\det A\neq 0$ bo‘lsa, u holda uning uchun A^{-1} teskari matrisa mavjud. (1.23) matrisali tenglamaning ikkala qismini A^{-1} ga chapdan ko‘paytirib, quyidagini hosil qilamiz:

$$A^{-1}\cdot(A\cdot X)=A^{-1}\cdot B$$

yoki

$$(A^{-1}\cdot A)\cdot X=A^{-1}\cdot B.$$

$A^{-1}\cdot A=E$, $E\cdot X=X$ ekanligini hisobga olib,

$$\boxed{X=A^{-1}\cdot B} \qquad (1.24)$$

ni topamiz. (1.24) formula A matrisa xosmas bo‘lganda n no‘malumli n ta chiziqli tenglamalar ***sistemasi yechimining matritsali yozuvidan*** iborat bo‘ladi.

1.35-misol

Ushbu sistemani yeching:

$$\begin{cases} x_1-2x_2+x_3=5 \\ 2x_1-x_3=0 \\ -2x_1+x_2+x_3=-1. \end{cases}$$

►Bu yerda

$$A=\begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}, \qquad B=\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}, \qquad X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$detA=\begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{vmatrix}=-4+2+1+4=3\neq 0,$$

$A_{11}=1,\ A_{12}=0,\ A_{13}=2,\ A_{21}=3,\ A_{22}=3,\ A_{23}=3,\ A_{31}=2,\ A_{32}=3,\ A_{33}=4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$x_1 = \frac{1}{3} \cdot 5 + 1 \cdot 0 + \frac{2}{3} \cdot (-1) = \frac{5}{3} - \frac{2}{3} = \frac{3}{3} = 1;$$

$$x_2 = 0 \cdot 5 + 1 \cdot 0 + 1 \cdot (-1) = 0 - 1 = -1;$$

$$x_3 = \frac{2}{3} \cdot 5 + 1 \cdot 0 + \frac{4}{3} \cdot (-1) = \frac{10}{3} - \frac{4}{3} = \frac{6}{3} = 2.$$

Bundan $x_1 = 1, \ x_2 = -1, \ x_3 = 2.$ ◄

1.5.3. Chiziqli algebraik tenglamalar sistemasini yechishning Gauss usuli.

Bizga n ta noma'lumli n ta chiziqli (1.18) tenglamalar sistemasi berilgan bo'lsin. Uning asosiy matritsasi A ning rangi $rang(A) = r \le n$ bo'lsa, kengaytirilgan matritsasi B ni har doim, elementar almashtirishlar yordamida, quyidagi ekvivalent matritsaga almashtirish mumkin.

$$\begin{bmatrix} 1 & \bar{a}_{12} & \dots & \bar{a}_{1r} & \bar{a}_{1\,r+1} & \dots & \bar{a}_{1n} & \bar{b}_1 \\ 0 & 1 & \dots & \bar{a}_{2r} & \bar{a}_{2\,r+1} & \dots & \bar{a}_{2n} & \bar{b}_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \bar{a}_{r\,r+1} & \dots & \bar{a}_{rn} & \bar{b}_r \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & \bar{b}_{r+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & \bar{b}_n \end{bmatrix} \qquad (1.25)$$

Agar bu matritsada $\bar{b}_{r+1}.\bar{b}_{r+2},....,\bar{b}_n$ lardan kamida bittasi noldan farqli bo'lsa, (1.18) sistema yechimga ega emas, chunki $rang(A) \ne rang(B)$ bo'ladi. Agar $\bar{b}_{r+1} = \bar{b}_{r+2} = ... = \bar{b}_n = 0$ bo'lsa, berilgan (1.18) chiziqli algebraik tenglamalar sistemasi birgalikda bo'ladi. Bu holda (1.25) matritsaning bazis satrlariga mos tenglamalarni tuzamiz.

$$\begin{cases} x_1+\bar{a}_{12}x_2+\ldots+\bar{a}_{1r}x_r+\bar{a}_{1r+1}x_{r+1}+\ldots+\bar{a}_{1n}x_n=\bar{b}_1 \\ \qquad x_2+\ldots+\bar{a}_{2r}x_r+\bar{a}_{2r+1}x_{r+1}+\ldots+\bar{a}_{2n}x_n=\bar{b}_2 \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\ \ldots\quad\ldots\quad\ldots\quad x_r+\bar{a}_{r\,r+1}x_{r+1}+\ldots+\bar{a}_{rn}x_n=\bar{b}_r \end{cases} \tag{1.26}$$

Hosil bo‘lgan (1.26) sistemaning yechimlari berilgan (1.18) chiziqli algebraik tenglamalar sistemasining ham yechimlaridir. (1.26) da $x_r, x_{r-1},\ldots,x_2,x_1$ bazis noma’lumlarni $x_{r+1},x_{r+2},\ldots,x_n$ erkli nomalumlar orqali, oxirgi tenglamadan boshlab ketma-ket aniqlanadi. Agar $r=n$ bo‘lsa, chiziqli algebraik tenglamalar sistemasining yechimi yagona bo‘ladi.

Gauss usuli n ta noma’lumli m ta chiziqli tenglamalar sistemasi bo‘lgan holda ham o‘rinli bo‘ladi.

1.36-misol

Ushbu sistemani Gauss usulida yeching:

$$\begin{cases} x+y+z=6 \\ 2x+2y-3z=-3 \\ 3x-y+2z=7 \end{cases}$$

► Berilgan tenglamalar sistemasining kengaytirilgan matritsasini tuzamiz va ekvivalent almashtirishlar yordamida quyidagi ekvivalent matritsani hosil qilamiz.

$$B=\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 2 & -3 & -3 \\ 3 & -1 & 2 & 7 \end{bmatrix}\sim\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & -5 & -15 \\ 0 & -4 & -1 & -11 \end{bmatrix}\sim\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Bundan sistema birgalida ekani kelib chiqadi, chunki $rang(A)=rang(B)=3$.

$$\begin{cases} x+y+z=6 \\ \quad 4y+z=11 \\ \qquad z=3 \end{cases}$$

sistemaga ega bo‘ldik. Oxirgi tenglamadan boshlab, $z=3$ ni ikkinchi tenglamaga qo‘yib, y ni topamiz:

$$4y+3=11,\ y=2.$$

Endi $y=2$ va $z=3$ ni 1- tenglamaga qo‘yib, x ni topamiz:

$$x+2+3=6,\ x=1.$$

Demak, $x=1, y=2, z=3$.◄

5-Auditoriya topshiriqlari

1. Berilgan chiziqli algebraik tenglamalar sistemasini Kramer usulida yeching

a) $\begin{cases} x_1 + 2x_2 + 3x_3 = 7 \\ 2x_1 - 5x_2 + x_3 = 4 \\ 3x_1 + 3x_2 - 5x_3 = -7 \end{cases}$ b) $\begin{cases} 3x_1 + 2x_2 + 5x_3 = 0 \\ 5x_1 + x_3 = 4 \\ 2x_1 + 3x_2 = 5 \end{cases}$ d) $\begin{cases} x_1 - 2x_2 + 4x_3 = 6 \\ 2x_1 + 5x_2 - 6x_3 = 7 \\ 3x_1 + 3x_2 - 2x_3 = 8 \end{cases}$

2. Berilgan chiziqli algebraik tenglamalar sistemasini matritsa usulida yeching

a) $\begin{cases} x_1 + 2x_2 - x_3 = -2 \\ 2x_1 - x_2 = -1 \\ x_2 + x_3 = -2 \end{cases}$ b) $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 4x_1 + 2x_2 - x_3 = 0 \\ x_2 - x_3 = -3 \end{cases}$

3. Berilgan chiziqli algebraik tenglamalar sistemasini Gauss usulida yeching.

a) $\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + 3x_2 - 2x_3 = -1 \\ x_1 + x_2 - 5x_3 = 6 \end{cases}$ b) $\begin{cases} 4x_1 + 2x_2 - 3x_3 + 2x_4 = 3 \\ 2x_1 + 3x_2 - 2x_3 + 3x_4 = 2 \\ 3x_1 + 2x_2 - 3x_3 + 4x_4 = 1 \end{cases}$

c) $\begin{cases} 2x_1 + 5x_2 - 8x_3 = 8 \\ 2x_1 + 3x_2 - 5x_3 = 7 \\ x_1 + 8x_2 - 7x_3 = 12 \\ 4x_1 + 3x_2 - 9x_3 = 9 \end{cases}$ d) $\begin{cases} 2x_1 + x_2 + 3x_3 + 2x_4 = -3 \\ x_1 + x_2 + 5x_3 + 2x_4 = 1 \\ 3x_1 + 3x_2 + 9x_3 + 5x_4 = -2 \\ 2x_1 + 3x_2 + 11x_3 + 5x_4 = 2 \end{cases}$

1-Shaxsiy uy topshiriqlari

1

Berilgan Δ determinant uchun a_{i2}, a_{3j} elementlarning minorlari va algebraik to‘ldiruvchilarini toping. Δ determinantni:

a) *i*-satr elementlari bo‘yicha yoyib;

b) *j*-ustun elementlari bo‘yicha yoyib;

d) *i*-satr elementlarini nollarga aylantirib hisoblang.

1.1. $\begin{vmatrix} 1 & 1 & -2 & 0 \\ 3 & 6 & -2 & 5 \\ 1 & 0 & 6 & 4 \\ 2 & 3 & 5 & -14 \end{vmatrix}$ $i = 4,\ j = 1$

1.2. $\begin{vmatrix} 2 & 0 & -1 & 3 \\ 6 & 3 & -9 & 0 \\ 0 & 2 & -1 & 3 \\ 4 & 2 & 0 & 6 \end{vmatrix}$ $i = 1,\ j = 3$

1.3. $\begin{vmatrix} 2 & 7 & 2 & 1 \\ 1 & 1 & -1 & 0 \\ 3 & 4 & 0 & 2 \\ 0 & 5 & -1 & -3 \end{vmatrix}$ $i = 4,\ j = 1$

1.4. $\begin{vmatrix} 4 & -5 & -1 & -5 \\ -3 & 2 & 8 & -2 \\ 5 & 3 & 1 & 3 \\ -2 & 4 & -6 & 8 \end{vmatrix}$ $i = 1,\ j = 3$

1.5. $\begin{vmatrix} 3 & 5 & 3 & 2 \\ 2 & 4 & 1 & 0 \\ 1 & -2 & 2 & 1 \\ 5 & 1 & -2 & 4 \end{vmatrix}$ $i=2,\ j=4$

1.6. $\begin{vmatrix} 3 & 2 & 0 & -5 \\ 4 & 3 & -5 & 0 \\ 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 4 \end{vmatrix}$ $i=1,\ j=2$

1.7. $\begin{vmatrix} 2 & -1 & 2 & 0 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$ $i=2,\ j=3$

1.8. $\begin{vmatrix} 3 & 2 & 0 & -2 \\ 1 & -1 & 2 & 3 \\ 4 & 5 & 1 & 0 \\ -1 & 2 & 3 & -3 \end{vmatrix}$ $i=3,\ j=1$

1.9. $\begin{vmatrix} 0 & -1 & 2 & 0 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$ $i=4,\ j=3$

1.10. $\begin{vmatrix} 0 & -2 & 1 & 7 \\ 4 & -8 & 2 & -3 \\ 10 & 1 & -5 & 4 \\ -8 & 3 & 2 & -1 \end{vmatrix}$ $i=4,\ j=2$

1.11. $\begin{vmatrix} 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \\ 2 & 1 & 4 & -6 \\ 3 & -2 & 9 & 4 \end{vmatrix}$ $i=1,\ j=4$

1.12. $\begin{vmatrix} 4 & -1 & 1 & 5 \\ 0 & 2 & -2 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 1 & -2 \end{vmatrix}$ $i=2,\ j=4$

1.13. $\begin{vmatrix} 1 & 8 & 2 & -3 \\ 3 & -2 & 0 & 4 \\ 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \end{vmatrix}$ $i=1,\ j=4$

1.14. $\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ 3 & 0 & 2 & 1 \\ 3 & -1 & 4 & 3 \end{vmatrix}$ $i=2,\ j=4$

1.15. $\begin{vmatrix} 3 & 1 & 2 & 3 \\ 4 & -1 & 2 & 4 \\ 1 & -1 & 1 & 1 \\ 4 & -1 & 2 & 5 \end{vmatrix}$ $i=1,\ j=3$

1.16. $\begin{vmatrix} 3 & 1 & 2 & 0 \\ 5 & 0 & -6 & 1 \\ -2 & 2 & 1 & 3 \\ -1 & 3 & 2 & 1 \end{vmatrix}$ $i=3,\ j=2$

1.17. $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 3 & 2 & 1 & -1 \\ 1 & 2 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{vmatrix}$ $i=3,\ j=1$

1.18. $\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$ $i=2,\ j=4$

1.19. $\begin{vmatrix} 6 & 2 & -10 & 4 \\ -5 & -7 & -4 & 1 \\ 2 & 4 & -2 & -6 \\ 3 & 0 & -5 & 2 \end{vmatrix}$ $i=2,\ j=3$

1.20. $\begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 0 & 6 \\ 2 & -2 & 1 & 4 \\ 3 & 1 & -2 & -1 \end{vmatrix}$ $i=4,\ j=3$

1.21. $\begin{vmatrix} 2 & 7 & 1 & 1 \\ 2 & 4 & 1 & 0 \\ 1 & -2 & 2 & 1 \\ 5 & 1 & -2 & 4 \end{vmatrix}$ $i=4,\ j=2$

1.22. $\begin{vmatrix} 1 & 2 & 0 & -5 \\ 0 & 1 & -5 & 5 \\ 1 & 0 & -2 & 3 \\ -1 & 1 & -3 & 4 \end{vmatrix}$ $i=3,\ j=3$

1.23. $\begin{vmatrix} 1 & 5 & -1 & 2 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$ $i=2,\ j=4$

1.24. $\begin{vmatrix} 2 & 4 & 3 & -5 \\ 1 & -1 & 2 & 3 \\ 4 & 5 & 1 & 0 \\ -1 & 2 & 3 & -3 \end{vmatrix}$ $i=1,\ j=4$

1.25. $\begin{vmatrix} 0 & -1 & 2 & 0 \\ 3 & 4 & 1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 \end{vmatrix}$ $i=3,\ j=1$

1.26. $\begin{vmatrix} 0 & -2 & 1 & 7 \\ 4 & -8 & 2 & -3 \\ 10 & 1 & -5 & 4 \\ -8 & 3 & 2 & -1 \end{vmatrix}$ $i=1,\ j=3$

1.27. $\begin{vmatrix} 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \\ 2 & 1 & 4 & -6 \\ 3 & -2 & 9 & 4 \end{vmatrix}$ $i=2,\ j=4$

1.28. $\begin{vmatrix} 4 & -1 & 1 & 5 \\ 0 & 2 & -2 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 1 & -2 \end{vmatrix}$ $i=3,\ j=3$

1.29. $\begin{vmatrix} 1 & 8 & 2 & -3 \\ 3 & -2 & 0 & 4 \\ 5 & -3 & 7 & -1 \\ 3 & 2 & 0 & 2 \end{vmatrix}$ $i=3,\ j=1$

1.30. $\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ 3 & 0 & 2 & 1 \\ 3 & -1 & 4 & 3 \end{vmatrix}$ $i=4,\ j=1$

2

Ikkita A va B matritsalar berilgan. Quyidagilarni toping:

a) $A\cdot B$; b) $B\cdot A$; d) A^{-1}.

2.1. $A=\begin{bmatrix} 2 & -1 & -3 \\ 8 & -7 & -6 \\ -3 & 4 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 2 & -1 & -2 \\ 3 & -5 & 4 \\ 1 & 2 & 1 \end{bmatrix}$.

2.2. $A=\begin{bmatrix} 3 & 5 & -6 \\ 2 & 4 & 3 \\ -3 & 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 8 & -5 \\ -3 & -1 & 0 \\ 4 & 5 & -3 \end{bmatrix}$.

2.3. $A=\begin{bmatrix} 2 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 6 & 0 \\ 2 & 4 & -6 \\ 1 & 2 & 1 \end{bmatrix}$.

2.4. $A=\begin{bmatrix} -6 & 1 & 11 \\ 9 & 2 & 5 \\ 0 & 4 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 7 \\ 1 & -3 & 2 \end{bmatrix}$.

2.5. $A=\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 7 & 1 \end{bmatrix}$.

2.6. $A=\begin{bmatrix} 2 & 3 & 2 \\ 1 & 3 & -1 \\ 4 & 1 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 2 & -1 \\ 3 & 1 & 2 \\ 5 & 3 & 0 \end{bmatrix}$.

2.7. $A=\begin{bmatrix} 6 & 7 & 3 \\ 3 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & -2 \\ 4 & 3 & 7 \end{bmatrix}$.

2.8. $A=\begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & -4 \\ -1 & 2 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 3 & 1 \\ 0 & 6 & 2 \\ 1 & 9 & 2 \end{bmatrix}$.

2.9. $A=\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 1 & 1 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 0 & 5 \\ 1 & 2 & 7 \\ 4 & 3 & 7 \end{bmatrix}$.

2.10. $A=\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 4 & -3 & 2 \\ -4 & 0 & 5 \\ 3 & 2 & -3 \end{bmatrix}$.

2.11. $A=\begin{bmatrix} 6 & 9 & 4 \\ -1 & -1 & 1 \\ 10 & 1 & 7 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 0 & 5 & 2 \end{bmatrix}$.

2.12. $A=\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 7 \\ 2 & 1 & 8 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 5 & 4 \\ -3 & 0 & 1 \\ 5 & 6 & -4 \end{bmatrix}$.

2.13. $A=\begin{bmatrix} 5 & 1 & -2 \\ 1 & 3 & -1 \\ 8 & 4 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 5 & 5 \\ 7 & 1 & 2 \\ 1 & 6 & 0 \end{bmatrix}$

2.14. $A=\begin{bmatrix} 2 & 2 & 5 \\ 3 & 3 & 6 \\ 4 & 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & -1 \end{bmatrix}$.

2.15. $A=\begin{bmatrix} 1 & -2 & 5 \\ 3 & 0 & 6 \\ 4 & 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & -1 \end{bmatrix}$.

2.16. $A=\begin{bmatrix} 5 & 4 & 2 \\ 1 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 5 & 1 & -5 \\ 3 & -7 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

2.17. $A=\begin{bmatrix} 3 & 1 & 0 \\ 4 & 3 & 2 \\ 2 & 2 & -7 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 7 & 0 \\ 5 & 3 & 1 \\ 1 & -6 & 1 \end{bmatrix}$.

2.18. $A=\begin{bmatrix} 8 & -1 & -1 \\ 5 & -5 & -1 \\ 10 & 3 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 2 & 5 \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

2.19. $A=\begin{bmatrix} 3 & -7 & 2 \\ 1 & -8 & 3 \\ 4 & -2 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 5 & -3 \\ 2 & 4 & 1 \\ 2 & 1 & -5 \end{bmatrix}$.

2.20. $A=\begin{bmatrix} 3 & -1 & 0 \\ 3 & 5 & 1 \\ 4 & -7 & 5 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 0 & 2 \\ 1 & -8 & 4 \\ 3 & 0 & 2 \end{bmatrix}$.

2.21. $A=\begin{bmatrix} 0 & -3 & 6 \\ 2 & 1 & -2 \\ -3 & 4 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} -2 & 1 & 2 \\ 3 & -5 & 4 \\ 5 & -1 & 6 \end{bmatrix}.$

2.22. $A=\begin{bmatrix} 3 & 1 & -3 \\ 2 & 0 & 5 \\ -3 & 7 & -2 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 4 & 5 \\ -3 & 5 & 0 \\ 4 & -3 & 3 \end{bmatrix}.$

2.23. $A=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 3 & 6 & 0 \\ 2 & 4 & -6 \\ 1 & 2 & 1 \end{bmatrix}.$

2.24. $A=\begin{bmatrix} -6 & 1 & 11 \\ 9 & 2 & 5 \\ 0 & 4 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}.$

2.25. $A=\begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 0 & 3 \\ 2 & 1 & 1 \\ 1 & 6 & 0 \end{bmatrix}.$

2.26. $A=\begin{bmatrix} 0 & -5 & -1 \\ 1 & 3 & -1 \\ 3 & -2 & 4 \end{bmatrix}, \quad B=\begin{bmatrix} 3 & 2 & -1 \\ 3 & 1 & 2 \\ 5 & 3 & 0 \end{bmatrix}.$

2.27. $A=\begin{bmatrix} 1 & 7 & 3 \\ -2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & -2 \\ 2 & 3 & 2 \end{bmatrix}.$

2.28. $A=\begin{bmatrix} 4 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 3 & 6 & 0 \\ 2 & 4 & -6 \\ 1 & 2 & 1 \end{bmatrix}.$

2.29. $A=\begin{bmatrix} 1 & 5 & 6 \\ 3 & -1 & -4 \\ -1 & 2 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & -6 & 1 \\ 0 & 6 & 2 \\ 1 & 3 & 0 \end{bmatrix}.$

2.30. $A=\begin{bmatrix}2 & 5 & 0\\ 1 & 3 & 2\\ 0 & 4 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & -5 & 5\\ 0 & -3 & 7\\ 3 & 2 & -3\end{bmatrix}$.

3. Chiziqli algebraik tenglamalar sistemasi birgalikda ekanligini tekshiring. Agar birgalikda bo‘lsa, uni

a) Kramer formulalari bo‘yicha;

b*)* matritsa usulida ;

d) Gauss usulida yeching.

3.1. $\begin{cases}2x_1+x_2+3x_3=7,\\ 2x_1+3x_2+x_3=1,\\ 3x_1+2x_2+x_3=6;\end{cases}$ 3.2. $\begin{cases}2x_1-x_2+2x_3=3,\\ x_1+x_2+2x_3=-4,\\ 4x_1+x_2+4x_3=-3;\end{cases}$

3.3. $\begin{cases}3x_1-x_2+x_3=12,\\ x_1+2x_2+4x_3=6,\\ 5x_1+x_2+2x_3=3;\end{cases}$ 3.4. $\begin{cases}2x_1-x_2+3x_3=-4,\\ x_1+3x_2-x_3=11,\\ x_1-2x_2+2x_3=-7;\end{cases}$

3.5. $\begin{cases}3x_1-2x_2+4x_3=12,\\ 3x_1+4x_2-2x_3=6,\\ 2x_1-x_2-x_3=-9;\end{cases}$ 3.6. $\begin{cases}8x_1+3x_2-6x_3=-4,\\ x_1+x_2-x_3=2,\\ 4x_1+x_2-3x_3=-5;\end{cases}$

3.7. $\begin{cases}4x_1+x_2-3x_3=9,\\ x_1-x_2-x_3=-2,\\ 8x_1+3x_2-6x_3=0;\end{cases}$ 3.8. $\begin{cases}2x_1+3x_2+4x_3=33,\\ 7x_1-5x_2=24,\\ 4x_1+x_3=39;\end{cases}$

3.9. $\begin{cases}2x_1+3x_2+4x_3=12,\\ 7x_1-5x_2+x_3=-33,\\ 4x_1+x_3=-7;\end{cases}$ 3.10. $\begin{cases}x_1+4x_2-x_3=6,\\ 5x_2+4x_3=-20,\\ 3x_1-2x_2+5x_3=-22;\end{cases}$

3.11. $\begin{cases}3x_1-2x_2+4x_3=21,\\ 3x_1+4x_2-2x_3=9,\\ 2x_1-x_2-x_3=10;\end{cases}$ 3.12. $\begin{cases}3x_1-2x_2-5x_3=5,\\ 2x_1+3x_2-4x_3=12,\\ x_1-2x_2+3x_3=-1;\end{cases}$

3.13. $\begin{cases}4x_1+x_2+4x_3=19,\\ 2x_1-x_2+2x_3=11,\\ x_1+x_2+2x_3=8;\end{cases}$ 3.14. $\begin{cases}2x_1-x_2+2x_3=0,\\ 4x_1+x_2+4x_3=6,\\ x_1+x_2+2x_3=4;\end{cases}$

3.15. $\begin{cases}2x_1-x_2+2x_3=8,\\ x_1+x_2+2x_3=11,\\ 4x_1+x_2+4x_3=22;\end{cases}$ 3.16. $\begin{cases}2x_1-x_2-3x_3=-9,\\ x_1+5x_2+x_3=20,\\ 3x_1+4x_2+2x_3=15;\end{cases}$

3.17. $\begin{cases} 2x_1 - x_2 - 3x_3 = 0, \\ 3x_1 + 4x_2 + 2x_3 = 1, \\ x_1 + 5x_2 + x_3 = -3; \end{cases}$

3.18. $\begin{cases} -3x_1 + 5x_2 + 6x_3 = -8, \\ 3x_1 + x_2 + x_3 = -4, \\ x_1 - 4x_2 - 2x_3 = -9; \end{cases}$

3.19. $\begin{cases} 3x_1 + x_2 + x_3 = -4, \\ -3x_1 + 5x_2 + 6x_3 = 36, \\ x_1 - 4x_2 - 2x_3 = -19; \end{cases}$

3.20. $\begin{cases} 3x_1 - x_2 + x_3 = 11, \\ 5x_1 + x_2 + 2x_3 = 8, \\ x_1 + 2x_2 + 4x_3 = 16; \end{cases}$

3.21. $\begin{cases} x_1 - 3x_2 - 7x_3 = 0 \\ x_1 + 2x_2 + 4x_3 = 6, \\ 4x_1 - x_2 - 2x_3 = -3; \end{cases}$

3.22. $\begin{cases} 3x_1 + 2x_2 + 2x_3 = 7, \\ x_1 + 3x_2 - x_3 = 11, \\ 3x_1 + 4x_2 = 15; \end{cases}$

3.23. $\begin{cases} x_1 - x_2 + 5x_3 = 21, \\ x_1 + 5x_2 - x_3 = 15, \\ 2x_1 - x_2 - x_3 = -9; \end{cases}$

3.24. $\begin{cases} 6x_1 + x_2 - 4x_3 = -8, \\ x_1 + x_2 - x_3 = 2, \\ 4x_1 + x_2 - 3x_3 = -5; \end{cases}$

3.25. $\begin{cases} x_1 - x_2 - x_3 = -2, \\ 4x_1 + x_2 - 3x_3 = 9, \\ 4x_1 + 2x_2 - 3x_3 = -9; \end{cases}$

3.26. $\begin{cases} 2x_1 - 3x_2 - 3x_3 = 6, \\ 5x_1 - 8x_2 - 4x_3 = -9, \\ 4x_1 + x_3 = 39; \end{cases}$

3.27. $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 12, \\ 7x_1 - 8x_2 + 4x_3 = 38, \\ 4x_1 + x_3 = -7; \end{cases}$

3.28. $\begin{cases} 7x_2 + 4x_3 = 20, \\ x_1 - x_2 - 5x_3 = 26, \\ x_1 - 3x_2 + 3x_3 = -14; \end{cases}$

3.29. $\begin{cases} x_1 - x_2 + 5x_3 = 11, \\ 3x_1 + 4x_2 - 2x_3 = 9, \\ 2x_1 + 5x_2 + 4x_3 = 10; \end{cases}$

3.30. $\begin{cases} x_1 + x_2 + 7x_3 = 5, \\ x_1 - 9x_2 + 13x_3 = -15, \\ x_1 - 2x_2 + 3x_3 = -39. \end{cases}$

II BOB. VEKTORLAR ALGEBRASI ELEMENTLARI

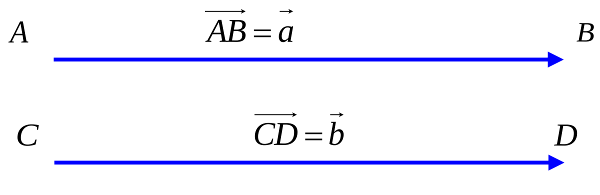
2.1. Vektorlar. Vektorlar ustida chiziqli amallar. Chiziqli bog‘liq va chiziqli erkli vektorlar. Bazis

Vektorlar. Asosiy tushunchalar. Yo‘nalgan kesma yoki nuqtalarning tartiblangan $\{A, B\}$ jufti ***vektor*** deyiladi; odatda birinchi nuqtani vektorning boshi, ikkinchi nuqtani esa uning ***oxiri (uchi)*** deyiladi (2.1-chizma) va $\overrightarrow{AB}$ kabi belgilanadi. Boshi va oxiri ko‘rsatilmagan vektor lotin alifbosining kichik harflari bilan belgilanadi: $\vec{a}$, $\vec{b}$, …

2.1-chizma

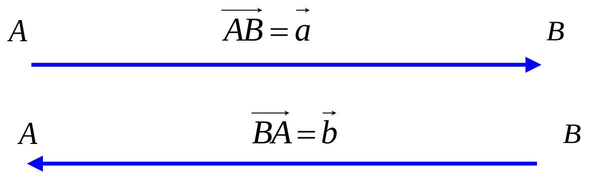
Vektorning ***moduli*** yoki ***uzunligi*** deb, vektorning boshi va oxiri orasidagi masofaga aytiladi. $|\overrightarrow{AB}|$ yoki $|\vec{a}|$ kabi belgilanadi. Bir to‘g‘ri chiziqda yoki parallel to‘g‘ri chiziqlarda yotuvchi vektorlar ***kollinear vektorlar*** deyiladi. Bir tekislikda yoki parallel tekisliklarda yotuvchi vektorlarga ***komplanar vektorlar*** deyiladi. Boshi va oxiri bir nuqtada bo‘lgan vektor ***nol vektor*** deyiladi.

Uzunliklari teng, kollinear va yo‘nalishlari bir xil bo‘lgan ikki vektor ***teng vektorlar*** deb ataladi, boshqacha aytganda, agar $\vec{a}$ va $\vec{b}$ vektorlar uchun quyidagi uchta shart $\left(|\vec{a}| = |\vec{b}|,\ \vec{a} \,||\, \vec{b},\ \vec{a} \uparrow\uparrow \vec{b}\right)$ bajarilsa, u holda $\vec{a}$ va $\vec{b}$ vektorlar teng deyiladi va $\vec{a} = \vec{b}$ deb yoziladi (2.2-chizma).



2.2-chizma

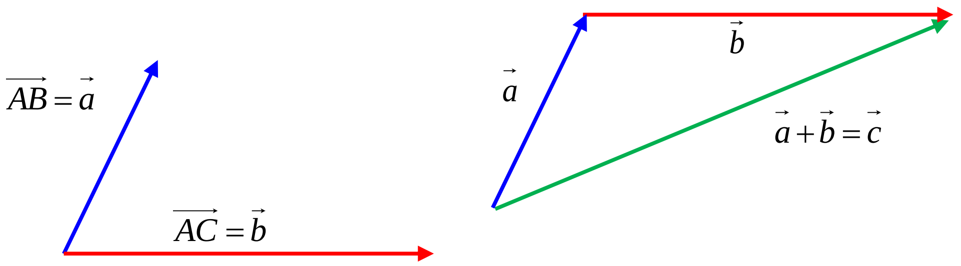
Uzunliklari teng, kollinear va yo‘nalishlari har xil bo‘lgan ikki vektorga ***qarama-qarshi vektorlar*** deyiladi, boshqacha aytganda, agar $\vec{a}$ va $\vec{b}$ vektorlar uchun quyidagi uchta shart $\left(|\vec{a}| = |\vec{b}|,\ \vec{a} \,||\, \vec{b},\ \vec{a} \uparrow\downarrow \vec{b}\right)$ bajarilsa, u holda $\vec{a}$ va $\vec{b}$ vektorlar qarama-qarshi vektorlar deyiladi va $\vec{a} = -\vec{b}$ deb yoziladi.



2.3-chizma

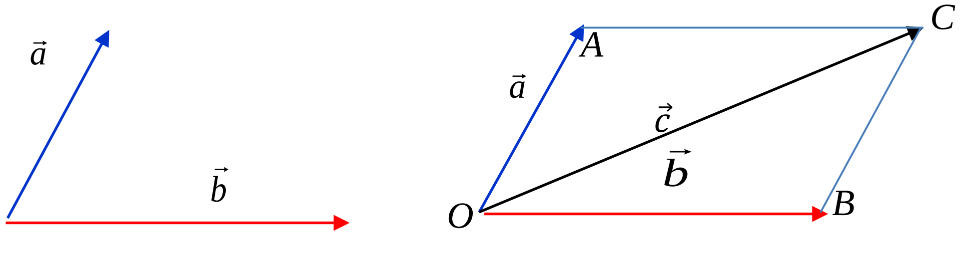
Vektorlar ustida chiziqli amallar.

1) Vektorlarni qo‘shish va ayrish. Vektorlar o‘z-o‘ziga parallel ko‘chirilsa, berilgan vektorga teng vektor hosil bo‘ladi. Ikkita $\vec{a}$ va $\vec{b}$ vektorning yig‘indisini topish uchun $\vec{a}=\overrightarrow{OA}$ vektorning oxiri $\vec{b}$ vektorning boshi bilan ustma-ust tushadigan qilib $\vec{b}$ vektorni o‘z-o‘ziga parallel ko‘chiramiz. Hosil bo‘lgan vektorni $\vec{b}=\overrightarrow{AB}$ deb belgilaymiz (2.4-chizma). O nuqta bilan B nuqtani tutashtiramiz. Natijada hosil bo‘lgan $\overrightarrow{OB}=\vec{c}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarning yig‘indisi deyiladi va $\vec{c}=\vec{a}+\vec{b}$ kabi yoziladi. Vektorlarni bunday qo‘shish qoidasi ***«uchburchak qoidasi»*** deb ataladi(2.4-chizma).



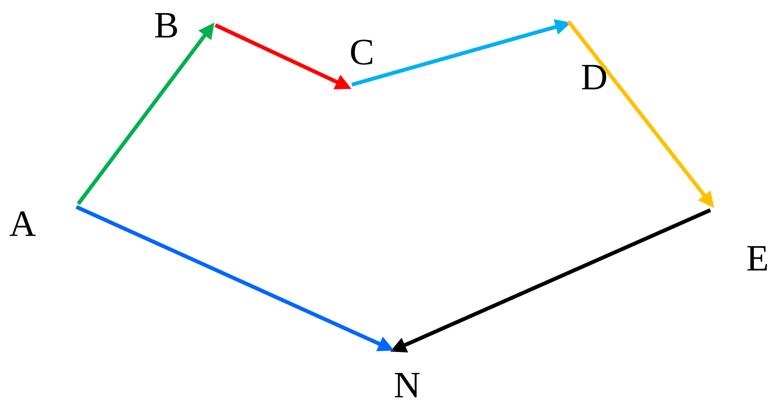
2.4-chizma

$\vec{a},\vec{b}$ vektorlar o‘zaro kollinear bo‘lmagan vektor bo‘lsin. Ularning boshini bitta O nuqtaga o‘z-o‘ziga parallel ravishda ko‘chiramiz, so‘ngra tomonlari $\vec{a}$ va $\vec{b}$ vektorlardan iborat parallelogramm chizamiz. Uning O nuqtaga qarama-qarshi uchini C deb $\overrightarrow{OC}$ vektorni qaraymiz. Ravshanki, $\overrightarrow{OC}=\vec{c}=\vec{a}+\vec{b}$. Vektorlar yig‘indisini bunday geometrik yasashga odatda «***parallelogramm qoidasi***» deb yuritiladi (2.5-chizma).



2.5-chizma.

Bizga bir necha $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EN}$ vektorlar berilgan bo‘lsin. Bu vektorlarning har biri ketma-ket kelgan jufti uchun birinchisining oxiri bilan ikkinchisining boshi ustma-ust tushsin (2.6-chizma). Bu holda vektorlar siniq chiziq tashkil qilib, yig‘indi vektor ularning yopuvchisiga teng, ya’ni $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EN}=\overrightarrow{AN}$



2.6-chizma

$\vec{a}, \vec{b}$ vektorlarning ayirmasi deb shunday $\vec{x}$ vektorga aytiladki, uni $\vec{b}$ vektorga qo‘shganda $\vec{a}$ vektor hosil bo‘ladi, ya’ni agar $\vec{x}$ vektor uchun ushbu $\vec{x}+\vec{b}=\vec{a}$ munosabat o‘rinli bo‘lsa, u holda $\vec{x}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarning ayirmasi deyiladi hamda $\vec{x}=\vec{a}-\vec{b}$ deb yoziladi.

Agar «*kamayuvchi*» $\vec{a}$ va «*ayriluvchi*» $\vec{b}$ vektorlar berilsa, u holda ushbu $\vec{b}+\vec{x}=\vec{a}$ munosabatni qanoatlantiruvchi $\vec{x}$ vektor doim mavjud. $\overrightarrow{BC}=\vec{x}$, $\overrightarrow{AC}=\vec{a}$, $\overrightarrow{AB}=\vec{b}$. Demak, $\vec{a}-\vec{b}$ ayirma vektorni chizish uchun bir nuqtadan chiquvchi $\vec{a}$ va $\vec{b}$ vektorlarni chizib, $\vec{b}$ vektorning uchidan $\vec{a}$ vektorning uchiga boruvchi vektorni chizish kifoya. Shunday qilib, vektorlarni ayirish amali hamma vaqt ma’noga ega.

2) Vektorni songa ko‘paytirish.

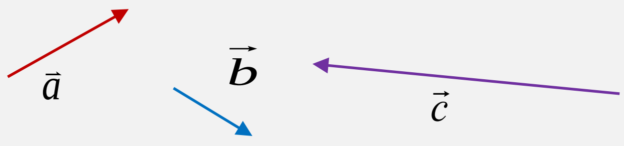
$\vec{a}$ vektorni $\lambda \in R$ soniga ko‘paytmasi deb shunday $\vec{b}$ vektorga aytiladiki, bu vektorning uzunligi $|\vec{b}|=\lambda\cdot|\vec{a}|$ teng bo‘lib, yo‘nalishi esa $\lambda>0$ bo‘lganda $\vec{a}$ vektor bilan bir xil yo‘nalgan, $\lambda<0$ bo‘lganda $\vec{a}$ vektorga qarama-qarshi yo‘nalgan bo‘ladi.

2.1-misol.

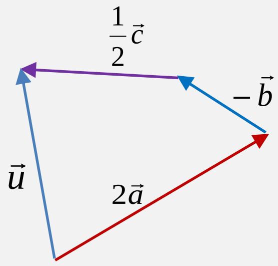
Berilgan $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlarga asosan quyidagi vektorni yasang:

$$\vec{u} = 2\vec{a} - \vec{b} + \frac{1}{2}\vec{c}\,.$$

▶



Bir nuqtadan boshlab $2\vec{a}$, $-\vec{b}$ va $\frac{1}{2}\vec{c}$ vektorlarni ketma- ket joylashtiramiz:



Izlangan $2\vec{a}$ vektor hosil bo‘ladi,◀

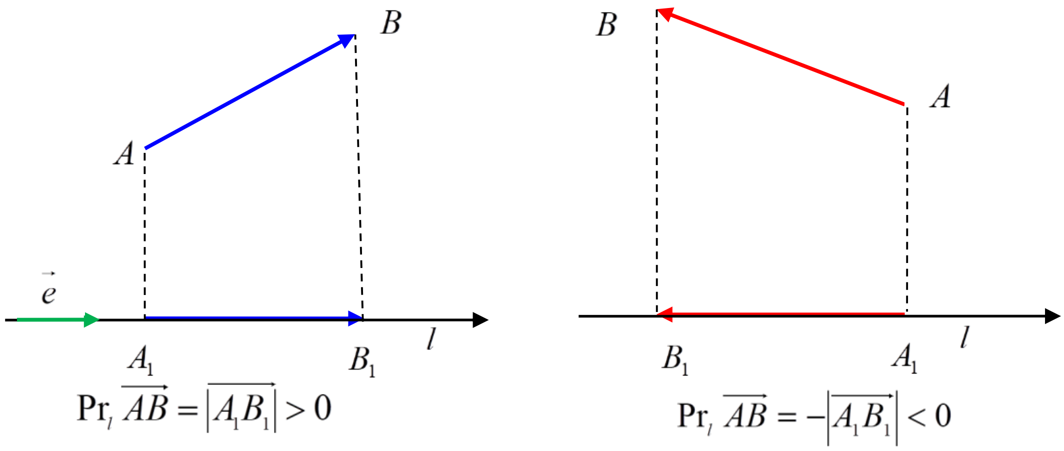
Vektorning koordinatalari. Musbat yo‘nalishi tanlab olingan l to‘g‘ri chiziq *o‘q* deb ataladi. O‘qning yo‘nalishini odatda strelka bilan ko‘rsatiladi (2.7-chizma), bu strelkaning yo‘nalishi l to‘g‘ri chiziqdagi munosabat yo‘nalishni aniqlovchi $\vec{e}$ vektor yo‘nalishi bilan bir xil bo‘ladi.

2.7-chizma.

$$\overrightarrow{OE} = \vec{e}\,, \qquad \left|\overrightarrow{OE}\right| = |\vec{e}| = 1\,.$$

Yo‘nalish o‘qdagi musbat yo‘nalish bilan bir xil bo‘lgan hamda uzunligi birga teng bo‘lgan vektor ($\vec{e}$ vektor) o‘qning ***orti (bazisi)*** deyiladi.

$\overrightarrow{AB}$ vektorning l o‘qdagi proeksiyasi deb, shunday $\overrightarrow{A_1B_1}$ vektorning uzunligiga aytiladiki, unda A_1 va B_1 lar mos ravishda A va B nuqtalarning l o‘qdagi ortogonal proeksiyalari bo‘lib, bu uzunlik $\overrightarrow{A_1B_1}$ va $\vec{e}$ vektorlarning yo‘nalishlari bir xil bo‘lganda musbat ishora bilan, aks holda manfiy ishora bilan olinadi (2.8-chizma).



2.8-chizma.

$\overline{AB}$ vektorning l o‘qdagi proeksiyasini $Pr_l\overline{AB}=\pm|A_1B_1|$. (2.1)

Bundan $\overrightarrow{AB}$ vektor o‘qqa perpendikulyar bo‘lgandagina uning proeksiyasi nolga teng degan xulosa kelib chiqadi. $\overrightarrow{A_1B_1}=x\cdot\vec{e}$ tenglikdagi x son $\overrightarrow{AB}$ vektorning proeksiyasidir, ya’ni $x=Pr_l\overrightarrow{AB}$.

Vektorning o‘qdagi proeksiyasining xossalari:

1. $Pr_l\left(\vec{a}+\vec{b}+\vec{c}+\ldots+\vec{d}\right)=Pr_l\vec{a}+Pr_l\vec{b}+Pr_l\vec{c}+\ldots+Pr_l\vec{d}$
2. $Pr_l\left(\lambda\cdot\vec{a}\right)=\lambda\cdot Pr_l\vec{a}$, $\lambda\neq 0$.
3. Teng vektorlarning bitta o‘qqa proeksiyalari o‘zaro tengdir.
4. $Pr_l\vec{a}=|\vec{a}|\cdot cos\varphi$, bu yerda φ - $\vec{a}$ va $\vec{e}$ vektorlar orasidagi burchak, $0\leq\varphi\leq\pi$.

Agar tekislikda (yoki fazoda) koordinatalar boshi deb ataluvchi nuqta, o‘zaro perpendikulyar to‘g‘ri chiziqlar, ularda musbat yo‘nalish hamda uzunlik birligi (umuman aytganda, har bir yo‘nalishdagi o‘qda har xil) tanlangan bo‘lsa, tekislikda (fazoda) ***dekart koordinatalar sistemasi*** berilgan deyiladi. O‘qlar mos ravishda abssissalar o‘qi, ordinatalar o‘qi, (aplikatalar o‘qi) deb yuritiladi. Tegishli o‘qlar koordinatalar o‘qlari deyiladi. Faraz qilaylik, tekislikda Dekart koordinatalar sistemasi berilgan bo‘lsin (uni qisqacha ***Oxy sistema*** deb ham yuritiladi) va $\vec{a}$ vektor koordinatalar boshi O nuqtadan chiqqan bo‘lsin.

$\vec{a}$ ***vektorning koordinatalari*** deb uning koordinata o‘qlaridagi proeksiyalariga aytiladi, ya’ni

$$x=Pr_{Ox}\vec{a}\,,\quad y=Pr_{Oy}\vec{a}\,.$$

Agar *Oxy* sistemada $\vec{a}=\{x_1,y_1\}$, $\vec{b}=\{x_2,y_2\}$ bo‘lsa, $\vec{a}+\vec{b}=\vec{c}=\{x_1+x_2,\ y_1+y_2\}$ bo‘ladi.

Agar *Oxy* sistemada $\vec{a}$ vektorning koordinatalari $\{x,y\}$ bo‘lsa, $\lambda\cdot\vec{a}$ vektorning shu sistemadagi koordinatalari $\{\lambda x,\lambda y\}$ bo‘ladi.

Agar *Oxy* sistemada $\overrightarrow{AB}$ vektor boshining koordinatalari $\{x_1,y_1\}$ va oxiri $\{x_2,y_2\}$ bo‘lsa, $\overrightarrow{AB}$ vektorning koordinatalari $\{x_2-x_1\ ,\ y_2-y_1\}$ bo‘ladi , ya’ni

$$\overrightarrow{AB}=\{x_2-x_1\ ,y_2-y_1\} \tag{2.2}$$

2.2-misol

Agar $\vec{a}\{5,4\}$ vektor boshining koordinatalari $A(-2,3)$ bo‘lsa, uning oxirining koordinatalarini aniqlang.

► $\vec{a}\{5,4\}$ vektor oxirining koordinatalari $B(x,y)$ bo‘lsin. U holda $x-(-2)=5,\ y-3=4 \Leftrightarrow x=5-2=3,\ \ y=4+3=7$ bo‘ladi. Demak, $B(3,7)$.◄

2.3-misol

Agar $\vec{b}\{2,-1\}$ vektor oxirining koordinatalari B$(3,2)$ bo‘lsa, uning boshining koordinatalarini aniqlang.

► $\vec{b}\{2,-1\}$ dan

$$3-x=2,\quad 2-y=-1,\quad x=3-2=1,\quad y=2+1=3.$$

Bundan $A(1,3)$.◄

2.1.1. Chiziqli bog‘liq va chiziqli erkli vektorlar sistemasi. Bazis.

Bizga n ta $\vec{a_1},\ \vec{a_2},\ \vec{a_3},\ldots,\vec{a_n}$ vektorlar va n ta $\alpha_1,\ \alpha_2,\ \alpha_3,\ldots,\alpha_n$ sonlar berilgan bo‘lsin bu sonlarning mos vektorlarga ko‘paytmalarining yig‘indisini tuzamiz.

$\alpha_1\cdot\vec{a_1}+\alpha_2\cdot\vec{a_2}+\alpha_3\cdot\vec{a_3}+\ldots+\alpha_n\cdot\vec{a_n}$ ko‘paytmaga vektorlar sistemasining ***chiziqli kombinatsiyasi*** deyiladi.

$\vec{a_1},\ \vec{a_2},\ \vec{a_3},\ldots,\vec{a_n}$ vektorlar sistemasi uchun kamida bittasi noldan farqli shunday n ta $\alpha_1\ ,\ \alpha_2\ ,\ \alpha_3\ ,\ldots,\alpha_n$ sonlar mavjud bo‘lsaki, ular uchun vektorlar sistemasining chiziqli kombinatsiyasi nolga teng, ya’ni

$$\alpha_1\cdot\vec{a_1}+\alpha_2\cdot\vec{a_2}+\alpha_3\cdot\vec{a_3}+\ldots+\alpha_n\cdot\vec{a_n}=0 \tag{2.3}$$

bo‘lsa, bunday vektorlar sistemasiga ***chiziqli bog‘liq sistema*** deb ataladi. Aks holda $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \ldots, \overrightarrow{a_n}$ vektorlar ***chiziqli erkli*** deyiladi, ular uchun (2.3) tenglik faqat $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$ bo‘lgandagina o‘rinli bo‘ladi.

Agar vektorlar chiziqli bog‘liq bo‘lsa, (2.3) dagi biror vektorni boshqa vektorlar orqali ifodalab olish mumkin. $\alpha_1 \cdot \overrightarrow{a_1}$ ifodani qoldirib qolgan ifodalarni tenglikning o‘ng tomoniga o‘tkazib $\alpha_1 \neq 0$ ga bo‘lsak,

$$\vec{a}_1 = -\frac{\alpha_2}{\alpha_1} \cdot \overrightarrow{a_2} - \frac{\alpha_3}{\alpha_1} \cdot \overrightarrow{a_3} - \frac{\alpha_4}{\alpha_1} \cdot \overrightarrow{a_4} - \ldots - -\frac{\alpha_n}{\alpha_1} \cdot \overrightarrow{a_n}$$

va belgilash kiritsak, bu vektor qolgan vektorlarning chiziqli kombinatsiyasidan iborat bo‘ladi:

$$\overrightarrow{a_1} = \beta_2 \cdot \overrightarrow{a_2} + \beta_3 \cdot \overrightarrow{a_3} + \beta_4 \cdot \overrightarrow{a_4} + \ldots + \beta_n \cdot \overrightarrow{a_n} \quad . \qquad (2.4)$$

Agar vektorlardan kamida biri qolgan vektorlarning chiziqli kombinatsiyasidan iborat bo‘lsa, u holda bu vektorlar chiziqli bog‘liqdir. Aks holda barcha vektorlar chiziqli erkli bo‘ladi.

Ixtiyoriy $\vec{a}$ vektorni n ta chiziqli erkli $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \ldots, \overrightarrow{e_n}$ vektorlarning chiziqli kombinasiyasi ko‘rinishida ifodalash mumkin bo‘lsa, u holda shu n ta vektorlar fazoning ***bazisi*** deyiladi.

Bazisni hosil qiladigan vektorlar soni ***fazoning o‘lchami*** deb ataladi. Bazisga kiruvchi vektorlar ***bazis vektorlar*** deb ataladi.

1. To‘g‘ri chiziqning o‘lchami 1 ga teng, chunki to‘g‘ri chiziqda istalgan $\vec{e}$ vektor bazis hosil qiladi, qolgan vektorlar shu bazis vektor orqali ifodalanadi:

$$\vec{a} = \alpha \cdot \vec{e}, \quad \alpha \neq 0. \qquad \text{(1 o‘lchovli fazo)}$$

2. Tekislikning o‘lchami 2 teng, chunki tekislikda kollinear bo‘lmagan istalgan ikkita $\overrightarrow{e_1}$ va $\overrightarrow{e_2}$ vektor chiziqli erkli bo‘lib, bazis hosil qiladi, qolgan vektorlarni esa ular orqali ushbu ko‘rinishda ifodalash mumkin:

$$\vec{a} = \alpha \cdot \overrightarrow{e_1} + \beta \cdot \overrightarrow{e_2}, \quad \left(\alpha^2 + \beta^2 \neq 0\right). \qquad \text{(2 o‘lchovli fazo)}$$

3. Fazoda

$$\vec{a} = \alpha \cdot \vec{e}_1 + \beta \cdot \vec{e}_2 + \gamma \cdot \vec{e}_3, \quad \left(\alpha^2 + \beta^2 + \gamma^2 \neq 0\right). \quad \text{(3 o‘lchovli fazo)}$$

Vektorlarni bazis vektorlarning chiziqli kombinatsiyasi ko‘rinishida ifodalashga ***bazis bo‘yicha yoyish*** deyiladi.

Ba’zis vektorning uzunliklari har xil bo‘ladi Biz amaliyotda birlik uzunlikka ega bo‘lgan birlik vektorlardan tashkil topgan bazislar bilan shug‘ullanamiz. Bazis vektorlar bir biriga nisbatan har xil joylashgan (har xil burchak ostida) bo‘ladi. Biz koordinata o‘qlarida yotuvchi, yo‘nalishi koordinata o‘qlarining musbat yo‘nalishi bilan ustma-ust tushuvchi birlik uzunlikka ega bo‘lgan va o‘zaro perpendikulyar bo‘lgan $\vec{i}, \vec{j}, \vec{k}$ birlik bazis vektorlar bilan shug‘ullanamiz Bu vektorlar ***ortonormal vektorlar*** yoki ***ortlar*** deyiladi.

$\vec{a}=\overrightarrow{OA}$ vektorning o‘qlaridagi proeksiyalari mos ravishda a_x, a_y, a_z bilan belgilasak, uning birlik–bazis vektorlar(ortlar) orqali yozuvi

$$\boxed{\vec{a}\left(a_x, a_y, a_z\right)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}} \tag{2.5}$$

dan iborat bo‘ladi.

Bu ifodaga $\vec{a}$ vektorning $\vec{i}, \vec{j}, \vec{k}$ ba’zis vektorlar yoki koordinata o‘qlari bo‘yicha yoyilmasi deyiladi

Koordinata boshidan chiqqan vektorga ***radius vektor*** deyiladi.

2.4-misol

Agar $\vec{a}\{-1,4\}$, $\vec{b}\{2,-1\}$, $\vec{c}\{3, 5\}$ vektorlar koordinatalari bilan berilgan bo‘lsa quyidagi vektorlarning koordinatalari aniqlansin:

$$a)\ \frac{\vec{c}-2\vec{b}}{2},\quad b)\ \frac{\vec{a}+\vec{b}}{2}-\vec{c}.$$

► $$a)\frac{\vec{c}-2\vec{b}}{2}=\vec{d}\left\{\frac{3-2\cdot 2}{2},\ \frac{5-2\cdot(-1)}{2}\right\}=\vec{d}\{-0.5, 3.5\},$$

$$b)\frac{\vec{a}+\vec{b}}{2}-\vec{c}=\vec{s}\left\{\frac{-1+2}{2}-3,\ \frac{4+(-1)}{2}-5\right\}=\vec{s}\{-2.5, -3.5\}.$$ ◄

5-Auditoriya topshiriqlari

1. Agar $\vec{c}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AC}$ va $\vec{a}=\overrightarrow{BC}$ vektorlar ABC_uchburchakning tomonlari

bo‘lsa, u holda bu uchburchakning $\overrightarrow{AN}$, $\overrightarrow{BM}$ va $\overrightarrow{CP}$ medianalarini $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar orqali ifodalang.

2. $\vec{a}=\vec{i}+4\vec{j}-5\vec{k}$ va $\vec{b}=3\vec{i}-2\vec{j}+3\vec{k}$ vektorlar berilgan bo‘lsa,

$\vec{u}=2\vec{a}-3\vec{b}$ va $\vec{v}=-\frac{3}{4}\vec{a}+\frac{1}{2}\vec{b}$ vektorlarni aniqlang. Dekart koordinatalar sistemasida $\vec{u}$ va $\vec{v}$ vektorlarni yasang.

3. $\vec{a}(2;-3;4)$, $\vec{b}(5;3;-2)$ vektorlarga qurilgan parellelogramning diagonallarini ifodalovchi vektorlarni toping.

4. $ABCD$ to‘g‘rito‘rtburchakning tomonlari uzunliklari $AB=4$, $BC=3$ bo‘lib, A va B uchidan $\overrightarrow{AB}$ va $\overrightarrow{BC}$ vektorlar yo‘nalishida $\vec{a}$ va $\vec{b}$ birlik vektorlar qo‘yilgan.

1) $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{AC}$, $\overrightarrow{CB}$ va $\overrightarrow{DB}$ vektorlarni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang.

2) N va P nuqtalar mos ravishda BC va CD tomonlarning o‘rtasi bo‘lsa, $\overrightarrow{AN}$, $\overrightarrow{AP}$ va $\overrightarrow{PN}$ vektorlarni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang.

5. Radiusi $R=3$ bo‘lgan aylananing $90°$ li AB yoyini C nuqta orqali $AC:CB=3:2$ nisbatda AC va CB yoylarga bo‘lingan. Agar $\overrightarrow{OA}=\vec{a}$ va $\overrightarrow{OB}=\vec{b}$ bo‘lsa, $\overrightarrow{OC}$ vektorni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang.

6. To‘g‘riburchakli $ABCD$ trapetsiyaning asoslari $AD=4$, $BC=2$ bo‘lib, D buchagi $45°$ ga teng. $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$ va $\overrightarrow{AD}$ vektorlarni $\overrightarrow{CD}$ vektor bilan aniqlangan l o‘qga proyeksiyalarini toping.

7. Asosi ychburchakdan iborat bo‘lgan SABC piramidada $\overrightarrow{SA}=\vec{a}$, $\overrightarrow{SB}=\vec{b}$ va $\overrightarrow{SC}=\vec{c}$. Agar M nuqta ΔABC ning og‘irlik markazi bo‘lsa, $\overrightarrow{SM}$ vektorni bu vektorlar orqali ifoda qiling.

8. Uchburchakning $A(1;2;-1)$ uchi, $\overrightarrow{AB}=\{-2;1;4\}$ va $\overrightarrow{BC}=\{3;-1;4\}$ tomonlari yotgan vektorlar berilgan bo‘lsa, uchburchakning qolgan uchlari va $\overrightarrow{AC}$ vektorni toping.

5-Mustaqil yechish uchun testlar

1. Agar $A(2;0;4)$, $B(5;2;4)$, $C(-2;6;5)$, $D(-5;6;3)$ berilgan bo‘lsa, $\vec{a}=\overrightarrow{AB}+\overrightarrow{CD}$ vektorni toping

A) $\vec{a}(0;2;-2)$; B) $\vec{a}(5;14;10)$; C) $\vec{a}(4;7;-2)$; D) $\vec{a}(7;1;0)$

2. Agar $A(2;0;4)$, $B(5;2;4)$, $C(-2;6;5)$, $D(0;6;3)$ berilgan bo‘lsa, $\vec{a}=\overrightarrow{AB}-\overrightarrow{CD}$ vektorni toping

A) $\vec{a}(6;2;2)$; B) $\vec{a}(0;-2;-2)$; C) $\vec{a}(4;7;-2)$; D) $\vec{a}(7;1;0)$

3. $A(1,-2,3)$, $B(3,4,-6)$ berilgan bo‘lsa, $\overrightarrow{AB}$ vektor uzunligini toping

A) 7; B) 11; C) 13; D) 8

4. A(-4;0;2), B(-1;2;-2), C(6;-2;4) uchburchak uchlari koordinatalari bo‘lsa, mediana chizig‘ini ifodalovchi $\overline{BE}$ vektor koordinatalarini aniqlang

A) $\{2;-3;5\}$; B) $\{2;3;-5\}$; C) $\{-2;3;-5\}$; D) $\bar{a}(7;1;0)$

5. $\vec{a_1},\ \vec{a_2},\ \vec{a_3},\ldots,\vec{a_n}$ vektorlarning … bittasini qolganlarining chiziqli kombinatsiyasi shaklida ifodalash mumkin …, bu sistema chiziqli bog‘liq sistema bo‘ladi.

A) kamida, bo‘lsa; B) ixtiyoriy, bo‘lsa;

D) kamida, bo‘lmasa; D) ixtiyoriy, bo‘lmasa;

6. $\vec{a_1},\ \vec{a_2},\ \vec{a_3},\ldots,\vec{a_n}$ vektorlarning … bittasini qolganlarining chiziqli kombinatsiyasi shaklida ifodalash mumkin …, bu sistema chiziqli erkli sistema bo‘ladi

A) kamida, bo‘lsa; B) ixtiyoriy, bo‘lsa;

D) kamida, bo‘lmasa; D) ixtiyoriy, bo‘lmasa;

2.2 . Kesmani berilgan nisbatda bo‘lish. Vektorlarning skalyar ko‘paytmasi

2.2.1. Ikki nuqta orasidagi masofa. Kesmani berilgan nisbatda bo‘lish

a) ***Ikki nuqta orasidagi masofa*****.** Fazoda ikkita ixtiyoriy $A(x_1, y_1, z_1),\ B(x_2, y_2, z_2)$ nuqta berilgan bo‘lsin. Bu nuqtalar orasidagi masofani topish bilan shug‘ullanamiz. A, B nuqtalarni koordinatalar boshi O nuqta bilan tutashtirib, bu nuqtalarning radius-vektorlarini yasaymiz. Izlanayotgan masofani $d(A,B)$ bilan belgilaymiz, ya’ni $\left|\overrightarrow{AB}\right| = d(A,B)$. Bu holda $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\overrightarrow{OA}$ va $\overrightarrow{OB}$ radius-vektorlarning koordinatalari mos ravishda $\overrightarrow{OA} = \{x_1, y_1, z_1\}$, $\overrightarrow{OB} = \{x_2, y_2, z_2\}$ bo‘lgani uchun $\overrightarrow{AB}$ vektorning to‘g‘ri burchakli $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ bazisga nisbatan koordinatalari quyidagicha bo‘ladi:

$$\overrightarrow{AB} = \{x_2 - x_1\ , y_2 - y_1\ , z_2 - z_1\} \quad \Leftrightarrow \quad \overrightarrow{AB} = (x_2 - x_1)\cdot \vec{e}_1 + (y_2 - y_1)\cdot \vec{e}_2 + (z_2 - z_1)\cdot \vec{e}_3$$

bundan

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad (2.6)$$

ni hosil qilamiz. $\left|\overrightarrow{AB}\right|$ esa A va B nuqtalar orasidagi $d(A,B)$ masofa bo‘lgani uchun

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Agar tekislikda ikkita $A(x_1, y_1),\ B(x_2, y_2)$ nuqta berilgan bo‘lsa, ular orasidagi masofa

$$d(A,B)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \qquad (2.7)$$

formula bilan aniqlanadi.

b) ***Kesmani berilgan nisbatda bo‘lish.*** $A(x_1,y_1,z_1)$ va $B(x_2,y_2,z_2)$ nuqtalar fazodagi ikkita ixtiyoriy har xil nuqta bo‘lsin.

A va B nuqtalardan o‘tuvchi to‘g‘ri chiziqning ixtiyoriy nuqtasi C uchun

$$\overrightarrow{AC}=\lambda\cdot\overrightarrow{CB} \qquad (2.8)$$

tenglik o‘rinli. (2.8) da C nuqta $[AB]$ kesmaning ichki nuqtasi bo‘lsa, $\lambda>0$, C nuqta $[AB]$ kesmaning tashqi nuqtasi bo‘lsa, $\lambda<0$ bo‘ladi.

$[AB]$ kesmani berilgan nisbatda bo‘lish masalasi quyidagicha aniqlanadi: $A(x_1,y_1,z_1)$ va $B(x_2,y_2,z_2)$ nuqtalar va λ son berilgan. (A,B) to‘g‘ri chiziqda yotuvchi va (2.3) tenglikni qanoatlantiruvchi C nuqtaning koordinatalari topilsin.

Ravshanki, (2.8) dan $\left|\overrightarrow{AC}\right|=|\lambda|\cdot\left|\overrightarrow{CB}\right|$, bundan

$$|\lambda|=\frac{\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{CB}\right|} \qquad (2.9)$$

Shuning uchun biz qarayotgan masala $(AB)$to‘g‘ri chiziqda yotib, $[AB]$ kesmani $\lambda>0$ bo‘lganda ichkarida, $\lambda<0$ bo‘lganda tashqaridan $\lambda:1$ nisbatda bo‘luvchi C nuqtaning koordinatalarini topishdan iboratdir.

C nuqtaning Dekart koordinatalarini $\{x,y,z\}$ bilan belgilaylik. U holda (2.8) tenglikka ko‘ra ushbu tengliklar sistemasini hosil qilamiz:

$$x-x_1=\lambda\cdot(x_2-x),\ y-y_1=\lambda\cdot(y_2-y),\ z-z_1=\lambda\cdot(z_2-z)$$

$\lambda\neq-1$ ekanini hisobga olib, C nuqtaning koordinatalari uchun bundan quyidagi formulalarni hosil qilamiz:

$$x=\frac{x_1+\lambda\cdot x_2}{1+\lambda},\quad y=\frac{y_1+\lambda\cdot y_2}{1+\lambda},\quad z=\frac{z_1+\lambda\cdot z_2}{1+\lambda} \qquad (2.10)$$

Agar $\lambda=1$ bo‘lsa, (2.10) dan ushbu formulaga ega bo‘lamiz.

$$x=\frac{x_1+x_2}{2},\quad y=\frac{y_1+y_2}{2},\quad z=\frac{z_1+z_2}{2} \qquad (2.11)$$

Bu berilgan kesma o‘rtasining koordinatalarini beradi. Agar $[AB]$ kesma tekislikda berilgan bo‘lsa, uni λ nisbatda bo‘lish formulalari

$$x=\frac{x_1+\lambda\cdot x_2}{1+\lambda},\qquad y=\frac{y_1+\lambda\cdot y_2}{1+\lambda}$$

ko‘rinishda bo‘ladi.

2.5-misol

Oxirgi nuqtalari $A(-1;\,8;-3)$ va $B(9;-7;\,2)$ bo‘lgan kesma P_1, P_2, P_3 va P_4 nuqtalar bilan teng beshta bo‘lakka bo‘lingan bo‘lsa P_1 va P_3 nuqtalarning koordinatalarini toping.

► $AP_1 : P_1B = 1:4$ bo‘lgani uchun, $\lambda = \frac{1}{4}$. (6.5) formulaga ko‘ra, $P_1(x_1;\, y_1; z_1)$ koordinatalari

$$x_1 = \frac{4\cdot(-1)+9}{4+1} = 1,\;\; y_1 = \frac{4\cdot 8+(-7)}{4+1} = 5,\;\; z_1 = \frac{4\cdot(-3)+2}{4+1} = -2.$$

$AP_3 : P_3B = 3:2$ bo‘lgani uchun, $\lambda = \frac{3}{2}$. (2.10) formulaga ko‘ra, $P_3(x_3;\, y_3; z_3)$ koordinatalari

$$x_3 = \frac{2\cdot(-1)+3\cdot 9}{2+3} = 5,\;\; y_1 = \frac{2\cdot 8+3\cdot(-7)}{2+3} = -1,\;\; z_1 = \frac{2\cdot(-3)+3\cdot 2}{2+3} = 0.$$

Demak, $P_1(1;\,5;-2)$ va $P_3(5;-1;\,0)$.◄

2.2.2 Vektorlarni skalyar ko‘paytirish

Ikki $\vec{a}$ va $\vec{b}$ ***vektorning skalyar ko‘paytmasi*** deb, bu vektorlar uzunliklarini ular orasidagi burchak kosinusi bilan ko‘paytmasiga teng bo‘lgan songa aytiladi va $(\vec{a},\vec{b})$ yoki $\vec{a}\cdot\vec{b}$ bilan belgilanadi.

Ta’rifga ko‘ra,

$$\boxed{(\vec{a},\vec{b}) = \vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot cos\varphi} \qquad (2.12)$$

Skalyar ko‘paytma tushunchasining manbai mexanikadir. Haqiqatan, agar $\vec{a}$ ozod vektor qo‘yilgan nuqta $\vec{b}$ vektorning boshidan oxiriga siljuvchi kuchni tasvirlasa, bu kuch bajargan A ish ushbu tenglik bilan aniqlanadi:

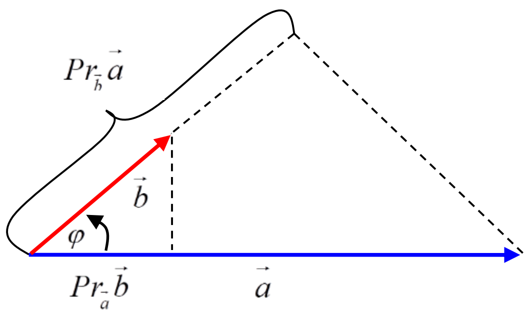
$$\boxed{A = |\vec{a}|\cdot|\vec{b}|\cdot cos\varphi}$$

Agar $\vec{a}\cdot\vec{b}$ ko‘paytmani $|\vec{a}|\cdot|\vec{b}|\cdot cos\varphi$ ko‘rinishda yozib, $|\vec{b}|\cdot cos\varphi = Pr_{\vec{a}}\vec{b}$ ekanini e’tiborga olsak, $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot Pr_{\vec{a}}\vec{b}$ ni hosil qilamiz.

$|\vec{a}|\cdot cos\varphi = Pr_{\vec{b}}\vec{a}$ ekanligini e’tiborga olsak, $\vec{a}\cdot\vec{b} = |\vec{b}|\cdot Pr_{\vec{b}}\vec{a}$ ni hosil qilamiz. Demak,

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot Pr_{\vec{a}}\vec{b}=|\vec{b}|\cdot Pr_{\vec{b}}\vec{a} \qquad (2.13)$$

formulalar o‘rinli. Boshqacha aytganda, ikki vektorning skalyar ko‘paytmasi ulardan birining uzunligi miqdori bilan ikkinchisining shu vektor yo‘nalishidagi proeksiysi ko‘paytmasiga teng.



2.9-chizma.

Agar ikki vektor orasidagi burchak $\frac{\pi}{2}$ ga teng bo‘lsa, ular ***ortogonal vektorlar*** deyiladi.

2.6-misol

Agar $\vec{a}, \vec{b}$ va $\vec{c}$ vektorlar koordinatalari bilan berilgan, ya’ni:

$$\vec{a}=\vec{i}-4\vec{j}+8\vec{k}\ ;\ \vec{b}=4\vec{i}+4\vec{j}-2\vec{k}\ ;\ \vec{c}=2\vec{i}+3\vec{j}+6\vec{k}\,.$$

bo‘lsa $\left(\vec{b}+\vec{c}\right)$ vektorning $\vec{a}$ vektordagi proyeksiyasini toping.

► $\vec{b}+\vec{c}=6\vec{i}+7\vec{j}+4\vec{k}=\vec{d}$, (2.13) dan $\mathrm{Pr}_{\vec{a}}(\vec{b}+\vec{c})=\mathrm{Pr}_{\vec{a}}\,\vec{d}=\dfrac{\vec{d}\cdot\vec{a}}{|\vec{a}|}$

formulani hosil qilamiz. $\mathrm{Pr}_{\vec{a}}(\vec{b}+\vec{c})=\dfrac{6\cdot 1+7\cdot(-4)+4\cdot 8}{\sqrt{1^2+(-4)^2+8^2}}=\dfrac{10}{9}$. ◄

Skalyar ko‘paytmaning bir qator eng sodda xossalarini keltiramiz:

2.1-Teorema. *Agar* $\vec{a}\cdot\vec{b}=0$ *bo‘lsa, u holda* $\vec{a}$ *va* $\vec{b}$ *vektorlar ortogonal bo‘ladi.*

2.2-Teorema. *Har qanday vektorning shu vektorning o'ziga skalyar ko'paytmasi bu vektorning uzunligi kvadratiga teng, ya'ni*

$$\vec{a}\cdot\vec{a}=\left|\vec{a}\right|^2 \tag{2.14}$$

2.3-Teorema. *Skalyar ko'paytma o'rin almashtirish qonuniga bo'ysunadi, ya'ni ixtiyoriy ikki* $\vec{a}$ *va* $\vec{b}$ *vektorlar uchun* $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ *munosabat o'rinli.*

2.4-Teorema. *Skalyar ko'paytma skalyar ko'paytuvchiga nisbatan gruppalash qonuniga bo'ysunadi, ya'ni* $\left(\lambda\cdot\vec{a}\right)\cdot\vec{b}=\vec{a}\cdot\left(\lambda\cdot\vec{b}\right)=\lambda\cdot\left(\vec{a}\cdot\vec{b}\right)$ *munosabatlar o'rinli.*

2.5-Teorema. *Skalyar ko'paytma qo'shishga nisbatan taqsimot qonuniga bo'ysunadi, ya'ni ixtiyoriy uchta* $\vec{a}$, $\vec{b}$ *va* $\vec{c}$ *vektorlar uchun ushbu tenglik o'rinli:*

$$\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$$

Skalyar ko'paytmaning Dekart koordinatalar sistemasidagi formulasi:

2.6-Teorema. *Dekart koordinatalar sistemasida* $\vec{a}=\{x_1,y_1,z_1\}$ *va* $\vec{b}=\{x_2,y_2,z_2\}$ *vektorlar berilgan bo'lsa, bu vektorlarning skalyar ko'paytmasi ularning mos koordinatalar ko'paytmalarining yig'indisiga teng, ya'ni*

$$\vec{a}\cdot\vec{b}=x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2 \tag{2.15}$$

Agar $\vec{a}=\{x_1,y_1\}$ *va* $\vec{b}=\{x_2,y_2\}$ *bo'lsa,*

$$\vec{a}\cdot\vec{b}=x_1\cdot x_2+y_1\cdot y_2 \tag{2.16}$$

bo'ladi.

$\vec{a}=\{x_1,y_1\}$ vektorning uzunligi koordinatalarda

$$\left|\vec{a}\right|=\sqrt{x^2+y^2} \tag{2.17}$$

$\vec{a}=\{x_1,y_1,z_1\}$ vektorning uzunligi esa

$$|\vec{a}|=\sqrt{x^2+y^2+z^2} \qquad (2.18)$$

formuladan topiladi.

Vektorlar orasidagi burchak koordinatalari orqali (Dekart sisitemasida), ya'ni skalyar ko'paytma ta'rifiga ko'ra osongina topiladi: $\vec{a}=\{x_1,y_1\}$ va $\vec{b}=\{x_2,y_2\}$ vektorlar uchun

$$\cos\left(\vec{a},\vec{b}\right)=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{x_1\cdot x_2+y_1\cdot y_2}{\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}} \qquad (2.19)$$

$\vec{a}=\{x_1,y_1,z_1\}$ va $\vec{b}=\{x_2,y_2,z_2\}$ vektorlar uchun

$$\cos\left(\vec{a},\vec{b}\right)=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\cdot\sqrt{x_2^2+y_2^2+z_2^2}} \qquad (2.20)$$

formulalar o'rinli.

2.7-misol

Ikki $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak $\varphi=\pi/4$ ga teng va $|\vec{a}|=\sqrt{2}$, $|\vec{b}|=3$ ekanligi ma'lum bo'lsa $\vec{c}=2\vec{a}+3\vec{b}$ vektorning uzunligini hisoblang.

► $\vec{c}$ vektorning uzunligini topish uchun vektorlarning skalyar ko'paytmasidan foydalanamiz. $\vec{a}\cdot\vec{a}=\vec{a}^2$ deb belgilab va $\vec{a}^2=|\vec{a}|^2$ ni e'tiborga olib, berilgan vektorning har ikki tomonini kvadratga ko'taramiz:

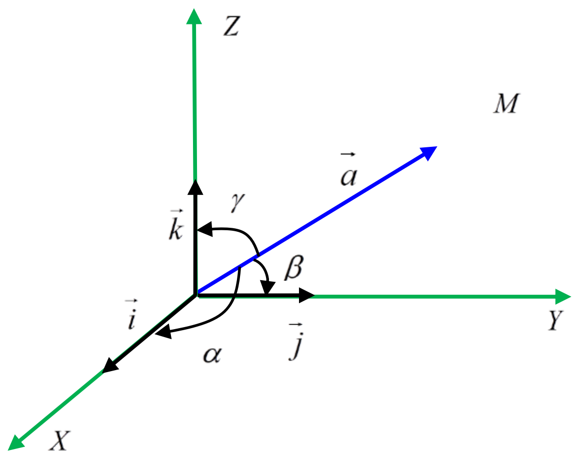
$$\vec{c}^2=\left(2\vec{a}+3\vec{b}\right)^2=4\vec{a}^2+12\vec{a}\cdot\vec{b}+9\vec{b}^2$$

berilganlarga asosan:

$$\vec{a}^2=|\vec{a}|^2=2;\ \vec{b}^2=|\vec{b}|^2=9;\ \vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot cos\varphi=\sqrt{2}\cdot3\cdot\frac{\sqrt{2}}{2}=3.$$

Demak, $\vec{c}^2=4\cdot2+12\cdot3+9\cdot9=125$ yoki $|\vec{c}|=\sqrt{125}=5\sqrt{5}$.◄

Odatda vektorning koordinata o'qlari bilan tashkil qilgan α,β,γ burchaklarning kosinuslari uning ***yo'naltiruvchi kosinuslari*** deyiladi(2.10-chizma).



2.10-chizma.

$\vec{a}=\{x,y,z\}$ vektorning yo‘naltiruvchi kosinuslari uning koordinatalari orqali quyidagicha aniqlanadi:

$$\cos\alpha=\frac{x}{\sqrt{x^2+y^2+z^2}},\ \cos\beta=\frac{y}{\sqrt{x^2+y^2+z^2}},\ \cos\gamma=\frac{z}{\sqrt{x^2+y^2+z^2}} \qquad (2.21)$$

Birlik vektorlarning koordinatalari uning yo‘naltiruvchi kosinuslaridan iborat, ya’ni agar $\left|\vec{a^0}\right|=1$, bo‘lsa,

$$\vec{a^0}=\{\cos\alpha,\cos\beta,\cos\gamma\} \qquad (2.22)$$

(2.21) ga ko‘ra,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \qquad (2.23)$$

formulani hosil qilish mumkin, ya’ni vektorning yo‘naltiruvchi kosinuslari kvadratlarining yig‘indisi birga teng.

6-Auditoriya topshiriqlari

1. $C(2;\,0;2)$ va $D(5;-2;\,0)$ nuqtalar yordamida teng uch qismga bo‘lingan kesmaning oxirlari A va B nuqtalarning koordinatalarini toping.

Javob: $A(-1;\,2;4)$, $B(8;-4;\,2)$

2. $\vec{a}$ va $\vec{b}$ vektorlar koordinatalari bilan berilgan:

$$\vec{a}=7\vec{i}+2\vec{j}+3\vec{k}\ ;\ \vec{b}=2\vec{i}-2\vec{j}+4\vec{k}$$

Bu vektorlarlarning skalyar ko‘paytmasini toping.

Javob: $\vec{a}\cdot\vec{b}=22$

3. Agar $|\vec{a}|=7\sqrt{2}$, $|\vec{b}|=4$ va $(\vec{a},\wedge\vec{b})=45^0$ bo‘lsa, $3\vec{a}+\alpha\vec{b}$ va $\vec{a}-2\vec{b}$ vektorlar α ning qanday qiymatlarida o‘zaro perpendikulyar bo‘ladi?

Javob: $\alpha=31{,}5$

4. Uchlari $A(-1;5;1)$, $B(1;1;-2)$ va $C(-3;3;2)$ nuqtalarda bo‘lgan uchburchak berilgan. AC tomonni davom ettirishdan hosil bo‘lgan tashqi burchakni aniqlang.

Javob: $\varphi=arccos(4/9)$

5. Uchlari $A(-2;3;1)$, $B(-2;-1;4)$ va $C(-2;-4;0)$ nuqtalarda bo‘lgan uchburchak berilgan. Bu uchburchakning C ichki burchagini hisoblang.

Javob: $\angle BCA=\pi/4$

6. Agar $A(-4;0;4)$, $B(-1;2;-2)$, $C(6;-2;4)$ chburchak uchlari koordinatalari bo‘lsa, $\overline{BA}$ vektorni mediana chizig‘ini ifodalovchi $\overline{BE}$ vektorga proyeksiyasini aniqlang.

Javob: $5\frac{1}{7}$

7. Rombning tomonlari umumiy uchdan chiquvchi ***a*** va ***b*** vektorlarda joylashgan. Uning diagonallari perpendikulyar ekanligini isbotlang.

8. Agar $\overrightarrow{OA}=\vec{a}$ va $\overrightarrow{OB}=\vec{b}$ vektorlar berilgan hamda $|\vec{a}|=2$, $|\vec{b}|=4$ v va $(\vec{a},\wedge\vec{b})=60^0$ bo‘lsa, *AOB* uchburchakning $\overrightarrow{OA}$ tomoni $\overrightarrow{OM}$ medianasi orasidagi φ burchak kosinusini toping.

Javob: $\cos\phi=\frac{2}{\sqrt{7}}$

6-Mustaqil yechish uchun testlar

1. Agar $A(-4;1)$, $B(2;4)$ nuqtalar uchun $AC:CB=2:1$ o‘rinli bo‘lsa, *C*-?

A) $C(-1;2)$; B) $C(-1;3)$; C) $C(0;3)$; D) $C(-2;2)$

2. Proyeksiyalar bilan berilgan ***a*** va ***b*** vektorlarning skalayar ko‘paytmasi qaysi javobda berilgan?

A) $|\vec{a}|Pr_{\vec{a}}\vec{b}$; B) $|\vec{b}|\text{Pr}_{\vec{a}}\,\vec{b}$; D) $|\vec{a}|Pr_{\vec{b}}\vec{b}$ E) $Pr_{\vec{a}}\vec{b}\cdot Pr_{\vec{b}}\vec{a}$

3. $\vec{a}(2;1;6)$ va $\vec{b}(1;-2;-1)$ vektorlarning skalyar ko‘paytmasini hisoblang.

A) 0; B) -4; C) -6; D) 4

4. $\vec{a}(4;-7;4)$, $\vec{b}(4,-2,-3)$ vektorlar berilgan. U holda $pr_{\vec{a}}\vec{b}$ ni toping

A) 2; B) 3; C) 4; D) 5

5. Agar $|\bar{a}|=3, |\bar{b}|=2, (\vec{a},\wedge\vec{b})=60^0$ berilgan bo‘lsa, $(\vec{a}+\vec{b})\cdot(2\vec{a}-3\vec{b})$ skalyar ko‘paytma topilsin.

A) 2; B) 3; C) 6; D) 4

6. $A(1,-2,3),\ B(3,4,-6), C(-3,1,3)$ berilgan bo‘lsa, $\overrightarrow{AB}$ va $\overrightarrow{AC}$ vektorlar orasidagi burchak kosinusini toping

A) $\frac{1}{2}$; B) $\frac{2}{11}$; C) 1; D) 0

2.3. Vektorlarning vektor va aralash ko‘paytmalari

2.3.1. Ikki vektorning vektor ko‘paytmasi

Vektor ko‘paytma ta’rifini kiritishdan avval, biz uchta o‘zaro nokomplanar vektor uchligining fazoda joylashishi bilan bog‘liq bo‘lgan zarur bir tushunchani kiritamiz. Shuni aytib o‘tamizki, keyingi punktlarda yuritiladigan mulohazalar faqat uch o‘lchovli fazoga doir bo‘ladi.

Agar komplanar $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar boshi umumiy nuqtaga keltirilgandan so‘ng $\vec{n}$ vektorning oxiridan (uchidan) qaraganda $\vec{a}$ vektordan $\vec{b}$ vektoga qarab π dan kichik burchakka burish soat miliga qarama-qarshi bo‘lsa, bu $\vec{a}$, $\vec{b}$, $\vec{c}$ uchlik ***o‘ng uchlik,*** aks holda ***chap uchlik*** deyiladi. Chap va o‘ng uchlikni tashkil etadigan uchlik ***tartiblangan uchlik*** deb yuritiladi.

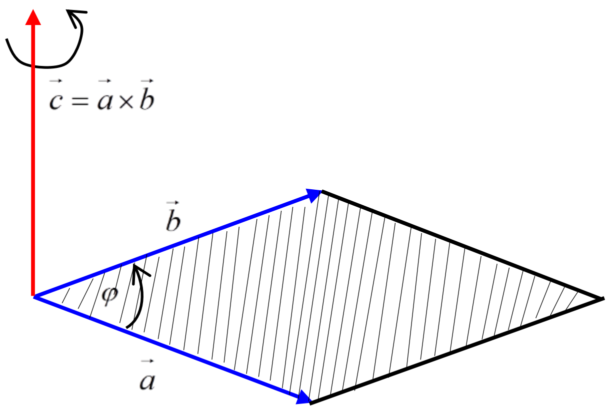
Biz o‘ng uchlikdan foydalanamiz.

$\vec{a}$ va $\vec{b}$ vektorlarning ***vektor ko‘paytmasi*** deb quyidagi shartlarni qanoatlantiradigan $\vec{c}$ vektorga aytiladi.

1) $\vec{c}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarga perpendikulyar (ortogonal)

2) $$\left|\vec{c}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin\left(\vec{a}, \vec{b}\right); \qquad (2.24)$$

3) $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning tartiblangan uchligi o‘ng uchlikni tashkil etadi (2.11-chizma).



2.11-chizma.

(Bu ta’rifda $\vec{a} \neq 0, \vec{b} \neq 0$ deb faraz qilinadi) $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasi $\vec{a} \times \vec{b}$ yoki $\left[\vec{a}, \vec{b}\right]$ ko‘rinishida yoziladi. Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo‘lmasa, u holda $\left|\vec{c}\right| = \left|\vec{a} \times \vec{b}\right|$ son $\vec{a}$ va $\vec{b}$ vektorlarga ysalgan parallelogrammning S yuziga teng bo‘ladi. Shunday qilib, $S=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot sin\left(\vec{a},\vec{b}\right)=\left|\vec{a}\times\vec{b}\right|$.

Agar $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo‘lsa, u holda $\vec{a}\times\vec{b}=0$, chunki $\varphi=\left(\vec{a},\vec{b}\right)=0$ yoki $\varphi=\pi$ da $sin\left(\vec{a},\vec{b}\right)=0$.

2.8-misol

Agar $\left|\vec{a}\right|=8$, $\left|\vec{b}\right|=15$, $\vec{a}\cdot\vec{b}=96$ bo‘lsa, $\left|\vec{a}\times\vec{b}\right|$ ni hisoblang.

► $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasi uzunligi, shu vektorlar uzunliklari ko‘paytmasi bilan ular orasidagi burchak sinusi ko‘paytmasiga teng. $\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko‘paytmasi ga asosan:

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot\left|\vec{b}\right|\cos\left(\vec{a},\wedge\vec{b}\right)$$

Bundan

$$\cos\left(\vec{a},\wedge\vec{b}\right)=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|}=\frac{96}{8\cdot15}=\frac{4}{5}$$

U holda

$$\sin\left(\vec{a},\wedge\vec{b}\right)=\sqrt{1-\cos^2\left(\vec{a},\wedge\vec{b}\right)}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

Demak,

$\left|\vec{a}\times\vec{b}\right|=|\vec{a}|\cdot\left|\vec{b}\right|\sin\left(\vec{a},\wedge\vec{b}\right)=8\cdot15\cdot\frac{3}{5}=72$ ◄

Vektor ko‘paytma quyidagi qonunlarga bo‘ysunadi:

1. Vektor ko‘paytmada ko‘paytuvchilar o‘rnini almashtirilsa, uning ishorasi o‘zgaradi, ya’ni

$$\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$$

2. Vektor ko‘paytma skalyar ko‘paytuvchiga nisbatan guruhlash qonuniga bo‘ysunadi, ya’ni

$$\left(\lambda\cdot\vec{a}\right)\times\vec{b}=\vec{a}\times\left(\lambda\cdot\vec{b}\right)=\lambda\cdot\left(\vec{a}\times\vec{b}\right)$$

2. $\vec{a}$ va $\vec{b}$ vektorlar yig‘indisi bilan $\vec{c}$ vektorning vektor ko‘paytmasi taqsimot qonuniga bo‘ysunadi, ya’ni

$$\left(\vec{a}+\vec{b}\right)\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$$

Endi vektor ko‘paytmaning koordinatalar orqali yozilishini ko‘rib o‘tamiz. Avvalo koordinata o‘qlarning $\vec{i},\vec{j},\vec{k}$ ortlar uchun quyidagi munosabatlar o‘rinli bo‘lishini eslatib o‘tamiz:

$$\begin{aligned} &\vec{i}\times\vec{i}=0\,, \quad \vec{i}\times\vec{j}=\vec{k}\,, \quad \vec{i}\times\vec{k}=-\vec{j}\,,\\ &\vec{j}\times\vec{j}=0, \quad \vec{j}\times\vec{k}=\vec{i}\,, \quad \vec{j}\times\vec{i}=-\vec{k}\,,\\ &\vec{k}\times\vec{k}=0\,, \quad \vec{k}\times\vec{i}=\vec{j}\,, \quad \vec{k}\times\vec{j}=-\vec{i}\,. \end{aligned} \tag{2.25}$$

Buni qisqacha quyidagi sxema orqali ham berish mumkin.

$$\left.\begin{aligned} &\overrightarrow{\vec{i}\times\vec{j}\times\vec{k}\times\vec{i}\times\vec{j}} \;\to +\\ &\overleftarrow{\vec{i}\times\vec{j}\times\vec{k}\times\vec{i}\times\vec{j}} \;\leftarrow - \end{aligned}\right\} \tag{2.26}$$

$\vec{a}$ va $\vec{b}$ vektorlar Dekart koordinatalar sistemasida mos ravishda $\vec{a}\{a_x;a_y;a_z\}$ va $\vec{b}\{b_x;b_y;b_z\}$ koordinatalarga ega bo‘lsin, ya’ni

$$\vec{a}\{a_x;a_y;a_z\}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}\,,\quad \vec{b}\{b_x;b_y;b_z\}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$$

$\vec{a}\times\vec{b}$ ko‘paytma uchun formulani (2.25) ni hamda vektor ko‘paytmaning xossalarini e’tiborga olib topamiz:

$$\vec{a}\times\vec{b}=a_xb_x\cdot(\vec{i}\times\vec{i})+a_yb_x\cdot(\vec{j}\times\vec{i})+a_zb_x\cdot(\vec{k}\times\vec{i})+\,+a_xb_y\cdot(\vec{i}\times\vec{j})+a_yb_y\cdot(\vec{j}\times\vec{j})+a_zb_y\cdot(\vec{k}\times\vec{j})+$$

$$+a_xb_z\cdot(\vec{i}\times\vec{k})+a_yb_z\cdot(\vec{j}\times\vec{k})+a_zb_z\cdot(\vec{k}\times\vec{k}).$$

yoki

$$\vec{a}\times\vec{b}=-a_yb_x\cdot\vec{k}+a_zb_x\cdot\vec{j}+a_xb_y\cdot\vec{k}-a_zb_y\cdot\vec{i}-a_xb_z\cdot\vec{j}+a_yb_z\cdot\vec{i}$$

Bir xil ortlarga ega bo‘lgan qo‘shiluvchilarni gruppalab yozamiz:

$$\vec{a}\times\vec{b}=(a_yb_z-a_zb_y)\cdot\vec{i}+(a_zb_x-a_xb_z)\cdot\vec{j}+(a_xb_y-a_yb_x)\cdot\vec{k}$$

Buni yana ushbu ko‘rinishda yozish mumkin:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \tag{2.27}$$

Bu formuladan quyidagi ikki tasdiq kelib chiqadi

1. (ikki vektorning kolleniar bo‘lish sharti). $\vec{a}$ va $\vec{b}$ vektorlar kolleniar bo‘lishi uchun $\vec{a}\times\vec{b}=0$ bo‘lishi zarur va etarli.
2. (uchburchak yuzining formulasi). $\vec{a}$ va $\vec{b}$ vektorlarga uchburchak yasalgan bo‘lsin, u holda bu uchburchakning yuzi:

$$S=\frac{1}{2}\left|\vec a\times\vec b\right|=\frac{1}{2}\,mod\begin{vmatrix}\vec i & \vec j & \vec k\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \qquad (2.28)$$

(2.24) va (2.28) formulalar vektor ko‘paytmaning geometrik tatbiqlari hisoblanadi.

2.9-misol

Berilgan $\vec a\{2;0;3\}=2\vec i+3\vec k$ va $\vec b\{0;-4;1\}=-4\vec j+\vec k$ vektorlardan tuzilgan parallelogramning yuzini hisoblang.

► (2.24) ga binoan, $\left|\vec a\times\vec b\right|=\left|\vec a\right|\cdot\left|\vec b\right|\sin\left(\vec a,\wedge\vec b\right)$. Vektor ko‘paytma xossalari va (2.25)ga asosan esa, $\vec a\times\vec b=\left(2\vec i+3\vec k\right)\times\left(-4\vec j+\vec k\right)=12\vec i-2\vec j-8\vec k$ bo‘ladi. Demak, parallelogramm yuzi

$S=\left|\vec a\times\vec b\right|=\sqrt{12^2+(-2)^2+(-8)^2}=\sqrt{212}=2\sqrt{53}\,(kv.b.)$ ◄

Quyida aralash ko‘paytmaning fizik tatbiqiga bir masala ko‘ramiz:

2.10-misol

Agar $N(1,2,3)$ nuqtaga $\vec F=\vec e_1-2\vec e_2+4\vec e_3$ kuch qo‘yilgan bo‘lsa bu kuchning $M(3,\ 2,-1)$ nuqtaga nisbatan momenti topilsin.

► $\overrightarrow{MN}$ vektorni aniqlaymiz: $\overrightarrow{MN}=\{1-3,2-2,\ 3-(-1)\}$, $\overrightarrow{MN}=\{-2,\ 0,4\}$. N nuqtaga qo‘yilgan $\vec F$ kuchning momenti

$$m_N\left(\vec F\right)=\overrightarrow{MN}\times\vec F=\begin{vmatrix}\vec e_1 & \vec e_2 & \vec e_3\\ \vec F_x & \vec F_y & \vec F_z\\ \left(\overrightarrow{MN}\right)_x & \left(\overrightarrow{MN}\right)_y & \left(\overrightarrow{MN}\right)_z\end{vmatrix}$$

formula bilan topiladi. Bu formulaga asosan quyidagini topamiz:

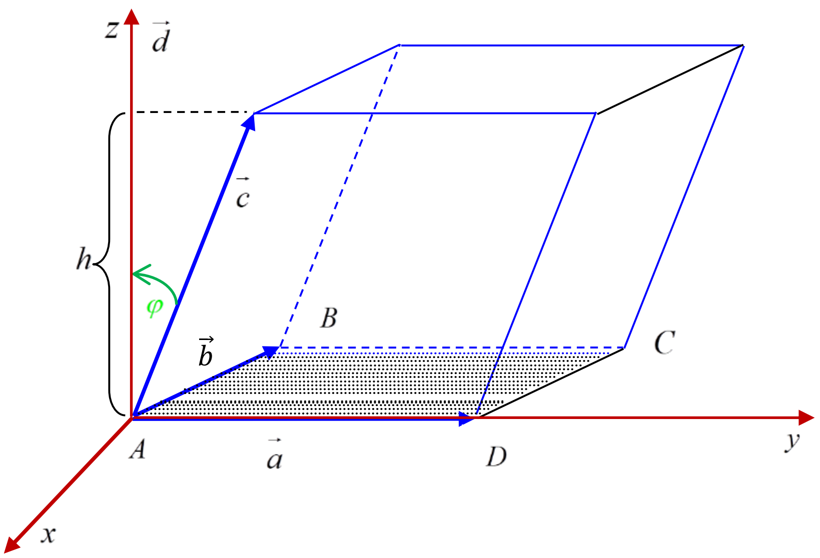
$$m_N\left(\vec F\right)=\overrightarrow{MN}\times\vec F=\begin{vmatrix}\vec e_1 & \vec e_2 & \vec e_3\\ 1 & -2 & 4\\ -2 & 0 & 4\end{vmatrix}=-8\vec e_1-12\vec e_2-4\vec e_3.$$ ◄

2.3.2. Vektorlarning aralash ko‘paytmasi

$\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar tartiblangan uchligining aralash ko‘paytmasi deb, $\vec{a}\times\vec{b}$ vektor bilan $\vec{c}$ vektorning skalyar ko‘paytmasiga teng songa aytiladi va $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ yoki $\left[\vec{a},\vec{b}\right]\cdot\vec{c}$ kabi belgilanadi

Aralash ko‘paytmaning moduli nuqtai nazardan ma’nosini tekshiramiz. $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar komplanar bo‘lmagan vektorlar bo‘lsin. $\vec{a}\times\vec{b}=\vec{d}$ deb belgilasak, $\vec{d}$ vektor moduli $\vec{a}$ va $\vec{b}$ vektorlardan yasalgan parallelogram yuziga teng (2.12-chizma) $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{d}\cdot\vec{c}$ bo‘lgani uchun skalyar ko‘paytma ta’rifiga ko‘ra

$$\vec{d}\cdot\vec{c}=\left|\vec{d}\right|\cdot Pr_{\vec{d}}\,\vec{c}$$



2.12-chizma.

Ammo $Pr_{\vec{d}}\,\vec{c}=h$ miqdorning moduli, ya’ni $|h|$ son $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarga yasalgan parallelepipedning balandligini anglatadi.

Aralash ko‘paytmaning absolyut qiymati shu $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarga yasalgan parallelepiped hajmiga teng, ya’ni

$$V_{parallelepiped}=\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|. \tag{2.29}$$

Aralash ko‘paytmaning ba’zi xossalarini keltiramiz:

1) Ko‘paytmada ikki vektorning o‘rinlari almashtirilsa, aralash ko‘paytmaning ishorasi teskariga almashadi, ya’ni quyidagi tengliklar o‘rinli:

$$(\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{b}\times\vec{a})\cdot\vec{c},$$
$$(\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{a}\times\vec{c})\cdot\vec{b},$$
$$(\vec{a}\times\vec{b})\cdot\vec{c}=-(\vec{c}\times\vec{b})\cdot\vec{a}.$$

2) $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning o‘rinlari “doiraviy shaklda” almashtirilsa, aralash ko‘paytma o‘z ishorasini o‘zgartirmaydi, ya’ni ushbu tengliklar o‘rinli:

$$(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}=(\vec{c}\times\vec{a})\cdot\vec{b}.$$

3) Agar $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlardan istalgan ikkitasi bir-biriga teng yoki parallel (kollinear) bo‘lsa, ularning aralash ko‘paytmasi nolga teng bo‘ladi.

4) Agar $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar o‘zaro komplanar vektorlar bo‘lsa, ularning aralash ko‘paytmasi nolga teng.

Endi aralash ko‘paytmani $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning koordinatalari orqali ifodalashga o‘tamiz. Dekart koordinatalar sistemasiga nisbatan $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlarning yoyilmasi berilgan bo‘lsin:

$$\vec{a}=\{x_1,y_1,z_1\}\Leftrightarrow\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$$
$$\vec{b}=\{x_2,y_2,z_2\}\Leftrightarrow\vec{b}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$$
$$\vec{c}=\{x_3,y_3,z_3\}\Leftrightarrow\vec{c}=x_3\vec{i}+y_3\vec{j}+z_3\vec{k}$$

U holda

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\x_1&y_1&z_1\\x_2&y_2&z_2\end{vmatrix}=\begin{vmatrix}y_1&z_1\\y_2&z_2\end{vmatrix}\vec{i}+\begin{vmatrix}z_1&x_1\\z_2&x_2\end{vmatrix}\vec{j}+\begin{vmatrix}x_1&y_1\\x_2&y_2\end{vmatrix}\vec{k}.$$

Shuning uchun

$$(\vec{a}\times\vec{b})\cdot\vec{c}=x_3\begin{vmatrix}y_1&z_1\\y_2&z_2\end{vmatrix}+y_3\begin{vmatrix}z_1&x_1\\z_2&x_2\end{vmatrix}+z_3\begin{vmatrix}x_1&y_1\\x_2&y_2\end{vmatrix}=\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}.$$

Shunday qilib, uch vektor aralash ko‘paytmasining uchinchi tartibli determinant orqali ifodasi ushbu ko‘rinishda bo‘ladi:

$$\Delta=(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}. \qquad (2.30)$$

Formuladan kelib chiqadigan ba’zi natijalarni keltiramiz.

2.1-Natija. $\vec{a}$, $\vec{b}$, $\vec{c}$ *vektorlar komplanar bo'lishi uchun*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0 \qquad (2.31)$$

tenglikning bajarilishi zarur va yetarli.

2.11-misol

Berilgan $\vec{a} = \{2;-1;3\}$, $\vec{b} = \{3;0;2\}$, $\vec{c} = \{1;-1;4\}$ vektorlarni chiziqli erklilikka tekshiring.

► Agar uch vektor komplanar bo'lsa, ular chiziqli bog'liq bo'ladi. Chunki tekislikda har qanday uch vektor chiziqli bog'liqdir. Berilgan vektorlarni komplanarlikka tekshirish kifoya.

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 0 - 2 - 9 - 0 + 4 + 12 = 5 \neq 0.$$

Demak, berilgan vektorlar chiziqli erkli ekan.◄

2.2-Natija. *Agar* $\vec{a} = \{x_1, y_1, z_1\}$, $\vec{b} = \{x_2, y_2, z_2\}$, $\vec{c} = \{x_3, y_3, z_3\}$ *bo'lib, bu vektorlar komplanar bo'lmasa, u holda ularga qurilgan parallelepiped hajmi* $V = \pm\Delta$ *formula o'rinli. Unda musbat ishora* $\vec{a}$, $\vec{b}$, $\vec{c}$ *o'ng uchlikni, manfiy ishora shu* $\vec{a}$, $\vec{b}$, $\vec{c}$ *lar chap uchlikni tashkil etganda olinadi.*

2.12-misol

Berilgan $\vec{a} = 4\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{b} = -2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} + x\vec{k}$ vektorlardan tuzilgan piramidaning hajmi 8 ga teng bo'lsa, x ni toping.

► (2.30) ko'ra, aralash ko'paytmani topamiz.

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 4 & 3 & -2 \\ -2 & -1 & 2 \\ 2 & 1 & x \end{vmatrix} = -4x + 12 + 4 - 4 - 8 + 6x = 2x + 4.$$

$V_{pir.}=\frac{1}{6}V_{par-d}$ bo‘lgani uchun va (2.29) dan, $V_{pir.}=\frac{1}{6}\left|2x+4\right|=8$, $\left|2x+4\right|=48$.

U holda, $x_1=-26$ va $x_2=22$.◀

2.3.3. Vektorlar algebrasinig mexanik masalalarga tatbiqi

2.13-masala

Quyidagi $\vec{F}=\{6,-2,1\}$ kuchning $A(3,4,-2)$ nuqtadan togri chiziq bo`ylab $B(4,-2,-3)$ ga siljishida bajargan ishni hisoblang.

▶ $\overrightarrow{AB}=\{x,y,z\}$ vektorning koordinatalarini aniqlaymiz . buning uchun $x=x_B-x_A,\ y=y_B-y_A,\ z=z_B-z_A$ formulalarga A va B nuqtalarning koordinatalarini qo‘yib

$$x=4-3=1,\quad y=-2-4=-6,\quad z=-3+2=-1$$

larni topamiz.

Demak, $\overrightarrow{AB}=\{1,-6,-1\}$. $\vec{F}$ kuch ta’siri ostida bajarilgan ish o‘tilgan $\overrightarrow{AB}$ yo‘l bilan $\vec{F}$ kuchning skalyar ko‘paytmasiga tengligidan, ya’ni ish $\vec{F}\cdot\overrightarrow{AB}$ ga teng. Shuni hisoblaymiz:

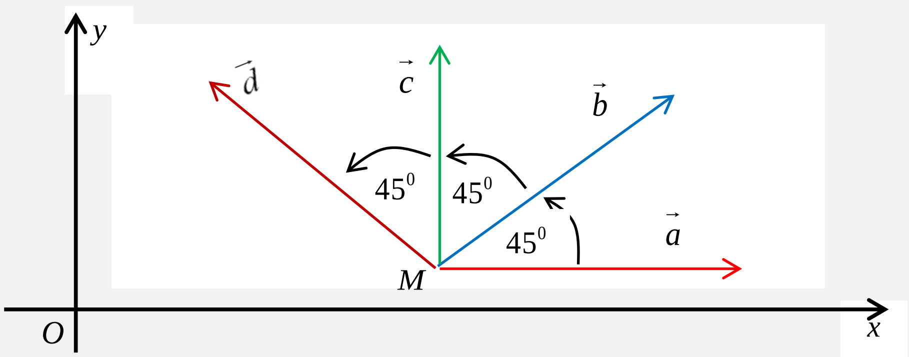
$$\vec{F}\cdot\overrightarrow{AB}=\left(6\vec{e}_1-2\vec{e}_2+\vec{e}_3\right)\cdot\left(\vec{e}_1-6\vec{e}_2-\vec{e}_3\right)=6\cdot1+(-2)\cdot(-6)+1\cdot(-1)=6+12-1=17.$$

Demak, $A=\vec{F}\cdot\overrightarrow{AB}=17$. ◀

2.14-masala

To‘rtta komplanar kuchlar bitta O nuqtaga qo‘yilgan. Bu kuchlarning har birining kattaligi 10 kg va o‘zaro qo‘shni bo‘lgan har ikki ketma-ket kelgan vektorlar orasidagi burchak 45° bo‘lsa, bu kuchlarning teng ta`sir etuvchisi topilsin.

▶



Javob: $R=\left|\overrightarrow{MN}\right|=\sqrt{\left(\vec{a}+\vec{b}+\vec{c}+\vec{d}\right)^2}=10\sqrt{4+2\sqrt{2}}\approx25,3kg.$ ◀

2.15-masala

Asosi ychburchakdan iborat bo‘lgan SABC piramidada $\overrightarrow{SA}=\vec{a}$, $\overrightarrow{SB}=\vec{b}$ va $\overrightarrow{SC}=\vec{c}$. Agar M nuqta ΔABC ning og‘irlik markazi bo‘lsa, $\overrightarrow{SM}$ vektorni bu vektorlar orqali ifoda qiling.

Javob: $\overrightarrow{SM}=\frac{1}{3}\left(\vec{a}+\vec{b}+\vec{c}\right)$.

2.16-masala

Harakatlanayotgan moddiy nuqta ko‘chishining koordinata o‘qlaridagi proyeksiyalari $s_x=2$ metr, $s_y=1$ metr, koordinata o‘qlariga ta`sir etuvchi $\vec{F}$ kuchning proyeksiyalari esa $F_x=5$ kG, $F_y=4$ kG ga teng. $\vec{F}$ kuchning bajargan A ishi va $\vec{F}$ kuch bilan s ko‘chish(siljish) orasidagi burchak topilsin.

Javob: $A=8\frac{kg}{m}$, $cos\theta=\frac{4\sqrt{2}}{15}$

2.17-masala

Berilgan $N(1,2,3)$ nuqtaga $\vec{F}=\vec{e}_1-2\vec{e}_2+4\vec{e}_3$ kuch qo‘yilgan. Bu kuchning $M(3,\ 2,-1)$ nuqtaga nisbatan momenti topilsin.

► $\overrightarrow{MN}$ vektorni aniqlaymiz: $\overrightarrow{MN}=\{1-3, 2-2,\ 3-(-1)\}$, $\overrightarrow{MN}=\{-2,\ 0, 4\}$. N nuqtaga qo‘yilgan $\vec{F}$ kuchning momenti

$$m_N\left(\vec{F}\right)=\overrightarrow{MN}\times\vec{F}=\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \vec{F}_x & \vec{F}_y & \vec{F}_z \\ \left(\overrightarrow{MN}\right)_x & \left(\overrightarrow{MN}\right)_y & \left(\overrightarrow{MN}\right)_z \end{vmatrix}$$

formula bilan topiladi. Bu formulaga asosan quyidagini topamiz:

$$m_N\left(\vec{F}\right)=\overrightarrow{MN}\times\vec{F}=\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -2 & 4 \\ -2 & 0 & 4 \end{vmatrix}=-8\vec{e}_1-12\vec{e}_2-4\vec{e}_3 .$$ ◄

2.18-masala

Berilgan $A(-2,1,-3)$ nuqtaga $\vec{F}=\{3,\ 4,\ -2\}$ kuch qo‘yilgan. Bu kuchning koordinatalar boshiga nisbatan momenti va koordinata o‘qlari bilan hosil qilgan burchaklarini toping.

Javob: $m_0\left(\vec{F}\right)=\overrightarrow{AO}\times\vec{F}==-10\vec{e}_1+13\vec{e}_2+11\vec{e}_3$;

$$\cos\alpha = -\frac{10}{\sqrt{390}};\ \cos\beta = \frac{13}{\sqrt{390}};\ \cos\gamma = \frac{11}{\sqrt{390}}.$$

7-Auditoriya topshiriqlari

1. Uchlari $A(1;2;0)$, $B(3;0;-3)$, $C(5;2;6)$ nuqtalarda bo‘lgan uchburchak yuzini hisoblang.

Javob: $S_{\Delta ABC} = 0.5\cdot\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right| = 14$ kv. birlik.

2. $\overrightarrow{AB} = -3\vec{i} - 22\vec{j} + 6\vec{k}$; $\overrightarrow{BC} = -2\vec{i} + 4\vec{j} + 4\vec{k}$ vektorlar ΔABC ning tomonlari. $\overrightarrow{AD}$ balandlikning uzunligini hisoblang.

Javob: $\left|\overrightarrow{AD}\right| = \frac{2S_{\Delta ABC}}{\left|\overrightarrow{BC}\right|} = \frac{8\sqrt{5}}{3}$.

3. $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar koordinatalari bilan berilgan:

$$\vec{a} = 2\vec{i} - \vec{j} - \vec{k};\ \vec{b} = \vec{i} + 3\vec{j} - \vec{k};\vec{c} = 3\vec{i} - 4\vec{j} + 7\vec{k}.$$

Bu vektorlarning aralash ko‘paytmasini toping.

Javob: $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = 33$.

4. Ushbu $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$; $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$; $\vec{c} = 3\vec{i} - 4\vec{j} + 7\vec{k}$ vektorlarning komplanarligini isbotlang.

5. Uchlari $A(1;2;3)$, $B(2;4;1)$, $C(7;6;3)$ va $D(2;-3;-1)$ nuqtalarda bo‘lgan piramida berilgan. Shu piramida uchun quyidagilarni: a) AB, AC, AD qirralarning uzunliklarini; b) ABC yoqning yuzini; d) piramidaning hajmini toping.

Javob:

$a)$ $\left|\overrightarrow{AB}\right| = \sqrt{17}$, $\left|\overrightarrow{AC}\right| = 2\sqrt{13}$, $\left|\overrightarrow{AD}\right| = 5\sqrt{2}$;

$b) S_{\Delta ABC} = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = 14$ kv. birlik;

$d)$ $V_{pir} = 30$ kub birlik.

6. Agar tekislikda $\vec{a}$ va $\vec{b}$ vektorlar nokollinear bo‘lsa α ning qanday qiymatida $\vec{p} = \alpha\vec{a} + 2\vec{b}$ va $\vec{q} = 3\vec{a} - \vec{b}$ vektorlar kollinear bo‘ladi.

Javob: $\alpha = -6$

7. Agar $|\vec{a}| = 3$, $\left|\vec{b}\right| = 4$ va $(\vec{a}\,\hat{}\,\vec{b}) = \frac{\pi}{3}$ bo‘lsa $\vec{p} = 3\vec{a} - 5\vec{b}$, $\vec{q} = \vec{a} + 7\vec{b}$ vektorlardan tuzilgan uchburchak yuzini toping.

Javob: $S_{\Delta} = \frac{1}{2}|\vec{p}\times\vec{q}| = 78\sqrt{3}$

8. $C(-1;\,4;-2)$ nuqtaga qo‘yilgan uchta $\vec{F}=\{2;-1;-2\}$, $\vec{Q}=\{3;\,2;-1\}$ va $\vec{P}=\{-4;\,1;\,3\}$ kuchlar berilgan. Bu kuchlar teng ta’sir etuvchisining $A(2;\,3;-1)$ nuqtaga nisbatan momentining yo‘naltiruvchi kosinuslarini toping.

Javob: $cos\alpha=\frac{1}{\sqrt{66}}$, $\cos\beta=-\frac{4}{\sqrt{66}}$, $\cos\gamma=-\frac{7}{\sqrt{66}}$

9. Berilgan $\vec{a}=2\vec{i}+3\vec{j}-\vec{k}$, $\vec{b}=-\vec{i}+3\vec{j}+2\vec{k}$, $\vec{c}=3\vec{i}+\vec{j}$ vektorlar uchun $(\vec{x},\vec{a})=5$, $(\vec{x},\vec{b})=-3$, $(\vec{x},\vec{c})=1$ shartlarni qanoatlantiruvchi $\vec{x}$ vektorni toping.

10. Berilgan $\vec{a}=4\vec{i}+3\vec{j}-2\vec{k}$, $\vec{b}=-2\vec{i}-\vec{j}+2\vec{k}$, $\vec{c}=2\vec{i}+\vec{j}+x\vec{k}$ vektorlardan tuzilgan piramidaning hajmi 8 ga teng bo‘lsa x ni toping.

11. Agar $A(x,2,1)$, $B(1;2;4)$, va $C(-1;3;1)$ uchburchakning uchlari hamda B uchidagi burchagi 60^0 bo‘lsa x ni toping.

12. Agar $|\vec{a}|=5$, $|\vec{b}|=8$, va $(\vec{a}\,\hat{}\,\vec{b})=\frac{\pi}{6}$ bo‘lsa $|(2\vec{a}+7\vec{b})\times(5\vec{a}-\vec{b})|$ ni hisoblang

13. Agar $\vec{a}=\{x;-1;3\}$, $\vec{b}=\{3;x;2\}$, $\vec{c}=\{1;-1;4\}$ vektorlar komplanar bo‘lsa x ning qiymatini toping.

14. Uchlari $A(2;3;-1), B(1;4;2), C(-2;2;0), D(-1;3;4)$ nuqtalarda bo‘lgan piramidaning B uchidan tushirilgan balandligi hisoblansin.

15. Uchburchakning $A(1;2;-1)$ uchi, $\overrightarrow{AB}=\{-2;1;4\}$ va $\overrightarrow{BC}=\{3;-1;4\}$ tomonlari yotgan vektorlar berilgan bo‘lsa uchburchakning qolgan uchlari va $\overrightarrow{AC}$ vektorni toping.

7-Mustaqil yechish uchun testlar

1. Berilgan $\vec{a}=2\vec{i}+3\vec{j}-\vec{k}$, $\vec{b}=-\vec{i}+3\vec{j}+2\vec{k}$ vektorlarning vektor ko‘paytmasi $\vec{a}\times\vec{b}$ ni toping

A) $\{9;-3;9\}$; B) $\{6;3;-9\}$; C) $\{-9;3;-5\}$; D) $\{9;3;6\}$

2. Agar $|\vec{a}|=5$, $|\vec{b}|=8$ va $(\vec{a}\,\hat{}\,\vec{b})=\frac{\pi}{6}$ bo‘lsa, $\left|\left(2\vec{a}+3\vec{b}\right)\times\left(\vec{a}-2\vec{b}\right)\right|$ ni hisoblang

A) $75\sqrt{3}$; B) 105; C) 140; D) $60\sqrt{3}$

3. Agar $\bar{a}(1;\,2;-3)$, $\bar{b}(-2;1;-1)$ bo‘lsa, $(\bar{a}-2\bar{b})\times(2\bar{a}-\bar{b})$ vektor ko‘paytmani toping

A) $(3;-21;15)$; B) $(-5;-35;-25)$; C) $(3;21;15)$; D) $(5;-35;25)$

4. $\vec{a}(2;1;6)$, $\vec{b}(1;-2;-1)$ va $\vec{c}(2;-4;-2)$ vektorlarning aralash ko'paytmasini hisoblang

A) 0; B) -4; C) 6; D) 4

5. Agar $\vec{a}=\{x;-1;2\}$, $\vec{b}=\{1;x;-3\}$, $\vec{c}=\{1;-3;5\}$ vektorlar chiziqli bog'liq bo'lsa, x ning qiymatini toping

A) $x_1=-2,\ x_2=0{,}2$ B) $x_1=2,\ x_2=0{,}2$

C) $x_1=-2,\ x_2=-0{,}2$ D) $x_1=2,\ x_2=-0{,}2$

2-Shaxsiy uy topshiriqlari

1. Agar $|\vec{a}|=13,\ |\vec{b}|=19$, va $|\vec{a}+\vec{b}|=24$ bo'lsa, $|\vec{a}-\vec{b}|$ ni hisoblang.

Javob: $|\vec{a}-\vec{b}|=22$.

2. Agar ΔABC da $\overrightarrow{AB}=\vec{m}$, $\overrightarrow{AC}=\vec{n}$ ekanligi ma'lum bo'lsa, quyidagi vektorlarni yasang:

1) $\dfrac{\vec{m}+\vec{n}}{2}$, 2) $\dfrac{\vec{m}-\vec{n}}{2}$, 3) $\dfrac{\vec{n}-\vec{m}}{2}$, 4) $-\dfrac{\vec{m}+\vec{n}}{2}$

3. $\vec{a}$, $\vec{b}$ vektorlarga yasalgan parallelogrammdan foydalanib quyidagi ayniyatlarning to'g'riligini chizmada tekshiring:

1) $(\vec{a}+\vec{b})+(\vec{a}-\vec{b})=2\vec{a}$, 2) $\vec{a}+(\vec{b}-\vec{a})=\vec{b}$ 3) $\dfrac{\vec{a}-\vec{b}}{2}+\vec{b}=\dfrac{\vec{a}+\vec{b}}{2}$;

4. Teng yonli $ABCD$ trapesiyaning pastki asosi $\overrightarrow{AB}=\vec{a}$, yon tomoni $\overrightarrow{AD}=\vec{b}$ va ular orasidagi burchagi $\alpha=\dfrac{\pi}{3}$ berilgan. Trapesiyaning qolgan tomonlari va diagonallarini tashkil etuvchi vektorlarni $\vec{a}$ va $\vec{b}$ vektorlar orqali ifodalang

Javob: $\overrightarrow{BC}=-\dfrac{b}{a}\vec{a}+\vec{b}$; $\overrightarrow{CD}=\dfrac{b-a}{a}\vec{a}$; $\overrightarrow{AC}=\dfrac{a-b}{a}\vec{a}+\vec{b}$; $\overrightarrow{BD}=-\vec{a}+\vec{b}$; bu yerda a, b mos ravshda $\vec{a}$, $\vec{b}$ vektorlarning uzunliklarini bildiradi.

5. Koordinatalar boshidan $M(12;-3;4)$ nuqtagacha bo'lgan masofani hisoblang. $\vec{r}(0;2;-3)$ radius vektorning ortlar bo'yicha yoyilmasini yozing va modulini hisoblang.

Javob: $\vec{r}=2\vec{j}-3\vec{k}$, $|\vec{r}|=\sqrt{13}$.

6. $M(-2;\ 1;\ 3)$ va $N(0;-1;\ 2)$ nuqtalar orasidagi masofani toping.

Javob: 3.

7. $\vec{a}\{3,\ 2,\ 7\}$ va $\vec{b}\{4,\ 1,\ -5\}$ vektorlarning yig'indisi va ayirmasini ort vektorlar yordamida yozing.

Javob: $\vec{a}+\vec{b}=7\vec{i}+3\vec{j}+2\vec{k}$

$\vec{a}-\vec{b}=-\vec{i}+\vec{j}+12\vec{k}$

8.Uchlari $A(5;\,2;\,6)$, $B(6;\,4;\,4)$, $C(4;\,3;\,2)$ va $D(3;\,1;\,4)$ nuqtalarda bo‘lgan to‘rtburchakning kvadrat ekanligini tekshiring.

9. α va β larning qanday qiymatlarida $\vec{a}=2\vec{i}+\alpha\vec{j}+\vec{k}$ va $\vec{b}=3\vec{i}-6\vec{j}+\beta\vec{k}$ vektorlar kollinear bo‘ladi?

Javob: $\alpha=-4;\quad \beta=\dfrac{3}{2}$.

10.Uchlari $A(2;\,1;\,-4)$, $B(1;\,3;\,5)$, $C(7;\,2;\,3)$ va $D(8;\,0;\,-6)$ nuqtalarda bo‘lgan to‘rtburchakning parallelogramm ekanligini isbotlang va parallelogramm tomonlari uzunliklarini toping.

Javob. $\overrightarrow{AB}=\overrightarrow{DC}$ bo‘lgani uchun parallelogramdir.

$\left|\overrightarrow{AB}\right|=\sqrt{86}\approx 9,3;\quad \left|\overrightarrow{DC}\right|=\sqrt{41}\approx 6,4.$

11. Uchlari $A(-1;\,2;\,3)$, $B(2;\,-1;\,1)$, $C(1;\,-3;\,-1)$ va $D(-5;\,3;\,3)$ nuqtalarda bo‘lgan to‘rtburchakning trapesiya ekanligini isbotlang

Ko‘rsatma . $\overrightarrow{AB}$ va $\overrightarrow{CD}$ vektorlarning kollinear. $\overrightarrow{AD}$ va $\overrightarrow{BC}$ vektorlarning kollinear emasligini tekshirish zarur.

12.Boshlang‘ich nuqtasi $M(-1;\,3;\,2)$ va oxirgi nuqtasi $N(0;\,1;\,4)$ bo‘lgan $\overrightarrow{MN}$ vektorning yo‘naltiruvchi kosinuslarini toping.

Javob. $\cos\alpha=\dfrac{1}{3};\quad \cos\alpha=-\dfrac{2}{3};\quad \cos\alpha=\dfrac{2}{3}$.

13. $\vec{a}$ vektor Ox o‘qi bilan $\alpha=45^\circ$, Oy o‘qi bilan $\beta=60^\circ$ burchak hosil qiladi. Agar $\left|\vec{a}\right|=6$ bo‘lsa, uning koordinatalari topilsin.

Javob. $\vec{a}\left\{3\sqrt{2};\,3;\,3\right\}$.

14. $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak $\varphi=\dfrac{\pi}{3}$ ga teng. $\left|\vec{a}\right|=4;$ $\left|\vec{b}\right|=3$ bo‘lsa, $\vec{c}=3\vec{a}+2\vec{b}$ vektorning uzunligini toping.

Javob. $\left|\vec{c}\right|=2\sqrt{63}.$.

15. $\vec{a}$ va $\vec{b}$ vektorlar koordinatalari bilan berilgan: $\vec{a}=7\vec{i}+2\vec{j}+3\vec{k};$ $\vec{b}=2\vec{i}+2\vec{j}+4\vec{k}$. Bu vektorlarning skalyar ko‘paytmasini toping.

16.Uchlari $A(-1;\,5;\,1)$, $B(1;\,1;\,-2)$, $C(-3;\,3;\,2)$ nuqtalarda bo‘lgan uchburchak berilgan. AC tomonni davom ettirishdan hosil bo‘lgan tashqi burchakni aniqlang.

Javob. $\varphi=arccos\left(\dfrac{4}{9}\right)$.

III BOB ANALITIK GEOMETRIYA ASOSLARI

3.1. Tekislikda to‘g‘ri chiziq tenglamalari

To‘g‘ri burchakli Dekart koordinatalar sistemasi Oxy tekislikda har qanday to‘g‘ri chiziq x va y ga nisbatan birinchi darajali

$$Ax+By+C=0 \tag{3.1}$$

tenglama bilan beriladi, bu yerda A, B, C –haqiqiy sonlar, $A^2+B^2>0$ va har qanday (3.1) tenglama to‘g‘ri chiziqni aniqlaydi.

(3.1) tenglama *to‘g‘ri chiziqning* ***umumiy tenglamasi*** deyiladi. To‘g‘ri chiziqqa perpendikulyar $\vec{n}=\{A;B\}$ vektor to‘g‘ri chiziqning ***normal vektori*** deyiladi.

Agar $B\neq 0$ bo‘lsa, (3.1)ni y ga nisbatan yechib,

$$y=kx+b\ (k=tg\alpha) \tag{3.2}$$

ko‘rinishda ifodalash mumkin. (3.2) tenglama *to‘g‘ri chiziqning* ***burchak koeffitsientli*** *tenglamasi* deyiladi. α - to‘g‘ri chiziq bilan Ox o‘qining musbat yo‘nalishi orasidagi burchak, k - to‘g‘ri chiziqning burchak koeffitsienti, b - to‘g‘ri chiziqning Oy o‘qidan kesadigan kesmasi.

To‘g‘ri chiziqning yana quyidagi tenglamalari mavjud:

1. $M_0(x_0;y_0)$ nuqtadan o‘tuvchi va $\vec{n}=\{A;B\}$normal vektorga ega to‘g‘ri chiziq tenglamasi:

$$A(x-x_0)+B(y-y_0)=0 \tag{3.3}$$

2. $M_0(x_0;y_0)$ nuqtadan o‘tuvchi va k - burchak koeffitsientli to‘g‘ri chiziq tenglamasi:

$$y-y_0=k(x-x_0) \tag{3.4}$$

3. To‘g‘ri chiziqning parametrik tenglamasi:

$$\begin{cases} x=x_0+mt \\ y=y_0+nt \end{cases} \tag{3.5}$$

Bu yerda, $\vec{s}(m;n)$ - to‘g‘ri chiziqning ***yo‘naltiruvchi vektori.***

4. To‘g‘ri chiziqning kanonik tenglamasi:

$$\frac{x-x_0}{m}=\frac{y-y_0}{n} \tag{3.6}$$

5. To‘g‘ri chiziqning “kesma”lardagi tenglamasi:

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (3.7)$$

Bu yerda a va b to‘g‘ri chiziqning mos ravishda Ox va Oy koordinata o‘qlaridan ajratgan kesmalari.

6. Ikki $M_1(x_1;y_1)$ va $M_2(x_2;y_2)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasi:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad (3.8)$$

3.1-misol

Quyidagi $2x-3y+6=0$ tenglama bilan berilgan to‘g‘ri chiziqning burchak koeffitsientini va o‘qlardan ajratgan kesmalarini aniqlang.

► $2x\ -3y+6=0$ ni y ga nisbatan yechamiz: $y=\frac{2}{3}x+2$, $k=\frac{2}{3}$.

Berilgan tenglamani quyidagicha almashtiramiz:

$$2x\ -3y=-6|:(-6)$$

$$\frac{x}{-3}+\frac{y}{2}=1$$

Demak, $a=-3$, $b=2$, $k=\frac{2}{3}$.◄

3.2-misol

ABC uchburchakning uchlari $A(-3;1)$, $B(5;-3)$ va $C(7;5)$ berilgan.

CD balandlik va AE medianalari kesishgan nuqtasini toping.

► CD balandlik AB tomonga perpendikulyar bo‘lishi kerak. Avval (3.8) ni qo‘llab, AB tomon tenglamasini tuzamiz.

$$\frac{x+3}{5+3}=\frac{y-1}{-3-1},\ y=-\frac{1}{2}x-\frac{1}{2},\ k_1=-\frac{1}{2}.$$

CD balandlik tenglamasida $k_2=2$, u holda (1.2)ga ko‘ra, $y-5=2(x-7)$ yoki $y=2x-9$.

E nuqta $B(5;-3)$ va $C(7;5)$ nuqtalarning o‘rtasi bo‘lgani uchun

$$E\left(\frac{5+7}{2};\frac{-3+5}{2}\right)=E(6;1).$$

$A(-3;1)$ va $E(6;1)$ nuqtalardan o‘tuvchi AE mediana tenglamasi: $y=1$.

CD balandlik va *AE* medianalar tenglamalarini birgalikda yechamiz:

$$\begin{cases} y = 2x - 9 \\ y = 1 \end{cases}; \quad \begin{cases} x = 5 \\ y = 1 \end{cases}.$$

Demak, $M(5;1)$ - *CD* balandlik va *AE* medianalar kesishgan nuqta.◀

Tekislikda to‘g‘ri chiziqlarning o‘zaro joylashish holatlarini ko‘rib chiqamiz.

1. Agar tekislikda to‘g‘ri chiziqlar

$$A_1x + B_1y + C_1 = 0 \text{ va } A_2x + B_2y + C_2 = 0$$

umumiy tenglamalar bilan berilgan bo‘lsin. U holda ular orasidagi φ burchaklardan biri ularning $\vec{n}_1 = \{A_1; B_1\}$ va $\vec{n}_2 = \{A_2; B_2\}$ normallari orasidagi burchakga teng va quyidagi formula bilan hisoblanadi:

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2}{\sqrt{A_1^2 + B_1^2}\sqrt{A_2^2 + B_2^2}} \tag{3.9}$$

To‘g‘ri chiziqlarning *perpendikulyarlik sharti*

$$A_1A_2 + B_1B_2 = 0 \tag{3.10}$$

formula bilan aniqlanadi.

To‘g‘ri chiziqlarning *parellellik sharti*

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2} \tag{3.11}$$

formula bilan aniqlanadi.

Agar

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \tag{3.12}$$

tenglik bajarilsa, to‘g‘ri chiziqlar ustma-ust tushadi.

2. Tekislikda to‘g‘ri chiziqlar $y = k_1x + b_1$ va $y = k_2x + b_2$ burchak koeffitsientli tenglamalar bilan berilgan bo‘lsin. U holda ular orasidagi φ burchak

$$tg\varphi = \frac{k_2 - k_1}{1 + k_1k_2} \tag{3.13}$$

formula bilan hisoblanadi. Bu holda to‘g‘ri chiziqlar parallel bo‘lishi uchun $k_1 = k_2$ tenglik bajarilishi va perpendikulyar bo‘lishi uchun $k_1k_2 = -1$ shart bajarilishi zarur va yetarli.

$M_0(x_0;y_0)$ nuqtadan $Ax+By+C=0$ to‘g‘ri chiziqgacha bolgan d masofa quyidagi

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}} \tag{3.14}$$

formula bilan hisoblanadi.

3.3-misol

Berilgan $x-2y+4=0$ to‘g‘ri chiziqqa nisbatan $M(5;\,2)$ nuqtaga simmetrik nuqtani toping.

►Avval $M(5;\,2)$ nuqtadan o‘tuvchi va $\vec{n}=\{1;-2\}$ normal vektorli $x-2y+4=0$ to‘g‘ri chiziqqa perpendikulyar to‘g‘ri chiziq tenglamasini tuzamiz. Bu holda $\vec{n}=\{1;-2\}$ izlanayotgan to‘g‘ri chiziqning yo‘naltiruvchi vektori bo‘ladi. (1.6)ga ko‘ra,

$$\frac{x-5}{1}=\frac{y-2}{-2},\ y=-2x+12.$$

Bu to‘g‘ri chiziqlar kesishish nuqtasini topamiz.

$$\begin{cases}x-2y+4=0\\ y=-2x+12\end{cases}\Rightarrow\ \begin{cases}x=4\\ y=4\end{cases}.$$

Topilgan $M_0(4;4)$ nuqta $M(5;\,2)$ nuqta va unga simmetrik $M'(x;\,y)$ nuqtalarning o‘rtasi bo‘lgani uchun

$$\frac{x+5}{2}=4,\qquad \frac{y+2}{2}=4$$

tenglik o‘rinli. Bundan, $x=3,\ y=6$. Demak, $M'(3;\,6)$.◄

3.4-misol

Kvadratning ikkita tomoni $5x-12y-65=0$ va $5x-12y+26=0$ to‘g‘ri chiziqlarda yotsa, kvadratning yuzini toping.

► Berilgan to‘g‘ri chiziqlar o‘zaro parallel bo‘lgani uchun ular kvadratning qarama-qarshi tomonlari bo‘lib, orasidagi masofa kvadrat tomonining uzunligiga teng. Buning uchun $5x-12y-65=0$ to‘g‘ri chiziqdan ixtiyoriy nuqta tanlanadi, masalan, $M_0(1;-5)$ va ikkinchi $5x-12y+26=0$ to‘g‘ri chiziqgacha masofa (3.14)ga asosan topiladi.

$$d=\frac{|5\cdot 1-12\cdot(-5)+26|}{\sqrt{5^2+(-12)^2}}=\frac{91}{13}=7\ .$$

Demak, $S_{kv}=49$.◄

8-Auditoriya topshiriqlari

1. $2x-5y+8=0$ tenglama bilan berilgan to‘g‘ri chiziqning burchak koeffitsientini va o‘qlardan ajratgan kesmalarini aniqlang.

2. $A(5;-3)$ nuqtadan o‘tuvchi va *a*) Ox o‘qiga; *b*) Oy o‘qiga; c) 1-chorak bissektrisasiga; d) $y=-2x+7$ to‘g‘ri chiziqga; e) $2x-5y+8=0$ to‘g‘ri chiziqga parallel to‘g‘ri chiziq tenglamalarini tuzing.

3. $A(-1;3)$ va $B(2;-5)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

4. A$(-2;5)$ nuqtadan o‘tib, $3x+5y-8=0$ to‘g‘ri chiziqga perpendikulyar to‘g‘ri chiziq tenglamasini tuzing.

5. Kvadratning bir uchi $A(-1;2)$ nuqtada, bir tomoni esa $4x-3y-15=0$ to‘g‘ri chiziqda yotadi. Kvadratning yuzini hisoblang.

6. $4x-3y-12=0$ to‘g‘ri chiziqqa parallel va undan $d=2$ masofada joylashgan to‘g‘ri chiziq tenglamasini tuzing.

7. Agar $M(4;\,2)$ nuqta to‘g‘ri chiziqning koordinatalar orasidagi kesmasining o‘rtasi ekani ma’lum bo‘lsa to‘g‘ri chiziq tenglamasini tuzing.

8. $A(1;-2)$, $B(5;4)$ va C$(-2;0)$ nuqtalar uchburchakning uchlari bo‘lsa, uning bissektrisalari tenglamalarini tuzing.

9. To‘g‘ri chiziqning $A(3;-4)$ nuqtasi unga koordinata boshidan tushirilgan perpendikulyar asosi ekani ma’lum bo‘lsa, bu to‘g‘ri chiziq tenglamasini tuzing.

10. $5x-y+4=0$ va $3x+2y-1=0$ to‘g‘ri chiziqlar orasidagi burchakni toping.

8-Mustaqil yechish uchun testlar

1. $3x+5y-8=0$ to‘g‘ri chiziqning burchak koeffitsientini va Oy o‘qidan ajratgan kesmasini aniqlang

$A)\ k=\frac{3}{5};\ b=\frac{8}{5}$ $B)\ k=-\frac{3}{5};\ b=\frac{8}{5};$ $C)\ k=\frac{5}{3};\ b=\frac{8}{3};$ $D)\ k=\frac{5}{3};\ b=-\frac{8}{3}$

2. Berilgan $A(3;-4)$ va $B(1;-3)$ va nuqtalardan o‘tuvchi to‘g‘ri chiziqning umumiy tenglamasini toping

A) $\frac{x-3}{-2}=\frac{y+4}{1};$ B) $y=-\frac{1}{2}x-\frac{5}{2}$ C) $x+2y+5=0;$ D) $\begin{cases} x=3-2t \\ y=-4+t \end{cases}$

3. Berilgan $A(3;-4)$ va $B(1;-3)$ va nuqtalardan o‘tuvchi to‘g‘ri chiziqning parametrik tenglamasini toping

A) $\frac{x-3}{-2}=\frac{y+4}{1};$ B) $y=-\frac{1}{2}x-\frac{5}{2}$ C) $x+2y+5=0;$ D) $\begin{cases} x=3-2t \\ y=-4+t \end{cases}$

4. Quyidagilardan qaysi biri $M(1;-3)$ nuqtadan o‘tib, $\vec{s}=\{-3;5\}$ vektorga parallel bo‘lgan to‘g‘ri chiziq bo‘ladi?

A) $\frac{x+3}{-1}=\frac{y-5}{3};$ B) $y=-\frac{1}{2}x-\frac{5}{2}$ C) $\frac{x-1}{3}=\frac{y+3}{-5};$ D) $\begin{cases} x=1-3t \\ y=-3-5t \end{cases}$

5. Trapetsiya asoslarining tenglamalari berilgan: $3x-4y-15=0$, $3x-4y-35=0$. Trapetsiyaning balandligini aniqlang

A) $\frac{3}{5};$ B) 3 C) 4 D) 5

3.2. Fazoda tekislik tenglamalari

To‘g‘ri burchakli Dekart koordinatalar sistemasida ixtiyoriy tekislik

$$Ax+By+Cz+D=0 \tag{3.15}$$

tenglama bilan beriladi, bu yerda A, B, C, D – ma’lum sonlar, $A^2+B^2+C^2>0$ va (3.15) ko‘rinishdagi har qanday tenglama biror tekislikni aniqlaydi. (3.15) tenglama ***tekislikning umumiy tenglamasi*** deb ataladi. (3.15) tenglama bilan berilgan tekislikka perpendikulyar $\vec{n}(A;B;C)$ vektor tekislikning ***normal vektori*****(yoki *****normali*****)** deyiladi.

Tekislikning bir nechta berilish usullari mavjud

1. Berilgan $M_0(x_0,y_0,z_0)$ nuqtadan o‘tuvchi va $\vec{n}(A;B;C)$ normal vektorga ega tekislik tenglamasi:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \tag{3.16}$$

2. Tekislikning o‘qlardan ajratgan kesmalar bo‘yicha tenglamasi:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \tag{3.17}$$

Agar (3.15)da $D\neq 0$ bo‘lsa, $-D$ga bo‘lish orqali (3.17) tenglama hosil qilinadi va bu yerda a, b, c tekislikning mos ravishda Ox, Oy, Oz o‘qlardan ajratgan kesmalaridir.

3. Uch nuqtadan o‘tuvchi tekislik tenglamasi. Agar tekislik bir to‘g‘ri chiziqda yotmaydigan $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ va $M_3(x_3,y_3,z_3)$ nuqtalardan o‘tsa, uning tenglamasi quyidagicha yoziladi:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \qquad (3.18)$$

Determinantni 1-satr elementlari bo‘yicha yoyish orqali (3.16) formulani hosil
qilish mumkin.

Tekisliklar orasidagi φ burchak deganda ular hosil qiladigan ikki yoqli burchaklardan biri tushuniladi.

(P_1): $A_1x+B_1y+C_1z+D_1=0$ va (P_2): $A_2x+B_2y+C_{21}z+D_2=0$ tekisliklar fazoda har qanday joylashganda ham ular orasidagi burchaklardan biri ularning $\vec{n}_1=\{A_1;B_1;C_1\}$ va $\vec{n}_2=\{A_2;B_2;C_2\}$normallari orasidagi burchakka teng (3.1-chizma). Shuning uchun tekisliklar orasidagi burchak quyidagi formula yordamida hisoblanadi:

$$cos\varphi=\cos\left(\vec{n}_1,\hat{\ }\vec{n}_2\right)=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}} \qquad (3.19)$$

Tekisliklarning ***perpendikulyarlik sharti***

$$A_1A_2+B_1B_2+C_1C_2 \qquad (3.20)$$

formula bilan aniqlanadi.

Tekisliklarning ***parellellik sharti***

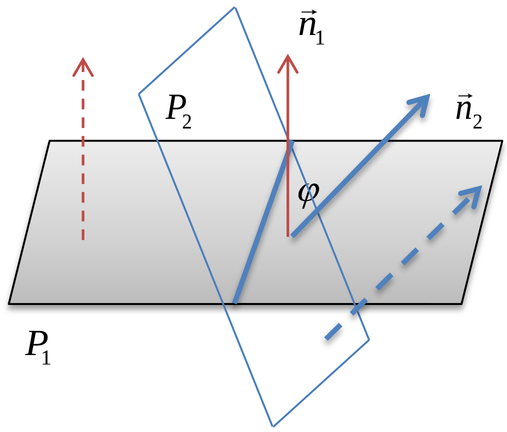
$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \qquad (3.21)$$

formula bilan aniqlanadi.

Agar

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=\frac{D_1}{D_2} \qquad (3.22)$$

tenglik o‘rinli bo‘lsa, tekisliklar ustma-ust tushadi. $\vec{n}_1$



3.1-chizma.

Tekislikning umumiy tenglamasidagi ba'zi koeffitsientlar nolga aylanganda tekislikning koordinata o'qlariga nisbatan vaziyati quyidagicha bo'ladi:

1. Agar $D=0$ bo'lsa, koordinatalar boshidan o'tadi.

2. Agar a) $A=0$ bo'lsa, $\vec{n}=B\vec{j}+C\vec{k}$ normal vektori Ox o'qiga perpendikulyar bo'ladi. Demak , tekislik Ox o'qiga parallel bo'ladi.

Xuddi shu kabi

b) $B=0$ bo'lsa, tekislik Oy o'qiga parallel bo'ladi;

c) $C=0$ bo'lsa, tekislik Oz o'qiga parallel bo'ladi.

3. Agar a) $D=0$, $C=0$ bo'lsa, $Ax+By=0$ koordinatalar boshidan o'tib Oz o'qiga parallel bo'ladi. Demak , tekislik Oz o'qidan o'tuvchi tekislik bo'ladi.

Xuddi shu kabi

b) $D=0$, $B=0$ bo'lsa, tekislik Oy o'qidan o'tuvchi tekislik bo'ladi;

c) $D=0$, $A=0$ bo'lsa, tekislik Ox o'qidan o'tuvchi tekislik bo'ladi.

4. Agar a) $A=0$, $B=0$ bo'lsa, $\vec{n}=C\vec{k}$ normal vektori Oz o'qiga parallel bo'ladi. Demak , $Cz+D=0$ tekislik Oxy tekisligiga parallel bo'ladi.

Xuddi shu kabi

b) $A=0$, $C=0$ bo'lsa, tekislik Oxz tekisligiga parallel bo'ladi;

c) $B=0$, $C=0$ bo'lsa, tekislik Oyz tekisligiga parallel bo'ladi.

5. Agar a) $A=0$, $B=0$ va $D=0$ bo'lsa, $Cz=0$ yoki $z=0$ tekislik Oxy tekisligiga parallel va koordinata boshidan o'tadi. Demak, Oxy koordinata tekisligining o'zi hosil bo'ladi. Xuddi shu kabi

b) $A=0$, $C=0$ va $D=0$ bo'lsa, Oxz tekisligi hosil bo'ladi;

c) $B=0$, $C=0$ va $D=0$ bo'lsa, Oyz tekisligi hosil bo'ladi.

Berilgan $M_0(x_0, y_0, z_0)$ nuqtadan $Ax + By + Cz + D = 0$ tekislikkacha bo‘lgan d masofa

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \qquad (3.23)$$

formula bilan hisoblanadi.

3.1-misol

Agar $M_1(2,0,4)$ va $M_2(5,5,1)$ nuqtalar berilgan bo‘lsa M_1 nuqtadan o‘tuvchi va $\overrightarrow{M_1M_2}$ vektorga perpendikulyar tekislik tenglamasini tuzing.

► $M_1(2,0,4)$ nuqtadan o‘tib, $\overrightarrow{M_1M_2} = \vec{n}(3,5,-3)$ normal vektorga ega bo‘lgan tekislik tenglamasi (3.16) ga ko‘ra,

$$3(x-2) + 5(y-0) - 3(z-4) = 0,$$

$$3x + 5y - 3z + 6 = 0 .$$ ◄

3.2-misol

Ox o‘qiga parallel, hamda $M_1(0,2,11)$ va $M_2(2,3,4)$ nuqtalardan o‘tuvchi tekislik tenglamasini tuzing.

► Ox o‘qiga parallel bo‘lgani uchun tekislikning umumiy tenglamasida $A = 0$ bo‘lib, normal vektori $\vec{n}(0;B;C)$ ko‘rinishda bo‘ladi. $\overrightarrow{M_1M_2}(2;1;-7) \perp \vec{n}$ dan va (3.16) formuladan foydalanib quyidagi tenglamalarni tuzamiz:

$$2\cdot 0 + 1\cdot B - 7\cdot C = 0,$$

$$B(y-3) + C(z-4) = 0.$$

Bu tenglamalarni birgalikda yechib, izlanayotgan tekislik tenglamasini hosil qilamiz.

$7(y-3) + (z-4) = 0$ yoki $7y + z - 25 = 0$.◄

3.3-misol

Berilgan $6x + 2y - 4z - 7 = 0$ va $9x + 3y - 6z + 13 = 0$ tekisliklar orasidagi burchakni toping.

► $\vec{n}_1 = \{6, 2, -4\}$, $\vec{n}_2 = \{9, 3, -6\}$

$$\cos\varphi = \frac{6\cdot 9 + 2\cdot 3 + (-4)\cdot(-6)}{\sqrt{6^2 + 2^2 + (-4)^2}\cdot\sqrt{9^2 + 3^2 + (-6)^2}} = \frac{84}{\sqrt{56}\cdot\sqrt{126}} = \frac{84}{\sqrt{7056}} = \frac{84}{84} = 1,$$

$\varphi = \arccos 1 = 0.$

Demak, berilgan tekisliklar o‘zaro parallel. ◀

3.4-misol.

Berilgan $M_0(4;3;0)$ nuqtadan, berilgan $M_1(1;3;0)$, $M_2(3;0;1)$ va $M_3(4;-1;2)$ nuqtalardan o‘tuvchi tekislikkacha bo‘lgan masofani toping.

► Dastlab (3.17) formuladan foydalanib, uch nuqtadan o‘tuvchi tekislik tenglamasini tuzamiz:

$$\begin{vmatrix} x-1 & y-3 & z-0 \\ 3-1 & 0-3 & 1-0 \\ 4-1 & -1-3 & 2-0 \end{vmatrix} = 0 \quad \text{yoki} \quad \begin{vmatrix} x-1 & y-3 & z \\ 2 & -3 & 1 \\ 3 & -4 & 2 \end{vmatrix} = 0.$$

Determinantni hisoblab, $2x+y-z-5=0$ tekislik tenglamasi hosil qilalamiz. $M_0(4;3;0)$ nuqtadan $2x+y-z-5=0$ tekislikkacha masofa

$$d = \frac{|2\cdot 4+1\cdot 3-1\cdot 0-5|}{\sqrt{2^2+1^2+(-1)^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}. \text{ ◀}$$

9-Auditoriya topshiriqlar

1. Quyidagi shartlarni qanoatlantiruvchi

a) berilgan $M_0(2;-3;0)$ nuqtadan o‘tib, $\vec{n}(1,5,-2)$ vektorga perpendikulyar;

b) Berilgan $M_0(3;-1;2)$ nuqtadan o‘tib, Oxz tekisligiga parallel;

c) Berilgan $M_1(1;2;-5)$ va $M_2(2;0;-1)$ va nuqtalardan o‘tib, Oy o‘qiga parallel;

d) $M_0(0;3;4)$ nuqtadan va Oz o‘qidan o‘tuvchi;

e) $A(3;5;-2)$ nuqtadan o‘tib, $\vec{n}_1(2;1;-3)$ va $\vec{n}_2(4;-3;-1)$ vektorlarga parallel tekislik tenglamalarini tuzing va ularni yasang.

2. $M_1(1;2;-5)$ va $M_2(2;0;-1)$ nuqtalardan o‘tib, $3x+5y-3z+6=0$ tekisligiga perpendiulyar tekislik tenglamasini tuzing.

3. $A(4;-3;5)$ nuqtadan o‘tib, koordinata o‘qlaridan $1:2:2$ nisbatdagi musbat kesmalar ajratadigan tekislik tenglamasini tuzing.

4. $7x-y-5z+6=0$ va $2x-y+3z-13=0$ tekisliklar orasidagi burchakni toping.

5. $A(1;3;-5)$ nuqtadan o‘tuvchi va $3x+2y-6z+7=0$, $2x-6y+3z-13=0$ tekisliklarga perpendikulyar tekislik tenglamasini tuzing.

6. $2x+6y-3z+15=0$ va $2x+6y-3z-13=0$ tekisliklar orasidagi masofani toping.

7. $2x - y + 4z + 21 = 0$ tekisligiga perpendikulyar va Ox, Oy koordinata o‘qlaridan mos ravishda $a = 2, b = -3$ kesma ajratuvchi tekislik tenglamasini tuzing.

8. Uchlari $A(-3;0;2)$, $B(1;2;-2)$, $C(0;1;-2)$ va $D(3;-3;2)$ nuqtalarda bo‘lgan piramidaning A uchidan BCD yog‘iga tushirilgan balandligi uzunligini toping.

9-Mustaqil yechish uchun testlar

1. $A(1;2;1)$ va $B(4;0;-5)$ nuqtalar berilgan. $A(1;2;1)$ nuqtadan o‘tib, $\overrightarrow{AB}$ vektorga perpendikulyar bo‘lgan tekislik tenglamasini toping

$A)$ $2x + 3y - 6z + 2 = 0$ $B)$ $4x - 6y - 12z - 3 = 0$

$C)$ $3x - 2y - 6z - 1 = 0$ $D)$ $6x - 2y - 3z + 5 = 0$

2. Ox o‘qidan o‘tuvchi tekislik tenglamasi berilgan javobni aniqlang

$A)$ $3y - 6z + 5 = 0$ $B)$ $5y + 12z = 0$

$C)$ $3x - 7 = 0$ $D)$ $6x - 2y - 3z = 0$

3. Oyz koordinata tekisligiga parallel tekislik tenglamasi berilgan javobni aniqlang

$A)$ $3y - 6z + 5 = 0$ $B)$ $5y + 12z = 0$

$C)$ $3x - 7 = 0$ $D)$ $6x - 2y - 3z = 0$

4. $2x - 3y + 6z - 7 = 0$ tekislikka perpendiklyar tekislikni toping

$A)$ $2x + 3y - 6z + 5 = 0$ $B)$ $4x - 5y + 12z - 7 = 0$

$C)$ $3x - 2y - 6z - 7 = 0$ $D)$ $6x - 2y - 3z + 5 = 0$

5. Berilgan $M(-9;-1;3)$ nuqtadan $3x + 6y + 2z - 8 = 0$ tekislikkacha bo‘lgan masofani toping

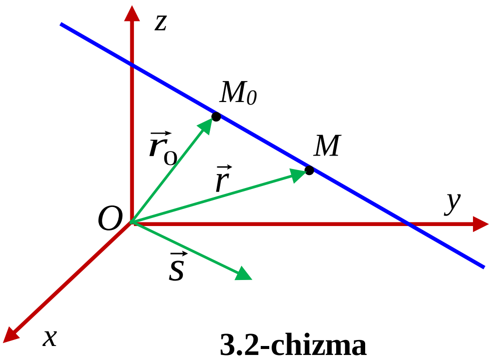
$A)$ 3 $B)$ 4 $C)$ 5 $D)$ 6

6. Berilgan $3x - 2y - 6z - 7 = 0$ va $6x - 3y + 2z = 0$ tekisliklar orasidagi burchak kosinusini toping

$A)$ $15/49$ $B)$ $18/49$ $C)$ $12/49$ $D)$ $16/49$

3.3. Fazoda to‘g‘ri chiziq. To‘g‘ri chiziq va tekislikning o‘zaro joylashuvi

Agar to‘g‘ri chiziqda yotuvchi $M_0(\vec{r_0}) = M_0(x_0, y_0, z_0)$ nuqta va to‘g‘ri chiziqga parallel $\vec{s}(m,n,p)$, $(|\vec{s}| \neq 0)$ vektor berilgan bo‘lsa, fazoda to‘g‘ri chiziqning vaziyati aniqlangan bo‘ladi. $M(\vec{r}) = M(x,y,z)$ nuqta to‘g‘ri chiziqdagi o‘zgaruvchan nuqta bo‘lsin. U holda $\overrightarrow{M_0M} = t \cdot \vec{s}$ bo‘ladi. Bu yerda t M nuqtaning vaziyatiga qarab ixtiyoriy haqiqiy son qiymati qabul qilishi mumkin. t to‘g‘ri chiziqning o‘zgaruvchan ***parametri*** deyiladi. $\overrightarrow{M_0M} = \vec{r} - \vec{r_0}$ dan to‘g‘ri chiziqning ***vektor tenglamasi*** hosil bo‘ladi:



3.2-chizma

$$\vec{r} = \vec{r_0} + t \cdot \vec{s} \qquad (3.24)$$

Bu tenglamadan koordinatalarga o‘tsak,

$$\begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases} \qquad (3.25)$$

to‘g‘ri chiziqning parametrik tenglamasi hosil bo‘ladi. (3.25) dan ***to‘g‘ri chiziqning kanonik tenglamasini*** hosil qilamiz

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}. \qquad (3.26)$$

$\vec{s}(m,n,p)$ vektor to‘g‘ri chiziqning ***yo‘naltiruvchi vektori*** deyiladi.

Ikki $M_1(x_1, y_1, z_1)$ *va* $M_2(x_2, y_2, z_2)$ ***nuqtalardan o‘tuvchi to‘g‘ri chiziq*** tenglamasi

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \qquad (3.27)$$

Har qanday ikkita parallel bo‘lmagan tekisliklarning tenglamalari birgalikda

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \tag{3.28}$$

to'g'ri chiziqning ***umumiy tenglamasi*** deyiladi. To'g'ri chiziqning $\bar{s}$ yo'naltiruvchi vektori sistemadagi tekisliklarning normal vektori $\vec{n_1}(A_1, B_1, C_1)$ va $\vec{n_2}(A_2, B_2, C_2)$ ning har biriga perpendikulyar, demak, $\vec{s} = \vec{n_1} \times \vec{n_2}$.

To'g'ri chiziqning umumiy tenglamasidan kanonik tenglamani hosil qilish mumkin. Buning uchun to'g'ri chiziqda yotuvchi bitta nuqta koordinatalarini va yo'naltiruvchi vektorni bilish yetarli, yoki avval to'g'ri chiziqning proyeksiyalardagi tenglamasiga o'tish lozim.

To'g'ri chiziqning proyeksiyalardagi tenglamasi uning umumiy tenglamasidan avval y ni, keyin x ni yo'qotib topiladi:

$$\begin{cases} x = mz + a \\ y = nz + b. \end{cases} \tag{3.29}$$

3.5-misol.

Ushbu $\begin{cases} x - 2y - z - 5 = 0 \\ 2x + y - 3z - 5 = 0 \end{cases}$ umumiy tenglama bilan berilgan to'g'ri chiziqning kanonik tenglamasini yozing.

► Bu yerda $\bar{n}_1(1,-2,-1)$ va $\bar{n}_2(2,1,-3)$, u holda

$$\bar{s} = \bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 2 & 1 & -3 \end{vmatrix} = 7\vec{i} + \vec{j} + 5\vec{k}, \qquad \vec{s}(7,1,5).$$

To'g'ri chiziqda yotuvchi bitta nuqtani topish uchun $z = 0$ deb, $x = 3,\ y = -1$ larni topamiz. $M_0(3,-1,0)$ berilgan to'g'ri chiziqda yotadi. Demak, to'g'ri chiziqning kanonik tenglamasi

$$\frac{x-3}{7} = \frac{y+1}{1} = \frac{z}{5}. \quad ◄$$

Ikkita to'g'ri chiziq kanonik tenglamalari bilan berilgan bo'lsin:

$$\frac{x - x_1}{m_1} = \frac{y - y_1}{n_1} = \frac{z - z_1}{p_1}; \qquad \frac{x - x_2}{m_2} = \frac{y - y_2}{n_2} = \frac{z - z_2}{p_2}. \tag{3.30}$$

Bu ***to'g'ri chiziqlar orasidagi burchak*** ularning yo'naltiruvchi $\bar{s}_1(m_1, n_1, p_1)$ va $\bar{s}_2(m_2, n_2, p_2)$ vektorlari orasidagi φ burchakka teng

$$\cos\varphi = \pm\frac{\bar{s}_1 \cdot \bar{s}_2}{|\bar{s}_1| \cdot |\bar{s}_2|} = \frac{|m_1m_2 + n_1n_2 + p_1p_2|}{\sqrt{m_1^2 + n_1^2 + p_1^2} \cdot \sqrt{m_2^2 + n_2^2 + p_2^2}}. \tag{3.31}$$

a) to'g'ri chiziqlarlarning ***perpendikulyarlik sharti***

$$m_1m_2+n_1n_2+p_1p_2=0 \quad (3.32)$$

b) to‘g‘ri chiziqlarning ***parallelik sharti***

$$\frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{p_1}{p_2} \quad (3.33)$$

d) to‘g‘ri chiziqlarning ***ayqash bo‘lish sharti***

$$\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \end{vmatrix} \neq 0, \quad (3.34)$$

e) parallel bo‘lmagan to‘g‘ri chiziqlarning ***kesishish sharti***

$$\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \end{vmatrix} = 0. \quad (3.35)$$

Berilgan $M_1(x_1,y_1,z_1)$ nuqtadan $\bar{s}(m,n,p)$ vektor bo‘ylab yo‘nalgan $M_0(x_0,y_0,z_0)$ nuqtadan o‘tuvchi to‘g‘ri chiziqgacha bo‘lgan ***masofa***

$$\boxed{d=\left|\vec{s}\times\overrightarrow{M_0M_1}\right|/\left|\vec{s}\right|} \quad (3.36)$$

formula bilan hisoblanadi.

3.6-misol.

Agar $A(0,-2,8)$, $B(4,3,2)$, $C(1,4,3)$ nuqtalar berilgan bo‘lsa, A nuqtadan o‘tib BC to‘g‘ri chiziqga parallel bo‘lgan to‘g‘ri chiziq tenglamasini tuzing.

► Izlanayotgan to‘g‘ri chiziq BC to‘g‘ri chiziqga parallel bo‘lgani uchun $\vec{s}=\overrightarrow{BC}(-3,1,1)$ deb tanlash kifoya. U holda $A(0,-2,8)$ nuqtadan o‘tuvchi yo‘naltiruvchisi $\vec{s}(-3,1,1)$ bo‘lgan to‘g‘ri chiziqning kanonik tenglamasini tuzamiz:

$$\frac{x}{-3}=\frac{y+2}{1}=\frac{z-8}{1} \blacktriangleleft$$

3.7-misol.

Berilgan $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z}{2}$ va $\frac{x-7}{3}=\frac{y-1}{4}=\frac{z-3}{2}$ to‘g‘ri chiziqlar orasidagi masofani toping.

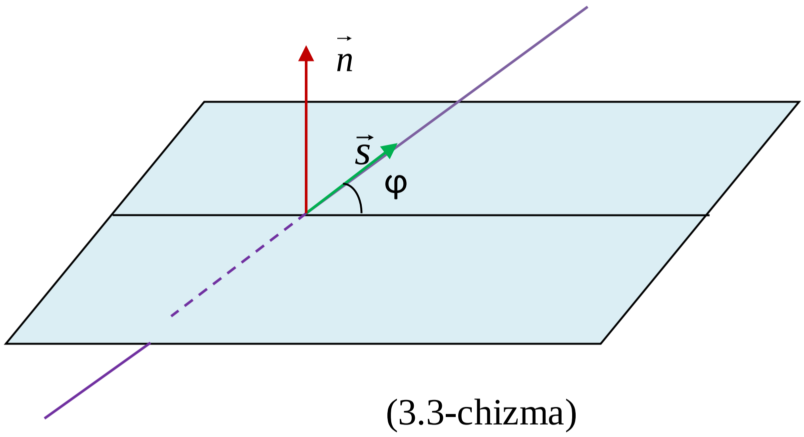
►Birinchi to‘g‘ri chiziqda yotgan ixtiyoriy nuqtadan, masalan, $M_1(2,-1,0)$ dan ikkinchi $\frac{x-7}{3}=\frac{y-1}{4}=\frac{z-3}{2}$ to‘g‘ri chiziqgacha masofa topiladi.

$$\overrightarrow{M_1M_0}(5,2,3),\ \vec{s}(3,4,2),\ |\vec{s}|=\sqrt{3^2+4^2+2^2}=\sqrt{29}\quad,$$

$$\vec{s}\times\overrightarrow{M_1M_0}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 4 & 2\\ 5 & 2 & 3\end{vmatrix}=8\vec{i}+\vec{j}-14\vec{k},\ \left|\vec{s}\times\overrightarrow{M_1M_0}\right|=3\sqrt{29}.$$

To‘g‘ri chiziqlar orasidagi masofa $d=\left|\vec{s}\times\overrightarrow{M_1M_0}\right|/\left|\vec{s}\right|=3$. ◄

To‘g‘ri chiziq (L): $\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$ va tekislik (T): $Ax+By+Cz+D=0$ tenglamalari berilgan bo‘lsin. To‘g‘ri chiziq va tekislik orasidagi burchak deb, to‘g‘ri chiziq va uning tekislikdagi orthogonal proyeksiyasi orasidagi φ burchakga aytiladi (3.3-chizma).



(3.3-chizma)

To‘g‘ri chiziq va tekislik orasidagi burchak quyidagi formula bilan hisoblanadi:

$$\left|\cos\left(\widehat{\vec{n},\vec{s}}\right)\right|=\sin\varphi=\frac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\sqrt{m^2+n^2+p^2}}.$$

To‘g‘ri chiziqning kanonik tenglamasidan parametrik tenglamasiga o‘tib, tekislik tenglamasiga qo‘yamiz

$$(Am+Bn+Cp)t+Ax_0+By_0+Cz_0+D=0.$$

Bunda uch hol bo‘lishi mumkin

1. Agar $Am+Bn+Cp\neq 0$ bo‘lsa, *to‘g‘ri chiziq va tekislik kesishadi.* Bu holda $t=-(Ax_0+By_0+Cz_0+D)/(Am+Bn+Cp)$ ni to‘g‘ri chiziq parametrik tenglamasiga qo‘yib, *to‘g‘ri chiziq va tekislikning kesishish nuqtasi M* topiladi.

Xususan, $\frac{A}{m}=\frac{B}{n}=\frac{C}{p}$ bo‘lsa, *to‘g‘ri chiziq va tekislik perpendikulyar* bo‘ladi.

2. Agar $Am+Bn+Cp=0$ va $Ax_0+By_0+Cz_0+D\neq 0$ bo‘lsa, *to‘g‘ri chiziq va tekislik parallel*.

3. Agar $Am+Bn+Cp=0$ va $Ax_0+By_0+Cz_0+D=0$ bo'lsa, *to'g'ri chiziq tekislikda yotadi*(to'g'ri chiziq tekislikga tegishli).

3.8-misol.

Berilgan $\frac{x-1}{-1}=\frac{y-1,5}{0}=\frac{z-3}{1}$ to'g'ri chiziqqa nisbatan $M(3,3,3)$ nuqtaga simmetrik $M`$ nuqtani toping.

▶ $M(3,3,3)$ nuqtadan o'tuvchi $\frac{x-1}{-1}=\frac{y-1,5}{0}=\frac{z-3}{1}$ to'g'ri chiziqqa perpendikulyar tekislik tenglamasini topamiz.

$-1(x-3)+0(y-3)+1(z-3)=0,$

$-x+z=0.$

To'g'ri chiziq va tekislik kesishgan nuqtani topamiz.

$$\frac{x-1}{-1}=\frac{y-1,5}{0}=\frac{z-3}{1}\Rightarrow\begin{cases}x=-t+1,\\ y=1,5,\\ z=t+3.\end{cases}$$

$$-(-t+1)+(t+3)=0,$$
$$2t+2=0,$$
$$t=-1.$$

$M_0(2;1,5;2)$- kesishish nuqtasi. Bundan

$$x_{M_0}=\frac{x_M+x_{M'}}{2}\Rightarrow x_{M'}=2x_{M_0}-x_M=2\cdot2-3=1,$$
$$y_{M_0}=\frac{y_M+y_{M'}}{2}\Rightarrow y_{M'}=2y_{M_0}-y_M=2\cdot1,5-3=0,$$
$$z_{M_0}=\frac{z_M+z_{M'}}{2}\Rightarrow z_{M'}=2z_{M_0}-z_M=2\cdot2-3=1.$$

Natijada, $M'(1,0,1)$ izlangan nuqtaga ega bo'lamiz. ◀

10-Auditoriya topshiriqlari

1. $\begin{cases}x-3y+2z+2=0\\ x+3y+z+14=0\end{cases}$ umumiy tenglama bilan berilgan to'g'ri chiziqning kanonik tenglamasini yozing. (Javob: $\frac{x+8}{-9}=\frac{y+2}{1}=\frac{z}{6}$.)

2. Uchburchakning $A(1,-2,3)$, $B(4,3,-2)$, $C(2,1,-2)$ uchlari berilgan bo'lsa, AD medianasining parametrik tenglamasini yozing.(Javob: $x=1+3t, y=-2+2t, z=3-2t.$)

3. A va B ning qanday qiymatlarida $Ax+By+6z-5=0$ tekislik va $\frac{x-3}{2}=\frac{y+4}{-5}=\frac{z+2}{3}$ to‘g‘ri chiziq perpendikulyar bo‘ladi? (Javob: $A=4,\ B=-10.$)

4. To‘g‘ri chiziq va tekislik orasidagi burchakni toping:

$a)\ \begin{cases} 3x-y-1=0, \\ 3x+2z-2=0 \end{cases}$ va $2x+y+z-4=0;$

$b)\ \begin{cases} x-2y+3=0, \\ 3y-z-1=0 \end{cases}$ va $2x+3y+z+1=0.$

(Javob: $a)\ \varphi=\arcsin\frac{1}{\sqrt{6}};$ $b)\ \varphi=\arcsin\frac{5}{7}.$)

5. To‘g‘ri chiziq va tekislikning o‘zaro joylashuvini aniqlang. Agar ular kesishuvchi bo‘lsa, kesishish nuqtasini toping:

$a)\ \frac{x-3}{2}=\frac{y+4}{4}=\frac{z}{3}$ va $3x-3y+2z-5=0;$

$b)\ \frac{x-13}{5}=\frac{y-1}{2}=\frac{z-4}{3}$ va $x+2y-3z-3=0;$

$d)\ \frac{x-5}{1}=\frac{y-4}{1}=\frac{z-7}{3}$ va $2x-y+3z-7=0.$

(Javob: $a)$ parallel; $b)$ to‘g‘ri chiziq tekislikda yotadi; $d)$ $M(3,2,1)$ nuqtada kesishadi.)

6. $A(3,\ 4,\ 0)$ nuqtadan va $\frac{x-2}{1}=\frac{y-3}{2}=\frac{z+1}{3}$ to‘g‘ri chiziqdan o‘tuvchi tekislik tenglamasini yozing. (Javob: $x-2y+z+5=0$.)

7. $\frac{x-2}{1}=\frac{y+3}{2}=\frac{z+1}{3}$ to‘g‘ri chiziqdan o‘tuvchi va $2x-y-z-3=0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x+7y-5z+14=0$.)

8. $\frac{x+3}{2}=\frac{y-2}{-3}=\frac{z+1}{1}$ va $\frac{x-2}{2}=\frac{y+3}{-3}=\frac{z}{1}$ parallel to‘g‘ri chiziqlardan o‘tuvchi tekislik tenglamasini yozing. (Javob: $2x+3y+5z+5=0$.)

9. $A(3,\ 1,-1)$ nuqtaning $x-2y+z+6=0$ tekislikdagi proyeksiyasini toping. (Javob: (2, 3, -2).)

10. $A(3,1,-2)$ nuqtaning $\frac{x+3}{2}=\frac{y-2}{-3}=\frac{z+1}{1}$ to‘g‘ri chiziqdagi proyeksiyasini toping. (Javob: (-1, -1, 0).)

11. $\begin{cases} x=z-2 \\ y=2z+1 \end{cases}$ va $\frac{x-2}{3}=\frac{y-4}{1}=\frac{z-2}{1}$ to‘g‘ri chiziqlarning kesishuvchi

ekanligini ko‘rsating, hamda ular joylashgan tekislik tenglamasini yozing. (Javob: $x+2y-5z=0$.)

10-Mustaqil yechish uchun testlar

1. $A(3,-2,0)$ va $B(5,-4,3)$ nuqtalardan o‘tuvchi to‘g‘ri chiziqning parametrik tenglamasini yozing.

A) $\begin{cases} x=3+3t, \\ y=-2+2t, \\ z=-2t. \end{cases}$ B) $\begin{cases} x=5+2t, \\ y=-4-2t, \\ z=3+3t. \end{cases}$ C) $\begin{cases} x=3+2t, \\ y=-2+2t, \\ z=3t. \end{cases}$ D) $\begin{cases} x=1+3t, \\ y=-2+2t, \\ z=3-2t. \end{cases}$

2. A ning qanday qiymatida $\frac{x-2}{3}=\frac{y+1}{A}=\frac{z}{2}$ va $\frac{x-7}{3}=\frac{y-1}{5}=\frac{z-3}{-2}$ to‘g‘ri chiziqlar perpendikulyar bo‘ladi?

A) 1; B) -2; C) 3; D) -1.

3. $\frac{x-2}{1}=\frac{y+3}{2}=\frac{z+1}{3}$ to‘g‘ri chiziq va $x+7y-5z+14=0$ tekislik qanday joylashgan?

A) parallel; B) perpendikulyar; C) to‘g‘ri chiziq tekislikda yotadi; D) kesishadi.

4. $\frac{x+1}{3}=\frac{y+1}{-2}=\frac{z+2}{2}$ to‘g‘ri chiziq va $2x+3y+5z+5=0$ tekislik kesishgan nuqtani toping.

A) $(3,-2,0)$, B) $(3,-2,-1)$, C) $(4,-1,-2)$, D) $(2,-3,0)$.

5. $\frac{x+2}{-1}=\frac{y-3}{2}=\frac{z-4}{3}$ va $\frac{x-3}{3}=\frac{y+2}{2}=\frac{z-8}{5}$ to‘g‘ri chiziqlar qanday joylashgan?

A) parallel; B) perpendikulyar; C) ayqash; D) kesishadi.

3-Shaxsiy uy topshiriqlari

1.1. $M(4,-1,-2)$ nuqtadan o‘tuvchi va $2x-y-3z+5=0$ tekislikga parallel bo‘lgan tekislikning o‘qlardan ajratgan kesmalarini toping.

1.2. $A(-1,3,2)$, $B(1,1,0)$ nuqtalardan o‘tuvchi va $x+2y-3z-3=0$ tekislikga perpendikulyar bo‘lgan tekislik tenglamasini yozing.

1.3. Agar $M_1(3,-2,4)$, $M_2(-1,4,2)$ nuqtalar berilgan bo‘lsa, M_1M_2 kesmaning o‘rtasidan o‘tuvchi va shu kesmaga perpendikulyar tekislik tenglamasini yozing.

1.4. Ox o‘qidan va $A(-1,3,-3)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing va $x-2y+2z+5=0$ tekislik bilan hosil qilgan burchagini aniqlang.

1.5. $M(4,-1,-2)$ nuqtadan $2x+2y-z+4=0$ tekislikgacha bo‘lgan masofani toping.

1.6. $A(-1, 3, 2), B(1,1,0)$ va $C(2, 0, -1)$ nuqtalardan o‘tuvchi tekislik tenglamasini yozing.

1.7. $A(4, 1, 2), B(2,-1,3)$ nuqtalardan o‘tuvchi va $\vec{a}(1,2,-5)$ vektorga parallel bo‘lgan tekislik tenglamasini yozing.

1.8. $A(3, 2,-3), B(-1, 4, 2)$ nuqtalardan o‘tuvchi va Oy o‘qiga parallel bo‘lgan tekislik tenglamasini yozing.

1.9. $M(5, 4, -8)$ nuqtadan $3x+6y-2z+15=0$ tekislikgacha bo‘lgan masofani toping.

1.10. $A(1, 2, 1), B(3,0,3)$ nuqtalardan o‘tuvchi va Ox o‘qidan $a=2$ kesma ajratuvchi tekislik tenglamasini yozing.

1.11. $A(-1, 2, 3)$ nuqtadan o‘tuvchi, $3x-y+2z+7=0$ va $2x+y+3z-5=0$ tekisliklarga perpendikulyar bo‘lgan tekislik tenglamasini yozing.

1.12. $A(2, -5, 2), B(1,0,1)$ va $C(2, 4, -1)$ nuqtalardan o‘tuvchi tekislik tenglamasini yozing.

1.13. O‘zaro parallel bo‘lgan $2x-9y+6z+17=0$ va $2x-9y+6z-16=0$ tekisliklar orasidagi masofani toping.

1.14. $x-3y+6=0$ va $x+2y-7=0$ tekisliklar orasidagi burchakni toping.

1.15. Oz o‘qidan o‘tuvchi va $2x+y-2z+7=0$ tekislik bilan 45° burchak tashkil etuvchi tekislik tenglamasini yozing.

1.16. $3x+6y-2z+15=0$ tekislikdan 4 birlik masofada yotuvchi tekislik tenglamasini yozing.

1.17. $C(2, 0, -1)$ nuqtadan o‘tuvchi va $\vec{a}(1, 3,-2), \vec{b}(1,-1,1)$ vektorlarga perpendikulyar tekislik tenglamasini yozing.

1.18. $x-2y-z-14=0$ va $x+y+z-3=0$ tekisliklarning kesishish chizig‘idan hamda $A(2, 4, -2)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing.

1.19. $x-2y+z-7=0$, $2x+y-3z+16=0$ tekisliklarning kesishish chizig‘idan o‘tuvchi va $4x+3y+z-15=0$ tekislikga perpendikulyar tekislik tenglamasini yozing.

1.20. $A(-1, 3, 2), B(1,1,0)$ va $C(2, 0, -1)$ nuqtalardan o‘tuvchi tekislik bilan Oxz tekislik orasidagi burchakni toping.

1.21. $2x-y+2z-7=0$ va $x-2y+2z-2=0$ tekisliklarning kesishish chizig‘idan o‘tuvchi hamda Ox o‘qiga parallel bo‘lgan tekislik tenglamasini yozing.

1.22. O‘zaro parallel bo‘lgan $2x-3y+6z-3=0$ va $2x-3y+6z-24=0$ tekisliklar orasidagi masofani toping.

1.23. $A(2,1,3)$ nuqtadan o‘qlardan $a=1, b=2, c=3$ kesma ajratuvchi tekislikgacha bo‘lgan masofani toping.

1.24. Ox o‘qidan o‘tuvchi va $2x+y-2z+7=0$ tekislik bilan 45° burchak tashkil etuvchi tekislik tenglamasini yozing.

1.25. $A(2,-3,1)$ nuqtadan o‘tuvchi, $2x+y-2z+7=0$ va $2x+y+3z-5=0$ tekisliklarga perpendikulyar bo‘lgan tekislik tenglamasini yozing.

1.26. $M(3,-1,7)$ nuqtadan o‘tuvchi va $3x+y-2z+15=0$ tekislikga parallel bo‘lgan tekislikning o‘qlardan ajratgan kesmalarini toping.

1.27. $A(2,-3,-1)$ nuqtadan o‘tuvchi, $x-3y+6=0$ va $2x+y-2z-5=0$ tekisliklarga perpendikulyar bo‘lgan tekislik tenglamasini yozing.

1.28. $M(-3,1,-9)$ nuqtaning $4x-3y-z-7=0$ tekislikga nisbatan simmetrik bo‘lgan M' nuqta koordinatalarini toping.

1.29. $5x+3y+z-18=0$ va $2x+z-9=0$ tekisliklar orasidagi burchakni toping.

1.30. O‘zaro parallel bo‘lgan $3x-2y-6z-13=0$ va $3x-2y-6z+15=0$ tekisliklar orasidagi masofani toping.

2.Quyidagi umumiy tenglama bilan berilgan to‘g‘ri chiziqlarning kanonik tenglamalarini yozing.

2.1 $\begin{cases} 2x+y+z-2=0 \\ 2x-y-3z+6=0 \end{cases}$ 2.2 $\begin{cases} 6x-7y-4z-2=0 \\ x+7y-z-5=0 \end{cases}$

2.3 $\begin{cases} x+3y+z+14=0 \\ x-3y+2z+2=0 \end{cases}$ 2.4 $\begin{cases} x-2y+3z-2=0 \\ 2x+3y-8z+3=0 \end{cases}$

2.5 $\begin{cases} x+y+z-2=0 \\ x-y-2z+2=0 \end{cases}$ 2.6 $\begin{cases} 6x-5y-4z+8=0 \\ 6x+5y+3z+4=0. \end{cases}$

2.7 $\begin{cases} 2x+2y-z-8=0 \\ x-2y+z-4=0 \end{cases}$ 2.8 $\begin{cases} 2x-5y+2z+5=0 \\ x+5y-z-5=0. \end{cases}$

2.9 $\begin{cases} 3x+y-z-6=0 \\ 3x-y+2z=0 \end{cases}$ 2.10 $\begin{cases} 2x-3y+z+6=0 \\ x-3y-2z+3=0 \end{cases}$

2.11 $\begin{cases} x+5y+2z+11=0 \\ x-y-z-1=0 \end{cases}$

2.12 $\begin{cases} 4x+y+z+2=0 \\ 2x-y-3z-5=0 \end{cases}$

2.13 $\begin{cases} 2x+3y+z+6=0 \\ x-3y-2z+3=0 \end{cases}$

2.14 $\begin{cases} 2x+y-3z-2=0 \\ 2x-y+z+6=0 \end{cases}$

2.15 $\begin{cases} 3x+4y-2z+1=0 \\ 2x-4y+3z+4=0. \end{cases}$

2.16 $\begin{cases} x+y-2z-2=0 \\ x-y+z+2=0 \end{cases}$

2.17 $\begin{cases} 5x+y-3z+4=0 \\ x-y+2z+2=0. \end{cases}$

2.18 $\begin{cases} x+5y-z++11=0 \\ x-y+2z-1=0. \end{cases}$

2.19 $\begin{cases} x-y-z-2=0 \\ x-2y+z+4=0. \end{cases}$

2.20 $\begin{cases} x-2y-z+4=0 \\ x-y+z-2=0 \end{cases}$

2.21 $\begin{cases} 4x+y-3z+2=0 \\ 2x-y+z-8=0. \end{cases}$

2.22 $\begin{cases} 6x-7y-z-2=0 \\ x+7y-4z-5=0 \end{cases}$

2.23 $\begin{cases} 3x+3y-2z-1=0 \\ 2x-3y+z+6=0. \end{cases}$

2.24 $\begin{cases} x+5y+2z-2=0 \\ 2x-5y-z+5=0 \end{cases}$

2.25 $\begin{cases} 6x-7y-4z-2=0 \\ x+7y-z-5=0 \end{cases}$

2.26 $\begin{cases} x+3y+2z+14=0 \\ x-3y+z+2=0. \end{cases}$

2.27 $\begin{cases} 2x+3y-2z+6=0 \\ x-3y+z+3=0. \end{cases}$

2.28 $\begin{cases} 3x+3y+z-1=0 \\ 2x-3y-2z+6=0. \end{cases}$

2.29 $\begin{cases} 3x+4y+3z+1=0 \\ 2x-4y-2z+3=0 \end{cases}$

2.30 $\begin{cases} 6x+5y-4z+4=0 \\ 6x-5y+3z+8=0. \end{cases}$

3. Quyidagi misollarni yeching.

3.1. $M(3,\ -1,\ 7)$ nuqtadan o‘tuvchi va $\begin{cases} 3x+y+3z+1=0 \\ x-2y-z+4=0 \end{cases}$ to‘g‘ri chiziqga parallel to‘g‘ri chiziq tenglamasini toping.(Javob: $\frac{x-3}{5}=\frac{y+1}{6}=\frac{z-7}{-7}$.)

3.2. m va C ning qanday qiymatlarida $\frac{x-3}{m}=\frac{y+2}{2}=\frac{z-8}{-5}$ to‘g‘ri chiziq $3x-2y+Cz-7=0$ tekislikga perpendikulyar bo‘ladi? .(Javob: $m=-3, C=5.$)

3.3. p ning qanday qiymatida $\frac{x-3}{1}=\frac{y+2}{2}=\frac{z-4}{p}$ va $\begin{cases} 3x+y-5z+1=0 \\ x-2y+3z-2=0 \end{cases}$ to‘g‘ri chiziqlar perpendikulyar bo‘ladi? .(Javob: $p=-5.$)

3.4. $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+1}{-1}$ to‘g‘ri chiziqdan o‘tuvchi va $x+2y+3z-5=0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x+7y-5z+6=0$.)

3.5. D ning qanday qiymatida $\begin{cases} x+y-2z+D=0 \\ x-2y+3z+12=0 \end{cases}$ to‘g‘ri chiziq Oz o‘qini kesib o‘tadi? (Javob: $D=-8.$)

3.6. $M(1, 3, -2)$ nuqtaning $\begin{cases} x+2y-z+3=0 \\ x+y+z+2=0 \end{cases}$ to‘g‘ri chiziqga nisbatan simmetrik nuqtasini toping.(Javob: $(-3, -5, 2).$)

3.7. $M(2, 0, 1)$ nuqtadan va $\frac{x-3}{3}=\frac{y+1}{5}=\frac{z+1}{-2}$ to‘g‘ri chiziqdan o‘tuvchi tekislik tenglamasini yozing. (Javob: $3x-y+2z-8=0$.)

3.8. $\frac{x-3}{3}=\frac{y+1}{5}=\frac{z+1}{-2}$ va $\frac{x+3}{3}=\frac{y+1}{-5}=\frac{z-8}{-7}$ to‘g‘ri chiziqlarning kesishuvchi ekanini isbotlang va shu to‘g‘ri chiziqlardan o‘tuvchi tekislik tenglamasini yozing.(Javob: $3x-y+2z-8=0$.)

3.9. $\frac{x+6}{3}=\frac{y-4}{-5}=\frac{z-15}{-7}$ va $\begin{cases} 2x+3z-3=0 \\ 2y+5z+7=0 \end{cases}$ to‘g‘ri chiziqlarning kesishish nuqtasini toping. (Javob: $(0, -6, 1).$)

3.10. $M(3, 0, -2)$ nuqtadan $\frac{x-1}{2}=\frac{y+6}{-1}=\frac{z-2}{3}$ to‘g‘ri chiziqga tushirilgan perpendikulyar tenglamasini yozing. (Javob: $\frac{x-3}{4}=\frac{y}{5}=\frac{z+2}{-1}$.)

3.11. $\frac{x-3}{3}=\frac{y-2}{2}=\frac{z+5}{-3}$ va $\frac{x-2}{3}=\frac{y-1}{2}=\frac{z+3}{-3}$ parallel to‘g‘ri chiziqlardan o‘tuvchi tekislik tenglamasini yozing.(Javob: $x-3y-z-2=0$.)

3.12. $\frac{x+2}{2}=\frac{y-1}{3}=\frac{z+6}{6}$ va $\begin{cases} 4x-y-z-2=0 \\ 2x+y-2z+17=0 \end{cases}$ to‘g‘ri chiziqlar orasidagi burchakni toping. (Javob: $\varphi=\arccos\frac{20}{21}\approx 17°48'$.)

3.13. $M(3, 4, 0)$ nuqtadan va $\frac{x+1}{1}=\frac{y+2}{2}=\frac{z+1}{2}$ to‘g‘ri chiziqgacha bo‘lgan masofani toping. (Javob: $d=\sqrt{17}.$)

3.14. $\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z-3}{3}$ va $\begin{cases} x-2y+3z-2=0 \\ 2x+3y-8z+3=0 \end{cases}$ to‘g‘ri chiziqlar orasidagi burchakni toping. (Javob: $\varphi=90°.$)

3.15. $A(-5, -3, 2)$ nuqtaning $\frac{x+1}{2}=\frac{y+1}{-3}=\frac{z}{1}$ to‘g‘ri chiziqga nisbatan simmetrik nuqtasini toping. (Javob: (3, 1,-2).)

3.16. $M(3,\ -1,\ 5)$ nuqtadan o‘tuvchi va $\begin{cases} 3x+y+3z+1=0 \\ x-2y-z+4=0 \end{cases}$ to‘g‘ri chiziqga perpendikulyar tekislik tenglamasini toping. (Javob: $5x+6y-7z+26=0$.)

3.17. $M(5,-1,3)$ nuqtaning $3x-y+2z-8=0$ tekislikdagi proyeksiyasini toping.(Javob: $(2,\ 0,\ 1)$.)

3.18. $\frac{x}{3}=\frac{y-1}{3}=\frac{z-1}{-1}$ to‘g‘ri chiziqdan o‘tuvchi va $x-7y+5z-5=0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x-2y-3z+5=0$.)

3.19. $x=2t+1, y=4t+2, z=5t+3$ to‘g‘ri chiziqga nisbatan $M(4,3,10)$ nuqtaga simmetrik bo‘lgan M' nuqtani toping. (Javob: $M'(2,9,6)$.)

3.20. $\frac{x-5}{2}=\frac{y+3}{-1}=\frac{z+2}{-1}$ to‘g‘ri chiziqdan va $M(4,-1,3)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing. (Javob: $x+3y-z+2=0$.)

3.21. $\frac{x-3}{2}=\frac{y+1}{-1}=\frac{z-2}{-1}$ to‘g‘ri chiziqdan va $M(3,1,8)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing. (Javob: $x+3y-z+2=0$.)

3.22. $\frac{x+4}{3}=\frac{y-2}{1}=\frac{z}{1}$ va $\begin{cases} x-y+z+3=0 \\ 3x-y-z+7=0 \end{cases}$ to‘g‘ri chiziqlar kesishuvchi ekanini isbotlang, kesishish nuqtasini toping. (Javob: $(-1,3,1)$.)

3.23. $\begin{cases} x+y-z-5=0 \\ x-2y+z=0 \end{cases}$ to‘g‘ri chiziq bilan $x+2y+3z-30=0$ tekislik perpendikulyar ekanini isbotlang va kesishish nuqtasini toping. (Javob: $(5,5,5)$.

3.24. $\begin{cases} 2x-y-2z-2=0 \\ x-y-4=0 \end{cases}$ va $\begin{cases} 3x-2y-2z+2=0 \\ y-2z-1=0 \end{cases}$ to‘g‘ri chiziqlar o‘zaro parallel ekanini isbotlang, ular orasidagi masofani toping. (Javob: $d=\sqrt{17}.$)

3.25. $M(1,-4,-5)$ nuqtadan $x=4t+6, y=3t+4, z=2t+2$ to‘g‘ri chiziqgacha bo‘lgan masofani toping. (Javob: $\sqrt{22}.$)

3.26. $A(2,\ 6,\ 9)$ nuqtaning $\begin{cases} x-2y+2z+1=0 \\ 3x-2y+z+1=0 \end{cases}$ to‘g‘ri chiziqdagi proyeksiyasini toping. (Javob: $(3,8,6)$.)

3.27. $\begin{cases} x+2y+2z-1=0 \\ 3x+y-4z+2=0 \end{cases}$ to‘g‘ri chiziq bilan $A(2,-2,0)$ va $B(3,-3,-1)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq orasidagi burchakni toping. (Javob: $\varphi=\arccos\frac{\sqrt{3}}{3}$.)

3.28. $x=2t-3,\ y=3t+1,\ z=-t-2$ to‘g‘ri chiziqdan o‘tuvchi va $3x-2y+z=0$ tekislikga perpendikulyar tekislik tenglamasini yozing. (Javob: $x-5y-13z-18=0$.)

3.29. $A(2,1,-3)$ nuqtadan o‘tub, $\begin{cases} x-2y+2z=0 \\ 3x-2y+z+1=0 \end{cases}$ to‘g‘ri chiziqga parallel bo‘lgan to‘g‘ri chiziqning parametrik tenglamasini yozing. (Javob: $x=2t+2,\ y=5t+1,\ z=4t-3.$)

3.30. $\begin{cases} 5x-2y+2=0 \\ 2x-z+1=0 \end{cases}$ to‘g‘ri chiziq va $A(4,\ 6,\ 1)$ va $B(0,\ -4,-7)$ nuqtalardan o‘tuvchi to‘g‘ri chiziqlarning parallelligini isbotlang va ulardan o‘tuvchi tekislik tenglamasini yozing. (Javob: $10x-8y+5z+3=0.$)

IV BOB. MATEMATIK ANALIZ ASOSLARI

4.1. Kompleks sonlar va ular ustida amallar. Muavr va Eyler formulalari

Ushbu $z = x + iy$ ko‘rinishdagi songa ***kompleks son*** deyiladi, bu yerda x, y - haqiqiy sonlar, i esa $i^2 = -1$ bo‘lgan ***mavhum birlik***. x- kompleks sonning ***haqiqiy qismi***, y esa ***mavhum qismi*** deb ataladi va $\operatorname{Re} z = x$, $\operatorname{Im} z = y$ kabi belgilanadi. Agar $y = 0$ bo‘lsa, $z = x \in R$ agar $x = 0$ bo‘lsa, $z = ty$ ***sof mavhum son*** hosil bo‘ladi.

Geometrik nuqtai nazardan, har bir $z = x + iy$ kompleks songa koordinatalar tekisligida bitta M(x, y) nuqta (yoki $\overrightarrow{OM}$ vektor) va, aksincha, har bir M(x, y) nuqtaga bitta $z = x + iy$ kompleks son mos keladi. Barcha kompleks sonlar to‘plami C harfi bilan belgilanadi va $R \subset C$.

$z = x + iy$ va $\bar{z} = x - iy$ sonlar ***qo‘shma kompleks sonlar*** deyiladi.

$z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ ikkita kompyleks sonlar uchun quyidagi amallar o‘rinli:

1) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$;

2) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$;

3) $\dfrac{z_1}{z_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$.

Ma’lumki, har bir $z = x + iy$ kompleks son uchun $x = r\cos\varphi$, $y = r\sin\varphi$ formulalar o‘rinli. $r = \left|\overrightarrow{OM}\right| = \sqrt{x^2 + y^2}$ son $z = x + iy$ ***kompleks sonning moduli*** deyiladi, $\overrightarrow{OM}$ vektor va Ox o‘qining musbat yo‘nalishi bilan hosil qilgan φ burchagi esa ***kompleks sonning argumenti*** deyiladi va $\varphi = \arg z$ kabi belgilanadi. U quyidagi

$$\varphi = \begin{cases} arctg\dfrac{y}{x}, \text{ agar } x > 0,\ y > 0 \text{ bo`lsa;} \\ \pi + arctg\dfrac{y}{x}, \text{ agar } x < 0 \text{ bo`lsa;} \\ 2\pi + arctg\dfrac{y}{x}, \text{ agar } x > 0,\ y < 0 \text{ bo`lsa} \end{cases} \qquad (4.1)$$

formula bilan hisoblanadi. Har qanday $z = x + iy$ kompleks son

$$z = r(\cos\varphi + i\sin\varphi) \qquad (4.2)$$

trigonometrik shaklda yoki $z = re^{i\varphi}$ ***ko'rsatkichli shaklda*** ifodalanadi(chunki $e^{i\varphi} = \cos\varphi + i\sin\varphi$ Eyler formulasi o'rinli).

Agar $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ kompleks sonlar bo'lsa,

$$z_1 z_2 = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)); \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)). \qquad (4.3)$$

$z = r(\cos\varphi + i\sin\varphi)$ kompleks sonni *n*-***darajaga oshirish*** uchun ***Muavr formulasi***

$$z^n = r^n(\cos n\varphi + i\sin n\varphi) \qquad (4.4)$$

o'rinli. *n-ildiz chiqarish* uchun esa

$$z_k = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right), \quad k = 0,1,2,\dots,n-1 \qquad (4.5)$$

formula qo'llanadi.

11-Auditoriya topshiriqlari

1. $z_1 = 2 + 3i$, $z_2 = 3 - 4i$ va $z_3 = 5 - 2i$ bo'lsa, $(2z_1 + z_2)z_3$ ni hisoblang. (Javob: $39 - 4i$.)

2. $z_1 = 2 + 3i$, $z_2 = 3 - 4i$ va $z_3 = 1 - 2i$ bo'lsa, $((z_1 + z_3)z_2)/z_3$ ni hisoblang. (Javob: $\frac{31}{5} + \frac{17}{5}i$.)

3. Kompleks sonlarni trigonometrik shaklda ifodalang: *a*) $z_1 = -1 + i$, *b*) $z_2 = 3i$ *d*) $z_3 = 2 - 2i$, *e*) $z_4 = -4$.

4. $z_1 = -2 + 3i$ bo'lsa, $|z - z_1| < 1$ shartni qanoatlantiruvchi nuqtalarning geometrik o'rni qanday sohani ifodalaydi? (Javob: Markazi z_1 nuqtada bo'lgan R=1 radiusli doiraning ichki qismi.)

5. $z_1 = -1 + 3i$ bo'lsa, $|z + z_1| > 2$ shartni qanoatlantiruvchi nuqtalarning geometrik o'rni qanday sohani ifodalaydi? (Javob: Markazi $-z_1$ nuqtada bo'lgan *R*=2 radiusli doiraning tashqi qismi.)

6. $1 < |z - i| < 3$ shartni qanoatlantiruvchi nuqtalarning geometrik o'rni qanday sohani ifodalaydi? (Javob: Markazi $z = i$ nuqtada bo'lgan R_1=1 va R_2=3 radiusli aylanalar orasidagi halqa.)

7. $0 < \mathrm{Re}(3iz) < 2$ shartni qanoatlantiruvchi nuqtalarning geometrik o'rni qanday sohani ifodalaydi? (Javob: $y = 0$ va $y = -\frac{2}{3}$ to'g'ri chiziqlar orasidagi gorizontal polosa.)

8. $(1 - \sqrt{3}i)^5$ ni hisoblang.(Javob: $16 + 16i\sqrt{3}$.)

9. $z^3+1=0$ tenglamaning ildizlarini toping. (Javob: $z_0=\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ z_1=-1,$ $z_2=\frac{1}{2}-\frac{\sqrt{3}}{2}i$.)

10. Hisoblang: 1) $\sqrt[4]{-8-8i\sqrt{3}}$, 2) $\sqrt[3]{-2+2i}$..

11. Tenglamalarni yeching: 1) $z^3-8=0$, 2) $z^6+64=0$.

12. Eyler formulasidan foydalanib

$$cosx+\cos 2x+\cos 3x+...+\cos nx$$

yig‘indini hisoblang. ($\text{Javob}:\left(\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}\right)\Big/\sin\frac{x}{2}$.)

11-Mustaqil yechish uchun testlar

1. $z_1=1+3i$ va $z_2=3-2i$ uchun $2z_1+z_2$ ni hisoblang.

A) $5+4i$, B) $5-8i$, C) $5-4i$, D) $2+3i$.

2. $z_1=2-3i$ va $z_2=3-4i$ berilgan bo‘lsa, z_1z_2 ni hisoblang.

A) $18-17i$, B) $-6-i$, C) $-6-17i$, D) $6+17i$.

3. $z_1=3-4i$ va $z_2=2-i$ berilgan bo‘lsa, z_1/z_2 ni hisoblang.

A) $3-2i$, B) $-2-i$, C) $2-i$, D) $0{,}4-i$.

4. $z=2-2i\sqrt{3}$ kompleks sonning modulini toping.

A) $r=3$; B) $r=4$; C) $r=5$; D) $r=1$.

5. $z=-2+2i\sqrt{3}$ kompleks sonning argumentini toping.

A) $\varphi=\frac{\pi}{6}$; B) $\varphi=\frac{\pi}{3}$; C) $\varphi=\frac{5\pi}{6}$; D) $\varphi=\frac{2\pi}{3}$.

6. $z=2\sqrt{3}-2i$ kompleks sonning trigonometrik shaklini toping.

A) $4(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6})$; B) $4(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6})$;

C) $4(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$; D) $4(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$.

7. $z=2\sqrt{3}+2i$ kompleks sonning ko‘rsatkichli shaklini toping.

A) $z=4e^{i\frac{\pi}{3}}$; B) $z=4e^{i\frac{\pi}{6}}$; C) $z=4e^{i\frac{5\pi}{6}}$; D) $z=4e^{-i\frac{\pi}{3}}$.

8. $z=-\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}$ kompleks sonning ko‘rsatkichli shaklini toping.

A) $z=e^{i\frac{6\pi}{7}}$; B) $z=e^{i\frac{\pi}{7}}$; C) $z=e^{-i\frac{\pi}{7}}$; D) $z=e^{-i\frac{6\pi}{7}}$.

9. $(\sqrt{3}-i)^4$ ni hisoblang.

A) $8+8i\sqrt{3}$, B) $8-8i\sqrt{3}$, C) $-8+8i\sqrt{3}$, D) $-8-8i\sqrt{3}$.

10. $z^3+8=0$ tenglamaning yechimi noto‘g‘ri berilgan javobni aniqlang.

A) $-1+i\sqrt{3}$, B) $1+i\sqrt{3}$, C) -2, D) $1-i\sqrt{3}$.

4.2. Funksiya va uning berilish usullari

Agar ixtiyoriy $x\in D$ elementga biror f qoida bilan yagona $y-$ element mos qo‘yilgan bo‘lsa, u holda $y=f(x)$ funksiya berilgan deyiladi. $x-$ ***erkli o‘zgaruvchi*** yoki ***argument*** deyiladi. $D-$ ***aniqlanish soha***, y ning qabul qiladigan qiymatlari esa ***qiymatlar to‘plami*** (yoki ***o‘zgarish sohasi***) deyiladi va E harfi bilan belgilanadi.

Funksiya *jadval usulda, grafik usulda* va *analitik usulda* beriladi. Analitik usulda berilgan $y=f(x)$ funksiyaning D va E sohalari ko‘p hollarda ko‘rsatilmaydi, ammo tabiiy ravishda $y=f(x)$ funksiya xossalariga ko‘ra aniqlanadi.

4.1-misol

Ushbu $y=\sqrt{7-6x-x^2}$ funksiyaning aniqlanish sohasi va qiymatlar to‘plamini toping.

► Kvadrat funksiya $7-6x-x^2\geq 0$ da aniqlangan. Kvadrat uchhadning ildizlari $x_1=-7,\ x_2=1$. Yuqoridagi tengsizlik $-(x+7)(x-1)\geq 0$ tengsizlikga teng kuchli bo‘lib, $-7\leq x\leq 1$ yechimga ega. Funksiyaning aniqlanish sohasi $D=[-7;1]$. D sohada $0\leq 7-6x-x^2\leq 16$ bo‘lgani uchun qiymatlar to‘plami $E=[0;4]$. ◄

$u=\varphi(x)$ funksiya D to‘plamda aniqlangan bo‘lib, uning qiymatlar to‘plami G bo‘lsin. Agar $y=f(u)$ funksiya G to‘plamda aniqlangan funksiya bo‘lsa, u holda $y=f(\varphi(x))$ ***murakkab funksiya*** deyiladi. $y=f(\varphi(x))$ funksiya ikkita $y=f(u)$ va $u=\varphi(x)$ funksiyalarning kompozitsiyasi yoki φ funksiyaning f funksiyasi deb ataladi. Murakkab funksiya ikki yoki undan ortiq funksiyadan tuzilgan bo‘ladi.

4.2-misol

Quyidagi murakkab funksiyalar nechta funksiyadan tashkil topgan:

$a)\ y=\sin(x^2+1)$, $b)\ y=\ln\sin 3^x$.

► *a*) $y=\sin(x^2+1)$ ikkita $y=\sin u$ va $u=x^2+1$ funksiyalardan tashkil topgan.

b) $y=\ln\sin 3^x$ funksiya uchta $y=\ln u$, $u=\sin v$ va $v=3^x$ funksiyalardan tashkil topgan.◄

$y=f(x)$ ***funksiyaning grafigi*** deb Oxy tekisligidagi koordinatalari f qoida bilan bog'langan $M(x;y)$ nuqtalar to'plamiga aytiladi.

$\forall x\in D$ uchun $-x\in D$ bo'lsin. Agar $\forall x\in D$ uchun $f(-x)=f(x)$ bo'lsa, $y=f(x)$ ***juft funksiya*** deyiladi. Agar $\forall x\in D$ uchun $f(-x)=-f(x)$ bo'lsa, $y=f(x)$ ***toq funksiya*** deyiladi.

Agar $y=f(x)$ funksiya D sohani E ga bir qiymatli akslantirsa, u holda x ni y orqali ifodalovchi funksiya $x=g(y)$ mavjud va u $y=f(x)$ ga ***teskari funksiya*** deyiladi. $x=g(y)$ funksiyaning aniqlanish sohasi E , qiymatlar to'plami esa D ga teng. $y\equiv f(g(y))$ va $x\equiv g(f(x))$ bolgani uchun $y=f(x)$ va $x=g(y)$ funksiyalar o'zaro teskari. O'z aro teskari $y=f(x)$ va $x=g(y)$ funksiyalarning grafigi birinchi va uchinchi chorak bissektrisa chizig'i $y=x$ ga nisbatan simmetrikdir.

4.3-misol

Ushbu $y=x^2-6x+11$ funksiyaga $(-\infty;3]$ oraliqdagi teskari funksiyani toping.

► Berilgan funksiyadan to'la kvadrat ajratamiz

$$y=(x-3)^2+2.$$

Bu tenglikdan x ni topamiz

$$x=3\pm\sqrt{y-2}$$

va $x\in(-\infty;3]$ ekanini e'tiborga olgan holda tanlaymiz

$$x=3-\sqrt{y-2}.$$

x ni y ga almashtirib, izlangan funksiyani topamiz

$$y=3-\sqrt{x-2}.$$ ◄

12-Auditoriya topshiriqlari

1. Quyidagi funksiyalarning aniqlanish sohasini toping:

1) $y=\sqrt{x^2+6x+5}$; 2) $y=\lg(5-4x-x^2)$; 3) $y=\arcsin\frac{1-x}{3}$; 4) $y=1/\sqrt{2^{3x}-4}$.

(Javob: 1) $(-\infty;-1]\cup[-5;\ \infty)$; 2) $(-5;\ 1)$; 3) $[-2;\ 4]$; 4) $(2/3;\ \infty)$.)

2. Quyidagi murakkab funksiyalar nechta funksiyadan tashkil topgan:

1) $y=\lg\sin x^2$; 2) $y=2^{|\cos x|}$; 3) $y=\sqrt[3]{arctg3^{x^2}}$; 4) $y=\cos^3\sqrt{\arcsin e^x}$?

(Javob: 1) 3 ta; 2) 3 ta; 3) 4 ta; 4) 5 ta.)

3. $y=\begin{cases} x, \text{ agar } x\le 0 \text{ bo`lsa,} \\ x^2, \text{ agar } x>0 \text{ bo`lsa} \end{cases}$ funksiyaning teskari funksiyasini toping. Berilgan funksiya va uning teskari funksiyasi grafigini yasang.

4. Quyidagi funksiyalarning grafiklarini yasang:

1) $y=\frac{2x+3}{x-1}$; 2) $y=|5-4x-x^2|$; 3) $y=2\sin 2x$; 4) $y=|x-2|+|x+1|$.

5. Quyidagi funksiyalarning juft yoki toqligini aniqlang:

1) $y=\frac{2x^2+3}{x^2-1}$; 2) $y=|5-4x-x^3|$; 3) $y=2\sin 2x+x$; 4) $y=|x-2|+|x+2|$

(Javob: 1) Juft ; 2) Juft ham, toq ham emas; 3) Toq; 4) Juft funksiya.)

6. Agar $f(x)=\lg x,\ \varphi(x)=\sin 2x,\ g(x)=x^2+x$ bo‘lsa, funksiyani $y=g(f(\varphi(x)))$ toping. (Javob: $y=\lg^2\sin 2x+\lg\sin 2x$.)

12-Mustaqil yechish uchun testlar

1. $y=\arccos\frac{2x}{x+3}$ funksiyaning aniqlanish sohasini toping.

A) $(-1;\ 1)$; B) $[-1;\ 1]$; C) $(-\infty;-1]\cup[3;\ \infty)$; D) $[-1;\ 3]$.

2. $y=\sqrt[3]{\lg|\sin x^3|}$ murakkab funksiya nechta funksiyadan tashkil topgan?

A) 3 ta; B) 4 ta; C) 5 ta; D) 2 ta.

3. Quyidagi funksiyalardan qaysilari juft ekanligini aniqlang:

1) $y=x^3\sin 2x$; 2) $y=x\frac{e^x-1}{e^x+1}$; 3) $y=|5-4x+x^2|$; 4) $y=x^2cos(x+1)$.

A) 1), 3) B) 1), 2) C) 1), 4) D) 3), 4).

4. $y=x^2-4x+7$ funksiyaga $(-\infty;2]$ oraliqdagi teskari funksiyani toping.

A) $y=2+\sqrt{x-3}$; B) $y=2-\sqrt{x-3}$; C) $y=2+\sqrt{x+3}$; D) $y=3-\sqrt{x-2}$

5. $y=\dfrac{x-1}{2-3x}$ funksiyaga teskari funksiyani toping.

A) $y=\dfrac{2x+1}{3x+1}$; B) $y=\dfrac{2-3x}{x-1}$; C) $y=\dfrac{2x-1}{3x+1}$; D) $y=\dfrac{x+1}{2+3x}$.

4.3. Sonli ketma-ketlik va funksiya limiti

Natural sonlar to‘plamida aniqlangan funksiya ***sonli ketma-ketlik*** deyiladi. $x_n=f(n), n\in N$. x_n – ketma-ketlikning n- hadi uning ***umumiy hadi*** deb ataladi. Sonli ketma-ketlik $\{x_n\}$orqali belgilanadi.

Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $N=N(\varepsilon)>0$ son mavjud bo‘lsaki, barcha $n\geq N$ lar uchun $|x_n-a|<\varepsilon$ tengsizlik bajarilsa, a o‘z garmas son $\{x_n\}$ ***ketma-ketlikning limiti*** deyiladi va quyidagicha belgilanadi:

$$\lim_{n\to\infty}x_n=a\,.$$

4.4-misol

Ushbu $x_n=\dfrac{1-2n^2}{2+4n^2}$ ketma ketlikning limiti $a=-\dfrac{1}{2}$ ekanini isbotlang.

► $\forall\varepsilon>0$ son uchun unga mos $N=N(\varepsilon)>0$ son mavjudligini ko‘rsatamiz:

$$|x_n-a|=\left|\frac{1-2n^2}{2+4n^2}+\frac{1}{2}\right|<\varepsilon \text{ yoki } \left|\frac{2-4n^2+2+4n^2}{2(2+4n^2)}\right|<\varepsilon,\quad \frac{1}{1+2n^2}<\varepsilon.$$

Bundan, $n>\sqrt{\dfrac{1}{2\varepsilon}-\dfrac{1}{2}}$, demak, $N(\varepsilon)=\left[\dfrac{1}{2\varepsilon}-\dfrac{1}{2}\right]$ deb tanlash kifoya. ◄

$y=f(x)$ funksiya $x=x_0$ nuqtaning biror atrofida aniqlangan bo‘lsin. Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son mavjud bo‘lsaki, $0<|x-x_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha x larda $|f(x)-A|<\varepsilon$ tengsizlik o‘rinli bo‘lsa, A son *$y=f(x)$ funksiyaning $x\to x_0$ dagi* ***limiti*** deyiladi va quyidagicha belgilanadi:

$$\lim_{x\to x_0}f(x)=A.$$

Xuddi shu kabi

$$\lim_{\substack{x\to x_0\\ x<x_0}} f(x)=A \quad \left(\lim_{\substack{x\to x_0\\ x>x_0}} f(x)=A\right)$$

limit mavjud bo‘lsa, bu limit $y=f(x)$ funksiyaning x_0 ***nuqtadagi chap(o‘ng) limiti*** deyiladi va $\lim_{x\to x_0-0} f(x)=f(x_0-0)\left(\lim_{x\to x_0+0} f(x)=f(x_0+0)\right)$ kabi belgilanadi.

Agar ixtiyoriy $\varepsilon>0$ son uchun shunday $N=N(\varepsilon)>0$ son mavjud bo‘lsaki, $|x|>N$ tengsizlikni qanoatlantiruvchi barcha x larda $|f(x)-A|<\varepsilon$ tengsizlik o‘rinli bo‘lsa, A son $y=f(x)$ *funksiyaning* $x\to\infty$ *dagi* ***limiti*** deb ataladi va $\lim_{x\to\infty} f(x)=A$ kabi belgilanadi.

$f(x)$ va $g(x)$ funksiyalar $x=x_0$ nuqtaning biror atrofida aniqlangan bo‘lib, $\lim_{x\to x_0} f(x)=A,\ \lim_{x\to x_0} g(x)=B$ bo‘lsin. U holda quyidagi tengliklar o‘rinli:

1) $\lim_{x\to x_0}(f(x)\pm g(x))=A\pm B$

2) $\lim_{x\to x_0}(f(x)\cdot g(x))=A\cdot B$

3) $\lim_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{A}{B},\ B\neq 0$

Limitlarni hisoblashda quyidagilardan foydalanamiz:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{a}{0}=\infty,\quad \lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\infty}{a}=\infty,\quad \lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{a}{\infty}=0,\quad a-\text{chekli son.}$$

4.5-misol

Quyidagi limitlarni hisoblang:

$$1)\ \lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right);\qquad 2)\ \lim_{x\to\pm\infty}\frac{2x^3+x-2}{x^3+4x+5}.$$

► 1) Bu yerda ∞ - ∞ tipidagi aniqmaslik, uni yechish uchun kasrga umumiy maxraj beriladi va sodda ko‘rinishga olib kelinadi:

$$\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)=\lim_{x\to 2}\frac{2-x}{(x-2)(x+2)}=\lim_{x\to 2}\left(-\frac{1}{x+2}\right)=-\frac{1}{4}.$$

2) Bu holda ∞/∞ tipidagi aniqmaslik, uni yechish uchun kasrning surat va maxrajini x ning yuqori darajasi x^3 ga bo'lamiz:

$$\lim_{x\to\infty}\frac{2x^3+x-2}{x^3+4x+5}=\lim_{x\to\infty}\frac{2+\frac{1}{x^2}-\frac{2}{x^3}}{1+\frac{4}{x^2}+\frac{5}{x^3}}=\frac{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\frac{1}{x^2}-\lim\limits_{x\to\infty}\frac{2}{x^3}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{4}{x^2}+\lim\limits_{x\to\infty}\frac{5}{x^3}}=\frac{2+0-0}{1+0+0}=2.$$ ◄

13-Auditoriya topshiriqlari

1. $x_n=\frac{3n-1}{n+2}$ ketma ketlikning limiti $a=3$ ekanini isbotlang.

Berilgan funksiyalarning limitlarini hisoblang.

2. $\lim\limits_{n\to\infty}\frac{2n^2-5n+2}{4-n^2}$. (Javob: -2)

3. $\lim\limits_{n\to\infty}\frac{\left(\sqrt{n^2+5}+2n\right)^2}{\sqrt[3]{n^6+2}}$.(Javob: 9)

4. $\lim\limits_{n\to\infty}\frac{(n+1)!+(n+2)!}{(n+3)!}$. (Javob: 0)

5. $\lim\limits_{x\to\frac{1}{2}}\frac{8x^3-1}{6x^2-5x+1}$. (Javob: 6)

6. $\lim\limits_{x\to1}\left(\frac{x+2}{x^2-5x+4}+\frac{x-4}{3\left(x^2-3x+2\right)}\right)$.(Javob:0)

7. $\lim\limits_{x\to2}\frac{\sqrt{x+2}-2}{3-\sqrt{x+7}}$. (Javob: $-\frac{3}{2}$)

8. $\lim\limits_{x\to+\infty}x\left(\sqrt{x^2+4}-x\right)$. (Javob: 2)

13-Mustaqil yechish uchun testlar

Sonli ketma-ketlik limitlarini hisoblang.

1. $\lim\limits_{n\to\infty}\frac{n^2-\sqrt{n^3+1}}{\sqrt[3]{n^6+2}-n}$.

A) 1; B) 2; C) 3; D) 0.

2. $\lim\limits_{n\to\infty}\left[\sqrt{n(n+2)}-\sqrt{n^2-2n+3}\right]$.

A) 1; B) 2; C) 3; D) 0.

3. $\lim\limits_{n\to\infty}\dfrac{2^{n+1}+3^{n+1}}{2^n+3^n}$.

A) 2; B) 2,5; C) 3; D) 5.

Funksiya limitlarini hisoblang.

4. $\lim\limits_{x\to 2}\dfrac{x^2-7x+10}{x^2-5x+6}$.

A) 1; B) 2; C) 3; D) 24.

5. $\lim\limits_{x\to 3}\dfrac{x^2-9}{\sqrt{x+1}-2}$.

A) 0; B) 12; C) 24; D) ∞.

6. $\lim\limits_{x\to+\infty}x\left(\sqrt{x^2+3}-\sqrt{x^2-1}\right)$.

A) 1; B) 2; C) 3; D) 0.

7. $\lim\limits_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)$.

A) 0; B) -2; C) 3; D) -1.

8. $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{1+x}-\sqrt{2x}}$.

A) $1{,}5\sqrt{2}$; B) $2\sqrt{2}/3$; C) $-2\sqrt{2}/3$ D) $-1{,}5\sqrt{2}$

4.4. Ajoyib limitlar

Funksiyalarning limitlarini hisoblashda ***1- ajoyib limit*** va ***2- ajoyib limit*** deb ataluvchi

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \tag{4.6}$$

va

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e\approx 2{,}71828 \tag{4.7}$$

limitlar, hamda ularga asoslangan quyidagi formulalar keng qo'llanadi:

1) $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\sin\beta x}=\dfrac{\alpha}{\beta}$, 2) $\lim\limits_{x\to 0}\dfrac{tgkx}{x}=k$, 3) $\lim\limits_{x\to 0}\dfrac{\arcsin kx}{x}=k$, 4) $\lim\limits_{x\to 0}\dfrac{arctgkx}{x}=k$,

5) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$, 6) $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\log_a e$, 7) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a\,(a>0)$,

8) $\lim\limits_{x\to 0}\dfrac{(1+x)^m-1}{x}=m$, 9) $\lim\limits_{x\to\infty}\left(1\pm\dfrac{n}{x}\right)^x=e^{\pm n}$, 10) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x+n}=e$.

4.6-misol

Quyidagi limitlarni hisoblang:

$$1)\ \lim_{x\to 0}\frac{\sin 7x}{\sin 3x};\quad 2)\ \lim_{x\to\infty}\left(\frac{3x+2}{3x-1}\right)^{2x+1};\quad 3)\ \lim_{x\to\pi}\frac{tg2x}{\sin 3x}.$$

► 1) Berilgan limitni hisoblashda 1-ajoyib limitdan foydalanamiz. Buning uchun quyidagicha almashtirish bajaramiz:

$$\lim_{x\to 0}\frac{\sin 7x}{\sin 3x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\frac{7x}{\frac{\sin 3x}{3x}\cdot 3x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\frac{7}{3\lim\limits_{x\to 0}\frac{\sin 3x}{3x}}=\frac{7}{3}.$$

2) Bu limit va shu kabi limitlarni hisoblashda berilgan funksiya asosiga birni qo‘shib ayriladi va 2-ajoyib limitga keltiriladi:

$$\lim_{x\to\infty}\left(\frac{3x+2}{3x-1}\right)^{2x+1}=\lim_{x\to\infty}\left(1+\frac{3x+2}{3x-1}-1\right)^{2x+1}=\lim_{x\to\infty}\left[\left(1+\frac{3}{3x-1}\right)^{\frac{3x-1}{3}}\right]^{\frac{6x+3}{3x-1}}=e^{\lim\limits_{x\to\infty}\frac{6x+3}{3x-1}}=e^{2}.$$

3) Bu limitni hisoblashda trigonometrik funksiyalarning davriyligidan va keltirish formulalaridan foydalanib, 1-ajoyib limitga keltiramiz:

$$\lim_{x\to\pi}\frac{tg2x}{\sin 3x}=\lim_{x\to\pi}\frac{tg(2x-2\pi)}{-\sin(3x-3\pi)}=-\lim_{x\to\pi}\frac{\frac{\sin(2x-2\pi)}{2x-2\pi}}{\frac{\sin(3x-3\pi)}{3x-3\pi}}\cdot\lim_{x\to\pi}\frac{2}{3\cos(2x-2\pi)}=-\frac{2}{3}.$$ ◄

14-Auditoriya topshiriqlari

Berilgan limitlarni hisoblang

1. $\lim\limits_{x\to 0}\dfrac{3x^2-5x}{\sin 3x}$ (Javob: $-\dfrac{5}{3}$)

2. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{\sin[\pi(x+2)]}$ (Javob: $\dfrac{1}{2\pi}$)

3. $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}$ (Javob: $\dfrac{3}{4}$)

4. $\lim\limits_{x\to 0}(1-x)tg\dfrac{\pi x}{2}$ (Javob: $\dfrac{2}{\pi}$)

5. $\lim\limits_{x\to 0}\dfrac{\sqrt{3x+4}-2}{\cos(\pi(x+1)/2)}$ (Javob: $-\dfrac{3}{2\pi}$)

6. $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ (Javob: $\dfrac{\sqrt{2}}{8}$)

7. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x+2tg^2 x}{x\sin 3x}$ (Javob: $\dfrac{4}{3}$)

8. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-\cos x}{\sin^2 x}$ (Javob: 1)

9. $\lim\limits_{x\to -\infty}\left(\dfrac{x-1}{2x+3}\right)^{3x}$ (Javob: ∞)

10. $\lim\limits_{x\to -\infty}\left(\dfrac{3x-2}{3x+2}\right)^{2-x}$ (Javob: $\sqrt[3]{e^4}$)

11. $\lim\limits_{x\to \infty}x\left(\ln(2x+3)-\ln(2x-1)\right)$ (Javob: 2)

12. $\lim\limits_{x\to \infty}x\left(\ln(1+3tg^2 x)\right)ctg^2 x$ (Javob: 3)

13. $\lim\limits_{x\to \frac{\pi}{4}}\left(\sin 2x\right)^{tg^2 x}$ (Javob: $\dfrac{1}{\sqrt{e}}$)

14. $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}$ (Javob: $\dfrac{1}{e}$)

15. $\lim\limits_{x\to e}\dfrac{e^{2x}-1}{\sin 3x}$ (Javob: $\dfrac{2}{3}$)

16. $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{x^2}$ (Javob: $-\dfrac{1}{2}$)

14-Mustaqil yechish uchun testlar

Berilgan limitlarni hisoblang.

1. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sqrt{x+1}-1}$.

A) 1,5; B) 2; C) 3; D) 6.

2. $\lim\limits_{x\to 0}\dfrac{1-\cos 3x}{x\sin x}$.

A) 3,5; B) 4; C) 4,5; D) 6.

3. $\lim\limits_{x\to \pi}\dfrac{1+\cos x}{\sin^2 2x}$.

A) 1,5; B) 0,25; C) 0,125; D) 0.5.

4. $\lim\limits_{x\to \pi}\dfrac{\cos\frac{x}{2}}{x-\pi}$.

A) 1,5; B) - 0,25; C) - 0,5; D) 0.5.

5. $\lim\limits_{x\to 0}\dfrac{\arcsin 2x}{\sqrt[3]{x+1}-1}$

A) 1,5; B) 2; C) 3; D) 6.

6. $\lim\limits_{x\to +\infty}\left(\dfrac{x+2}{2x-1}\right)^{x+3}$.

A) 0; B) 2; C) 0,5; D) ∞.

7. $\lim\limits_{x\to 0}(1-3x)^{\frac{x-1}{x}}$.

A) e; B) e^2; C) e^3; D) 0.

8. $\lim\limits_{x\to +\infty}\left(\frac{x^2+2}{x^2-x}\right)^x$.

A) e; B) e^2; C) e^3; D) 0.

9. $\lim\limits_{x\to +\infty}(x+1)(\ln(3x+2)-\ln(3x-1))$.

A) e; B) e^2; C) 1; D) 0.

10. $\lim\limits_{x\to 0}\frac{e^{\sin x}-1}{\sin 2x}$.

A) 1,5; B) 2; C) 2,5; D) 0.5.

4.5. Funksiya uzluksizligi. Uzilish turlari

Agar $y=f(x)$ funksiya $x=x_0$ nuqtaning biror atrofida aniqlangan bo‘lib, $\lim\limits_{x\to x_0} f(x)=f(x_0)$ bo‘lsa, u holda $y=f(x)$ funksiya $x=x_0$ ***nuqtada uzluksiz*** deyiladi. Bu ta’rif uzluksizlikning quyidagi shartlarini o‘z ichiga oladi:

1) $f(x)$ funksiya $x=x_0$ nuqtaning biror atrofida aniqlangan;

2) $\lim\limits_{x\to x_0-0} f(x)$, $\lim\limits_{x\to x_0+0} f(x)$ chekli limitlar mavjud;

3) ular o‘z aro teng $\lim\limits_{x\to x_0-0} f(x)=\lim\limits_{x\to x_0+0} f(x)$;

4) bu limit $f(x)$ funksiyaning $x=x_0$ nuqtadagi qiymatiga teng.

$y=f(x)$ funksiya x_0 nuqtada uzluksiz bo‘lishi uchun argumentning cheksiz kichik orttirmasi Δx ga funksiyaning cheksiz kichik orttirmasi Δy mos kelishi zarur va yetarli, ya’ni uzluksizlik $\lim\limits_{\Delta x\to 0}\Delta y=0$ shart bajarilishiga teng kuchli.

4.7-misol

Berilgan $y=x^2$ funksiya ixtiyoriy $x\in R$ nuqtada uzluksiz ekanini isbotlang.

► Argumentning ixtiyoriy Δx orttirmasida funksiya orttirmasi

$$\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+\Delta x^2.$$

U holda

$$\lim_{\Delta x\to 0}\Delta y=\lim_{\Delta x\to 0}\left(2x\Delta x+\Delta x^2\right)=0.$$

Bundan, $y=x^2$ funksiya butun sonlar o‘qida uzluksiz ekani kelib chiqadi.◄

x_0 nuqtada yuqoridagi shartlardan kamida bittasi bajarilmasa, x_0 nuqta $y=f(x)$ funksiyaning ***uzilish nuqtasi*** deyiladi. Agar x_0 nuqtada $f(x_0-0)$, $f(x_0+0)$

chekli limitlar mavjud va $f(x_0-0)\neq f(x_0+0)$ bo‘lsa, x_0 ***birinchi tur uzilish nuqtasi*** deyiladi. Agar $f(x_0-0)$ yoki $f(x_0+0)$ limitlardan hech bo‘lmaganda bittasi mavjud bo‘lmasa yoki cheksizlikka teng bo‘lsa, x_0 ***ikkinchi tur uzilish nuqtasi*** deyiladi. Agar x_0 nuqtada $f(x_0-0)$, $f(x_0+0)$ chekli limitlar mavjud va $f(x_0-0)=f(x_0+0)$ bo‘lib, x_0 nuqtada funksiya aniqlanmagan bo‘lsa, x_0 ***yo‘qotish mumkin bo‘lgan uzilish nuqtasi*** deyiladi.

4.8-misol

Ushbu $y=\frac{\sin x}{x}$ funksiyaning uzilish nuqtasini toping. Uzilish turini aniqlang.

► $\lim\limits_{x\to -0}\frac{\sin x}{x}=\lim\limits_{x\to +0}\frac{\sin x}{x}=1$ bo‘lgani holda $x=0$ da funksiya aniqlanmagan. Demak, $x=0$ yo‘qotish mumkin bo‘lgan uzilish nuqtadir. ◄

4.9-misol

Ushbu $y=\frac{x-2}{|x-2|}$ funksiyaning uzilish nuqtasini toping. Uzilish turini aniqlang.

► Berilgan funksiya $x=2$ nuqtada aniqlanmagan. $\lim\limits_{x\to 2-0} y=-1$, $\lim\limits_{x\to 2+0} y=1$ chekli limitlar mavjud va o‘z aro teng bo‘lmagani uchun $x=2$ birinchi tur uzilish nuqtasi bo‘ladi. ◄

15-Auditoriya topshiriqlari

1. $y=2-\frac{|x|}{x}$ funksiyani uzluksizlikga tekshiring va grafigini yasang.(Javob: $x=0$ birinchi tur uzilish nuqtasi.)

2. $y=\begin{cases} x+1, & \text{agar } x\le 1 \text{ bo`lsa,} \\ 3-ax^2, & \text{agar } x>1 \text{ bo`lsa} \end{cases}$ funksiya a ning qanday qiymatida uzluksiz bo‘ladi? (Javob: $a=1$.)

3. $y=\begin{cases} \frac{x-1}{|x-1|}, & \text{agar } x\neq 1 \text{ bo`lsa,} \\ 0, & \text{agar } x=1 \text{ bo`lsa} \end{cases}$ funksiyani uzluksizlikga tekshiring va uzilish turini aniqlang. (Javob: $x=1$ - birinchi tur uzilish nuqtasi.)

4. $x=0$ nuqta $y=\dfrac{1}{1+2^{\frac{1}{x}}}$ funksiyaning 1-tur uzilish nuqtasi ekanini isbotlang. $x=0$ nuqta atrofida grafigini chizing.

5. $y=2^{\frac{1}{3-x}}$ funksiyani $x=2$ va $x=3$ nuqtalarda uzluksizlikga tekshiring. (Javob: $x=2$ da uzluksiz, $x=3$ - ikkinchi tur uzilish nuqtasi.)

6. $y=\left(1+\dfrac{1}{x}\right)^{x}$ funksiyani uzluksizlikga tekshiring va uzilish turini aniqlang. (Javob: $x=-1$ - ikkinchi tur uzilish nuqtasi, $x=0$ - yo‘qotish mumkin bo‘lgan uzilish nuqtasi.)

7. $y=\dfrac{\sin x}{\pi^2-x^2}$ funksiyani uzluksizlikga tekshiring va uzilish turini aniqlang. (Javob: $x=\pi$ -yo‘qotish mumkin bo‘lgan uzilish nuqtasi.)

15-Mustaqil yechish uchun testlar

1. a ning qanday qiymatida $y=\begin{cases} x^2-1, & \text{agar } x\le 2 \text{ bo`lsa,} \\ 7-ax, & \text{agar } x>2 \text{ bo`lsa} \end{cases}$ funksiya uzluksiz bo‘ladi?

A) 1; B) 2; C) 3; D) 4.

2. $y=arctg\dfrac{1}{x}$ funksiyani $x=0$ nuqtada uzluksizlikka tekshirilsin.

A) Uzluksiz; B) 1-tur uzilish; C) 2-tur uzilish; D) Yo‘qotish mumkin bo‘lgan uzilish.

3. a va b ning qanday qiymatlarida $y=\begin{cases} x-1, & \text{agar } x\le -1 \text{ bo`lsa} \\ ax^2+b, & \text{agar } -1<x<2 \text{ bo`lsa} \\ 5-2x, & \text{agar } x\ge 2 \text{ bo`lsa} \end{cases}$ funksiya uzluksiz bo‘ladi?

A) -4 va 2; B) 2 va -4; C) 1 va -3; D) -3 va 1.

4. $y=\begin{cases} 3x+5, & \text{agar } x\le -1 \text{ bo`lsa,} \\ 1-x, & \text{agar } -1<x\le 1 \text{ bo`lsa,} \\ \ln x, & \text{agar } x>1 \text{ bo`lsa} \end{cases}$ funksiyaning uzilish nuqtalarini toping.

A) $x=1$; B) $x=\pm 1$; C) $x=-1$; D) funksiya uzluksiz.

5. $y=1+2^{\frac{1}{x}}$ funksiyani $x=0$ nuqtada uzluksizlikga tekshiring.

A) Uzluksiz; B) 1-tur uzilish; C) 2-tur uzilish; D) Yo‘qotish mumkin bo‘lgan uzilish.

6. $y=\frac{1}{\lg|x|}$ funksiyaning nechta uzilish nuqtasi mavjud?

A) 0; B) 1; C) 2; D) 3.

4-Shaxsiy uy topshiriqlari

Berilgan limitlarni hisoblang.

1

1.1. $\lim\limits_{x\to 2}\frac{x^2-7x+10}{2x^2+3x-14}$

1.2. $\lim\limits_{x\to -2}\frac{3x^2+7x+2}{x^2+5x+6}$

1.3. $\lim\limits_{x\to 3}\frac{x^3-7x-6}{3x^2-5x-12}$

1.4. $\lim\limits_{x\to 1}\frac{2x^2-x-1}{3x^2-2x-1}$

1.5. $\lim\limits_{x\to 1/3}\frac{3x^2+2x-1}{27x^3-1}$

1.6. $\lim\limits_{x\to -3}\frac{12+x-x^2}{x^3+27}$

1.7. $\lim\limits_{x\to 2}\frac{2x^2-x-6}{2x^3-5x^2+4}$

1.8. $\lim\limits_{x\to -4}\frac{x^2+3x-4}{2x^2-9x-14}$

1.9. $\lim\limits_{x\to 1/2}\frac{4x^2+4x-3}{2x^2+x-1}$

1.10. $\lim\limits_{x\to -1}\frac{3x^2-x-4}{1-4x^2-3x^3}$

1.11. $\lim\limits_{x\to 5}\frac{x^2-2x-15}{x^2-10x+25}$

1.12. $\lim\limits_{x\to -1}\frac{5x^2+4x-1}{3x^3-2x+1}$

1.13. $\lim\limits_{x\to 7}\frac{x^2-8x+7}{x^2-5x-14}$

1.14. $\lim\limits_{x\to -8}\frac{2x^2+15x-8}{3x^2+25x+8}$

1.15. $\lim\limits_{x\to 2}\dfrac{-5x^2+11x-2}{3x^2-x-10}$

1.16. $\lim\limits_{x\to 1}\dfrac{7x^2-4x-3}{3x^3-2x-1}$

1.17. $\lim\limits_{x\to 4}\dfrac{x^2-x-12}{3x^2-3x-12}$

1.18. $\lim\limits_{x\to 5}\dfrac{x^2-3x-10}{2x^2-13x+15}$

1.19. $\lim\limits_{x\to -2}\dfrac{2x^2+3x-2}{3x^3-13x-2}$

1.20. $\lim\limits_{x\to -2}\dfrac{4x^2+7x-2}{3x^2+8x+4}$

1.21. $\lim\limits_{x\to -3}\dfrac{x^3-7x+6}{3x^2+7x-6}$

1.22. $\lim\limits_{x\to 4}\dfrac{x^2-3x-4}{3x^2-2x-40}$

1.23. $\lim\limits_{x\to -3}\dfrac{2x^2+5x-3}{3x^2+10x+3}$

1.24. $\lim\limits_{x\to 1}\dfrac{4x^2-5x+1}{x^3-8x+7}$

1.25. $\lim\limits_{x\to 1/4}\dfrac{12x^2+13x-4}{4x^2-5x+1}$

1.26. $\lim\limits_{x\to 2}\dfrac{2x^2+5x-18}{3x^3-5x^2-4}$

1.27. $\lim\limits_{x\to -2}\dfrac{x^2+4x+4}{3x^2+5x-2}$

1.28. $\lim\limits_{x\to 1/3}\dfrac{27x^3-1}{3x^2+5x-2}$

1.29. $\lim\limits_{x\to -5}\dfrac{2x^2+7x-15}{x^2+10x+25}$

1.30. $\lim\limits_{x\to 1}\dfrac{2x^3-x-1}{5x^2-2x-3}$

2

2.1. $\lim\limits_{x\to\infty}\dfrac{2x^2-x-1}{3x^2-2x-1}$.

2.2. $\lim\limits_{x\to\infty}\dfrac{2x^3-5x-1}{5x^3-2x^2+3}$.

2.3. $\lim\limits_{x\to\infty}\dfrac{3x^2+5x-2}{2x^2-3x+1}$.

2.4. $\lim\limits_{x\to\infty}\dfrac{2x^2-3x-5}{2+5x-3x^2}$.

2.5. $\lim\limits_{x\to\infty}\dfrac{7x^3+4x}{x^3-3x+2}$.

$\lim\limits_{x\to\infty}\dfrac{3x^4+2x-5}{2x^3+x+7}$.

2.7. $\lim\limits_{x\to\infty}\dfrac{4-3x-2x^2}{x^2+12x+13}$.

$\lim\limits_{x\to\infty}\dfrac{5x^3-x^2-1}{3x^2-2x^3}$.

2.9. $\lim\limits_{x\to\infty}\dfrac{2x-x^3}{3x^3+2x^2+1}$.

2.10. $\lim\limits_{x\to\infty}\dfrac{4x^3-2x+1}{2x^2-5x-3}$.

2.11. $\lim\limits_{x\to\infty}\dfrac{2x^3+3x^2-1}{5-3x^3}$.

2.12. $\lim\limits_{x\to\infty}\dfrac{5x^3-7x^2+3}{2+2x-x^2}$.

2.13. $\lim\limits_{x\to\infty}\dfrac{2x^3+3x^2-5x}{5+3x^2-2x^3}$.

2.14. $\lim\limits_{x\to\infty}\dfrac{2x^2-x-1}{3x^3-2x+5}$.

2.15. $\lim\limits_{x\to\infty}\dfrac{x^3-7x^2+1}{3x^2-2x+4}$.

2.16. $\lim\limits_{x\to\infty}\dfrac{5+3x-4x^2}{x^2+2x+3}$.

2.17. $\lim\limits_{x\to\infty}\dfrac{4-3x^2}{x^3+5x-6}$.

2.18. $\lim\limits_{x\to\infty}\dfrac{7-2x^2}{2x^3+5x-7}$.

2.19. $\lim\limits_{x\to\infty}\dfrac{4-5x^2-3x^5}{x^5+6x+8}$.

2.20. $\lim\limits_{x\to\infty}\dfrac{2x^4+1}{8x^4+5x^2+13}$.

2.21. $\lim\limits_{x\to\infty}\dfrac{5-2x-3x^2}{3x^3+2x-5}$.

2.22. $\lim\limits_{x\to\infty}\dfrac{4x^3+3x-7}{1-2x^3}$.

2.23. $\lim\limits_{x\to\infty}\dfrac{4-3x-2x^2}{x^2+12x+13}$.

2.24. $\lim\limits_{x\to\infty}\dfrac{3-2x-2x^2}{2x^2+5x+3}$.

2.25. $\lim\limits_{x\to\infty}\dfrac{2x^2+8x-11}{-x^2+3x+4}$.

2.26. $\lim\limits_{x\to\infty}\dfrac{2x^4+5}{x^3+2x^2+3}$.

2.27. $\lim\limits_{x\to\infty}\dfrac{3x^2-5x+7}{3x^2+x+1}$.

2.28. $\lim\limits_{x\to\infty}\dfrac{3x^3-2x+5}{6x^2+5x+1}$.

2.29. $\lim\limits_{x\to\infty}\dfrac{4-x^2}{x^2+2x+1}$.

2.30. $\lim\limits_{x\to\infty}\dfrac{1+3x-2x^3}{4x^2+2x-6}$.

3

3.1. $\lim\limits_{x\to-3}\dfrac{6-x-x^2}{\sqrt{x+7}-2}$.

3.2. $\lim\limits_{x\to3}\dfrac{2x^2-5x-3}{\sqrt{x+2}-\sqrt{8-x}}$.

3.3. $\lim\limits_{x\to-1}\dfrac{3-\sqrt{10+x}}{3x^2+2x-1}$.

3.4. $\lim\limits_{x\to-2}\dfrac{\sqrt{6+x}-2}{3x^2+4x-4}$.

3.5. $\lim\limits_{x\to3}\dfrac{\sqrt{6+x}-3}{x^3-27}$.

3.6. $\lim\limits_{x\to0}\dfrac{5x}{\sqrt{5+x}-\sqrt{5-x}}$.

3.7. $\lim\limits_{x\to -2}\dfrac{\sqrt{7+x}-\sqrt{3-x}}{2x^2+5x+2}$.

3.8. $\lim\limits_{x\to 4}\dfrac{2x^2-9x+4}{\sqrt{x-3}-\sqrt{5-x}}$.

3.9. $\lim\limits_{x\to 0}\dfrac{\sqrt{7}x}{\sqrt{x+7}-\sqrt{7-x}}$.

3.10. $\lim\limits_{x\to 4}\dfrac{x^2-4x}{\sqrt{2x+1}-\sqrt{x+5}}$.

3.11. $\lim\limits_{x\to 0}\dfrac{x^2+5x}{\sqrt{2x+3}-\sqrt{3-x}}$.

3.12. $\lim\limits_{x\to 0}\dfrac{2-\sqrt{x^2+4}}{x^2}$.

3.13. $\lim\limits_{x\to 5}\dfrac{2x^2-7x-15}{\sqrt{x+4}-3}$.

3.14. $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{\sqrt{5x+1}-4}$.

3.15. $\lim\limits_{x\to -4}\dfrac{x^3+64}{4-\sqrt{20+x}}$.

3.16. $\lim\limits_{x\to 0}\dfrac{x^2+2x}{\sqrt{x+9}-3}$.

3.17. $\lim\limits_{x\to 5}\dfrac{5-\sqrt{4x+5}}{x^2-3x-10}$.

3.18. $\lim\limits_{x\to 1}\dfrac{\sqrt{2+7x}-3}{3-\sqrt{x+8}}$.

3.19. $\lim\limits_{x\to -2}\dfrac{\sqrt{x+7}-\sqrt{5}}{2-5x-3x^2}$.

3.20. $\lim\limits_{x\to 3}\dfrac{x^3-27}{3-\sqrt{x+6}}$.

3.21. $\lim\limits_{x\to -3}\dfrac{x^2+5x+6}{\sqrt{x+4}-\sqrt{7+2x}}$.

3.22. $\lim\limits_{x\to 4}\dfrac{\sqrt{3x+4}-4}{x^3-16x}$.

3.23. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^3-16x}$.

3.24. $\lim\limits_{x\to 4}\dfrac{\sqrt{4x}-x}{x^2-4x}$.

3.25. $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-x}{3x^2-x-10}$.

3.26. $\lim\limits_{x\to -2}\dfrac{\sqrt{4+x}-\sqrt{2}}{3x^2+5x-2}$.

3.27. $\lim\limits_{x\to -4}\dfrac{\sqrt{12-x}+x}{x^3+4x^2}$.

3.28. $\lim\limits_{x\to 2}\dfrac{\sqrt{3+x}-\sqrt{7-x}}{x^3-6x+4}$.

3.29. $\lim\limits_{x\to 9}\dfrac{\sqrt{2x+7}-5}{3-\sqrt{x}}$.

3.30. $\lim\limits_{x\to 3}\dfrac{x^3-27}{\sqrt{3x}-x}$.

4

4.1. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{3x+1}}{\cos\left(\pi(x+1)/2\right)}$.

4.2. $\lim\limits_{x\to 0}\dfrac{1-\cos 10x}{5x^2}$.

4.3. $\lim\limits_{x\to 0}\dfrac{3x^2-5x}{\sin 3x}$.

4.4. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{\cos 7x-\cos 3x}$.

4.5. $\lim\limits_{x\to 0}\dfrac{4x}{\mathrm{tg}(\pi(2+x))}$.

4.6. $\lim\limits_{x\to 0}\dfrac{2x}{\mathrm{tg}[2\pi(x+1/2)]}$.

4.7. $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{4x^2}$.

4.8. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{\sqrt{2+x}-\sqrt{2}}$.

4.9. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{\sin[\pi(x+2)]}$.

4.10. $\lim\limits_{x\to 0}\dfrac{arctg\, 2x}{\sin(2\pi(x+10))}$.

4.11. $\lim\limits_{x\to 0}\dfrac{x\sin\left(\pi(x+5)\right)}{1-\cos 2x}$.

4.12. $\lim\limits_{x\to 0}\dfrac{\cos(x+5\pi/2)tgx}{\arcsin 2x^2}$.

4.13. $\lim\limits_{x\to 0}\dfrac{2x\sin x}{1-\cos x}$.

4.14. $\lim\limits_{x\to 0}\dfrac{x\sin 5x}{1-\cos 4x}$.

4.15. $\lim\limits_{x\to 0}\dfrac{\sin 7x}{x^2+\pi x}$.

4.16. $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos x}{1-\cos x}$.

4.17. $\lim\limits_{x\to -2}\dfrac{\mathrm{tg}\pi x}{x+2}$.

4.18. $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-2}{3arctgx}$.

4.19. $\lim\limits_{x\to 2}\dfrac{\sin 7\pi x}{\sin 8\pi x}$.

4.20. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos x}}{x\sin x}$.

4.21. $\lim\limits_{x\to \pi/3}\dfrac{1-2\cos x}{\pi-3x}$.

4.22. $\lim\limits_{x\to \pi}\dfrac{1+\cos 3x}{\sin^2 7x}$.

4.23. $\lim\limits_{x\to 0}\dfrac{\sin^2 x-tg^2x}{x^4}$.

4.24. $\lim\limits_{x\to 1}\dfrac{3-\sqrt{10-x}}{\sin 3\pi x}$.

4.25. $\lim\limits_{x\to 0}\dfrac{tgx-\sin x}{x(1-\cos 2x)}$.

4.26. $\lim\limits_{x\to 1}\dfrac{1+\cos \pi x}{\mathrm{tg}^2\pi x}$.

4.27. $\lim\limits_{x\to \pi/4}\dfrac{1-\sin 2x}{(\pi-4x)^2}$.

4.28. $\lim\limits_{x \to \pi} \dfrac{1-\sin(x/2)}{\pi - x}$.

4.29. $\lim\limits_{x \to 2} \dfrac{\operatorname{arctg}(x^2 - 2x)}{\sin 3\pi x}$.

4.30. $\lim\limits_{x \to \pi} \dfrac{\cos 5x - \cos 3x}{\sin^2 x}$.

5

5.1. $\lim\limits_{x\to 0}\left(1+3tg^2 x\right)^{ctg^2 x}$.

5.2. $\lim\limits_{x\to\infty}\left(\dfrac{2x}{2x+3}\right)^{-4x}$.

5.3. $\lim\limits_{x\to\infty}\left(x\ln\dfrac{x}{x+3}\right)$.

5.4. $\lim\limits_{x \to \pi/2}\left(\operatorname{tg}\dfrac{x}{2}\right)^{1/(x-\pi/2)}$.

5.5. $\lim\limits_{x\to\infty}\left(\dfrac{2x-3}{2x+5}\right)^{5x-2}$.

5.6. $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin^2 2x)}{5x^2}$.

5.7. $\lim\limits_{x \to 3}\left(\dfrac{9-2x}{3}\right)^{\operatorname{tg}\frac{\pi x}{6}}$.

5.8. $\lim\limits_{x\to\infty}\left(\dfrac{3-2x}{1-2x}\right)^{3x-2}$.

5.9. $\lim\limits_{x \to 0}\left(5-\dfrac{4}{\cos x}\right)^{1/\sin^2 3x}$.

5.10. $\lim\limits_{x \to 0}\left(\operatorname{tg}\left(\dfrac{\pi}{4}-x\right)\right)^{\operatorname{ctg}x}$.

5.11. $\lim\limits_{x\to\pi/2}\left(\sin 2x\right)^{tg^2 2x}$.

5.12. $\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x+8}\right)^{-3x}$.

5.13. $\lim\limits_{x\to 0}\dfrac{2^x-3^x}{\sin x}$.

5.14. $\lim\limits_{x\to 0}\dfrac{\ln(1+4x^2)}{3x^2}$.

5.15. $\lim\limits_{x\to 0}\dfrac{tg\,2x}{\ln(1+3x)}$.

5.16. $\lim\limits_{x\to 1}\left(\dfrac{x-2}{2x-3}\right)^{ctg\,\pi x}$.

5.17. $\lim\limits_{x\to \pi}\dfrac{\ln(1+\sin 2x)}{x^2-\pi^2}$.

5.18. $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin 3x)}{x^2+5x}$.

5.19. $\lim\limits_{x\to 0}\dfrac{\ln(\cos 2x)}{\sin^2 x}$.

5.20. $\lim\limits_{x\to \pi}\dfrac{\ln(\cos 2x)}{(x-\pi)\sin x}$.

5.21. $\lim\limits_{x\to 4}(2x-7)^{\frac{3x}{x^2-16}}$.

5.22. $\lim\limits_{x\to 1}\dfrac{\ln(1+\sin \pi x)}{x^2+2x-3}$.

5.23. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{\ln(1+3x^2)}$.

5.24. $\lim\limits_{x\to 2}(2x-3)^{\frac{3x}{x-2}}$.

5.25. $\lim\limits_{x\to 3}\left(\dfrac{x-2}{2x-5}\right)^{2/(3-x)}$.

5.26. $\lim\limits_{x\to 0}\left(6-\dfrac{5}{\cos x}\right)^{\mathrm{ctg}^2 x}$.

5.27. $\lim\limits_{x\to 2}\left(\dfrac{4x-7}{5}\right)^{tg\frac{\pi x}{4}}$

5.28. $\lim\limits_{x\to \infty}x\big(\ln(2x+1)-\ln(2x+5)\big)$.

5.29. $\lim\limits_{x\to \infty}\left(\dfrac{x^2+2x-3}{x^2+3x+1}\right)^{2x-3}$.

5.30. $\lim\limits_{x\to\infty} x\left(\ln(3x-1)-\ln(3x-5)\right).$

6

Berilgan funksiyalarni uzluksizlikka tekshiring, uzluksizlik oraliqlarini va uzilish nuqtalarining turini aniqlang.

6.1. $a)\ f(x)=\begin{cases} x+4,\ x\le -2, \\ x^2-2,\ -2<x\le 2, \\ 3x-5,\ x>2, \end{cases}$ $\quad b)\ f(x)=2^{\frac{1}{x+3}}+1.$

6.2. $a)\ f(x)=\begin{cases} 0,\ x<-1, \\ x^2+1,\ -1\le x\le 2, \\ 3x,\ x>2, \end{cases}$ $\quad b)\ f(x)=3^{\frac{1}{x-2}}+2.$

6.3. $a)\ f(x)=\begin{cases} 3x,\ x\le -2, \\ x^2+2,\ -2<x\le 2, \\ x+3,\ x>2, \end{cases}$ $\quad b)\ f(x)=\dfrac{3}{2^x-1}.$

6.4. $a)\ f(x)=\begin{cases} 0,\ x\le 0, \\ \sin x,\ 0<x\le \pi, \\ 2x-6,\ x>\pi, \end{cases}$ $\quad b)\ f(x)=\dfrac{1}{2^{\frac{1}{x+3}}+1}.$

6.5. $a)\ f(x)=\begin{cases} \cos x,\ x\le -\pi/2, \\ 0,\ -\pi/2<x\le \pi/2, \\ 1,\ x>\pi/2, \end{cases}$ $\quad b)\ f(x)=4^{\frac{1}{4-x}}-1.$

6.6. $a)\ f(x)=\begin{cases} -x+1,\ x<-2, \\ x^2-1,\ -2\le x\le 2, \\ 3x-2,\ x>2, \end{cases}$ $\quad b)\ f(x)=6^{\frac{1}{x-5}}+3.$

6.7. $a)\ f(x)=\begin{cases} 1,\ x<-1, \\ -x^2+1,\ -1\le x\le 2, \\ x-5,\ x>2, \end{cases}$ $\quad b)\ f(x)=5^{\frac{1}{x+2}}-1.$

6.8. $a)\ f(x)=\begin{cases} 1,\ x<0, \\ 2^x+1,\ 0\le x\le 2, \\ 3x-1,\ x>2, \end{cases}$ $\quad b)\ f(x)=\dfrac{x-3}{x+2}.$

6.9. $a)\ f(x)=\begin{cases} \sqrt{1-x},\ x<0, \\ x^2-2x,\ 0\le x\le 2, \\ x-1,\ x>2, \end{cases}$ $\quad b)\ f(x)=3^{\frac{1}{x+1}}+1.$

6.10. a) $f(x)=\begin{cases}0, & x<1,\\ \ln x, & 1\le x\le 3,\\ x-1, & x>3,\end{cases}$ b) $f(x)=6^{\frac{x+2}{x-2}}-1.$

6.11. a) $f(x)=\begin{cases}-x+1, & x<-2,\\ x^3+5, & -2\le x\le 1,\\ x+5, & x>1,\end{cases}$ b) $f(x)=2arctg\dfrac{1}{x-2}.$

6.12. a) $f(x)=\begin{cases}3-2x, & x<-2,\\ x^3+1, & -2\le x\le 2,\\ 2x+5, & x>2,\end{cases}$ b) $f(x)=\dfrac{(1+x)^3-1}{x}.$

6.13. a) $f(x)=\begin{cases}2x-3, & x<-2,\\ x^3+1, & -2\le x\le 2,\\ 2^x+1, & x>2,\end{cases}$ b) $f(x)=\dfrac{1}{x}+\dfrac{1}{|x|}.$

6.14. a) $f(x)=\begin{cases}2^x-3, & x<2,\\ \log_2 x, & 2\le x\le 8,\\ 2x-15, & x>8,\end{cases}$ b) $f(x)=\dfrac{1}{1-e^{1-x}}.$

6.15. a) $f(x)=\begin{cases}2^x-1, & x<0,\\ 2\sin x, & 0\le x\le \pi,\\ 2x-5, & x>\pi,\end{cases}$ b) $f(x)=\dfrac{1}{1+e^{1/x}}.$

6.16. a) $f(x)=\begin{cases}x+\pi, & x<-\pi/2,\\ \sin x+1, & -\pi/2\le x\le \pi/2,\\ 2x+1, & x>\pi,\end{cases}$ b) $f(x)=\dfrac{1}{1+3^{1/(x+1)}}.$

6.17. a) $f(x)=\begin{cases}x+1, & x<0,\\ 2\cos x, & 0\le x\le \pi,\\ 1-x, & x>\pi,\end{cases}$ b) $f(x)=arctg3^{1/x}.$

6.18. a) $f(x)=\begin{cases}2x-1, & x<-1,\\ 3x^2, & -1\le x\le 1,\\ 2^x+1, & x>1,\end{cases}$ b) $f(x)=\dfrac{3^{1/x}-1}{3^{1/x}+1}.$

6.19. a) $f(x)=\begin{cases}3x+2, & x<-1,\\ -2x^2+1, & -1\le x\le 1,\\ x-2, & x>1,\end{cases}$ b) $f(x)=3^{1/(2-x)}+1.$

6.20. a) $f(x)=\begin{cases}2x+1, & x<-1,\\ -2x^2+1, & -1\le x\le 2,\\ x+5, & x>2,\end{cases}$ b) $f(x)=2^{1/(x-3)}+1.$

6.21. a) $f(x)=\begin{cases}1-x, & x<-1,\\ -x^2+3, & -1\le x\le 2,\\ 2x-3, & x>2,\end{cases}$ b) $f(x)=\dfrac{1}{3^{1/(2-x)}+1}$.

6.22. a) $f(x)=\begin{cases}x+2, & x<-2,\\ x^2-4, & -2\le x\le 3,\\ 2x-1, & x>3,\end{cases}$ b) $f(x)=\dfrac{2}{3^{tgx}+1}$.

6.23. a) $f(x)=\begin{cases}2x+1, & x<-1,\\ -2x^2+1, & -1\le x\le 2,\\ 2-3x, & x>2,\end{cases}$ b) $f(x)=\dfrac{3^{1/(2-x)}-1}{3^{1/(2-x)}+1}$.

6.24. a) $f(x)=\begin{cases}1-2x, & x<-1,\\ -2\cos\pi x+1, & -1\le x\le 2,\\ 2x-3, & x>2,\end{cases}$ b) $f(x)=5^{1/(x+3)}+1$.

6.25. a) $f(x)=\begin{cases}x-5, & x<-2,\\ -2x^2+1, & -2\le x\le 2,\\ 2x+3, & x>2,\end{cases}$ b) $f(x)=\dfrac{1}{3^{1/(2-x)}}+1$.

6.26. a) $f(x)=\begin{cases}x+1, & x<-1,\\ x^2-1, & -1\le x\le 2,\\ 4-x, & x>2,\end{cases}$ b) $f(x)=arctg\dfrac{2}{3-x}$.

6.27. a) $f(x)=\begin{cases}x+1, & x<-1,\\ x^2, & -1\le x\le 2,\\ \log_2 x+3, & x>2,\end{cases}$ b) $f(x)=\dfrac{x^2-1}{x^3-1}$.

6.28. a) $f(x)=\begin{cases}2x+1, & x<0,\\ 1-2^x, & 0\le x\le 2,\\ x-5, & x>2,\end{cases}$ b) $f(x)=\dfrac{3^{1/x}-2}{3^{1/x}+2}$.

6.29. a) $f(x)=\begin{cases}\sin x+1, & x<0,\\ 1-x^2, & 0\le x\le 2,\\ x+1, & x>2,\end{cases}$ b) $f(x)=2arctg\dfrac{1}{3-x}$.

6.30. a) $f(x)=\begin{cases}2x+1, & x<-1,\\ x^2-2, & -1\le x\le 2,\\ 8-3x, & x>2,\end{cases}$ b) $f(x)=\dfrac{1}{1+4^{tgx}}$.

V BOB. DIFFERENSIAL HISOB ELEMENTLARI

5.1. Funksiya hosilasi

Berilgan $y = f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo‘lib, $x,\ x+\Delta x \in [a,b]$ funksiyaning $\Delta y = f(x+\Delta x) - f(x)$ orttirmasini argument orttirmasi Δx ga nisbati $\frac{\Delta y}{\Delta x}$ ning Δx nolga intilgandagi limiti $y = f(x)$ funksiyaning x nuqtadagi ***hosilasi*** deyiladi hamda quyidagi belgilardan biri bilan belgilanadi: $y',\ f'(x),\ \frac{dy}{dx}$. Demak,

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Agar bu limit mavjud bo‘lsa, $y = f(x)$ funksiya x nuqtada ***differensiallanuvchi***, hosilani topish jarayoni esa ***differensiallash*** deyiladi.

Berilgan $y = f(x)$ funksiyaning x nuqtadagi hosilasi funksiya grafigiga $M(x, f(x))$ nuqtasida o‘ tkazilgan urinmaning burchak koeffitsientiga teng.

Fizik nuqtai nazardan $y' = f'(x)$ hosila funksiyaning x nuqtadagi argument x ga nisbatan o‘z garish tezligini aniqlaydi.

Agar $C-$o‘zgarmas son, $u(x)$ va $v(x)-$ differensiallanuvchi funksiyalar bo‘lsa, quyidagi ***differensiallash qoidalari*** o‘rinli:

$$1)\ C' = 0; \quad 2)\ (u \pm v)' = u' \pm v'; \quad 3)\ (Cu)' = Cu'; \quad 4)\ (uv)' = u'v + uv';$$

$$5)\ \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}; \quad 6)\ \left(\frac{C}{v}\right)' = -\frac{Cv'}{v^2};$$

7) agar $y = f(u)$, $u = \varphi(x)$, ya’ni $y = f(\varphi(x))-$ differensiallanuvchi funksiyalardan tashkil topgan murakkab funksiya bo‘lsa, u holda

$$y'_x = y'_u u'_x \text{ yoki } y' = f'(u)\varphi'(x);$$

8) agar $y = f(x)$ funksiya uchun differensiallanuvchi $x = g(y)$ teskari funksiya mavjud va $g'(y) \neq 0$ bo‘lsa, u holda

$$y'_x = \frac{1}{x'_y} \text{ yoki } f'(x) = \frac{1}{g'(x)};$$

5.1-misol

Ushbu $y = x^3$ funksiya hosilasini ta’rif bo‘ yicha toping.

► Argumentning ixtiyoriy Δx orttirmasida

$$\Delta y = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3,$$

u holda

$$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x} = \lim_{\Delta x\to 0}\left(3x^2 + 3x\Delta x + \Delta x^2\right) = 3x^2 .$$ ◄

Funksiya x nuqtada differensiallanuvchi bo‘lsa, u shu nuqtada uzluksiz bo‘ladi, aksinchasi har doim ham o‘rinli emas, ya’ni x nuqtada uzluksiz funksiya shu nuqtada differensiallanuvchi bo‘lmasligi ham mumkin.

5.2-misol

Ushbu $y = |x|$ funksiya $x = 0$ nuqtada differensiallanuvchi bo‘ladimi?

► Funksiya berilgan nuqtada uzluksiz. Argumentning $x = 0$ nuqtadagi ixtiyoriy Δx orttirmasida funksiya orttirmasi

$$\Delta y = \begin{cases} -\Delta x, \text{ agar } \Delta x < 0 \text{ bo`lsa,} \\ \Delta x, \text{ agar } \Delta x > 0 \text{ bo`lsa.} \end{cases}$$

Hosila ta’rifiga ko‘ra,

$$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \begin{cases} -1, \text{ agar } \Delta x < 0 \text{ bo`lsa,} \\ 1, \text{ agar } \Delta x > 0 \text{ bo`lsa.} \end{cases}$$

Bundan kelib chiqadiki, $y = |x|$ funksiya $x = 0$ nuqtada hosilaga ega emas.◄

5.3-misol

Ushbu $y = \ln^3(\arcsin\sqrt{x})$ funksiyaning hosilasini toping

► Avval $y = u^3$ murakkab funksiyadan hosila hisoblaymiz $y' = 3u^2u'$, bu yerda $u = \ln(\arcsin\sqrt{x})$, hamda $u = \ln v$ bo‘lgani uchun, $u' = \frac{1}{v}v'$, $v = \arcsin\sqrt{x}$. O‘z navbatida

$$v = \arcsin w,\quad v' = \frac{1}{\sqrt{1-w^2}}w',\quad w = \sqrt{x},\quad w' = (\sqrt{x})' = \frac{1}{2\sqrt{x}}.$$

Demak,

$$y' = 3\ln^2(\arcsin\sqrt{x})\frac{1}{\arcsin\sqrt{x}}\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}.$$ ◄

16-Auditoriya topshiriqlari

1. $y = \dfrac{3x-2}{1+4x}$ funksiya hosilasini ta’rifdan foydalanib toping.

2. $y=\sqrt[3]{x}$ funksiya $x=0$ nuqtada uzluksiz va differensiallanuvchi bo‘ladimi?

3. Quyidagi funksiyalarning hosilalarini toping:

$a)\ \ y=3x^3-4\sqrt[5]{x^4}+5/x^2;$ $\qquad b)\ \ y=x^2\cos x\cdot\ln x;$

$c)\ \ y=(x^3+2^x)tgx;$ $\qquad d)\ \ y=e^x/(x^2+1)$

4. Hosilalar jadvali va differensiallash qoidalaridan foydalanib quyidagi funksiyalarning hosilalarini hisoblang:

$a)\ \ y=arctg\sqrt{1+e^{-x}};$ $\qquad b)\ \ y=(2^{\cos 3x}+\sin 3x)^2;$

$c)\ \ y=x^3tg\sqrt[5]{x}+3^{\sin x};$ $\qquad d)\ \ y=\lg^2(x^5+\sin^2 x);$

5. $y=e^{tg\frac{x}{2}}$ funksiya $y'\sin x=y\ln y$ tenglamani qanoatlantirishini tekshiring.

16-Mustaqil yechish uchun testlar

Quyidagi funksiyalarning hosilalarini toping:

1. $y=6\sqrt[3]{x}-\dfrac{1}{2x^2}+\dfrac{1}{\sqrt[3]{x^2}};$

A) $2\sqrt[3]{x^2}+\dfrac{1}{x}-\dfrac{2}{3\sqrt{x}}$; B) $\dfrac{2}{\sqrt[3]{x^2}}+\dfrac{1}{x^3}-\dfrac{2}{3\sqrt[3]{x^5}}$; C) $2\sqrt[3]{x^2}-\dfrac{1}{x^3}+\dfrac{2}{3\sqrt[3]{x^5}}$; D) $\dfrac{2}{\sqrt[3]{x^2}}+\dfrac{1}{x}-\dfrac{2}{3\sqrt[3]{x^5}}$.

2. $y=\dfrac{2x+3}{x^2+1};$

A) $\dfrac{2(x^2-3x+1)}{(x^2+1)^2}$; B) $\dfrac{2(x^2+3x-1)}{(x^2+1)^2}$; C) $\dfrac{2(3x^2+3x+1)}{(x^2+1)^2}$; D) $\dfrac{2(1-3x-x^2)}{(x^2+1)^2}$.

3. $y=(2x^2-7)\ln(x+1)+\sqrt{a};$

A) $4x\ln(x+1)+\dfrac{2x^2-7}{x+1}$; B) $\dfrac{4x}{x-1}$; C) $4x+\dfrac{2x^2-7}{x+1}$; D) $4x\ln(x+1)+\dfrac{2x^2-7}{x+1}+\dfrac{1}{2\sqrt{a}}$.

4. $y=x^2\cos 3x+arctg\sqrt{x-1};$

A) $-6x\sin 3x+\dfrac{1}{x\sqrt{x-1}}$; B) $2x\cos 3x-3x^2\sin 3x+\dfrac{1}{2\sqrt{x^2-x}}$;

C) $-6x\sin 3x+\dfrac{1}{2x\sqrt{x-1}}$;   D) $2x\cos 3x-3x^2\sin 3x+\dfrac{1}{2x\sqrt{x-1}}$.

5. $y=\log_2 \sin^2 2^x$;

A) $\dfrac{2\cos 2^x}{\sin^2 2^x}\ln 2$;   B) $\dfrac{2^x ctg 2^x}{\ln 2}$;   C) $2^{x+1} ctg 2^x$;   D) $\dfrac{2^x ctg 2^x}{\sin 2^x}$.

5.2. Logarifmlab differensiallash. Oshkormas va parametrik funksiya hosilalari

Funksiyani ketma-ket logarifmlash va differensiallash jarayoniga ***logarifmlab differensiallash*** deyiladi: $(\ln f(x))'=f'(x)/f(x)$. Bu qoida funksiyani avval logarifmlash hosila topishni soddalashtiradigan hollarda qo‘llanadi.

5.4-misol

Ushbu $y=\dfrac{(x+3)^2\sqrt[3]{x-1}}{\sqrt[4]{(x+2)^3}}$ funksiya hosilasini toping.

► Avval logarifmlash maqsadga muvofiq,

$$\ln y=2\ln(x+3)+\frac{1}{3}\ln(x-1)-\frac{3}{4}\ln(x+2).$$

Tenglikdan hosila hisoblaymiz

$$\frac{y'}{y}=\frac{2}{x+3}+\frac{1}{3(x-1)}-\frac{3}{4(x+2)};$$

$$y'=\frac{(x+3)^2\sqrt[3]{x-1}}{\sqrt[4]{(x+2)^3}}\left(\frac{2}{x+3}+\frac{1}{3(x-1)}-\frac{3}{4(x+2)}\right);$$

$$y'=\frac{(x+3)(19x^2+26x+3)}{12(x+2)\sqrt[3]{(x-1)^2}\sqrt[4]{(x+2)^3}}.$$ ◄

$y=u^v$, bu yerda $u=u(x), v=v(x)$, ko‘rinishdagi funksiyaning hosilasini hisoblashda avval logarifmlash quyidagi formulaga olib keladi:

$$y'=u^v\ln u\cdot v'+vu^{v-1}\cdot u'.$$

5.5-misol

Ushbu $y=x^{\sin x^3}$ funksiya limitini hisoblang.

► $\ln y=\sin x^3\cdot\ln x,\quad y'=x^{\sin x^3}\ln x\cdot 3x^2\cos x^3+\sin x^3\cdot x^{\sin x^3-1}$. ◄

Agar y va x orasidagi bog‘lanish oshkormas ko‘rinishda, $F(x,y)=0$ tenglama bilan berilgan bo‘lsa, bunday funksiya ***oshkormas funksiya*** deyiladi. y' hosila $F(x,y)=0$ tenglikning ikki tarafidan, y x ning funksiyasi ekanligini e’tiborga olgan holda, hosila olib topiladi.

5.6-misol

Agar $x^3+y^3-3xy=0$ bo‘lsa, y' hosilani hisoblang.

► Tenglikning ikki tarafidan hosila olamiz

$$3x^2+3y^2y'-3y-3xy'=0,$$

so‘ ngra tenglamadan y' ni topamiz

$$y'=(y-x^2)/(y^2-x). \blacktriangleleft$$

Agar y funksiyaning x argumentga bog‘liqligi parametrik ko‘rinishda, $x=x(t),\ y=y(t)$ tenglamalar bilan berilgan bo‘lsa, bunday funksiya ***parametrik funksiya*** deyiladi. y' yoki y'_x hosila $y'_x=\dfrac{y'_t}{x'_t}$ formula bilan hisoblanadi.

5.7-misol

Quyidagi $\begin{cases} x=\sqrt{1-t^2}, \\ y=tg\sqrt{1+t} \end{cases}$ tenglama bilan berilgan funksiyaning y'_x hosilasini toping.

► $y'_x=\dfrac{y'_t}{x'_t}$, $x'(t)=\dfrac{-t}{\sqrt{1-t^2}}$, $y'(t)=\dfrac{1}{\cos^2\sqrt{1+t}}\cdot\dfrac{1}{2\sqrt{1+t}}=\dfrac{1}{2\sqrt{1+t}\cos^2\sqrt{1+t}}$.

$$y'_x=-\frac{\sqrt{1-t^2}}{2t\sqrt{1+t}\cos^2\sqrt{1+t}}=-\frac{\sqrt{1+t}}{2t\cos^2\sqrt{1+t}}. \blacktriangleleft$$

17-Auditoriya topshiriqlari

1. Quyidagi hosilalarni logarifmlab differerensiallash qoidasi asosida hisoblang:

a) $y=\dfrac{(x-2)^2}{(x+1)^3\sqrt{x+2}}$; d) $y=\dfrac{(x-2)^2\sqrt[3]{x+1}}{\sqrt{x-3}}$;

b) $y=(x^3+2)^{\cos x}$; e) $y=(\ln x)^{tg2x}$.

2. Oshkormas ko‘rinishda berilgan funksiyaning y'_x hosilasini toping.

a) $x^4+y^4=x^2y^2$; d) $x^y=y^x$;

b) $y=(x^2+3)y+x\cos y=0$; e) $y=x+arctgy$.

3. Parametrik ko‘rinishda berilgan funksiyaning y'_x hosilasini toping.

a) $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases}$ b) $\begin{cases} x = 1 - t^2, \\ y = t - t^3, \end{cases}$

d) $\begin{cases} x = \dfrac{1-t}{1+t^2}, \\ y = \dfrac{2t}{1+t^2}, \end{cases}$ e) $\begin{cases} x = \arcsin \dfrac{1}{\sqrt{1+t^2}}, \\ y = \arccos \dfrac{t}{\sqrt{1+t^2}}. \end{cases}$

17-Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biriga logarifmlab differensiallash qoidasi qo‘llanadi:

A) $y = tg^2(\sin^2 x)$; B) $y = \sqrt{xtgx\sin^3 x}$;

D) $y = \sqrt{tgx + \sin\sqrt{x}}$; E) $y = xtg(x\sin x)$.

2. Quyidagilardan qaysi biriga logarifmlab differensiallash qoidasi qo‘llanadi:

A) $y = 2^{x^2} + y^{\ln 2}$; B) $y = x^y + (\cos x)^{\ln 2}$; D) $y = x^{1+\ln 2}$; E) $y = 2^x x^{\ln x}$.

3. $y = 2^{x^2} + y^{\ln 2}$ tenglama bilan berilgan berilgan funksiya …

A) logarifmlab differensiallanadi. B) murakkab funksiya bo‘ladi.

D) oskormas funksiya bo‘ladi. E) parametrik funksiya bo‘ladi.

4. $x^2 + y^2 + 4x - 2y - 3 = 0$ tenglama bilan berilgan egri chiziqga $(0,3)$ nuqtasida o‘ tkazilgan urinma tenglamasini yozing.

A) $y = 3 - x$, B) $y = 2x + 3$, D) $y = x + 3$, E) $y = 3 - 2x$.

5. $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t \end{cases}$ tenglama bilan berilgan funksiya hosilasini toping.

A) $y = -ctgx$, B) $y = -tgx$, D) $y = ctg^2 x$, E) $y = tg^2 x$.

5-Shaxsiy uy topshiriqlari

Berilgan funksiyalarning hosilalarini toping.

1

1.1. $y = \sqrt[3]{x^2} + x\arcsin x - \dfrac{\ln x}{\cos x}$.

1.2. $y = \dfrac{1}{2\sqrt[3]{x^2}} + xarctgx - \dfrac{\sin x}{\ln x}$.

1.3. $y = \dfrac{1}{3x^3} - 2^x ctgx - \dfrac{\arcsin x}{x}$.

1.4. $y = \dfrac{5}{\sqrt[5]{x^2}} + 3^x arcctgx + \dfrac{\log_2 x}{tgx}$.

1.5. $y = \dfrac{1}{2x^2} + 2^x \arcsin x - \dfrac{\cos x}{\lg x}$.

1.6. $y = 3^x \arcsin x - \dfrac{\ln x}{\sqrt[3]{x^2}} - \dfrac{1}{x^2}$.

1.7. $y=\arcsin x\log_3 x-\frac{3e^x}{\sqrt[3]{x^2}}-ctgx.$

1.8. $y=\frac{\lg x}{\sin x}-3\sqrt[3]{x^2}-2^x arctgx.$

1.9. $y=\sin x\ln x-\frac{3^x}{arctgx}-\frac{3}{x^4}.$

1.10. $y=\frac{1}{3x^3}+3^x\arccos x-\frac{tgx}{\lg x}.$

1.11. $y=\frac{1}{3\sqrt[5]{x^4}}+5^x\arccos x-\frac{ctgx}{\log_5 x}.$

1.12. $y=tgx\log_3 x-\frac{5^x}{\sin x}-\frac{5}{2\sqrt[5]{x^2}}.$

1.13. $y=\frac{\lg x}{\cos x}-4\sqrt[4]{x^3}-e^x\arcsin x.$

1.14. $y=4^x\log_2 x-\frac{\sin x}{arctgx}-\frac{2}{3x^3}.$

1.15. $y=3^x\log_3 x-\frac{arctgx}{\cos x}-\frac{2}{3\sqrt[4]{x^3}}.$

1.16. $y=\frac{\lg x}{\cos x}-5\sqrt[5]{x^2}-(e^x+x)tgx.$

1.17. $y=arcctgx\lg x-\frac{e^x}{\sin x}-\frac{5}{4\sqrt[5]{x^3}}.$

1.18. $y=\frac{1}{3\sqrt{x^3}}-3^x\cos x-\frac{arctgx}{x}.$

1.19. $y=2e^x\sin x-\frac{\ln x}{arctgx}-\frac{3}{\sqrt[3]{x}}.$

1.20. $y=\arcsin x\log_5 x-\frac{3^x}{tgx}-\frac{1}{3\sqrt[5]{x^3}}.$

1.21. $y=\frac{1}{3x^5}+e^x\cos x-\frac{1+x^2}{arctgx}.$

1.22. $y=\frac{\ln x}{\sin x}-8\sqrt[4]{x^5}-e^x arcctgx.$

1.23. $y=x^2 arcctgx-\frac{4^x}{\cos x}-\frac{5}{x\sqrt[5]{x}}.$

1.24. $y=3^x ctgx+\frac{arctgx}{\lg x}-\frac{3}{2\sqrt[3]{x^2}}.$

1.25. $y=2^x\arcsin x-\frac{ctgx}{\ln x}-\frac{3}{x\sqrt[3]{x}}.$

1.26. $y=e^x\sin x-\frac{arcctgx}{\ln x}-\frac{3}{x\sqrt[3]{x}}.$

1.27. $y=e^x\cos x-\frac{\arcsin x}{\log_5 x}-\frac{3}{x^7}.$

1.28. $y=\sin x\ln x-\frac{arctgx}{1+x^2}+\frac{4}{3\sqrt[4]{x^3}}.$

1.29. $y=\frac{5}{x\sqrt[5]{x^2}}+4^x arctgx-\frac{\cos x}{\log_3 x}.$

1.30. $y=(x^2+1)arctgx-\frac{3^x}{\sin x}-\frac{4}{x\sqrt[4]{x}}.$

2

2.1. $y=\sin^3 2x\cdot tg(2x+1)^3.$

2.2. $y=2^{\sin x}\cdot tg^2(2x^3+1).$

2.3. $y=\lg(\sin^3 2x)\cdot tg\sqrt{2x+1}.$

2.4. $y=3^{-x^2}\cdot\arcsin\sqrt{2x+1}.$

2.5. $y=\ln^3(2x^2+1)\cdot arctg^2\sqrt{x}.$

2.6. $y=\log_5^{\,2}(3x+4)\cdot arcctg^3\sqrt{x+1}.$

2.7. $y=3^{2x^2+1}\cdot\arcsin^2\sqrt{\ln x}.$

2.8. $y=\arcsin^2 3x\cdot ctg5x^3.$

2.9. $y=3^{-\cos^2 x}\cdot arctg\sqrt{x^2+1}.$

2.10. $y=5^{\cos x}\cdot\arcsin 3x^3.$

2.11. $y=\log_3(\sin^2 x)\cdot\arccos^3\sqrt{x}.$

2.12. $y=e^{tgx}\cdot\arcsin(\ln^3 x).$

2.13. $y=tg(3e^x)\cdot\arccos^3\sqrt{\lg x}.$

2.14. $y=3^{\sin(2x^2+1)}\cdot\ln^2\sqrt{tgx}.$

2.15. $y=tg(x^2+1)\cdot\arccos^2\sqrt{\log_3 x}.$

2.16. $y=2^{ctgx^2}\cdot\lg^3(\sin^2 x).$

2.17. $y=e^{1/x}\cdot arctg^2(3^x).$

2.18. $y=5^{tgx}\cdot\sqrt{\arcsin(1/x)}.$

2.19. $y=5^{-1/x}\cdot\arcsin(2x+1)^3.$

2.20. $y=\log_2(\cos^3 x)\cdot arctg^2\sqrt{x}.$

2.21. $y = \lg\left(tg^2 3x\right) \cdot \arccos\sqrt{1-2x}.$

2.22. $y = arcctg 2x^3 \cdot \sin^2(e^x + x^3).$

2.23. $y = \lg^2(3x+4) \cdot \arcsin^3\sqrt{1-x^2}.$

2.24. $y = \cos(2e^{3x}) \cdot arctg^3\sqrt{\log_2 x}.$

2.25. $y = \sin\left(tg^2 3x\right) \cdot \ln\sqrt{1+2x^2}.$

2.26. $y = tg(1/x) \cdot \arcsin^3(e^x - x).$

2.27. $y = tg^2(2x+1) \cdot arctg^3\sqrt{\sin x^2}.$

2.28. $y = 2^{\sin x^2} \cdot \lg^3(\sin(1/x)).$

2.29. $y = \ln^2(3x+2) \cdot \arccos^3(1/x).$

2.30. $y = 2^{tgx} \cdot \sqrt{\arccos(1/x)}.$

3

3.1. $a)\quad y = (arctg 3x)^{\ln(x+3)}, \quad b)\ y = \dfrac{(x+5)^3\sqrt{x-2}}{\sqrt[3]{(x+2)^2}}.$

3.2. $a)\quad y = (tg 3x)^{\arccos(2x+3)}, \quad b)\ y = \dfrac{(x-3)^2\sqrt{x+2}}{\sqrt[5]{(x+3)^2}}.$

3.3. $a)\quad y = (\lg(3x+2))^{arctg(2x+1)}, \quad b)\ y = \dfrac{(x+3)^2\sqrt[3]{(x-3)^2}}{\sqrt{x+4}}.$

3.4. $a)\quad y = (\log_3(3x+2))^{tg 5x}, \quad b)\ y = \dfrac{(x-5)^3\sqrt[3]{(x+3)^2}}{\sqrt{x+2}}.$

3.5. $a)\quad y = (\cos(3x+2))^{\sin^2 3x}, \quad b)\ y = \dfrac{(x-2)^3\sqrt[5]{(x-3)^3}}{(x+5)^5}.$

3.6. $a)\quad y = (tg(2x+5))^{\ln(3x+2)}, \quad b)\ y = \dfrac{(x+3)^3(x-2)^2}{\sqrt[3]{(x-5)^2}}.$

3.7. $a)\quad y = (\arccos 3x)^{\lg 5x}, \quad b)\ y = \dfrac{(x-5)^3\sqrt{(x-2)^3}}{\sqrt[3]{(x+3)^2}}.$

3.8. $a)\quad y = (\log_2(x+2))^{\arcsin 2x}, \quad b)\ y = \dfrac{(x-5)^3(x+6)^5}{\sqrt[3]{(x-3)^2}}.$

3.9. $a)\quad y = (arctg(2x+1))^{\cos(3x+2)}, \quad b)\ y = \dfrac{(x+2)^3(x+5)^4}{\sqrt[5]{(x-5)^4}}.$

3.10. $a)\quad y = (\log_3(2x+7))^{\sin(x+3)}, \quad b)\ y = \dfrac{(x+3)^5}{\sqrt[3]{(x+5)^2}\,(x-2)^2}.$

3.11. $a)\quad y = (arcctg 2x)^{\ln(5x+3)}, \quad b)\ y = \dfrac{(x+3)^2\sqrt{(x-2)^3}}{\sqrt[5]{x+7}}.$

3.12. $a)\quad y = (\sin(2x+5))^{arctgx}, \quad b)\ y = \dfrac{(x+3)^5(x-5)^3}{\sqrt[3]{x-7}}.$

3.13. $a)\quad y = (\cos(3x+5))^{\arcsin 2x}, \quad b)\ y = \dfrac{(x+3)^2\sqrt[7]{(x-5)^3}}{(x+7)^5}.$

3.14. $a)\quad y = (\ln(3x+4))^{\sin(2x+5)}, \quad b)\ y = \dfrac{(x-3)^3\sqrt[5]{(x-5)^4}}{(x+5)^7}.$

3.15. $a)\quad y=(ctg3x)^{\ln(5x-2)},\quad b)\ y=\dfrac{\sqrt{x+3}\cdot\sqrt[3]{(x+2)^4}}{(x-2)^5}.$

3.16. $a)\quad y=(\arcsin 3x)^{\lg(7x+3)},\quad b)\ y=\dfrac{\sqrt{x+3}\cdot(x+7)^5}{\sqrt[6]{(x-2)^5}}.$

3.17. $a)\quad y=(tg3x)^{1/x^2},\quad b)\ y=\dfrac{\sqrt{(x+3)^3}\cdot\sqrt[3]{(x-3)^4}}{(x+2)^5}.$

3.18. $a)\quad y=(\arccos 3x)^{\log_3(2x+1)},\quad b)\ y=\dfrac{(x-6)^3(x+3)^5}{\sqrt[3]{(x+4)^4}}.$

3.19. $a)\quad y=(tg3x)^{\arccos(x-2)},\quad b)\ y=\dfrac{\sqrt[3]{(x+5)^4}}{(x-4)^5(x+3)^3}.$

3.20. $a)\quad y=(\sin(5x+3))^{\cos^2 x},\quad b)\ y=\dfrac{\sqrt[5]{(x+3)^4}}{(x-2)^3(x+5)^5}.$

3.21. $a)\quad y=(\sin(3x+1))^{arctg2x},\quad b)\ y=\dfrac{(x-3)^5\sqrt[4]{(x-5)^3}}{(x+7)^6}.$

3.22. $a)\quad y=(ctg2x)^{1/x^3},\quad b)\ y=\dfrac{\sqrt{(x+3)^3}\cdot\sqrt[4]{(x-2)^3}}{(x+2)^7}.$

3.23. $a)\quad y=(\sin 3x)^{\log_2 x},\quad b)\ y=\dfrac{\sqrt{(x-3)^5}\cdot\sqrt[3]{(x-5)^2}}{(x+3)^3}.$

3.24. $a)\quad y=(tg(2x+3))^{arcctgx},\quad b)\ y=\dfrac{\sqrt[5]{(x+3)^4}(x-5)^3}{\sqrt[3]{x-7}}.$

3.25. $a)\quad y=(1/x)^{arctg3x},\quad b)\ y=\dfrac{(x+5)^5(x-2)^3}{\sqrt[3]{(x-7)^2}}.$

3.26. $a)\quad y=(3x+5)^{tg^2 3x},\quad b)\ y=\dfrac{(x+7)^5\sqrt{(x+2)^3}}{\sqrt[3]{(x-5)^4}}.$

3.27. $a)\quad y=(\log_3(3x+1))^{\arccos 2x},\quad b)\ y=\dfrac{(x+3)^2\cdot\sqrt[4]{(x-5)^3}}{(x+1)^5}.$

3.28. $a)\quad y=(\cos^2 x)^{\log_2(3x+2)},\quad b)\ y=\dfrac{(x-3)^3(x+5)^5}{\sqrt[3]{(x+2)^5}}.$

3.29. $a)\quad y=(1/x)^{\arccos 5x},\quad b)\ y=\dfrac{(x+7)^4(x-2)^5}{\sqrt[5]{(x-5)^3}}.$

3.30. $a)\quad y=(\sin^2 x)^{\log_5(3x+5)},\quad b)\ y=\dfrac{(x-3)^3\sqrt{(x+5)^5}}{\sqrt[6]{(x+3)^5}}.$

4

Oshkormas shaklda berilgan funksiyalarning hosilalarini hisoblang.

4.1. $2^{x+y}=2^x+2^y.$

4.2. $y^2=\sin x+x\cos y.$

4.3. $y=7x+ctgy.$

4.4. $tgy=3x+5y.$

4.5. $y^2+x^2=\sin y.$

4.6. $xy=x^2+ctgy.$

4.7. $xy-6=\cos y.$

4.8. $e^y=4x-7y.$

4.9. $y^2x^2+x=5y.$

4.10. $y^2=(x-y)/(x+y).$

4.11. $xy=ctgy.$

4.12. $\sin y=xy^2+5.$

4.13. $\sin(xy)+\cos(xy)=tg(x+y).$

4.14. $y=x+arctgy.$

4.15. $x-y=\arcsin x-\arcsin y.$

4.16. $x\sin y-\cos y+\cos 2y=0.$

4.17. $x^3+5x^2y+4xy^2+.y^3=0$

4.18. $y=1+xe^y.$

4.19. $y^2=(x-y)/(x+y).$

4.20. $y=e^y+4x.$

4.21. $y^2=x+\ln(y/x).$

4.22. $y\sin x-\cos(x+y)=0.$

4.23. $x^{2/3}+y^{2/3}=a^{2/3}.$

4.24. $y=\cos(x+y).$

4.25. $\cos y=5x-3y.$

4.26. $x^3+y^3=7xy^2+2x^2y.$

4.27. $3^x+3^y=3^{x+y}.$

4.28. $xy=\cos y.$

4.29. $x^4+y^4=x^2y^2.$

4.30. $x^y=y^x.$

5

Parametrik shakldа berilgan funksiyalarning hosilalarini hisoblang.

5.1. $\begin{cases} x=\ln\left(t+\sqrt{t^2+1}\right), \\ y=t\sqrt{t^2+1}. \end{cases}$

5.2. $\begin{cases} x=\sqrt{1-t^2}, \\ y=tg\sqrt{t+1}. \end{cases}$

5.3. $\begin{cases} x=\sqrt{2t-t^2}, \\ y=\arcsin(t-1). \end{cases}$

5.4. $\begin{cases} x=\arcsin(\sin t), \\ y=\arccos(\cos t). \end{cases}$

5.5. $\begin{cases} x=\sqrt{2t-t^2}, \\ y=1/\sqrt[3]{(1-t)^2}. \end{cases}$

5.6. $\begin{cases} x=\ln(ctgt), \\ y=1/\cos^2 t. \end{cases}$

5.7. $\begin{cases} x=ctg(2e^t), \\ y=\ln(tge^t). \end{cases}$

5.8. $\begin{cases} x=arcctg(e^{t/2}), \\ y=\sqrt{e^t+1}. \end{cases}$

5.9. $\begin{cases} x=\left(3t^2+1\right)/(3t^2), \\ y=\sin\left(t^3/3+t\right). \end{cases}$

5.10. $\begin{cases} x=\arcsin(1/t), \\ y=\sqrt{t^2-1}+\arccos(1/t). \end{cases}$

5.11. $\begin{cases} x=1/\ln t, \\ y=\ln\left(\left(1+\sqrt{1-t^2}\right)/t\right). \end{cases}$

5.12. $\begin{cases} x=\arcsin\sqrt{t}, \\ y=\sqrt{1+\sqrt{t}}. \end{cases}$

5.13. $\begin{cases} x = \ln(tgt), \\ y = 1/\sin^2 t. \end{cases}$

5.14. $\begin{cases} x = \ln(1-t^2), \\ y = \arcsin\sqrt{1-t^2}. \end{cases}$

5.15. $\begin{cases} x = arctgt \\ y = \ln\left(\sqrt{1-t^2}/(t+1)\right). \end{cases}$

5.16. $\begin{cases} x = \ln((1-t)/(1+t)), \\ y = \sqrt{1-t^2}. \end{cases}$

5.17. $\begin{cases} x = \arccos(1/t), \\ y = \sqrt{t^2-1} + \arcsin(1/t). \end{cases}$

5.18. $\begin{cases} x = tg(2e^t), \\ y = \ln(ctge^t). \end{cases}$

5.19. $\begin{cases} x = (\arcsin t)^2, \\ y = t/\sqrt{1-t^2}. \end{cases}$

5.20. $\begin{cases} x = t(t\cos t - 2\sin t), \\ y = t(t\sin t + 2\cos t). \end{cases}$

5.21. $\begin{cases} x = \arcsin\left(t/\sqrt{1+t^2}\right), \\ y = \arccos(1/\sqrt{1+t^2}). \end{cases}$

5.22. $\begin{cases} x = 6t/\left(1+t^2\right), \\ y = 6t^2/\left(1+t^2\right). \end{cases}$

5.23. $\begin{cases} x = a(t\cos t + \sin t), \\ y = a(\sin t - t\cos t). \end{cases}$

5.24. $\begin{cases} x = tgt, \\ y = 1/\sin 2t. \end{cases}$

5.25. $\begin{cases} x = \ln\left(1+t^2\right), \\ y = arctg\sqrt{1+t^2}. \end{cases}$

5.26. $\begin{cases} x = (1+t^3)/(t^2-1), \\ y = t/(t^2-1). \end{cases}$

5.27. $\begin{cases} x = (2t+t^2)/(t^3+1), \\ y = \left(2t-t^2\right)/(t^3+1). \end{cases}$

5.28. $\begin{cases} x = 2tgt, \\ y = 2\sin^2 t + \sin 2t. \end{cases}$

5.29. $\begin{cases} x = a(t - \sin t), \\ y = a(1-\cos t). \end{cases}$

5.30. $\begin{cases} x = \ln(1+t^2), \\ y = t - arctgt. \end{cases}$

5.3. Funksiya differensiali. Yuqori tartibli hosilalar.Yuqori tartibli differensiallar

Berilgan $y=f(x)$ funksiyaning ***differensiali*** deb, funksiya orrtirmasining argument orttirmasi $\Delta x=dx$ ga nisbatan chiziqli bosh qismiga aytiladi. $dy=f'(x)dx$ kabi belgilanadi.

Differensial ta'rifidan va hosila hisoblash qoidalaridan foydalanib, quyidagi formulalarni hosil qilamiz ($u=u(x),\ v=v(x)$):

1) $dC=0$; 2) $d(u\pm v)=du\pm dv$; 3) $d(Cu)=Cdu$; 4) $d(uv)=vdu+udv$;

5) $d\left(\dfrac{u}{v}\right)=\dfrac{vdu-udv}{v^2}$; 6) $d\,f(u)=f'(u)u'dx=f'(u)du$, 7) $dx=\Delta x$.

Funksiya orttirmasi uning differensialidan Δx ga nisbatan yuqori tartibli cheksiz kichik miqdorga farq qiladi. Shuning uchun, argumentning x_0 nuqtadagi cheksiz kichik orttirmasida, funksiyaning orttirmasi uning shu nuqtadagi differensialiga taqriban teng bo'ladi, ya'ni $\Delta y\approx dz,\ f(x_0+\Delta x)-f(x_0)\approx f'(x_0)\Delta x$, bundan

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x \tag{5.1}$$

taqribiy hisoblash formulasiga ega bo'lamiz. Bu formula yordamida funksiyaning $x=x_0+\Delta x$ nuqtadagi qiymati taqribiy hisoblanadi. Hisoblashdagi funksiyaning nisbiy xatoligi

$$\delta=\left|\frac{f(x)-f(x_0)}{f(x_0)}\right|\cdot 100\% \tag{5.2}$$

formula bilan topiladi.

5.8-misol

Ushbu $tg44^\circ$ ni differensial yordamida taqribiy hisoblang va nisbiy xatolikni toping.

► $f(x)=tgx,\ f'(x)=\dfrac{1}{\cos^2 x},\ x=44^\circ,\ x_0=45^\circ,\ \Delta x=-1^\circ=-1^\circ\cdot\dfrac{\pi}{180^\circ}\approx-0{,}017.$

Taqribiy hisoblash formulasi (3.1)dan foydalansak,

$$tg44^\circ\approx tg45^\circ-\frac{1}{\cos^2 45^\circ}\cdot 0{,}017=1-2\cdot 0{,}017=0{,}966.$$

Nisbiy xatolik, $\delta=\left|\dfrac{f(x)-f(x_0)}{f(x_0)}\right|\cdot 100\%=\dfrac{0{,}034}{1}\cdot 100\%=3{,}4\%.$ ◄

Berilgan $y = f(x)$ funksiyaning hosilasidan olingan hosila ***ikkinchi tartibli hosila***, $(n-1)-$tartibli hosilasidan olingan hosila $n-$***tartibli hosila*** deyiladi va mos ravishda $y'' = (y')' = f''(x) = \dfrac{d^2y}{dx^2}$, $y^{(n)} = (y^{(n-1)})' = f^{(n)}(x) = \dfrac{d^ny}{dx^n}$ kabi belgilanadi.

Yuqori tartibli hosila hisoblashda quyidagi formulalar o‘rinli $u = u(x),\ v = v(x)$):

$$1)\ (u \pm v)^{(n)} = u^{(n)} \pm v^{(n)};\quad 2)\ (Cu)^{(n)} = Cu^{(n)};$$

$$3)\ (uv)^{(n)} = u^{(n)}\cdot v + \frac{n}{1!}u^{(n-1)}\cdot v' + \frac{n(n-1)}{2!}u^{(n-2)}\cdot v'' + \ldots + u\cdot v^{(n)}.$$

Oxirgi 3) formula ***Leybnits formulasi*** deb ataladi.

5.9-misol

Leybnits formulasi yordamida hisoblang: $y = (x^2+1)\cdot\log_2 x,\ y^{(10)} - ?$

► Qulaylik uchun quyidagicha belgilashlar kiritamiz va hosilalarini hisoblaymiz:

$$u = \log_2 x,\ v = x^2 + 1.$$

$$u' = \frac{1}{x\ln 2},\ u'' = -\frac{1}{x^2\ln 2}, u''' = \frac{2!}{x^3\ln 2}, u^{(4)} = -\frac{3!}{x^4\ln 2}, \ldots, u^{(9)} = \frac{8!}{x^9\ln 2}, u^{(10)} = -\frac{9!}{x^{10}\ln 2}.$$

$$v' = 2x,\ v'' = 2,\ v''' = v^{(4)} = \ldots = v^{(10)} = 0.$$

Leybnits formulasini $n = 10$ uchun yozib olamiz

$$(uv)^{(10)} = u^{(10)}\cdot v + \frac{10}{1!}u^{(9)}\cdot v' + \frac{10\cdot 9}{2!}u^{(8)}\cdot v'' + \ldots + u\cdot v^{(10)}.$$

Yuqoridagi hosilalarni hisobga olsak, yig‘indining birinchi uchta hadi qoladi, ya’ni

$$y^{(10)} = -\frac{9!}{x^{10}\ln 2}(x^2+1) + \frac{10\cdot 8!}{x^9\ln 2}\cdot 2x - \frac{90\cdot 7!}{x^8\ln 2},$$

$$y^{(10)} = -\frac{2\cdot 7!}{x^{10}\ln 2}(x^2+36). ◄$$

$F(x,y) = 0$ tenglama x ga bog‘liq y funksiyani aniqlasa, bu funksiyadan yuqori tartibli hosila olish uchun y va uning hosilalari x ning funksiyasi ekanini e’tiborga olgan holda, tegishli marta differensiallash kerak.

Parametrik ko‘rinishda berilgan $x = x(t),\ y = y(t)$ funksiyadan ikkinchi tartibli hosila quyidagi formula bilan hisoblanadi:

$$y'_x=\frac{y'_t}{x'_t},\ y''_{xx}=(y'_x)'_t\cdot x'_t=\left(\frac{y'_t}{x'_t}\right)'_t\cdot\frac{1}{x'_t}\ \text{yoki}\ y''_{xx}=\frac{y''_{tt}x'_t-y'_tx''_{tt}}{(x'_t)^3}.$$

5.10-misol

Quyidagi $\begin{cases}x=\arcsin t,\\ y=\ln(1-t^2)\end{cases}$ parametrik funksiya berilgan bo‘lsa y''_{xx} −?

▶ $y'_x=\frac{y'_t}{x'_t},\ x'(t)=\frac{1}{\sqrt{1-t^2}},\ y'_t=\frac{-2t}{1-t^2},\quad y'_x=\frac{-2t}{\sqrt{1-t^2}}.$

$$y''_{xx}=(y'_x)'_t\cdot\frac{1}{x'_t}=\frac{-2\cdot\frac{-t}{\sqrt{1-t^2}}+2t\sqrt{1-t^2}}{1-t^2}\cdot\sqrt{1-t^2}=\frac{2t+2t(1-t^2)}{1-t^2}=\frac{2t(2-t^2)}{1-t^2}.$$ ◀

Funksiyaning differensialidan olingan differensial ***ikkinchi tartibli differensial***, $(n-1)-$ tartibli differensialdan olingan differensial $n-$ ***tartibli differensial*** deyiladi va mos ravishda

$$d^2y=d(dy)=f''(x)dx^2,\ d^ny=d(d^{n-1}y)=f^{(n)}(x)dx^n$$

formulalar bilan hisoblanadi.

5.11-misol

Agar $y=4^{-x^2}$ bo‘lsa d^2y ni hisoblang

▶ $y'=-2x\cdot4^{-x^2}\ln4,\ y''=4^{1-x^2}x^2\ln^24-2\cdot4^{-x^2}\ln4,$

$d^2y=f''(x)dx^2,\ d^2y=(4^{1-x^2}x^2\ln^24-2\cdot4^{-x^2}\ln4)dx^2.$ ◀

18-Auditoriya topshiriqlari

1. $y=arctg\left(x+\sqrt{1+x^2}\right)$ funksiyaning ikkinchi tartibli hosilasini hisoblang.

2. Oshkormas tenglama bilan berilgan funksiyalarning y'' hosilasini toping.

$a)\ \ln y=x+y$, $\qquad b)\ arctgy=x+y.$

3. Parametrik tenglama bilan berilgan funksiyalardan y'' hosilasini hisoblang.

$a)\ \begin{cases}x=t^4\\ y=\ln t,\end{cases}\qquad b)\ \begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t).\end{cases}$

4. $y=2^{3x+1}$ funksiyaning $n-$tartibli hosilasini toping.

5. Quyidagi funksiyalarning birinchi va ikkinchi tartibli differensiallarini toping:

$a)\ y = xe^{x^2}$, $b)\ y = \sqrt{1-x^2}\arcsin x$, $d)\ y = x^3\ln x$, $e)\ y = \dfrac{\ln x}{x^2}$.

6. Leybnits formulasidan foydalanib ko‘rsatilgan tartibli hosilalarni toping.

$a)\ y = x^2e^x,\ y^{(5)} - ?$ $b)\ y = (x^2+1)\sin x,\ y^{(20)} - ?$

7. Differensial yordamida taqribiy hisoblang, absolyut va nisbiy xatolikni toping.

$a)\ y = \sqrt[3]{x^2+2x+5},\ x = 0{,}97$ $b)\ y = \sqrt{\dfrac{x^2-3}{x^2+5}},\ x = 2{,}037.$

18-Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y = x^3 + \ln x$ tenglama bilan berilgan funksiyaning uchinchi tartibli hosilasi bo‘ladi?

A) $y'' = 6 - 2/x^3$, B) $y'' = 6 + 1/(2x^3)$, C) $y'' = 6 - 4/x^3$, D) $y'' = 6 + 2/x^3$.

2. Quyidagilardan qaysi biri $y^2 = 4x$ tenglama bilan berilgan funksiyaning ikkinchi tartibli hosilasi bo‘ladi?

A) $y'' = -2/y^2$, B) $y'' = 2/y^2$, C) $y'' = -4/y^3$, D) $y'' = 4/y^3$.

3. Quyidagilardan qaysi biri $y = t^2 + 1,\ x = \ln t$ parametrik tenglamalar bilan berilgan funksiyaning ikkinchi tartibli hosilasi bo‘ladi?

A) $y'' = -2t^3$, B) $y'' = -4t^2$, C) $y'' = -4t$, D) $y'' = -2t^2$.

4. Quyidagilardan qaysi biri noto‘g‘ri ?

A) $d^2(\sin x) = \sin x dx^2$, B) $d^2(e^x) = e^x dx^2$, C) $d^2(cosx) = -\cos x dx^2$, D) $d^2(x^3) = 6xdx^2$.

5. $\sqrt{2{,}07^3+1}$ ni differensial yordamida taqribiy hisoblang.

A) 3,21 B) 3,14 C) 3,22 D) 3,15.

5.4. O‘rta qiymat haqidagi teoremalar. Lopital qoidasi

5.1-Teorema *(Roll teoremasi)*. *$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo‘lsin. Agar funksiya $(a;b)$ intervalda differensiallanuvchi bo‘lib, $f(a) = f(b)$ tenglik o‘rinli bo‘lsa, u holda kamida bitta shunday bir $c \in (a;b)$ nuqta topiladiki, $f'(c) = 0$ bo‘ladi.*

5.2-Teorema *(Lagranj teoremasi)*. *$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo‘lib, $(a;b)$ intervalda differensiallanuvchi bo‘lsa, u holda kamida bitta shunday bir $c \in (a;b)$ nuqta topiladiki, $f(b) - f(a) = f'(c)(b-a)$ tenglik o‘rinli bo‘ladi.*

Bu tenglikga ***Lagranjning chekli orttirmalar formulasi*** deyiladi.
Teoremani geometrik izohlaydigan bo‘lsak, uning har bir sharti o‘rinli bo‘lganda, $y = f(x)$ funksiya grafigida $A(a; f(a))$ va $B(b; f(b))$ nuqtalarini tutashturuvchi AB yoyiga tegishli hech bo‘lmaganda bitta nuqta topiladiki, chiziqning shu nuqtasiga o‘ tkazilgan urinma AB vatarga parallel bo‘ladi.

5.3-Teorema*(Koshi teoremasi)*. *$y = f(x)$ va $y = g(x)$ funksiyalar $[a;b]$ kesmada uzluksiz bo‘lib, $(a;b)$ intervalda differensiallanuvchi va $g'(x) \neq 0$, $x \in (a;b)$ bo‘lsa, u holda kamida bitta shunday bir $c \in (a;b)$ nuqta topiladiki, $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$ tenglik o‘rinli bo‘ladi.*

Lopital qoidasi($\frac{0}{0}$ va $\frac{\infty}{\infty}$ tipidagi aniqmasliklarni ochish uchun). $f(x), g(x)$ funksiyalar $x = x_0$ nuqtaning biror atrofida uzluksiz va differensiallanuvchi bo‘lsin. Agar $x \to x_0$ da $f(x), g(x)$ funksiyalar nolga (yoki $\pm\infty$ ga) intilsa va $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$ mavjud bo‘lsa, u holda $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$ ham mavjud va bu limitlar teng, ya’ni

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Lopital qoidasi $x_0 = \pm\infty$ da ham o‘rinli.

5.12-misol

Ushbu $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\sin 3x}$ limitni hisoblang

►Kasrning surati ham, maxraji ham uzluksiz, differensiallanuvchi va $x\to x_0$ da nolga intiluvchi funksiyalar bo‘lgani uchun Lopital qoidasini qo‘llaymiz,

$$\lim_{x\to 0}\frac{e^{2x}-1}{\sin 3x}=\lim_{x\to 0}\frac{2e^{2x}}{3\cos 3x}=\frac{2}{3}.$$ ◄

Agar $\dfrac{f'(x)}{g'(x)}$ nisbat $x\to x_0$ da yana $\dfrac{0}{0}$ va $\dfrac{\infty}{\infty}$ tipidagi aniqmaslik bo‘lsa va $f'(x), g'(x)$ funksiyalar ham yuqoridagi shartlarni qanoatlantirsa, qoidani yana bir bor qo‘llab ikkinchi tartibli hosilaga o‘tish mumkin, va hakozo. Lekin hosilalar nisbatining limiti mavjud bo‘lmasa ham funksiyalar nisbatining limiti mavjud bo‘lishi mumkin.

5.13-misol

Ushbu $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x+\cos x}$ limitni hisoblang

► $\lim\limits_{x\to\infty}\dfrac{(x+\sin x)'}{(x+\cos x)'}=\lim\limits_{x\to\infty}\dfrac{1+\cos x}{1-\sin x}$, bu limit mavjud emas, chunki kasrning surati va maxraji $[0;2]$ kesmadagi ixtiyoriy sonni, kasrning o‘z i esa ixtiyoriy musbat sonni qabul qila oladi. Demak, Lopital qoidasini qo‘llab bo‘lmaydi. Lekin

$$\lim_{x\to\infty}\frac{x+\sin x}{x+\cos x}=\lim_{x\to\infty}\frac{1+\sin x/x}{1+\cos x/x}=1.$$ ◄

$0\cdot\infty$ va $\infty-\infty$ tipidagi aniqmasliklar osonlik bilan $\dfrac{0}{0}$ yoki $\dfrac{\infty}{\infty}$ tipidagi aniqmasliklarga keltiriladi. Masalan, agar $f(x)g(x)$, $f(x)\to 0, g(x)\to\infty$ bo‘lsa, bu ko‘paytma $\dfrac{f(x)}{1/g(x)}$ yoki $\dfrac{g(x)}{1/f(x)}$ lardan biriga almashtiriladi, agar $f(x)-g(x)$, $f(x)\to\infty, g(x)\to\infty$ bo‘lsa, $f(x)-g(x)=f(x)\left(1-\dfrac{g(x)}{f(x)}\right)$, bu esa $0\cdot\infty$ tipidagi aniqmaslikdir.

5.14-misol

Ushbu $\lim\limits_{x\to\infty}x^2e^{-3x}$ limitni hisoblang

►Bu esa $0\cdot\infty$ tipidagi aniqmaslik.

$$\lim_{x\to\infty} x^2e^{-3x} = \lim_{x\to\infty}\frac{x^2}{e^{3x}} = \lim_{x\to\infty}\frac{2x}{3e^{3x}} = \lim_{x\to\infty}\frac{2}{9e^{3x}} = 0. \blacktriangleleft$$

$f(x)^{g(x)}$ ko'rinishdagi funksiya limitini hisoblashda $1^{\infty}, 0^0, \infty^0$ tipidagi aniqmasliklar mavjud. Bunday aniqmasliklarni avval logarifmlab, $0\cdot\infty$ tipiga keltiriladi: $A = \lim\limits_{x\to x_0} f(x)^{g(x)}$, $\ln A = \lim\limits_{x\to x_0}\ln f(x)^{g(x)} = \lim\limits_{x\to x_0}\left(g(x)\cdot\ln f(x)\right)$. So'ngra yuqoridagi kabi almashtitirish bajarilib, Lopital qoidasi qo'llanadi.

5.15-misol

Ushbu $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$ limitni hisoblang

► Bu yerda ∞^0 tipidagi aniqmaslik.

$$A = \lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}, \ \ln A = \lim_{x\to 0}\left(\sin x \ln\frac{1}{x}\right) = -\lim_{x\to 0}\frac{\ln x}{1/\sin x} = -\lim_{x\to 0}\frac{1/x}{-\cos x/\sin^2 x} =$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0}\frac{\sin 2x}{\cos x - x\sin x} = 0, \qquad A = \lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x} = 1. \blacktriangleleft$$

19-Auditoriya topshiriqlari

1. $y = x - x^3$ funksiya uchun $[-1;0]$ va $[0;1]$ kesmalarda Roll teoremasi shartlari bajarilishini ko'rsating va mos c nuqta qiymatlarini aniqlang.
2. $y = x^2$ parabolaninig $A(1;1)$ va $B(3;9)$ nuqtalari orasidagi yoyida yotuvchi shunday bir nuqtani topingki, bu nuqtadan o'tkazilgan urunma AB vatarga parallel bo'lsin.

Quyidagi limitlarni Lopital qoidasi yordamida hisoblang:

3. $\lim\limits_{x\to 2}\frac{x^4 - 3x^2 + x - 6}{x^2 - 4}$
4. $\lim\limits_{x\to\infty}(x(e^{1/x} - 1))$
5. $\lim\limits_{x\to\pi/3}\frac{1 - 2\cos x}{\sin 3x}$
6. $\lim\limits_{x\to 0}\frac{e^{x^2} + 3x^2 - 1}{\sin^2 x}$
7. $\lim\limits_{x\to 0}(\cos 2x)^{\frac{1}{x^2}}$
9. $\lim\limits_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$
10. $\lim\limits_{x\to+\infty}\frac{\pi - 2arctgx}{\ln\left(1 + \frac{1}{x}\right)}$
11. $\lim\limits_{x\to\frac{1}{2}}\left(\frac{1}{x} - 1\right)^{tg\pi x}$
11. $\lim\limits_{x\to 0}(e^x + x)^{1/x}$
12. $\lim\limits_{x\to\pi/2}(ctgx)^{2x-\pi}$

19-Mustaqil yechish uchun testlar

Quyidagi limitlarni hisoblang:

1. $\lim\limits_{x\to 3}\dfrac{x^4-6x^2-3x-18}{x^3-27}$;

A) $2\frac{5}{9}$, B) $2\frac{4}{9}$, C) $3\frac{5}{9}$, D) $3\frac{4}{9}$.

2. $\lim\limits_{x\to 0}\dfrac{\ln(1+3x)}{e^{2x}-1}$;

A) 2, B) $1\frac{1}{2}$, C) 1, D) $\frac{1}{6}$.

3. $\lim\limits_{x\to\pi/2}(tgx)^{2x-\pi}$;

A) 2, B) $1\frac{1}{2}$, C) 1, D) $\frac{1}{6}$.

4. $\lim\limits_{x\to\pi}(x-\pi)tg\dfrac{x}{2}$;

A) 0, B) 1, C) 2, D) ∞.

5. $\lim\limits_{x\to 3}\left(\dfrac{1}{x-3}-\dfrac{5}{x^2-x-6}\right)$;

A) 2, B) $1\frac{1}{3}$, C) 1, D) $\frac{1}{5}$.

6-Shaxsiy uy topshiriqlari

1

Quyidagi funksiyalarning $n-$tartibli hosilasini toping.

1.1. $y=\ln(2x+1)$

1.2. $y=x\sqrt{e^x}$

1.3. $y=\dfrac{1}{2x+1}$

1.4. $y=e^{4x}$

1.5. $y=\ln(3+x^2)$

1.6. $y=\dfrac{x}{3x+1}$

1.7. $y=\log_3(x+4)$

1.8. $y=\lg(5x+1)$

1.9. $y=\sin 3x$

1.10. $y=\sqrt[3]{e^{2x+1}}$

1.11. $y=\dfrac{1+x}{1-x}$

1.12. $y=\sqrt{x-7}$

1.13. $y=\cos 2x$

1.14. $y=\dfrac{5x+1}{13(2x+3)}$

1.15. $y=\dfrac{4}{x+3}$.

1.16. $y=\dfrac{x}{x^2-1}$

1.17. $y=\dfrac{4+15x}{5x+1}$

1.18. $y=xe^{6x}$

1.19. $y = \sin^2 x$

1.20. $y = \log_5(2x-1)$

1.21. $y = xe^x$

1.22. $y = \cos^2 x$

1.23. $y = \dfrac{1}{x^2-3x+2}$

1.24. $y = \dfrac{1}{x-7}$.

1.25. $y = \dfrac{1+x}{\sqrt{x}}$

1.26. $y = x\ln x$

1.27. $y = 3e^{-3x}$.

1.28. $y = \cos(3x+1)$

1.29. $y = 3^x$

1.30. $y = a^{2x}$

2

Berilgan parametrik funksiyalarning ikkinchi tartibli y''_{xx} hosilasini toping.

2.1. $\begin{cases} x = \cos 2t \\ y = 2\sec^2 t. \end{cases}$

2.2. $\begin{cases} x = \sqrt{1-t^2} \\ y = 1/t \end{cases}$

2.3. $\begin{cases} x = e^t \cos t, \\ y = e^t \sin t. \end{cases}$

2.4. $\begin{cases} x = \sin^2 t, \\ y = 1/ch^2 t. \end{cases}$

2.5. $\begin{cases} x = t + \sin t, \\ y = 2 - \cos t. \end{cases}$

2.6. $\begin{cases} x = 1/t, \\ y = 1/(1+t^2). \end{cases}$

2.7. $\begin{cases} x = \sqrt{t}, \\ y = 1/\sqrt{1-t}. \end{cases}$

2.8. $\begin{cases} x = \sin t \\ y = \sec t. \end{cases}$

2.9. $\begin{cases} x = sh^2 t, \\ y = th^2 t. \end{cases}$

2.10. $\begin{cases} x = tgt, \\ y = 1/\sin 2t. \end{cases}$

2.11. $\begin{cases} x = \sqrt{t-1}, \\ y = t/\sqrt{1-t}. \end{cases}$

2.12. $\begin{cases} x = \ln t, \\ y = arctgt. \end{cases}$

2.13. $\begin{cases} x = \sqrt{t} \\ y = \sqrt[3]{t-1} \end{cases}$

2.14. $\begin{cases} x = \cos t/(1+2\cos t), \\ y = \sin t/(1+2\cos t). \end{cases}$

2.15. $\begin{cases} x = \sqrt[3]{t-1}, \\ y = \ln t. \end{cases}$

2.16. $\begin{cases} x = t + \sin t, \\ y = 2 + \cos t. \end{cases}$

2.17. $\begin{cases} x = \cos^2 t, \\ y = tg^2 t. \end{cases}$

2.18. $\begin{cases} x = \sqrt{t-3} \\ y = \ln(t-2) \end{cases}$

2.19. $\begin{cases} x = \sin t \\ y = \ln\cos t. \end{cases}$

2.20. $\begin{cases} x = \cos t \\ y = \ln\sin t. \end{cases}$

2.21. $\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t. \end{cases}$

2.22. $\begin{cases} x = e^t \\ y = \arcsin t. \end{cases}$

2.23. $\begin{cases} x = t - \sin t, \\ y = 2 - \cos t. \end{cases}$

2.24. $\begin{cases} x = cht \\ y = \sqrt[3]{sh^2 t}. \end{cases}$

2.25. $\begin{cases} x = \cos t + t\sin t \\ y = \sin 2t. \end{cases}$

2.26. $\begin{cases} x = \cos t \\ y = \sin^4 (t/2). \end{cases}$

2.27. $\begin{cases} x = arctgt \\ y = t^2/2. \end{cases}$

2.28. $\begin{cases} x = 1/t^2, \\ y = 1/(t^2 + 1). \end{cases}$

2.29. $\begin{cases} x = \cos t - t\sin t \\ y = cost + t\sin t. \end{cases}$

2.30. $\begin{cases} x = 2(t - \sin t), \\ y = 4(2 + \cos t). \end{cases}$

3

Differensial yordamida $0{,}01$ aniqlikda taqribiy hisoblang va nisbiy xatolikni toping.

3.1. $a)\ \sqrt[3]{27{,}5};$ $b)\ arctg1{,}02.$

3.2. $a)\ \sqrt[7]{130};$ $b)\ \arcsin 0{,}54.$

3.3. $a)\ 2{,}9/\sqrt{2{,}9^2 + 16};$ $b)\ \sin 92°.$

3.4. $a)\ \sqrt[5]{200};$ $b)\ arctg\sqrt{3{,}2}.$

3.5. $a)\ 4{,}01^{1{,}5};$ $b)\ arctg\sqrt{0{,}97}.$

3.6. $a)\ \sqrt[3]{70};$ $b)\ \ln tg 46°.$

3.7. $a)\ \sqrt[4]{16.64};$ $b)\ \sin 29°.$

3.8. $a)\ \left(0{,}98 + \sqrt{5 - 0{,}98^2}\right)/2;$ $b)\ e^{0{,}2}.$

3.9. $a)\ 0{,}98^{1{,}5};$ $b)\ arctg\sqrt{1{,}02}.$

3.10. $a)\ \sqrt[3]{26{,}19};$ $b)\ \cos 59°.$

3.11. $a)\ 3{,}02^4 + 3{,}02^3;$ $b)\ ctg 29°.$

3.12. $a)\ \sqrt{(2{,}037^2 - 3)/(2{,}037^2 + 5)};$ $b)\ tg 44°.$

3.13. $a)\ \sqrt{(4 - 3{,}02)/(1 + 3{,}02)};$ $b)\ arctg\sqrt{3{,}1}.$

3.14. $a)\ 4{,}16^{-0{,}5};$ $b)\ \ln tg 47°15'.$

3.15. $a)\ 3{,}03^5;$ $b)\ \arcsin 0{,}4983.$

3.16. $a)\ \sqrt[3]{65};$ $b)\ arctg 0{,}98.$

3.17. $a)\ \sqrt[5]{237};$ $b)\ \sin 31°.$

3.18. $a)\ 4{,}1/\sqrt{4{,}1^2 + 9};$ $b)\ e^{0{,}25}.$

3.19. $a)\ \sqrt[3]{150};$ $b)\ arctg\sqrt{2{,}9}.$

3.20. $a)\ 4{,}01^3+4{,}01^2;$ $b)\ \ln tg44°.$

3.21. $a)\ 1{,}05+\sqrt{3+1{,}05^2};$ $b)\ \ln ctg46°.$

3.22. $a)\ \sqrt[4]{85};$ $b)\ \ln arctg\sqrt{0{,}97}.$

3.23. $a)\ \sqrt[3]{8{,}36};$ $b)\ \arcsin 0{,}08.$

3.24. $a)\ \sqrt[5]{1{,}03^2};$ $b)\ \sqrt[3]{0{,}01+3\cos 0{,}01}.$

3.25. $a)\ \sqrt{1{,}97^2+5};$ $b)\ \cos 61°.$

3.26. $a)\ 5{,}02^3+5{,}02^2;$ $b)\ ctg44°.$

3.27. $a)\ \sqrt{1+0{,}01+\sin 0{,}01};$ $b)\ arctg\sqrt[3]{1{,}02}.$

3.28. $a)\ \sqrt[3]{8{,}24};$ $b)\ arcctg\sqrt{3{,}1}.$

3.29. $a)\ 9{,}16^{-0{,}5};$ $b)\ \ln ctg47°15'.$

3.30. $a)\ 2{,}03^6;$ $b)\ \arcsin 0{,}512.$

4

Quyidagi limitlarni Lopital qoidasi yordamida hisoblang.

4.1. $a)\ \lim\limits_{x\to\pi/4}\dfrac{1/\cos^2 x-2tgx}{1+\cos 4x};$ $b)\ \lim\limits_{x\to 0}(\ln(x+e))^{1/x}.$

4.2. $a)\ \lim\limits_{x\to\infty}\dfrac{e^{1/x^2}-1}{2arctgx^2-\pi};$ $b)\ \lim\limits_{x\to 0}\sqrt[x]{\cos\sqrt{x}}.$

4.3. $a)\ \lim\limits_{x\to\pi/4}\dfrac{1+\cos 4x}{2tgx-\sec^2 x};$ $b)\ \lim\limits_{x\to\infty}\left(\dfrac{2}{\pi}arctgx\right)^x.$

4.4. $a)\ \lim\limits_{x\to 0}\dfrac{e^{x^2}-1-x^2}{tg^2 2x};$ $b)\ \lim\limits_{x\to\infty}(\cos(4/\sqrt{x}))^x$

4.5. $a)\ \lim\limits_{x\to 0}\dfrac{\ln(1+x^2)}{\cos 3x-e^{-x}};$ $b)\ \lim\limits_{x\to 1/2}(\ln 2x\ln(2x-1))$

4.6. $a)\ \lim\limits_{x\to 0}\dfrac{1-\cos 5x}{tg^2 2x};$ $b)\ \lim\limits_{x\to\infty}((\pi-2arctgx)\cdot\ln x)$

4.7. $a)\ \lim\limits_{x\to 0}\dfrac{3^x-2^x}{x\sqrt{1-x^2}};$ $b)\ \lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

4.8. $a)\ \lim\limits_{x\to 0}\dfrac{\arcsin 4x}{5-5e^{-3x}};$ $b)\ \lim\limits_{x\to a}(a^2-x^2)\cdot tg\dfrac{\pi x}{2a}$

4.9. $a)\ \lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt[3]{x}};$ $b)\ \lim\limits_{x\to 2}\left(2-\dfrac{x}{2}\right)^{tg\frac{\pi x}{4}}$

4.10. $a)\ \lim\limits_{x\to 0}\dfrac{\arcsin x-2\arcsin x}{x\sqrt{1-x^2}};$ $b)\ \lim\limits_{x\to 0}x^{1/\ln(e^x-1)}$

4.11. a) $\lim\limits_{x\to\pi}\dfrac{\sqrt[3]{\cos 2x}-1}{2\sin^2(x/4)-1}$; b) $\lim\limits_{x\to 0}\arcsin x\cdot ctgx$

4.12. a) $\lim\limits_{x\to 0}\dfrac{e^{3x}-\cos 3x}{e^{2x}-\cos 2x}$; b) $\lim\limits_{x\to 1}\left(\dfrac{1}{2\left(1-\sqrt{x}\right)}-\dfrac{1}{3\left(1-\sqrt[3]{x}\right)}\right)$

4.13. a) $\lim\limits_{x\to 0}\dfrac{\sin(e^{x^2}-1)}{\cos x-1}$; b) $\lim\limits_{x\to 1}(1-x)^{\cos(\pi x/2)}$

4.14. a) $\lim\limits_{x\to 0}\dfrac{e^{x}-x^2/2-x-1}{\cos x-x^2/2-1}$; b) $\lim\limits_{x\to\infty}x^{5/(1+2\ln x)}$

4.15. a) $\lim\limits_{x\to 0}\dfrac{\pi/x}{\cos(5x/2)}$; b) $\lim\limits_{x\to\infty}(\ln 2x)^{1/\ln x}$

4.16. a) $\lim\limits_{x\to 1}\dfrac{\ln(1-x)+tg(\pi x/2)}{ctg\pi x}$; b) $\lim\limits_{x\to\infty}x\sin\dfrac{5}{6x}$

4.17. a) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1-x^3}{\sin^2 2x}$; b) $\lim\limits_{x\to 1}(1-x)^{\log_2 x}$

4.18. a) $\lim\limits_{x\to 0}\dfrac{x-arctgx}{x^3}$; b) $\lim\limits_{x\to 1}\left(ctg\,\dfrac{\pi x}{4}\right)^{tg(\pi x/2)}$

4.19. a) $\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{2x}}{\sin 5x}$; b) $\lim\limits_{x\to 0}(ctg 2x)^{1/\ln x}$

4.20. a) $\lim\limits_{x\to -1}\dfrac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}-1}$; b) $\lim\limits_{x\to 1}\dfrac{1}{\cos(\pi x/2)\ln(1-x)}$

4.21. a) $\lim\limits_{x\to\pi/6}\dfrac{1-\sin 3x}{(6x-\pi)^2}$; b) $\lim\limits_{x\to\pi/4}(tg 2x)^{4x-\pi}$

4.22. a) $\lim\limits_{x\to 0}\dfrac{e^{3\sqrt{x}}-1}{\sqrt{\sin 2x}}$; b) $\lim\limits_{x\to\infty}\left(x^2 e^{1/x^2}\right)$

4.23. a) $\lim\limits_{x\to 0}\dfrac{3^x-3^{\sin x}}{x^3}$; b) $\lim\limits_{x\to\infty}\left(\dfrac{2}{\pi}arctgx\right)^{x}$

4.24. a) $\lim\limits_{x\to\pi}\dfrac{(1-\pi)^2}{1-tg(x/4)}$; b) $\lim\limits_{x\to 1}\ln x\ln(x-1)$

4.25. a) $\lim\limits_{x\to 0}\dfrac{e^{x}-e^{-x}-2x}{x-\sin x}$; b) $\lim\limits_{x\to 1}(tg\pi x)^{2arctgx-1}$

4.26. a) $\lim\limits_{x\to 0}\dfrac{e^{tgx}-e^{x}}{tgx-x}$; b) $\lim\limits_{x\to 0}x^{\sin 6x}$

4.27. a) $\lim\limits_{x\to 0}\dfrac{\ln\sin 3x}{\ln\sin 2x}$; b) $\lim\limits_{x\to\infty}(x-1)^{1/\ln 2(x-1)}$

4.28. a) $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{2+5x}+2}{\sqrt{3+x}-1}$; b) $\lim\limits_{x\to\infty}x^3\sin(a/x)$

4.29. a) $\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{2x}}{tg 2x}$; b) $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^{tgx}$

4.30. a) $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{\sin 3x}$; b) $\lim\limits_{x\to\infty} x\sin\dfrac{6}{7x}$

5.5. Funksiyaning monotonligi, ekstremumni topish. Funksiyaning eng katta va eng kichik qiymati

Differensial hisobning asosiy vazifalardan biri funksiyalarni tekshiririshning umumiy usullarini ishlab chiqishdir.

Agar $y = f(x)$ funksiya argumentining $(a;b)$ oraliqdagi katta qiymatiga funksiyaning katta(kichik) qiymati mos kelsa, ya'ni $x_1 < x_2$ bo'lganda $f(x_1) < f(x_2)\big(f(x_1) > f(x_2)\big)$ bo'lsa, u holda bu funksiya shu oraliqda ***o'suvchi(kamayuvchi)*** deyiladi.

5.4-Teorema. *Agar $[a;b]$ kesmada hosilaga ega bo'lgan $f(x)$ funksiya shu kesmada o'suvchi(kamayuvchi) bo'lsa, uning hosilasi $[a;b]$ kesmada manfiy(musbat) bo'lmaydi, ya'ni $f'(x) \ge 0\big(f'(x) \le 0\big)$.*

5.5-Teorema. *Agar $[a;b]$ kesmada uzluksiz va $(a;b)$ oraliqda differensiallanuvchi $f(x)$ funksiya uchun $a < x < b$ da $f'(x) > 0$ bo'lsa, bu funksiya $[a;b]$ da o'suvchi(kamayuvchi) bo'ladi.*

Biror oraliqdan olingan ixtiyoriy $x_1 < x_2$ uchun $f(x_1) \le f(x_2)\big(f(x_1) \ge f(x_2)\big)$ tengsizlik o'rinli bo'lsa, u holda $f(x)$ funksiya shu oraliqda ***kamaymaydigan(o'smaydigan)*** funksiya deyiladi.

Funksiyaning kamaymaydigan yoki o'smaydigan oraliqlari uning ***monotonlik oraliqlari*** deyiladi.

Funksiyaning hosilasi nolga teng bo'ladigan va mavjud bo'lmaydigan nuqtalar ***kritik nuqtalar*** deyiladi.

5.16-misol

$f(x) = 2x^2 - \ln x$

$x > 0$ $\qquad$ $f'(x) = 4x - \dfrac{1}{x}$

$f'(x) = 0, \quad 4x - \dfrac{1}{x} = 0, \quad x = \dfrac{1}{2}$

$f'(x) > 0$ $\qquad$ $4x - \frac{1}{x} > 0$

$(1/2;\infty)$

$f'(x) < 0$ $\qquad$ $4x - \frac{1}{x} < 0$

$(0;1/2)$

Agar absolyut miqdori bo'yicha yetarlicha kichik bo'lgan ixtiyoriy Δx uchun $f(x_0 + \Delta x) < f(x_0)$ $\left(f(x_0 + \Delta x) > f(x_0)\right)$ bo'lsa, $x = x_0$ nuqta $f(x)$ funksiyaning ***maksimum*** ***(minimum)*** ***nuqtasi*** deyiladi. Funksiyaning maksimum(minimum) nuqtalardagi qiymatlari esa ***maksimum*** ***(minimum)*** ***qiymatlari*** deyiladi.

Maksimum va minimum nuqtalari *funksiyaning **ekstremumlari***, maksimal va minimal qiymatlari esa ***funksiyaning ekstremal qiymatlari*** deyiladi.

5.6-Teorema. *Agar differensiallanuvchi $y = f(x)$ funksiya $x = x_0$ nuqtada maksimumga yoki minimumga ega bo'lsa, u holda $f'(x_0) = 0$ bo'ladi yoki $f'(x_0)$ mavjud bo'lmaydi.*

Bu ekstremumning zaruriy shartidir. Chunki funksiya biror nuqtada ekstremumga erishsa, shu nuqta har doim kritik nuqta bo'ladi. Ammo har bir kritik nuqta ham ekstremum nuqta bo'la olmaydi. Masalan, $y = x^3$ funksiyadagi $x = 0$ nuqta.

Quyida biz funksiya ekstremumining ikkita yetarlilik sharti bilan tanishamiz

5.7-Teorema *(funksiya ekstremumining 1-yetarlilik sharti).* *$f(x)$ funksiya $x = x_0$ kritik nuqtani o'z ichiga olgan birorta intervalda uzluksiz va shu intervalning hamma nuqtalarida differensiallanuvchi bo'lsin.Agar $f'(x)$ hosila $x < x_0$ da musbat, $x > x_0$ da manfiy bo'lsa, $x = x_0$ maksimum nuqta, $x < x_0$ da manfiy, $x > x_0$ da musbat bo'lsa, $x = x_0$ minimum nuqta bo'ladi.*

Bu yerda ko'rsatilgan tengsizlik $x = x_0$ nuqtaning yetarlicha kichik atrofida bajarilishi mumkin Bu teorema birinchi tartibli hosila yordamida funksiyani ekstremumga tekshirish qoidasini aniqlaydi, uni quyidagi sxemada ifodalaymiz:

Kritik nuqta x_0 dan o'tishda $f'(x)$ ning ishorasi	Kritik nuqtaning xarakteri

$x < x_0$	$x = x_0$	$x > x_0$	
+	$f'(x_0) = 0$ yoki mavjud emas	–	Maksimum nuqtasi
–	$f'(x_0) = 0$ yoki mavjud emas	+	Minimum nuqtasi
+	$f'(x_0) = 0$ yoki mavjud emas	+	Ekstremum mavjud emas (funksiya o‘suvchi)
–	$f'(x_0) = 0$ yoki mavjud emas	–	Ekstremum mavjud emas (funksiya kamayuvchi)

5.17-misol

Ushbu $f(x) = x^2 - 4\ln(1+x)$ funksiyani ekstremumga tekshiring.

►Funksiya $x > -1$ da aniqlangan. Funksiya hosilasini hisoblaymiz:

$$f'(x) = 2x - \frac{4}{1+x} = \frac{2(x^2 + x - 2)}{1+x}.$$

Funksiya aniqlanish sohasiga tegishli bitta $x = 1$-kritik nuqta mavjud ekan. $-1 < x < 1$ da $f'(x) < 0$ va $x > 1$ da $f'(x) < 0$ bo‘lgani uchun $x = 1$-minimum nuqta; $y_{\min} = 1 - 4\ln 2$. ◄

5.8-Teorema (*funksiya ekstremumining 2-yetarlilik sharti*). *$f(x)$ funksiya ikki marta differensiallanuvchi, $f'(x_0) = 0$ va $f''(x_0) \neq 0$ bo‘lsa, $x = x_0$da ekstremum mavjud. Agar $f''(x_0) < 0$ bo‘lsa, $x = x_0$ maksimum nuqta, $f''(x_0) > 0$ bo‘lsa $x = x_0$ minimum nuqta bo‘ladi.*

5.18-misol

Ushbu $f(x) = x^2 e^{-x}$ funksiyani ikkinchi tartibli hosila yordamida ekstremumga tekshiring.

►Funksiya $x \in R$ da aniqlangan. Funksiyaning birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = (2x - x^2)e^{-x},$$

$$f''(x) = (2 - 2x)e^{-x} - (2x - x^2)e^{-x} = (2 - 4x + x^2)e^{-x}.$$

$f'(x) = 0,\ (2x - x^2)e^{-x} = 0$ tenglamadan funksiyaning $x_1 = 0$ va $x_2 = 2$ kritik nuqtalari topiladi. Bu nuqtalardagi ikkinchi tartibli hosila qiymatlarini hisoblaymiz: $f''(0)=2>0$, ya’ni $x_1=0$ - minimum nuqta, $f''(2)=-2e^{-2}<0$, ya’ni $x_2=2$ - maksimum nuqta; $y_{\min}=0$, $y_{\max}=4e^{-2}$.◄

$[a;b]$ kesmada uzluksiz funksiya bu kesmada o‘z ining eng katta va eng kichik qiymatlaruga erishadi va bu qiymatlarga yoki $(a;b)$ intervalda yotuvchi kritik nuqtalarda, yoki $[a;b]$ kesma oxirlarida erishadi.

5.19-misol

Ushbu $y=2x+3\sqrt[3]{x^2}$ funksiyaning $[-3;1]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

►Funksiya hosilasi $y'=2+\frac{2}{\sqrt[3]{x}}=\frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$. U holda $x_1=-1$ hosila $y'=0$ bo‘ladigan, $x_2=0$ hosila y' mavjud bo‘lmagan, ya’ni uziladigan nuqtalar bo‘ladi. Ikkila kritik nuqtalar ham intervalga tegishli. Funksiyaning kritik nuqtalardagi va kesma oxirlaridagi qiymatlarini hisoblaymiz:

$y(-1)=1$, $y(0)=0$, $y(-3)=3(\sqrt[3]{9}-2)\approx 0{,}24$, $y(1)=5$.

Topilganlarni taqqoslab, berilgan funksiya $[-3;1]$ kesmadagi eng katta qiymatiga $x=1$ nuqtada, eng kichik qiymatiga $x=0$ nuqtada erishadi, degan xulosaga kelamiz. Demak, $[-3;1]$ kesmada $y_{eng\,kat.}=5$, $y_{eng\,kich.}=0$ bo‘lar ekan.◄

5.20-misol

Radiusi R ga teng bo‘lgan sharga ichki chizilgan eng katta hajmli aylanma konusning balandligini aniqlang.

►Konus hajmi: $V=\frac{1}{3}\pi r^2 H$. Bu yerda konus balandligi H sharning o‘qkesimida hosil bo‘lgan aylanaga ichki chizilgan teng yonli uchburchak balandligi hamdir, konus asosining radiusi esa shu uchburchak asosining yarmiga teng. Demak,

$$r^2=R^2-(H-R)^2=2RH-H^2.$$

Bundan

$$V=\frac{1}{3}\pi\left(2RH-H^2\right)H=\frac{\pi}{3}\left(2RH^2-H^3\right)=V(H).$$

H ning qanday qiymatida hajm eng katta bo‘lishini topish uchun hosil bo‘lgan funksiyadan hosila olib, nolga tenglaymiz:

$$V'=\frac{\pi}{3}\left(4RH-3H^2\right)=0.$$

$H=\frac{4}{3}R$ nuqtani topib, uni $V''=\frac{\pi}{3}(4R-6H)$ hosilaga qo‘ysak, $V''\left(\frac{4R}{3}\right)=-\frac{4\pi R}{3}<0$. Demak, $H=\frac{4}{3}R$ da konus hajmi eng katta bo‘lar ekan. ◀

20-Auditoriya topshiriqlari

1. $y=x^4-2x^2+3$ funksiyaning kritik nuqtalari va monotonlik oraliqlarini toping.

2. $y=\sqrt[3]{x^2(x+3)}$ funksiyaning kritik nuqtalari va monotonlik oraliqlarini toping.

3. $y=x/(x^2-6x-16)$ funksiyaning kritik nuqtalari va monotonlik oraliqlarini toping.

4. $y=\sqrt[3]{(x^2-6x+5)^2}$ funksiyani ekstremumga tekshiring.

5. $y=x\ln^2 x$ funksiyani ekstremumga tekshiring.

6. $y=(2x-1)/(x-1)^2$ funksiyani ekstremumga tekshiring.

7. $y=e^{-x^2/2}$ funksiyani ekstremumga tekshiring.

8. $f(x)=\frac{3}{4}x^4-x^3-9x^2+7$ funksiyaning $[-2;\,2]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

9. $y=x+3\sqrt[3]{x}$ funksiyaning $[-1;\,1]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

10. Sig‘imi $V=16\pi\approx 50m^3$ bo‘lgan silindr shakldagi yopiq idish tayyorlash talab qilingan bo‘lsin. Tayyorlashga eng kam material sarflash uchun idishning o‘lchamlari(R-radiusi va H-balandligi) qanday bo‘lishi kerak?

20-Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y=x^2-2\ln x$ funksiyaning kritik nuqtalari bo‘ladi?

A) $x=1$, B) $x=\pm1$, C) $x=1,\ x=0$, D) $x=\pm1,\ x=0$

2. $y=x^4-2x^3+x^2+1$ funksiyaning o‘sish oralig‘ini aniqlang.

A) $(-\infty;\ \infty)$, B) $(-\infty;\ 0)\cup(1/2;\ 1)$, C) $(0;\ 1/2)\cup(1;\ \infty)$, D) $(-\infty;\ 0)\cup(1/2;\ \infty)$

3. $y = x^4 - 2x^3 + x^2 + 1$ funksiyaning kamayish oralig‘ini aniqlang.

A) $(-\infty;\ \infty)$, B) $(-\infty;\ 0)\cup(1/2;\ 1)$, C) $(0;\ 1/2)\cup(1;\ \infty)$, D) $(-\infty;\ 0)\cup(1/2;\ \infty)$

4. $y = x^3 - 3x^2 - 9x + 7$ funksiyaning minimum nuqtasini aniqlang.

A) $x = -1$, B) $x = 3$, C) $x = -1,\ x = 3$, D) mavjud emas.

5. $y = x^3 - 3x^2 - 9x + 7$ funksiyaning $[-2;\ 2]$ kesmadagi eng katta qiymatini toping.

A) 5 B) 10 C) 12 D) 15

5.6. Funksiya grafigining qavariqligi va botiqligi. Assimptotalar. Funksiyani to‘la tekshirish va grafigini yasash

Agar $y = f(x)$ funksiyaning grafigi $(a;b)$ oraliqning ixtiyoriy nuqtasida o‘tkazilgan urunmadan pastda(yuqorida) yotsa u holda funksiya grafigi shu oraliqda ***qavariq(botiq)*** deyiladi.

Funksiya grafigining qavariq qismini botiq qismidan ajratuvchi $M(x_0, f(x_0))$ nuqta grafikning ***egilish nuqtasi*** deyiladi.

5.9-Teorema. *Agar $(a;b)$ oraliqning hamma nuqtalarida $f''(x) < 0$ $(f''(x) > 0)$ bo‘lsa, u holda bu oraliqning $y = f(x)$ funksiya grafigi qavariq (botiq) bo‘ladi.*

5.10-Teorema. *Agar $f''(x_0) = 0$ bo‘lsa yoki $f''(x_0)$ mavjud bo‘lmasa va $x = x_0$ nuqtadan o‘tishida $f''(x)$ ishorasini o‘zgartirsa, u holda absissasi x_0 ga teng bo‘lgan nuqta $y = f(x)$ funksiya grafigining egilish nuqtasi bo‘ladi.*

5.21-misol

Ushbu $y = x^4 - 12x^3 + 48x^2 - 50$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtasi topilsin.

► Funksiya $x \in R$ da aniqlangan. Funksiyaning birinchi va ikkinchi tartibli hosilalarini hisoblaymiz:

$$y' = 4x^3 - 36x^2 + 96x,\ \ y'' = 12x^2 - 72x + 96 = 12(x^2 - 6x + 8),$$

$$y'' = 0,\ x^2 - 6x + 8 = (x-2)(x-4) = 0.$$

$x_1 = 2$ va $x_2 = 4$ nuqtalar yordamida funksiya aniqlanish sohasini oraliqlarga ajratib, quyidagi jadvalni tuzamiz:

x	$(-\infty; 2)$	2	$(2; 4)$	4	$(4; \infty)$
$f''(x)$	+	0	–	0	+
$f(x)$	$\smile$	62	$\frown$	206	$\smile$
	botiq	egilish n.	qavariq	egilish n.	botiq

Javob: $(-\infty; 2)$ va $(4; \infty)$ -funksiya grafigining botiqlik oraliqlari, $(2; 4)$ -funksiya grafigining qavariqlik oralig‘i, $M(2; 62)$ va $N(4; 206)$ -egilish nuqtalari.◀

5.22-misol

Ushbu $y=\sqrt[3]{(x+3)x^2}$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtasi topilsin.

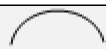
►Funksiya $x \in R$ da aniqlangan. Funksiyaning birinchi tartibli hosilasini hisoblaymiz:

$$y'=\left(\sqrt[3]{x+3}\cdot\sqrt[3]{x^2}\right)'=\frac{\sqrt[3]{x^2}}{3\sqrt[3]{(x+3)^2}}+\frac{2\sqrt[3]{x+3}}{3\sqrt[3]{x}}=\frac{x+2}{\sqrt[3]{x(x+3)^2}}.$$

Ikkinchi tartibli hosila hisoblashda logarifmlab differensiallash qoidasini qo‘llaymiz: $\ln y'=\ln\frac{x+2}{\sqrt[3]{x(x+3)^2}}=\ln(x+2)-\frac{1}{3}\ln x-\frac{2}{3}\ln(x+3),$

$$y''=\frac{x+2}{\sqrt[3]{x(x+3)^2}}\left(\frac{1}{x+2}-\frac{1}{3x}-\frac{2}{3(x+3)}\right)=-\frac{2}{\sqrt[3]{x^4(x+3)^5}}.$$

$f''(x)$ nolga teng bo‘la olmaydi, egilish nuqtalarini hosila mavjud bo‘lmagan $x_1=-3$ va $x_2=0$ nuqtalardan qidiramiz:

x	$(-\infty; -3)$	-3	$(-3; 0)$	0	$(0; \infty)$
$f''(x)$	$+$	mavjud emas	$-$	mavjud emas	$-$
$f(x)$		0		0	
	botiq	egilish nuqta	qavariq	egil.nuqta emas	qavariq

Javob: $(-\infty; -3)$ oraliqda funksiya grafigi botiq, $(-3; 0)$ va $(0; \infty)$ oraliqlarda funksiya grafigi qavariq, $M(-3; 0)$- funksiya grafigining egilish nuqtasi.◀

Agar $y=f(x)$ funksiya grafigining o‘zgaruvchi nuqtasi cheksiz uzoqlashganda undan biror to‘g‘ri chiziqqacha bo‘lgan masofa nolga intilsa, bu to‘g‘ri chiziq $y=f(x)$ funksiya grafigining ***assimptotasi*** deyiladi.

1) Agar $\lim_{x\to a} f(x) = \pm\infty$ bo‘lsa, $x = a$ to‘g‘ri chiziq funksiya grafigining *vertikal assimptotasi* deyiladi.

2) Agar $k = \lim_{x\to -\infty} \frac{f(x)}{x}$ va $b = \lim_{x\to -\infty}(f(x) - kx)$ yoki $k = \lim_{x\to +\infty} \frac{f(x)}{x}$ va $b = \lim_{x\to +\infty}(f(x) - kx)$ limitlar mavjud bo‘lsa, u holda $y = kx + b$ funksiya grafigining *og‘ma assimptotasi* deyiladi. Xususan, $k = 0$ da $y = b$ *gorizontal assimptota* hosil bo ladi.

5.23-misol

Ushbu $y = \frac{x^3}{x^2 - 4}$ funksiya grafigining assimptotalari topilsin.

►Funksiya $x \neq 2$ da aniqlangan. $\lim_{x\to \pm 2} \frac{x^3}{x^2 - 4} = \pm\infty$ bo‘lgani uchun, $x = -2$ va $x = 2$ to‘g‘ri chiziqlar funksiya grafigining vertikal assimptotalari bo‘ladi.

Og‘ma assimptotalarni qidiramiz:

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2}{x^2 - 4} = 1,\ b = \lim_{x\to\infty}(f(x) - kx) = \lim_{x\to\infty}\left(\frac{x^3}{x^2 - 4} - x\right) = \lim_{x\to\infty} \frac{4x}{x^2 - 4} = 0.$$

Demak, yagona $y = x$ og‘ma assimptotasi mavjud.◄

Quyida funksiyani to‘la tekshirish va grafigini yasash uchun umumiy sxemani keltiramiz:

1. Funksiya aniqlanish sohasi va uzilish nuqtalari topiladi.
2. Juft, toqligi, davriyligi tekshiriladi.
3. Koordinata o‘qlari bilan kesishish nuqtalari topiladi.
4. Assimptotalari topiladi.
5. O‘sish, kamayish oraliqlari, ekstremumlari topiladi.
6. Qavariqlik, botiqlik oraliqlari va egilish nuqtalari topiladi.
7. Ayrim nuqtalardagi qiymatlari hisoblanadi.
8. Funksiya grafigi yasaladi.

5.24-misol

Quyidagi $y = \frac{(x+3)^2}{x - 4}$ funksiyani to‘la tekshiring va grafigini yasang.

►Yuqoridagi sxema bo‘ yicha tekshiramiz:

1. Funksiya $x \in (-\infty; 4) \cup (4; \infty)$ da aniqlangan.

2. Funksiya juft ham, toq ham, davriy ham emas.

3. Koordinata o‘qlari bilan kesishish nuqtalari: $(-3;0)$va $(0;-2,25)$.

4. $x=4$ to‘g‘ri chiziq funksiya grafigining vertikal assimptosi, chunki

$$\lim_{x\to 4-0}\frac{(x+3)^2}{x-4}=-\infty \text{ va } \lim_{x\to 4+0}\frac{(x+3)^2}{x-4}=+\infty .$$

Og‘ma assimptotalarni qidiramiz:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{(x+3)^2}{x^2-4x}=\lim_{x\to\pm\infty}\frac{\left(1+\frac{3}{x}\right)^2}{1-\frac{4}{x}}=1$$

$$b=\lim_{x\to\pm\infty}(f(x)-kx)=\lim_{x\to\pm\infty}\left(\frac{(x+3)^2}{x-4}-x\right)=\lim_{x\to\pm\infty}\frac{10x+9}{x-4}=10 .$$

Demak, funksiya grafigining og‘ma assimptotasi $y=x+10$.

5. O‘sish, kamayish oraliqlari, ekstremumlari topamiz :

$$y'=\frac{x^2-8x-33}{(x-4)^2}, \quad x^2-8x-33=0 \text{ yoki } x_1=-3 \text{ va } x_2=11.$$

x	$(-\infty;-3)$	-3	$(-3;4)$	4	$(4;11)$	11	$(11;\infty)$
y'	$+$	0	$-$	mavjud emas	$-$	0	$+$
y	↗	0	↘	mavjud emas	↘	28	↗
	o‘suvchi		kamayuvchi		kamayuvchi		o‘suvchi

6. Qavariqlik, botiqlik oraliqlari va egilish nuqtalari topamiz:

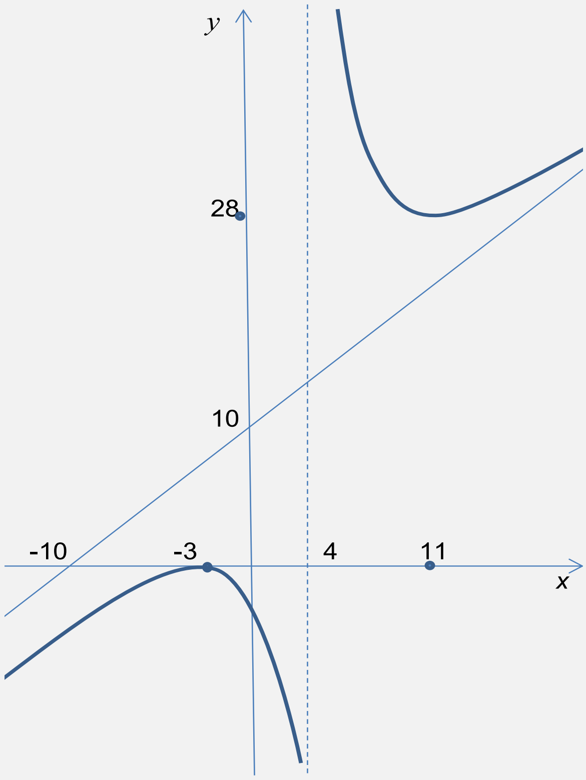
$$y''=\left(\frac{x^2-8x-33}{(x-4)^2}\right)'=\frac{98}{(x-4)^3} .$$

y'' nolga teng bo‘la olmaydi, y'' mavjud bo‘lmaydigan nuqta esa aniqlanish sohasiga tegishli emas. Bundan egilish nuqta mavjud emasligini aniqlaymiz. $x\in(-\infty;4)$da $y''<0$, funksiya grafigi qavariq, $x\in(4;\infty)$da $y''>0$, funksiya grafigi botiq bo‘ladi.

7. Ayrim nuqtalardagi funksiya qiymatlarini hisoblaymiz: $(-10;-3,5)$, $(-4;-1/8)$, $(2;-12,5)$, $\left(10;\ 28\frac{1}{6}\right)$ va $\left(12;\ 28\frac{1}{8}\right)$.

8. Yuqoridagi tekshirishlardan foydalanib funksiya grafigini yasaymiz

(5.1-chizma):



◀

(5.1-chizma)

21-Auditoriya topshiriqlari

1. $y = x^3 - 5x^2 + 3x - 5$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalari topilsin.

2. $y = \ln(1 + x^2)$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalari topilsin.

3. $y = arctgx - x$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalari topilsin.

4. $y = e^{-x^2/2}$ funksiyaning assimptotalarini toping.

5. $y = x^3 / \left(2(1+x)^2\right)$ funksiyaning assimptotalarini toping.

6. $y = x \ln\left(e + \frac{1}{x}\right)$ funksiyaning assimptotalarini toping.

7. $y = \sqrt[3]{x^2(x+3)}$ funksiyani to‘la tekshiring va grafigini yasang.

8. $y = x^3 / \left(4(2-x)^2\right)$ funksiyani to‘la tekshiring va grafigini yasang.

21-Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y = x^2 - 2\ln x$ funksiya grafigining egilish nuqtasi bo'ladi?

A) $(1;1)$ B) $(e; e^2 - 2)$ C) $(e^{-1}; e^{-2} + 2)$ D) mavjud emas.

2. Quyidagilardan qaysi biri $y = x^3 - 3x^2$ funksiya grafigining egilish nuqtalari bo'ladi?

A) $(0;0)$ va $(2;-4)$ B) $(2;-4)$ C) $(1;-2)$ D) $(0;0)$

3. $y = x^3 - 3x^2$ funksiya grafigining qavariqlik oraliqlarini toping.

A) $(1;\ \infty)$, B) $(-\infty;\ 0)\cup(2;\ \infty)$, C) $(0;\ 2)$, D) $(-\infty;\ 1)$

4. $y = xe^{-x}$ funksiya grafigining botiqlik oralig'ini toping.

A) $(1;\ \infty)$, B) $(2;\ \infty)$, C) $(-\infty;\ 2)$, D) $(-\infty;\ 1)$

5. $y = \dfrac{(x+1)^2}{x-2}$ funksiyaning og'ma asimptotasini toping.

A) $y = -x$, B) $y = x + 4$, C) $y = x$, D) $y = x - 4$

7-Shaxsiy uy topshiriqlari

1

Funksiyalarning berilgan oraliqdagi eng katta va eng kichik qiymatlarini toping.

1.1. $y = 2\sin x + \cos 2x,\ [0;\ \pi/2]$

1.2. $y = x^3 e^{x+1},\ [-4;\ 0]$

1.3. $y = e^{4x-x^2},\ [1;\ 3]$

1.4. $y = (x+1)\sqrt[3]{x^2},\ [-4/5;\ 3]$

1.5. $y = 4 - e^{-x^2},\ [0;\ 1]$

1.6. $y = \sqrt[3]{x - x^3},\ [-2;\ 2]$

1.7. $y = (x-2)e^x,\ [-2;1]$

1.8. $y = x/(9 - x^2),\ [-2;\ 2]$

1.9. $y = (1 + \ln x)/x,\ [1/e;\ e]$

1.10. $y = x^2 + 2x + 2/(x-1),\ [-1;\ 3]$

1.11. $y = (x^5 - 8)/x^4,\ [-3;\ 1]$

1.12. $y = (e^{2x} + 1)/e^x,\ [-1;\ 2]$

1.13. $y = e^{6x-x^2},\ [-3;\ 3]$

1.14. $y = ((x+1)/x)^3,\ [1;\ 2]$

1.15. $y = (x+2)e^{1-x},\ [-2;\ 2]$

1.16. $y = \ln(x^2 - 2x + 4),\ [-1;\ 3/2]$

1.17. $y = 3x^4 - 16x^3 + 2,\ [-3;\ 1]$

1.18. $y = \ln(x^2 - 2x + 2),\ [0;\ 3]$

1.19. $y = x^4/4 - 6x^3 + 7,\ [-2;\ 4]$

1.20. $y = (3 - x)e^{-x},\ [0;\ 5]$

1.21. $y = (x^3 + 4)/x^2,\ [1;\ 2]$

1.22. $y = 3x/(1 + x^2),\ [0;\ 5]$

1.23. $y = x^5 - 5x^4 + 5x^3 + 1,\ [-1;\ 2]$

1.24. $y = 108x - x^4,\ [-1;\ 4]$

1.25. $y = (x-1)e^{-x},\ [0;\ 3]$

1.26. $y = x^3/(x^2 - x + 1)$, $[-2;\ 2]$

1.27. $y = (2x-1)/(x-1)^2$, $[-1/2;\ 0]$

1.28. $y = \sqrt[3]{(x^2-1)^2}$, $[-3;\ 2]$

1.29. $y = 2x^3 + 3x^2 + 2x + 1$, $[-1;\ 5]$

1.30. $y = xe^x$, $[-2;\ 0]$.

Berilgan funksiyalarni to‘la tekshiring va grafigini yasang.

2

2.1. $y = \dfrac{4x - x^2 - 4}{x}$	**2.16.** $y = \dfrac{x^2 + 2x + 2}{x - 1}$
2.2. $y = \dfrac{x+1}{(x-1)^2}$	**2.17.** $y = \dfrac{x^2}{4x^2 - 1}$
2.3. $y = e^{\frac{1}{5+x}}$	**2.18.** $y = x + \dfrac{\ln x}{x}$
2.4. $y = \dfrac{x^2}{9 - x}$	**2.19.** $y = \dfrac{x^3}{x^2 - x + 1}$
2.5. $y = x - \ln(1 + x^2)$	**2.20.** $y = x^2 - 2\ln x$
2.6. $y = \dfrac{\ln x}{\sqrt{x}}$	**2.21.** $y = \dfrac{e^{2x} + 1}{e^x}$
2.7. $y = x^3 e^{-x^2/2}$	**2.22.** $y = (x-1)e^{3x+1}$
2.8. $y = \dfrac{4 - 2x}{1 - x^2}$	**2.23.** $y = \dfrac{5x}{4 - x^2}$
2.9. $y = \dfrac{x^3 + 4}{x^2}$	**2.24.** $y = \dfrac{x^3}{x^4 - 1}$
2.10. $y = \dfrac{(x+1)^2}{x - 2}$	**2.25.** $y = \dfrac{x^4}{x^3 - 1}$
2.11. $y = \dfrac{4x^3 + 1}{x^4}$	**2.26.** $y = x + \dfrac{4}{x + 2}$
2.12. $y = \dfrac{3x^2 - 1}{x^3}$	**2.27.** $y = x^2 e^{-x}$
2.13. $y = \sqrt{x} e^{-x/2}$	**2.28.** $y = \dfrac{e \ln x}{x}$
2.14. $y = \dfrac{1 + \ln x}{x}$	**2.29.** $y = \dfrac{x^3}{x^2 - 1}$
2.15. $y = \dfrac{3 - x^2}{x + 2}$	**2.30.** $y = \dfrac{(4-x)^3}{9(2-x)^2}$.

VI BOB . ANIQMAS INTEGRAL. ANIQ INTEGRAL. XOSMAS INTEGRAL

6.1. Boshlang‘ich funksiya va aniqmas integral

Agar $(a;b)$ oraliqda aniqlangan $y=f(x)$ funksiya uchun $F'(x)=f(x)$ tenglik o‘rinli bo‘lsa, $F(x)$ funksiya $f(x)$ funksiyaning ***boshlang‘ich funksiyasi*** deyiladi.

$f(x)$ funksiyaning ikkita boshlang‘ich funksiyasi bir biridan faqat o‘zgarmas songa farq qiladi.

Boshlang‘ich funksiyalar to‘plami $F(x)+C$, bu yerda C o‘z garmas son, $f(x)$ funksiyadan $(a;b)$ oraliq bo‘ yicha olingan ***aniqmas integral*** deyiladi va quyidagicha yoziladi:

$$\int f(x)\,dx=F(x)+C.$$

Quyida biz integrallashning asosiy qoidalari bilan tanishamiz:

1) $\int f'(x)\,dx=\int df(x)=f(x)+C,$

$d\int f(x)\,dx=d\big(F(x)+C\big)=f(x)dx;$

2) $\int\big(f(x)\pm g(x)\big)dx=\int f(x)dx\pm\int g(x)dx;$

3) $\int kf(x)\,dx=k\int f(x)dx,\ k$ - o‘z garmas son;

4) agar $\int f(x)\,dx=F(x)+C$ bo‘lsa, u holda $\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+C$, bu yerda a va b - o‘z garmas sonlar, $a\neq 0$;

5) agar $\int f(x)\,dx=F(x)+C$ va $u=\varphi(x)$ bo‘lsa, u holda $\int f(u)\,dx=F(u)+C$

Aniqmas integralning ta’rifi, integrallash qoidalari va asosiy elementar funksiyalarning hosilalari jadvalidan foydalanib aniqmas integrallar jadvalini tuzamiz:

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq -1.$
2. $\int\frac{du}{u}=\ln|u|+C.$
3. $\int a^u\,du=\frac{a^u}{\ln a}+C.$
4. $\int e^u\,du=e^u+C.$
5. $\int\sin u\,du=-\cos u+C.$
6. $\int\cos u\,du=\sin u+C.$
7. $\int\frac{du}{\cos^2 u}=tgu+C.$

8. $\int \frac{du}{\sin^2 u} = -ctgu + C$.

9. $\int \frac{du}{a^2+u^2} = \begin{cases} \frac{1}{a} arctgu + C \\ -\frac{1}{a} arcctgu + C. \end{cases}$

10. $\int \frac{du}{\sqrt{a^2-u^2}} = \begin{cases} \frac{1}{a} \arcsin u + C \\ -\frac{1}{a} \arccos u + C. \end{cases}$

11. $\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C$.

12. $\int \frac{du}{\sqrt{u^2+k}} = \ln\left|u+\sqrt{u^2+k}\right| + C$.

13. $\int \frac{du}{\sin u} = \ln\left|tg\frac{u}{2}\right| + C$.

14. $\int \frac{du}{\cos u} = \ln\left|tg\left(\frac{u}{2}+\frac{\pi}{4}\right)\right| + C$.

15. $\int shu\, du = chu + C$.

16. $\int chu\, du = shu + C$.

17. $\int \frac{du}{ch^2 u} = thu + C$.

18. $\int \frac{du}{ch^2 u} = thu + C$.

6.1-misol

Quyidagi $\int\left(3x^2+2\sqrt{x}-\frac{5}{x^2}\right)dx$ integralni hisoblang.

► $\int\left(3x^2+2\sqrt{x}-\frac{5}{x^2}\right)dx = 3\int x^2dx + 2\int x^{1/2}dx - 5\int x^{-2}dx = x^3 + \frac{1}{\sqrt{x}} + \frac{10}{x^3} + C$. ◄

6.2-misol

Quyidagi $\int \frac{1+2x^2}{x^2(1+x^2)}dx$ integralni hisoblang.

► $\int \frac{1+2x^2}{x^2(1+x^2)}dx = \int \frac{(1+x^2)+x^2}{x^2(1+x^2)}dx = \int \frac{1}{x^2}dx + \int \frac{1}{1+x^2}dx = -\frac{1}{x} + arctgx + C$. ◄

6.3-misol

Quyidagi $\int (3x-5)^7 dx$ integralni hisoblang.

► $\int (3x-5)^7 dx = \frac{1}{3}\int (3x-5)^7 d(3x-5) = \frac{1}{24}(3x-5)^8 + C$. ◄

6.4-misol

Ushbu $\int \frac{8x - arctg2x}{1+4x^2}dx$ integralni hisoblang.

▶

$$\int \frac{8x - arctg2x}{1+4x^2}dx = \int \frac{8x}{1+4x^2}dx - \int \frac{arctg2x}{1+4x^2}dx = \int \frac{d\left(1+4x^2\right)}{1+4x^2} + \int arctg2xd(arctg2x) =$$

$$= \ln\left|1+4x^2\right| - \frac{1}{2}arctg^2 2x + C. ◀$$

6.5-misol

Ushbu $\int \frac{\sin 2x}{4+\sin^2 x}dx$ integralni hisoblang.

▶ $\int \frac{\sin 2x}{4+\sin^2 x}dx = \int \frac{d(4+\sin^2 x)}{4+\sin^2 x} = \ln\left(4+\sin^2 x\right) + C$. ◀

Yuqorida biz biror ifodani differensial ostiga kiritib, yoddan bu ifodani u deb almashtirib ***bevosita integrallash*** usulidan foydalandik.

Bu yerda $\varphi(x) = u$ deb almashtirish olinib, u yangi oʻzgaruvchili integral $\int f[\varphi(x)]\varphi'(x)dx = \int f(u)du$ koʻrinishga keltirilgan boʻladi.

Agar $x = \varphi(u)$, $dx = \varphi'(u)du$ deb almastirsak, $\int f(x)dx = \int f(\varphi(u))\varphi'(u)du$ integralni hosil qilamiz. Bu ***oʻzgaruvchini almashtirish usuli*** deyiladi.

6.6-misol

Ushbu $\int x^2 \sqrt[3]{2-5x^3}dx$ integralni hisoblang.

▶ $2-5x^3 = u,\ -15x^2dx = du,\ x^2dx = -\frac{1}{15}du$ almashtirishlarni bajaramiz:

$$\int x^2 \sqrt[3]{2-5x^3}dx = -\frac{1}{15}\int \sqrt[3]{u}du = -\frac{1}{20}u^{\frac{4}{3}} + C = -\frac{1}{20}\sqrt[3]{\left(2-5x^3\right)^4} + C \text{ . ◀}$$

6.7-misol

Ushbu $\int x\sqrt{x-1}dx$ integralni hisoblang.

▶ $x-1 = t^2, \quad x = t^2+1,\ dx = 2tdt$ deb almashtiramiz.

$$\int x\sqrt{x-1}dx = \int \left(t^2+1\right)t \cdot 2tdt = 2\int \left(t^4+t^2\right)dt = \frac{2}{5}t^5 + \frac{2}{3}t^3 + C = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

. ◀

6.8-misol

Ushbu $\int\sqrt{a^2-x^2}dx$ integralni hisoblang.

► Bunday keyin har qanday almashtirishlarni vertikal chiziqlar orasida berib ketamiz.

$$\int\sqrt{a^2-x^2}dx=\left|\begin{matrix}x=a\sin t\\ dx=a\cos tdt\end{matrix}\right|=\int\sqrt{a^2-a^2\sin^2 t}\,a\cos tdt=a^2\int\cos^2 tdt=a^2\int\frac{1+\cos 2t}{2}dt=$$

$$=\frac{a^2}{2}\left(t+\frac{\sin 2t}{2}\right)+C=\left|\begin{matrix}\sin t=\frac{x}{a},\ t=\arcsin\frac{x}{a}\\ \cos t=\sqrt{1-\frac{x^2}{a^2}}\end{matrix}\right|=\frac{a^2}{2}\left(\arcsin\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)+C=$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C.◄$$

Bo'laklab integrallash usuli quyidagi formulaga asolangan:

$$\int udv=uv-\int vdu,$$

bu yerda $u(x)$ va $v(x)$ - differensiallanvchi funksiyalar. Bu formula ***bo'laklab integrallash formulasi*** deyiladi. Bo'laklab integrallash formulasi ko'pincha quyidagi ko'rinishdagi integrallarni hisoblashda ishlatiladi:

1) $\int p(x)e^{ax}dx,\ \int p(x)\sin mxdx,\ \int p(x)\cos axdx;$

2) $\int p(x)arctgxdx,\ \int p(x)arcctgxdx,\ \int p(x)\arcsin xdx,$

$\int p(x)\arccos xdx,\ \int p(x)\ln xdx$

Bu integrallarni hisoblashda, 1 – turdagi integrallarda ***u*** uchun ***p(x)*** ko'phad, qolgan qismi ***dv*** uchun olinib, 2 - turdagi integrallarda ***u*** uchun mos ravishda *arctgx*, *arcctgx*, *arcsinx*, *arccosx* va *ln*xlar, qolgan qismi ***dv*** uchun olinadi.

6.9-misol

Ushbu $\int x\cos xdx$ integralni hisoblang.

►Bu 1-turdagi integral bo'lgani uchun quyidagicha bo'laklab integrallaymiz:

$$\int x\cos xdx=\left|\begin{matrix}u=x; & du=dx\\ dv=\cos xdx; & v=\int\cos xdx=\sin x\end{matrix}\right|$$

$$= x\sin x - \int \sin x dx = x\sin x + \cos x + C \text{.} \blacktriangleleft$$

6.10-misol

Ushbu $\int xarctgxdx$ integralni hisoblang.

$$\blacktriangleright \int xarctgxdx = \left| \begin{matrix} u = arctgx, & du = \dfrac{dx}{1+x^2} \\ dv = xdx, & v = \dfrac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2}arctgx - \int \frac{x^2}{2(1+x^2)}dx =$$

$$= \frac{x^2}{2}arctgx - \frac{1}{2}x + \frac{1}{2}arctgx + C = \frac{x^2+1}{2}arctgx - \frac{x}{2} + C \text{.} \blacktriangleleft$$

Bo‘laklab integrallash qoidasini bir necha marta qo‘llash mumkin.

6.11-misol

Ushbu $\int x^2 e^x dx$ integralni hisoblang.

► Bu yerda ikki marta bo‘laklab integrallash qoidasi qo‘llanadi:

$$\int x^2 e^x dx = \left| \begin{matrix} u = x^2, \ dv = e^x dx \\ du = 2xdx, \quad v = e^x \end{matrix} \right| = x^2 \cdot e^x - \int e^x \cdot 2xdx = x^2 \cdot e^x - 2\int xe^x dx =$$

$$= \left| \begin{matrix} u = x, \ dv = e^x dx \\ du = dx, v = e^x \end{matrix} \right| = x^2 \cdot e^x - 2\left[xe^x - \int e^x dx\right] = x^2 \cdot e^x - 2xe^x + 2e^x + C.$$

◄

Ayrim integralni ikki marta bo‘laklab integrallansa o‘z iga qaytib keladi. Bu holda integralni noma’lum sifatida qarab, tenglama yechiladi.

6.12-misol

Ushbu $\int e^{2x} \sin x dx$ integralni hisoblang.

$$\blacktriangleright \int e^{2x} \sin x dx = \left| \begin{matrix} u = \sin x, & du = \cos x dx \\ dv = e^{2x} dx, & v = \dfrac{1}{2} e^{2x} \end{matrix} \right| = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx =$$

$$= \left| \begin{matrix} u = \cos x, & du = -\sin x dx \\ dv = e^{2x} dx, & v = \dfrac{1}{2} e^{2x} \end{matrix} \right| = \frac{1}{2} e^{2x} \sin x - \frac{1}{2}\left(\frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x dx \right) =$$

$$= \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x dx \text{.}$$

Oxirgi integralni chap tomonga o‘ tkazamiz

$$\frac{5}{4}\int e^{2x}\sin x dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x.$$

Demak,

$$\int e^{2x}\sin x dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C. ◀$$

Ko‘pincha bo‘laklashni vertikal chiziqlar orasida bermay, integral ostida ham bajarish mumkin. Buning uchun biror funksiyani differensial octiga kiritiladi va bu differensialni *dv* sifatida qaraladi.

6.13-misol

Ushbu $\int xe^{3x}dx$ integralni hisoblang.

$$▶ \int xe^{3x}dx = \int xd\left(\frac{1}{3}e^{3x}\right) = x\cdot\frac{1}{3}e^{3x} - \int\frac{1}{3}e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C. ◀$$

Ayrim misollarda differensial funksiya dv oshkor ko‘rinishda bo‘lmasligi mumkin.

6.14-misol

Ushbu $\int\frac{x^2}{\left(x^2+a^2\right)^2}dx$ integralni hisoblang.

$$▶ \int\frac{x^2}{\left(x^2+a^2\right)^2}dx = \int\frac{x\cdot x}{\left(x^2+a^2\right)^2}dx = \int xd\left(-\frac{1}{2}\cdot\frac{1}{x^2+a^2}\right) =$$

$$= -\frac{x}{2\left(x^2+a^2\right)} + \int\frac{1}{2\left(x^2+a^2\right)}dx = -\frac{x}{2\left(x^2+a^2\right)} + \frac{1}{2a}arctg\frac{x}{a} + C. ◀$$

22-Auditoriya topshiriqlari

1. Bevosita integrallab yoki o‘zgaruvchini almashtirib hisoblang

1. $\int \frac{\sqrt{\arcsin x} - 4x}{\sqrt{1-x^2}}dx.$ 5. $\int \frac{dx}{x^2\sqrt{x^2+9}}$
2. $\int x\sqrt[5]{(5x^2-3)^7}\,dx.$ 6. $\int \frac{e^x}{\sqrt{e^x+1}}dx$
3. $\int 2^x e^{2x}dx.$ 7. $\int \sqrt[3]{1+\sin x}\cos x dx$
4. $\int \frac{\sqrt{1+\ln x}}{x\ln x}dx.$ 8. $\int \frac{dx}{x^2\sqrt{x^2+9}}$

2. Bo‘laklab integrallash usuli yordamida hisoblang

1. $\int xc0s(2x-1)dx.$ 5. $\int e^{\sqrt{x}}dx$
2. $\int x\cdot 2^x dx.$ 6. $\int x^2\ln^2 xdx$
3. $\int \ln^2(x+1)dx.$ 7. $\int \sin(\ln x)dx$
4. $\int \arccos xdx.$ 8. $\int e^x\cos 2xdx$

22-Mustaqil yechish uchun testlar

1. Hisoblang: $\int \frac{x+3}{\sqrt{9-x^2}}dx$

A) $3\arcsin\frac{x}{3}+\sqrt{9-x^2}+C$, B) $\arcsin\frac{x}{3}-\sqrt{9-x^2}+C$,

C) $\arcsin\frac{x}{3}+\sqrt{9-x^2}+C$, D) $3\arcsin\frac{x}{3}-\sqrt{9-x^2}+C$.

2. Integralni hisoblang: $\int \frac{dx}{x\ln^2 x}$

A) $-\frac{1}{\ln x}+C$ B) $\ln(\ln x)+C$, C) $\frac{\ln^2 x}{2}+C$, D) $-\frac{1}{\ln^2 x}+C$.

3. Integralni hisoblang: $\int xe^{\frac{x}{4}}dx$

A) $4e^{\frac{x}{4}}(x-4)+C$, B) $4e^{\frac{x}{4}}(x-1)+C$, C) $e^{\frac{x}{4}}(x-4)+C$, D) $4e^{\frac{x}{4}}(x-16)+C$

4. $\int x^2e^{3x}dx$ integralni hisoblashda necha marta bo‘laklab integrallanadi?

A) 1 marta, B) 2 marta, C) 3 marta, D) Bo‘laklab integrallanmaydi.

5. $\int 2^x e^{3x}dx$ integralni hisoblashda necha marta bo‘laklab integrallanadi?

A) 1 marta, B) 2 marta, C) 3 marta, D) Bo‘laklanmaydi.

6.2. Kasr-ratsional funksiyalarni integrallash

Quyidagi

$$R(x)=\frac{Q_m(x)}{P_n(x)}=\frac{b_0x^m+b_1x^{m-1}+...+b_{m-1}x+b_m}{a_0x^n+a_1x^{n-1}+...+b_{n-1}x+b_n} \qquad (6.1)$$

ko‘rinishdagi kasrga ***kasr-ratsional funksiya*** yoki qisqacha ***ratsional funksiya*** deyiladi. Bu yerda $m,\ n\in N$ va $a_i, b_j\in R,\ i=\overline{1,\ n},\ j=\overline{1,\ m}$.

Agar $m<n$ bo‘lsa, ***to‘g‘ri*** kasr, $m\geq n$ bo‘lsa, ***noto‘g‘ri*** kasr deyiladi.

Har qanday noto‘g‘ri kasr ko‘phadlarni bo‘lish qoidasi yordamida qandaydir ko‘phad va to‘g‘ri kasr yig‘indisi shaklida ifodalanadi:

$$R(x)=\frac{Q_m(x)}{P_n(x)}=M_{m-n}(x)+\frac{r(x)}{P_n(x)}. \qquad (6.2)$$

Masalan, $\dfrac{x^4+2}{x^2+3x-1}=x^2-3x+10+\dfrac{-33x+12}{x^2-3x+10}$, chunki

$$\begin{array}{r|l} -\ \ x^4+2 & x^2+3x-1 \\ x^4+3x^3-x^2 & \overline{x^2-3x+10} \\ \hline -\ \ -3x^3+x^2+2 & \\ -3x^3-9x^2+3x & \\ \hline -\ \ 10x^2-3x+2 & \\ 10x^2+30x-10 & \\ \hline -33x+12 & \end{array}$$

$M_{m-n}(x)$ ko‘phadni integrallash oson bo‘lgani uchun, ratsional funksiyani integrallash $\dfrac{r(x)}{P_n(x)}$ to‘g‘ri kasrni integrallash masalasiga keltiriladi.

Quyidaqi to‘g‘ri kasrlar ***oddiy ratsional kasrlar*** deyiladi:

I. $\dfrac{A}{x-a}$,

II. $\dfrac{A}{(x-a)^k}\cdot(k\geq 2$ va butun son)

III. $\dfrac{Ax+B}{x^2+px+q}$(maxrajning diskreminanti $D=p^2-4q<0)$.

IV. $\dfrac{Ax+B}{(x^2+px+q)^k}$ $(k\geq 2$ va butun, $D<0)$.

Bu yerda A, B, a, p, q - haqiqiy sonlar.

Endi bu kasrlarning integrallarini hisoblaymiz:

1. $\int \frac{A}{x-a}dx = A\ln|x-2| + C$.

2. $\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}dx = -\frac{A}{(k-1)(x-a)^{k-1}} + C$.

3. $\int \frac{Ax+B}{x^2+px+q}dx$ integralda $A \neq 0$ bo‘lsa, suratida maxrajining hosilasini hosil qilib olamiz:

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\int \frac{(2x+p)+\left(\frac{2B}{A}-p\right)}{x^2+px+q}dx == \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx +$$

$$+\left(B-\frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q} = \frac{A}{2}\ln(x^2+px+q) + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2 + \left(q-\frac{p^2}{4}\right)}.$$

Oxirgi integralda $q-\frac{p^2}{4} = \frac{4q-p^2}{4} > 0 (D<0)$ bo‘lgani uchun, jadvaldagi $\int \frac{du}{u^2+a^2}$ integralga keladi. Demak,

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\ln(x^2+px+q) + \frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}} + C. \qquad (6.3)$$

4. $\int \frac{Ax+B}{(x^2+px+q)^k}dx = \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(\left(x+\frac{p}{2}\right)^2 + \frac{4q-p^2}{4}\right)^k}$.

Bunda

$$\frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx = -\frac{A}{2}\cdot\frac{1}{(k-1)(x^2+px+q)^{k-1}}, \qquad (6.4)$$

oxirgi integralda esa $u = x+\frac{p}{2}$, $a = \frac{\sqrt{4q-p^2}}{2}$ almashtirish bajaramiz.

$$\int \frac{du}{(u^2+a^2)^k} = \frac{1}{a^2}\int \frac{(u^2+a^2)-u^2}{(u^2+a^2)^k}du = \frac{1}{a^2}\int \frac{du}{(u^2+a^2)^{k-1}} - \frac{1}{a^2}\int \frac{u^2}{(u^2+a^2)^k}du.$$

Birinchi integral berilgan integralning tartibi bittaga kamaygan holi, ikkinchi integralni bo‘laklab integrallash mumkin. Natijada, quyidagi rekkurent formulani hosil qilamiz:

$$\int\frac{du}{\left(u^2+a^2\right)^k}=-\frac{u}{2a^2(k-1)\left(u^2+a^2\right)^{k-1}}+\frac{2k-3}{2a^2(k-1)}\int\frac{du}{\left(u^2+a^2\right)^{k-1}}\,. \qquad (6.5)$$

Eslatma. *Agar maxrajda* ax^2+bx+c *ko'phad bo'lsa, avval a qavsdan chiqariladi:* $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

6.15-misol

Ushbu $\int\frac{3x-2}{2x^2+8x+26}dx$ integralni hisoblang.

►Avval maxrajidan 2 ko'paytuvchi qavsdan chiqaramiz, suratida maxrajining hosilasini hosil qilib olamiz.

$$\frac{1}{2}\int\frac{3x-2}{x^2+4x+13}dx=\frac{3}{4}\int\frac{2x+4-4-\frac{4}{3}}{x^2+4x+13}dx=\frac{3}{4}\int\frac{2x+4}{x^2+4x+13}dx-4\int\frac{dx}{(x+2)^2+3^2}=$$

$$=\frac{3}{4}\ln\left(x^2+4x+13\right)-\frac{4}{3}arctg\frac{x+2}{3}+C\,.◄$$

6.16-misol

Ushbu $\int\frac{7x+3}{\left(x^2+2x+5\right)^2}dx$ integralni hisoblang.

$$► \int\frac{7x+3}{\left(x^2+2x+5\right)^2}dx=\frac{7}{2}\int\frac{2x+2-2+\frac{6}{7}}{\left(x^2+2x+5\right)^2}dx=\frac{7}{2}\int\frac{2x+2}{\left(x^2+2x+5\right)^2}dx-4\int\frac{dx}{\left((x+1)^2+2^2\right)^2}\,.$$

Birinchi qo'shiluvchi (4) formulaga ko'ra,

$$\frac{7}{2}\int\frac{2x+2}{\left(x^2+2x+5\right)^2}dx=-\frac{7}{2}\cdot\frac{1}{x^2+2x+5}\,.$$

Ikkinchi integral uchun (6.5) rekkurent formulani qo'llasak,

$$\int\frac{dx}{\left((x+1)^2+2^2\right)^2}=-\frac{x+1}{2\cdot 2^2(2-1)\left((x+1)^2+2^2\right)}+\frac{2\cdot 2-3}{2\cdot 2^2(2-1)}\int\frac{d(x+1)}{(x+1)^2+2^2}=$$

$$=-\frac{x+1}{8\left((x^2+2x+5\right)^2}+\frac{1}{8}\cdot\frac{1}{2}arctg\frac{x+1}{2}\,.$$

Demak,

$$\int\frac{7x+3}{\left(x^2+2x+5\right)^2}dx=-\frac{7}{2\left(x^2+2x+5\right)}-\frac{x+1}{8(x^2+2x+5)^2}+\frac{1}{16}arctg\frac{x+1}{2}+C\,.◄$$

Ma'lumki, har qanday haqiqiy koeffitsientli ko'phad quyidagi ko'paytma shaklida ifodalanadi:

$$P_n(x) = a_0(x-\alpha_1)^{k_1}\cdot\ldots\cdot(x-\alpha_\beta)^{k_\beta}(x^2+p_1x+q_1)^{t_1}\cdot\ldots\cdot(x^2+p_sx+q_s)^{t_s}, \qquad (6.6)$$

bu yerda $\alpha_1,\ldots,\alpha_\beta$ lar ko'phadning $k_1,\ldots,k_\beta$ karrali haqiqiy ildizlari, $p_i^2-4q_i<0,\ (i=\overline{1,s})$ va $k_1+\ldots+k_\beta+2t_1+\ldots+2t_s=n$.

6.1-Teorema (*to'g'ri kasrni oddiy kasrlar yig'ndisiga ajratish haqida*) *Maxraji (6.6) shaklda tasvirlangan har qanday to'g'ri ratsional kasrni I-IV turdagi oddiy kasrlar yig'indisiga yoyish mumkin. Bu yoyilmada* $P_n(x)$ *ko'phadning har bir* k_r *karrali* α_r *haqiqiy ildiziga (*$(x-\alpha_r)^{k_r}$ *ko'paytuvcisiga)*

$$\frac{A_1}{x-\alpha_r}+\frac{A_2}{(x-\alpha_r)^2}+\frac{A_3}{(x-\alpha_r)^3}\ldots+\frac{A_{k_r}}{(x-\alpha_r)^{k_r}} \qquad (6.7)$$

ko'rinishdagi k_r *ta oddiy kasrlar yig'indisi mos keladi.* $P_n(x)$ *ko'phadning har bir juft qo'shma- kompleks ildiziga (*$(x^2+p_\gamma x+q_\gamma)^{t_\gamma}$ *ko'paytuvchisiga)*

$$\frac{M_1x+N_1}{x^2+p_\gamma x+q_\gamma}+\frac{M_2x+N_2}{(x^2+p_\gamma x+q_\gamma)^2}+\frac{M_3x+N_3}{(x^2+p_\gamma x+q_\gamma)^3}+\ldots+\frac{M_{t_\gamma}x+N_{t_\gamma}}{(x^2+p_\gamma x+q_\gamma)^{t_\gamma}} \qquad (6.8)$$

ko'rinishdagi t_γ *ta oddiy kasrlar yig'indisi mos keladi.*

Demak, integral ostidagi $R(x)$ to'g'ri ratsional kasrni (6.7) va (6.8) formulalarni e'tiborga olib noma'lum koeffitsientli oddiy kasrlarga yoyiladi. So'ng bu kasrlarga umumiy maxraj beriladi. Yoyilmadagi A, M, N koeffitsientlarning qiymatlari esa

1) ***noma'lum koeffitsientlar usuli;***

2) ***o' rniga qo' yish usulidan*** biri yoki ikkalasini qo'llab aniqlanadi.

Noma'lum koeffitsientlari usulida $R(x)$ to'g'ri ratsional kasrning suratidagi ko'phad hosil bo'lgan kasrning suratidagi ko'phadga aynan tengligidan x ning bir xil daragalari oldidagi koeffitsientlar tenglab, n ta noma'lum uchun n ta tenglamalar sistemasi hosil qilinib noma'lum koeffitsientlar topiladi.

O'rniga qo'yish usulida ko'phadlar, x ning barcha qiymatlarida aynan teng bo'lgani uchun, x ning tayin xususiy qiymatlarida tenglab noma'lum koeffitsientlar topiladi.

6.17-misol

Ushbu $\int\frac{x^2-3x+2}{x(x+1)^2}dx$ integralni hisoblang.

► Maxrajdagi ko‘phadning $x=0$ bir karrali haqiqiy va $x=-1$ ikki karrali ildizlari bor bo‘lgani uchun

$$\frac{x^2-3x+2}{x(x+1)^2}=\frac{A}{x}+\frac{B}{(x+1)^2}+\frac{C}{x+1}.$$

mumiy maxraj berib, suratdagi ko‘phadlarni tenglaymiz

$x^2-3x+2\equiv Ax^2+2xA+A+Bx+Cx^2+Cx$ yoki

$x^2-3x+2\equiv x^2\left(A+C\right)+x\left(2A+B+C\right)+A.$

Noma’lum koeffitsientlari usulidan foydalanamiz, x ning darajalari oldidagi koeffitsintlarni tenglaymiz:

$$x^2: \quad A+C=1;$$
$$x: \quad 2A+B+C=-3;$$
$$x^0: \quad A=2.$$

Bundan , $A=2,\ B=-6,\ C=-1$.

Demak,

$$\int\frac{x^2-3x+2}{x(x+1)^2}dx=\int\frac{2}{x}dx-\int\frac{6}{(x+1)^2}dx-\int\frac{1}{x+1}dx=$$

$$=2\ln|x|+\frac{6}{x+1}-\ln|x+1|+C=\ln\frac{x^2}{|x+1|}+\frac{6}{x+1}+C. \blacktriangleleft$$

6.18-misol

Ushbu $\int\frac{(x^2+3)dx}{x(x-1)(x+2)}$ integralni hisoblang.

► Integral ostida to‘g‘ri ratsional kasr va u I turdagi sodda kasrlar yig‘indisiga ajraladi

$$\frac{x^2+3}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{D}{x+2},$$

bundan $x^2+3=A(x-1)(x+2)+Bx(x+2)+Dx(x-1).$

$A,\ B,\ D$ koeffutsientlarni topish uchun o‘rniga qo‘yish usulidan foydalanamiz:

$x=0$ bo‘lganda $3=-2A$, bundan $A=-\frac{3}{2}$;

$x=1$ bo‘lganda $4=3B,$ bundan $B=\frac{4}{3}$;

$x=-2$ bo‘lganda, $7=6D,$ bundan $D=\frac{7}{6}$.

Shunday qilib , quyidagini hosil qilamiz :

$$\int\frac{(x^2+3)dx}{x(x-1)(x+2)}=-\frac{3}{2}\int\frac{dx}{x}+\frac{4}{3}\int\frac{d(x-1)}{x-1}+\frac{7}{6}\int\frac{d(x+2)}{x+2}=$$

$$= -\frac{3}{2}\ln|x| + \frac{4}{3}\ln|x-1| + \frac{7}{6}\ln|x+2| + C = \ln\sqrt[6]{\frac{(x-1)^8|x+2|^7}{|x|^9}} + C. ◀$$

6.19-misol

Ushbu $\int \frac{dx}{x^3+8}$ integralni hisoblang.

▶ Integral ostida to‘g‘ri ratsional kasrning maxrajidagi ko‘phad ko‘paytuvchilarga ajratiladi va sodda kasrlar yig‘indisi shaklida ifodalanadi

$$\frac{1}{x^3+8} = \frac{1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Mx+N}{x^2-2x+4}.$$

Umumiy maxraj berib suratlari tenglanadi

$$A(x^2-2x+4) + Bx(x+2) + C(x+2) \equiv 1$$

A, M, N koeffitsientlarni topish uchun yuqoridagi usullarni birga qo‘llaymiz:

$$x=-2: \quad 12A=1$$
$$x^2: \quad A+B=0;$$
$$x^0: \quad 4A+2C=1.$$

Bundan, $A=1/12,\ B=-1/12,\ C=1/3$ va

$$\frac{1}{x^3+8} = \frac{1}{12(x+2)} - \frac{x-4}{12(x^2-2x+4)}.$$

Endi integralni hisoblaymiz:

$$\int\frac{dx}{x^3+8} = \frac{1}{12}\int\frac{dx}{x+2} - \frac{1}{12}\int\frac{x-4}{x^2-2x+4}dx = \frac{1}{12}\ln|x+2| - \frac{1}{12\cdot 2}\int\frac{(2x-2)-6}{x^2-2x+4}dx =$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\int\frac{(2x-2)dx}{x^2-2x+4} + \frac{1}{4}\int\frac{d(x-1)}{(x-1)^2+\left(\sqrt{3}\right)^2} =$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln\left|x^2-2x+4\right| + \frac{1}{4\sqrt{3}}\,arctg\frac{x-1}{\sqrt{3}} + C =$$

$$= \ln\sqrt[24]{\frac{(x+2)^2}{x^2-2x+4}} + \frac{1}{4\sqrt{3}}\,arctg\frac{x-1}{\sqrt{3}} + C. ◀$$

Shunday qilib ratsional kasrni integrallash uchun

1) uning to‘g‘ri yoki noto‘g‘ri kasr ekanligini tekshiriladi, aks holda(ya’ni noto‘g‘ri kasr bo‘lganda) butun qismi ajratiladi, ko‘phad va to‘g‘ri ratsional kasr hosil qilinadi;

2) to‘g‘ri ratsional kasrni oddiy kasrlar yig‘indisiga ajratiladi;

3) yoyilmaning koeffitsientlari topiladi;

4) ifoda integrallanadi.

23-Auditoriya topshiriqlari

Integrallarni hisoblang.

1. $\int \frac{x^3}{x-2}dx.$
2. $\int \frac{x^4}{x^2+2}dx.$
3. $\int \frac{3x+5}{x^2-4x+5}dx.$
4. $\int \frac{5x+2}{x^2+2x+10}dx.$
5. $\int \frac{(x+1)^3}{x^2-x}dx$
6. $\int \frac{2x^2-5x+1}{x^3-2x^2+x}dx$
7. $\int \frac{7x-15}{x^3-2x^2+5}dx$
8. $\int \frac{3x^2+2x+1}{(x+1)^2(x^2+1)}dx$
9. $\int \frac{2x+1}{(x^2+2x+5)^3}dx$
10. $\int \frac{x+1}{x^4+4x^2+4}dx$

23-Mustaqil yechish uchun testlar

1. $\frac{2x^2+x+3}{x^2(x+1)^3(x^2+4)}$ ratsional kasrning oddiy kasrlarga yoyilmasi to‘g‘ri ko‘rsatilgan variantni aniqlang

A) $\frac{A}{x^2}+\frac{B}{(x+1)^3}+\frac{Cx+D}{x^2+4}$,
B) $\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}+\frac{B_3}{(x+1)^3}+\frac{Cx+D}{x^2+4}$,

C) $\frac{A}{x^2}+\frac{B}{(x+1)^3}+\frac{C}{x^2+4}$,
D) $\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}+\frac{B_3}{(x+1)^3}+\frac{C}{x^2+4}$.

2. $\frac{x+3}{x^3+2x^2}$ ratsional kasrning oddiy kasrlarga yoyilmasi to‘g‘ri ko‘rsatilgan variantni aniqlang.

A) $\frac{A}{x}+\frac{Cx+D}{x^2+2}$,
B) $\frac{A}{x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$,
C) $\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x^2+2}$,
D) $\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+2}$.

3. Integralni hisoblang: $\int \frac{x^2+2x}{x+3}dx$.

A) $\frac{x^2}{2}-x+3\ln|x+3|+C$,
B) $\frac{x^2}{2}+x+3\ln|x+3|+C$,

C) $x^2-3x+\ln|x+3|+C$,
D) $\frac{x^2}{2}+2x+3\ln|x+3|+C$.

4. Integralni hisoblang: $\int \frac{x-4}{(x-2)(x-3)}dx$.

A) $\ln\frac{(x-3)^2}{|x-2|}+C$, B) $\ln\frac{(x-2)^2}{|x-3|}+C$, C) $\ln(x-3)^2|x-2|+C$, D) $\ln(x-2)^2|x-3|+C$.

5. Integralni hisoblang: $\int \frac{2x+1}{x^2+2x+5}dx$.

A) $\ln|x^2+2x+5|+C$, B) $\ln|x^2+2x+5|+\frac{1}{2}arctg\frac{x+1}{2}+C$,

C) $\frac{1}{2}\ln|x^2+2x+5|+C$, D) $\frac{1}{2}\ln|x^2+2x+5|+\frac{1}{2}arctg\frac{x+1}{2}+C$.

6.3. Trigonometrik funksiyalarni integrallash

Barcha trigonometrik funksiyalarni *sinx* va *cosx* orqali ifodalash mumkin. *sinx* va *cosx* ning ratsional funksiyasini $R(\sin x, \cos x)$ ko‘rinishda belgilaymiz.

Quyidagi

$$\int R(\sin x, \cos x)dx$$

integralni $tg\frac{x}{2}=z$ belgilash yordamida z o‘z garuvchili ratsional funksiyaning integraliga almashtirish mumkin. Integralni bunday almashtirish ratsionallashtirish deyiladi. Haqiqatdan ham, $tg\frac{x}{2}=z$ desak,

$$\cos x=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-z^2}{1+z^2};\ \sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$\frac{x}{2}=arctgz,\quad x=2arctgz,\ dx=\frac{2dz}{1+z^2}.$$

Shuning uchun

$$\int R(\sin x,\cos x)dx=\int R\left(\frac{2z}{1+z^2};\frac{1-z^2}{1+z^2}\right)\cdot\frac{2dz}{1+z^2}=\int R_1(z)dz$$

hosil bo‘ladi, bunda $R_1(z)$ - z o‘z garuvchili ratsional funksiya.

Bunday almashtirish $R(\sin x, \cos x)$ ko‘rinishdagi har qanday funksiyani integrallashga imkon beradi, shuning uchun bunday almashtirish ***universal trigonometrik almashtirish*** deyiladi.

6.20-misol

Ushbu $I=\int\frac{dx}{4\sin x+3\cos x+5}$ integralni hisoblang.

▶ $tg\frac{x}{2}=z$ almashtirishdan foydalanamiz:

$$\sin x=\frac{2z}{1+z^2};\quad \cos x=\frac{1-z^2}{1+z^2};\quad dx=\frac{2dz}{1+z^2}.$$

$$I=\int\frac{\frac{2dz}{1+z^2}}{4\cdot\frac{2z}{1+z^2}+3\cdot\frac{1-z^2}{1+z^2}+5}=\int\frac{2dz}{(1+z^2)\cdot\frac{8z+3-3z^2+5+5z^2}{1+z^2}}=\int\frac{2dz}{2z^2+8z+8}=$$

$$=\int\frac{2dz}{2(z^2+4z+4)}=\int\frac{dz}{(z+2)^2}=-\frac{1}{z+2}+C=-\frac{1}{tg\frac{x}{2}+2}+C.$$ ◀

Ko‘pincha, universal trigonometrik almashtirish murakkab ratsional funksiyaga olib keladi. Shuning uchun, xususiy sodda almashtirishlardan bir nechtasini keltirib o‘tamiz.

1. Agar $R(\sin x,\cos x)$ funksiya *sinx* ga nisbatan toq bo‘lsa, ya’ni

$R(-\sin x,\cos x)=-R(\sin x,\cos x)$ bo‘lsa, u holda $z=\cos x,\ dz=-\sin xdx$ almashtirish bu funksiyani ratsionallashtiradi.

2. Agar $R(\sin x,\cos x)$ funksiya *cosx* ga nisbatan toq bo‘lsa, ya’ni

$R(\sin x,-\cos x)=-R(\sin x,\cos x)$ bo‘lsa, u holda $z=\sin x,\ dz=\cos xdx$ almashtirish bu funksiyani ratsionallashtiradi.

3. Agar $R(\sin x,\cos x)$ funksiya *sinx* va *cosx* ga nisbatan juft bo‘lsa, ya’ni $R(-\sin x,-\cos x)=R(\sin x,\cos x)$ bo‘lsa. U holda $z=tgx$ almashtirish bu funksiyani ratsionallashtiradi.

Bu holda,

$$\sin^2 x=\frac{tg^2x}{1+tg^2x}=\frac{z^2}{1+z^2};\quad \cos^2 x=\frac{1}{1+tg^2x}=\frac{1}{1+z^2};$$

$$x = arctgz,\ dx = \frac{dz}{1+z^2}$$

almashtirishlar o‘rinli bo‘ladi.

6.21-misol

Ushbu $I = \int \frac{dx}{1+\sin^2 x}$ integralni hisoblang.

►Integral belgisi ostidagi funksiya juft funksiya, shuning uchun $z = tgx$ almashtirishni bajaramiz. U holda, $x = arctgz,\ dx = \frac{dz}{1+z^2};\ \ \sin^2 x = \frac{z^2}{1+z^2}$.

Natijada ,

$$I = \int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{dt}{1+z^2}}{1+\frac{z^2}{1+z^2}} = \int \frac{\frac{dz}{1+z^2}}{\frac{1+z^2+z^2}{1+z^2}} = \int \frac{dz}{1+2z^2} = \frac{1}{2}\int \frac{dz}{\frac{1}{2}+z^2} = \frac{1}{2}\int \frac{dz}{\left(\sqrt{\frac{1}{2}}\right)^2+z^2} =$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}}} arctg\frac{z}{\sqrt{\frac{1}{2}}} + C = \frac{\sqrt{2}}{2} arctg\sqrt{2}Z + C = \frac{\sqrt{2}}{2} arctg\sqrt{2}tgx + C\,.◄$$

6.22-misol

Ushbu $I = \int \frac{\sin^3 x}{2+\cos x}dx$ integralni hisoblang.

► Integral ostidagi funksiya *sinx* ga nisbatan toq funksiya . Shuning uchun $z = \cos x,\ dz = -\sin xdx$ almashtirishni bajaramiz:

$$I = \int \frac{\sin^2 x \cdot \sin xdx}{2+\cos x} = \int \frac{(1-\cos^2 x)\sin xdx}{2+\cos x} = -\int \frac{(1-z^2)dz}{2+z} = \int \frac{z^2-1}{2+z}dz =$$

$$= \int\left(z-2+\frac{3}{z+2}\right)dz = \frac{z^2}{2} - 2z + 3\ln|z+2| + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln|\cos x + 2| + C$$

.◄

4. Agar $R(\sin x, \cos x)$ funksiya *sinx* va *cosx* darajalarining ko‘paytmasi bo‘lsa, ya’ni $\int \sin^n x\cdot\cos^m xdx$ ko‘rinishdagi integralni hisoblashda, m va n ga bog‘liq holda turli almashtirishlar bajariladi:

a) agar $n > 0$ va toq bo‘lsa, u holda $\cos x = z,\ \sin xdx = -dz$ almashtirish bajariladi;

b) agar $m>0$ va toq bo‘lsa, u holda $\sin x = z, \cos x dx = dz$ almashtirish bajariladi.

6.23-misol

Ushbu $I = \int \frac{\sin^3 x}{\cos^4 x} dx$ integralni hisoblang.

► $\cos x = z, \sin x dx = -dz$ almashtirishni bajaramiz:

$$I = \int \frac{\sin^2 x \sin x}{\cos^4 x} dx = \int \frac{(1-\cos^2 x)\sin x dx}{\cos^4 x} = -\int \frac{(1-z)^2 dz}{z^4} = -\int \frac{dz}{z^4} + \int \frac{z^2}{z^4} dz =$$

$$= \frac{1}{3z^3} - \frac{1}{z} + C = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C. ◄$$

d) agar ikkala n va m ko‘rsatkichlar juft va nomanfiy bo‘lsa, u holda trigonometriyadan ma’lum bo‘lgan

$$\sin^2 x = \frac{1-\cos 2x}{2}; \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

darajani pasaytirish formulalaridan foydalanamiz.

6.24-misol

Ushbu $I = \int \sin^4 x dx$ integralni hisoblang.

► Darajani pasaytirish formulasidan foydalanamiz:

$$I = \int \sin^4 x dxd = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx =$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + \frac{1}{8}x + \frac{1}{8}\cdot\frac{\sin 4x}{4} + C =$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. ◄$$

e) agar $m+n=-2k \le 0$ (juft, nomusbat) bo‘lsa, u holda $tgx = z$ yoki $ctgx = z$ almashtirish integralni darajali funksiyalarning integrallari yig‘indisiga olib keladi. Xususan, $n<0$, $m<0$ va $m+n=-2k\le 0$ bo‘lsa, kasrning suratini $1=(\sin^2 x+\cos^2 x)^s$ ifodaga almashtirish mumkin, bu yerda $s = \frac{|m+n|}{2} - 1$.

6.25-misol

Ushbu $I = \int \frac{\sin^2 x}{\cos^6 x} dx$ integralni hisoblang.

►Bu yerda $n=2, m=-6,\ m+n=-4<0$, $tgx=z,\ x=arctgz,\ dx=\dfrac{dz}{1+z^2}$ almashtirishni bajaramiz.

$$\frac{\sin^2 x}{\cos^6 x}=\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^4 x}=tg^2x\left(\frac{1}{\cos^4 x}\right)=tg^2x(1+tg^2x)^2=z^2(1+z^2)^2,$$

Natijada ,

$$I=\int\frac{\sin^2 x}{\cos^6 x}dx=\int z^2(1+z^2)^2\frac{dz}{1+z^2}=\int(z^2+z^4)dz=$$

$$=\frac{z^3}{3}+\frac{z^5}{5}+C=\frac{tg^3x}{3}+\frac{tg^5x}{5}+C.$$ ◄

6.26-misol

Ushbu $I=\int\dfrac{dx}{\sin^3 x\cdot\cos x}$ integralni hisoblang.

►Bu yerda $n=-3, m=-1,\ m+n=-4<0$

$$I=\int\frac{dx}{\sin^3 x\cdot\cos x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cdot\cos x}dx=\int\frac{1}{\sin x\cdot\cos x}dx+\int\frac{\cos x}{\sin^3 x}dx=$$

$$=2\int\frac{dx}{\sin 2x}+\int\frac{d(\sin x)}{\sin^3 x}=\ln|tgx|-\frac{1}{2\sin^2 x}+C.$$ ◄

f) agar darajalardan biri nolga teng, ikkinchisi manfiy toq son bo‘lsa, u holda $tg\dfrac{x}{2}=z$ almashtirish bajariladi.

6.27-misol

Ushbu $I=\int\dfrac{dx}{\sin^3 x}$ integralni hisoblang.

►Quyidagicha almashtirish bajaramiz:

$$tg\frac{x}{2}=z;\ dx=\frac{2dz}{1+z^2};\ \sin x=\frac{2z}{1+z^2}$$

Natijada, $I=\int\dfrac{dx}{\sin^3 x}=\int\dfrac{\dfrac{2dz}{1+z^2}}{\left(\dfrac{2z}{1+z^2}\right)^3}=\dfrac{1}{4}\int\dfrac{(1+z^2)^2}{z^3}dz=$

$$=\frac{1}{4}\int\frac{1+2z^2+z^4}{z^3}dz=\frac{1}{4}\int\left(\frac{1}{z^3}+\frac{2}{z}+z\right)dz=-\frac{1}{8z^2}+\frac{1}{2}\ln|z|+\frac{1}{4}\cdot\frac{z^2}{2}+C=$$

$$=-\frac{1}{8}ctg^2\frac{x}{2}+\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+\frac{1}{8}tg^2\frac{x}{2}+C.$$ ◄

5. Quyidagi ko‘rinishdagi integrallarni qarab chiqamiz:

$$\int \cos nx\cdot\cos mxdx,$$

$$\int \sin nx\cdot\cos mxdx,$$

$$\int \sin nx\cdot\sin mxdx.$$

Bunday integrallarni hisoblash uchun trigonometrik funksiyalarning ko‘paytmasini yig‘indiga almashtiruvchi formulalar qo‘llanadi:

$$\cos\alpha\cdot\cos\beta=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\sin\alpha\cdot\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\sin\alpha\cdot\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

6.28-misol

Ushbu $I=\int\sin 3x\cdot\cos 2xdx$ integralni hisoblang.

►Integral ostidagi ko‘paytmani yig‘indiga almashtirib integrallaymiz.

$$I=\int\sin 3x\cdot\cos 2xdx=\frac{1}{2}\int(\sin 5x+\sin x)dx=-\frac{1}{2}\cdot\frac{\cos 5x}{5}-\frac{1}{2}\cdot\cos x+C=$$

$$=-\frac{1}{10}\cdot\frac{\cos 5x}{1}-\frac{1}{2}\cdot\cos x+C.$$ ◄

24-Auditoriya topshiriqlari

Integrallarni hisoblang.

1. $\int \frac{dx}{3+5\cos x}$.
2. $\int \frac{dx}{3\sin^2 x+5\cos^2 x}$.
3. $\int \frac{dx}{8-4\sin x+7\cos x}$.
4. $\int \cos^3 x \sin^{10} x dx$.
5. $\int \sin^4 3x dx$
6. $\int \frac{dx}{\cos^4 x} dx$
7. $\int tg^3 x dx$
8. $\int \frac{\cos^2 x}{\sin^4 x} dx$
9. $\int \frac{\cos^4 x+\sin^4 x}{\cos^2 x-\sin^2 x} dx$
10. $\int \sin 3x \sin 5x dx$

24-Mustaqil yechish uchun testlar

1. $\int \frac{dx}{2\cos^2 x+3\sin^2 x}$ integralni hisoblashda qaysi almashtirish qo‘llanadi?

   A) $tg\frac{x}{2}=t$, B) $\sin x=t$, C) $\cos x=t$, D) $tgx=t$.

2. $\int \frac{\sin^3 x}{\cos^4 x} dx$ integralni hisoblashda qaysi almashtirish qo‘llanadi?

   A) $tgx=t$, B) $\sin x=t$, C) $\cos x=t$, D) To‘g‘ri javob yo‘q.

3. $\int \frac{\sin^2 x}{\cos^4 x} dx$ integralni hisoblashda qaysi almashtirish qo‘llanadi?

   A) $tgx=t$, B) $tg\frac{x}{2}=t$, C) $\cos x=t$, D) $\sin x=t$.

4. Integralni hisoblang: $\int \frac{\cos x dx}{\sin^3 x}$

   A) $-\frac{2}{\sin^3 x}+C$, B) $-\frac{3}{\sin^2 x}+C$, C) $\frac{3}{\sin^2 x}+C$, D) $-\frac{1}{2\sin^2 x}+C$.

5. Integralni hisoblang: $\int cos 5x \sin 3x dx$.

   A) $\frac{1}{16}\cos 8x-\frac{1}{4}\cos 2x+C$, B) $\frac{1}{16}\sin 8x-\frac{1}{4}\cos 2x+C$,

   C) $-\frac{1}{16}\cos 8x+\frac{1}{4}\cos 2x+C$, D) $\frac{1}{16}\cos 8x-\frac{1}{4}\sin 2x+C$.

6.4. Ba’zi irratsional funksiyalarni integrallash

Har qanday irratsional funksiyalar ucnun ham elementar funksiyalar ko‘rinishidagi boshlang‘ich funksiyalarni aniqlab bo‘lmaydi. Biz quyida ayrim almashtirishlar yordamida ratsional funksiyalar integrallariga olib kelinadigan irratsional funksiyalarning integrallarini ko‘rib chiqamiz.

Quyidagi

$$\int R\left(x,\left(\frac{ax+b}{cx+d}\right)^{\frac{r_1}{s_1}},\left(\frac{ax+b}{cx+d}\right)^{\frac{r_2}{s_2}},\ldots\left(\frac{ax+b}{cx+d}\right)^{\frac{r_n}{s_n}}\right)dx, \qquad (6.9)$$

bu yerda R-ratsional funksiya, a, b, c , d - o‘zgarmas sonlar, r_i , s_i musbat butun sonlar, integral

$$\frac{ax+b}{cx+d}=t^m \qquad (6.10)$$

almashtirish yordamida ratsionallashtiriladi. Bu yerda m - $\frac{r_1}{s_1},\frac{r_2}{s_2},\ldots\frac{r_n}{s_n}$ kasrlarning umumiy maxraji, ya’ni $m=EKUB(s_1,s_2,\ldots s_n)$.

Xususan,

$$\int R\left(x,x^{\frac{r_1}{s_1}},x^{\frac{r_2}{s_2}},\ldots x^{\frac{r_n}{s_n}}\right)dx$$

Integral $x=t^m$ almashtirish yordamida ratsionallashtiriladi.

6.29-misol

Ushbu $\int\frac{\sqrt{x}}{\sqrt[4]{x^3+4}}dx$ integralni hisoblang.

► $EKUK(2,4)=4$ bo‘lgani uchun,

$$\int\frac{\sqrt{x}}{\sqrt[4]{x^3+4}}dx=\left|\begin{matrix}x=t^4\\ dx=4t^3dt\end{matrix}\right|=4\int\frac{t^5}{t^3+4}dt=4\int\left(t^2-\frac{4t^2}{t^3+4}\right)dt=\frac{4}{3}t^3-\frac{16}{3}\ln\left|t^3+4\right|+C=$$

$$=\left|t=\sqrt[4]{x}\right|=\frac{4}{3}\sqrt[4]{x^3}-\frac{16}{3}\ln\left|\sqrt[4]{x^3}+4\right|+C.$$ ◄

Integral

$$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

ko‘rinishida berilgan bo‘lsin. Avval kasr suratida ildiz ostidagi kvadrat uchhadning differensiali hosil qilinadi($A\neq 0$), kvadrat uchhaddan to‘la kvadrat ajratiladi va quyidagi amallar bajariladi:

$$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx=\frac{A}{2a}\int\frac{(2ax+b)dx}{\sqrt{ax^2+bx+c}}+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{\sqrt{ax^2+bx+c}}=$$

$$=\frac{A}{a}\sqrt{ax^2+bx+c}+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{\sqrt{a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)}}.$$

Agar $c\neq\frac{b^2}{4a}$, $a>0$ bo‘lsa, oxirgi integralni

$$\int \frac{du}{\sqrt{u^2+k}} = \ln\left|u+\sqrt{u^2+k}\right| + C$$

integralga keltirib hisoblash mumkin.

Agar $c > \frac{b^2}{4a}$, $a < 0$ bo‘lsa,

$$\int \frac{du}{\sqrt{k^2-u^2}} = \arcsin\frac{u}{k} + C$$

integralga keltirib hisoblash mumkin.

Eslatma. *Qulaylik uchun, kvadrat uchhadni to‘la kvadratga ajratishdan avval* a *ning modulini ildizdan chiqarish kerak.*

6.30-misol

Ushbu $\int \frac{5x+3}{\sqrt{x^2-4x+8}}dx$ integralni hisoblang.

► $\int \frac{5x+3}{\sqrt{x^2-4x+8}}dx = \frac{5}{2}\int \frac{2x-4}{\sqrt{x^2-4x+8}}dx + \left(3-\frac{5\cdot 4}{2}\right)\int \frac{dx}{\sqrt{x^2-4x+8}} =$

$= 5\sqrt{x^2-4x+8} - 7\int \frac{dx}{\sqrt{(x-2)^2+4}} = 5\sqrt{x^2-4x+8} - 7\ln\left|x-2+\sqrt{(x-2)^2+4}\right| + C$

.◄

6.31-misol

Ushbu $\int \frac{3x-2}{\sqrt{10-8x-2x^2}}dx$ integralni hisoblang.

► Qulaylik uchun, avval 2 ni ildizdan chiqarib olamiz.

$$\int \frac{3x-21}{\sqrt{5-8x-2x^2}}dx = \frac{1}{\sqrt{2}}\int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \frac{\sqrt{2}}{2}I_1.$$

Hosil bo‘lgan integralni hisoblaymiz.

$$I_1 = \int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \int \frac{-\frac{3}{2}(-4-2x)+8}{\sqrt{5-4x-x^2}}dx = -\frac{3}{2}\int \frac{(-4-2x)dx}{\sqrt{5-4x-x^2}} +$$

$$+8\int \frac{dx}{\sqrt{1-(x+2)^2}} = -3\sqrt{5-4x-x^2} + 8\arcsin(x+2) + C_1.$$

Demak,

$$\int \frac{3x-2}{\sqrt{10-8x-2x^2}}dx = -\frac{3\sqrt{2}}{2}\sqrt{5-4x-x^2} + 4\sqrt{2}\arcsin(x+2) + C.$$ ◄

Agar integral

$$\int \frac{Ax+B}{(x-\alpha)\sqrt{ax^2+bx+c}}dx$$

ko‘rinishda bo‘lsa, $x-\alpha=\frac{1}{t}$ almashtirish yordamida hisoblanadi.

6.32-misol

Ushbu $\int \frac{dx}{(x+1)\sqrt{x^2+2x+10}}$ integralni hisoblang.

$$\blacktriangleright \int \frac{dx}{(x+1)\sqrt{x^2+2x+10}} = \left| \begin{aligned} x+1&=\frac{1}{t} \\ dx&=-\frac{1}{t^2}dt \end{aligned} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+9}} = -\int \frac{dt}{\sqrt{9t^2+1}} =$$

$$= -\frac{1}{3}\ln\left|3t+\sqrt{9t^2+1}\right| + C = -\frac{1}{3}\ln\left|\frac{3}{x+1}+\sqrt{\frac{9}{(x+1)^2}+1}\right| + C. \blacktriangleleft$$

Agar integral

$$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$$

ko‘rinishda bo‘lsa, kvadrat uchhadni to‘la kvadratga ajratib, quyidagi

1) $\int R\left(u,\sqrt{k^2-u^2}\right)du$, $\quad u=k\sin t \, (u=k\cos t)$ almashtirish;

2) $\int R\left(u,\sqrt{k^2+u^2}\right)du$, $\quad u=ktgt \, (u=kctgt)$ almashtirish;

3) $\int R\left(u,\sqrt{u^2-k^2}\right)du$, $\quad u=\frac{k}{\sin t}\left(u=\frac{k}{\cos t}\right)$ almashtirish

yordamida hisoblanadigan integrallardan biriga keltirish mumkin.

6.33-misol

Ushbu $\int \sqrt{3+2x-x^2}\,dx$ integralni hisoblang.

$$\blacktriangleright \int \sqrt{3+2x-x^2}\,dx = \int \sqrt{4-(x-1)^2}\,dx = \left| \begin{aligned} x-1&=2\sin t \\ dx&=2\cos t dt \end{aligned} \right| =$$

$$= \int \sqrt{4-4\sin^2 t}\cdot 2\cos t dt = 4\int \cos^2 t dt = 2\int (1+\cos 2t)dt = 2t+\sin 2t + C =$$

$$= 2t + 2\sin t\sqrt{1-\sin^2 t} + C = 2\arcsin\frac{x-1}{2} + \frac{(x-1)\sqrt{3+2x-x^2}}{2} + C. \blacktriangleleft$$

6.34-misol

Ushbu $\int \frac{dx}{\sqrt{(x^2+4x+5)^3}}$ integralni hisoblang.

▶

$$\int \frac{dx}{\sqrt{(x^2+4x+5)^3}} = \int \frac{dx}{\sqrt{((x+2)^2+1)^3}} = \begin{vmatrix} x+2 = tgt \\ dx = \frac{dt}{\cos^2 t} \end{vmatrix} = \int \frac{dt}{\cos^2 t\sqrt{(tg^2 t+1)^3}} =$$

$$= \int \frac{dt}{\cos^2 t\sqrt{(tg^2 t+1)^3}} = \int \cos t dt = \sin t + C = \frac{tgt}{\sqrt{tg^2 t+1}} + C = \frac{x+2}{\sqrt{x^2+4x+5}} + C.$$ ◀

25-Auditoriya topshiriqlari

Aniqmas integrallarni hisoblang.

1. $\int \frac{dx}{\sqrt{2x+3}+3\sqrt[3]{2x+3}}$

2. $\int \sqrt{\frac{1-x}{1+x}} \cdot \frac{dx}{x}$

3. $\int \frac{(3x+5)dx}{\sqrt{x^2+6x+7}}$

4. $\int \frac{(2x-7)dx}{\sqrt{9-8x-x^2}}$

5. $\int \frac{(2x+3)dx}{\sqrt{2x^2-x+6}}$

6. $\int \frac{(3x+4)dx}{\sqrt{2+3x-x^2}}$

7. $\int \frac{dx}{x^2\sqrt{x^2+4}}$

8. $\int \sqrt{12-4x-x^2}$

9. $\int \sqrt{6x-x^2}dx$

10. $\int \frac{dx}{x\sqrt{x^2+x+1}}$

25-Mustaqil yechish uchun testlar

1. $\int \frac{dx}{\sqrt{x}+3\sqrt[3]{x}}$ integralda qanday almashtirish bajariladi?

A) $x=t^2$; B) $\sqrt[3]{x}=t$; C) $x=t^5$; D) $\sqrt[6]{x}=t$

2. Quyidagi $\int \frac{(2x-1)dx}{\sqrt{x^2-2x+5}}$ integralning yechimini toping

A) $2\sqrt{x^2-2x+5}+C$; B) $2\sqrt{x^2-2x+5}-\frac{1}{2}arctg\frac{x-1}{2}+C$;

C) $2\sqrt{x^2-2x+5}+\frac{1}{2}arctg\frac{x-1}{2}+C$; D) $2\sqrt{x^2-2x+5}-arctg\frac{x-1}{2}+C$;

3. $\int \sqrt{4-x^2}dx$ integral yechimini toping

A) $2\arcsin\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+C$; B) $2\arcsin\frac{x}{2}+x\sqrt{4-x^2}+C$;

C) $2\arcsin\frac{x}{2}-x\sqrt{4-x^2}+C$; D) $2\arcsin\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}+C$;

4. Quyidagi $\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}}$ integralda qanday almashtirish bajariladi?

A) $x+1=t^2$; B) $\sqrt{x^2+2x+1}=t$; C) $x+1=1/t$; D) $x=tgt$.

5. Quyidagi $\int \frac{dx}{x^2\sqrt{x^2-9}}$ integralda qanday almashtirish bajariladi?

A) $x=3/\cos t$; B) $x=3\sin t$; C) $x^2-9=t^2$; D) $x=3tgt$.

8-Shaxsiy uy topshiriqlari

Aniqmas integrallarni hisoblang.

1

1.1. a) $\int \frac{dx}{x\sqrt{x^2+1}}$.

b) $\int (4-3x)e^{-3x}dx$.

d) $\int \frac{12-6x}{(x+1)(x^2-4x+13)}dx$

1.2. a) $\int \frac{1+\ln x}{x}dx$.

b) $\int \operatorname{arctg}\sqrt{4x-1}dx$.

d) $\int \frac{x^3+6x^2+13x+8}{x(x+2)^3}dx$.

1.3. a) $\int \frac{dx}{x\sqrt{x^2-1}}$.

b) $\int (3x+4)e^{3x}dx$.

d) $\int \frac{x^3-6x^2+13x-6}{(x+2)(x-2)^3}dx$.

1.4. a) $\int \frac{x^2+\ln x^2}{x}dx$.

b) $\int (4x-2)\cos 2x dx$.

d) $\int \frac{2x^3-2x^2+5}{(x-1)^2(x^2+4)}dx$

1.5. a) $\int \frac{xdx}{\sqrt{x^4+x^2+1}}$.

b) $\int e^{-2x}(4x-3)dx$.

d) $\int \frac{x^3-6x^2+11x-10}{(x+2)(x-2)^3}dx$.

1.6. a) $\int \frac{(\arccos x)^3-1}{\sqrt{1-x^2}}dx$.

b) $\int (5x-2)e^{3x}dx$.

d) $\int \frac{x^3+6x^2+11x+7}{(x+1)(x+2)^3}dx$.

1.7. a) $\int \operatorname{tg} x \ln\cos x dx$.

b) $\int \frac{xdx}{\cos^2 x}$.

d) $\int \frac{x^3+8x-2}{x^2(x^2+4)}dx$

1.8. a) $\int \frac{\operatorname{tg}(x+1)}{\cos^2(x+1)}dx$.

b) $\int \ln(x^2+4)dx$.

d) $\int \frac{2x^3+x+1}{(x+1)x^3}dx$.

1.9. a) $\int \frac{x^3}{(x^2+1)^2}dx$.

b) $\int (2-4x)\sin 2x dx$.

d) $\int \frac{4x+2}{x^4+4x^2}dx$

1.10. a) $\int \frac{1-\cos x}{(x-\sin x)^2}dx$.

b) $\int \operatorname{arctg}\sqrt{6x-1}dx$.

d) $\int \frac{x^2-2x+4}{x^3(x^2+1)}dx$

1.11. a) $\int \frac{x\cos x+\sin x}{(x\sin x)^2}dx$.

b) $\int (4-16x)\sin 4x dx$.

d) $\int \frac{x^3+x+2}{(x+2)x^3}dx$.

1.12. a) $\int \frac{xdx}{\sqrt{x^4 - x^2 - 1}}$.

b) $\int e^{-3x}(2-9x)dx$.

d) $\int \frac{2x+22}{(x+2)(x^2-2x+10)}dx$

1.13. a) $\int \frac{dx}{\cos^2 x\sqrt{tg^3 x}}$

b) $\int \operatorname{arctg}\sqrt{3x-1}dx$.

d) $\int \frac{x^3-3x^2+5}{x^3(x^2+1)}dx$

1.14. a) $\int \frac{1/(2\sqrt{x})+1}{(\sqrt{x}+x)^2}dx$.

b) $\int \operatorname{arctg}\sqrt{5x-1}dx$.

d) $\int \frac{6x}{x^3-1}dx$

1.15. a) $\int \frac{(x^2+1)dx}{(x^3+3x+1)^5}$.

b) $\int (5x+6)\cos 2xdx$.

d) $\int \frac{x^3+3x^2-12x+4}{(x-1)^2(x^2+1)}dx$

1.16. a) $\int \frac{4\operatorname{arctg} x - x}{1+x^2}dx$.

b) $\int (3x-2)\cos 5xdx$.

d) $\int \frac{2x^3+6x^2+7x+1}{(x-1)(x+1)^3}dx$.

1.17. a) $\int \frac{x-(\operatorname{arctg} x)^4}{1+x^2}dx$.

b) $\int (x\sqrt{2}-3)\cos 2xdx$.

d) $\int \frac{x^3-6x^2+10x-10}{(x+1)(x-2)^3}dx$.

1.18. a) $\int \frac{x+\cos x}{x^2+2\sin x}dx$.

b) $\int (4x+7)\cos 3xdx$.

d) $\int \frac{2x^3+6x^2+7x}{(x-2)(x+1)^3}dx$.

1.19. a) $\int \frac{2\cos x + 3\sin x}{(2\sin x - 3\cos x)^3}dx$.

b) $\int \ln(\cos x)dx$

d) $\int \frac{x^2-2x+4}{x^3(x^2+1)}dx$

1.20. a) $\int \frac{3x - \arccos 2x}{\sqrt{1-4x^2}}dx$

b) $\int arctg\sqrt{2x+1}dx$

d) $\int \frac{2x^3+6x^2+5x}{(x+2)(x+1)^3}dx$.

1.21. a) $\int \frac{x^3+x}{x^4+1}dx$.

b) $\int \frac{\ln(\cos x)dx}{\sin^2 dx}$

d) $\int \frac{2x^3+6x^2+7x+4}{(x+2)(x+1)^3}dx$.

1.22. a) $\int \frac{5x-(\arcsin 3x)^3}{\sqrt{1-9x^2}}dx$

b) $\int cos(\ln x)dx$

d) $\int \frac{x^3+6x^2+10x+10}{(x-1)(x+2)^3}dx$.

1.23. a) $\int \frac{x+\cos 2x}{\sqrt{x^2+\sin 2x}}dx$

b) $\int \frac{\ln(\sin x)dx}{\cos^2 dx}$

d) $\int \frac{x^3+6x^2+13x+6}{(x-2)(x+2)^3}dx$.

1.24. a) $\int \frac{3x\sin 3x - \cos 3x}{(x\cos 3x)^3}dx$

b) $\int \sin(\ln x)dx$

d) $\int \frac{x^3-6x^2+13x-8}{x(x-2)^3}dx.$

1.25. a) $\int \frac{\sqrt[3]{ctg^2 3x}}{\sin^2 3x}dx$

b) $\int x arctg\, 2x dx$

d) $\int \frac{x^3-6x^2+13x-7}{(x+1)(x-2)^3}dx.$

1.26. a) $\int \frac{3x-(arcctgx)^3}{\sqrt{1-x^2}}dx$

b) $\int x^2 arctgx dx$

d) $\int \frac{x^3+6x^2+14x+10}{(x+1)(x+2)^3}dx.$

1.27. a) $\int \frac{3x+x^3}{x^4+2}dx$

b) $\int (1-6x)e^{2x}dx.$

d) $\int \frac{2x^3+6x^2+7x+2}{x(x+1)^3}dx.$

1.28. a) $\int \frac{x^2-\ln^2 x}{x}dx$

b) $\int \frac{\arcsin x}{\sqrt{x+1}}dx$

d) $\int \frac{3x^3+9x^2+10x+2}{(x-1)(x+1)^3}dx.$

1.29. a) $\int \frac{dx}{(1+x^2)\sqrt{(arctgx)^3}}$

b) $\int xe^{-6x}dx$

d) $\int \frac{x^3-6x^2+14x-6}{(x+1)(x-2)^3}dx.$

1.30. a) $\int \frac{3\arcsin^2 x+4x}{\sqrt{1-x^2}}dx$

b) $\int (2x-5)\cos 4x dx.$

d) $\int \frac{2x^3+6x^2+5x+4}{(x-2)(x+1)^3}dx.$

2

2.1. a) $\int \frac{\sin^3 2x}{cos^2 2x}dx$

b) $\int tg^4 3x dx$

d) $\int \frac{dx}{3+cosx+\sin x}$

2.2. a) $\int \frac{\cos^3 x}{\sqrt[3]{cos^4 x}}dx$

b) $\int \sin^4 2x dx$

d) $\int \frac{dx}{3cos^2 x+4\sin^2 x}$

2.3. a) $\int \sin 4x \sin x dx$

b) $\int ctg^4 5x dx$

d) $\int \frac{dx}{2-3cosx+\sin x}$

2.4. a) $\int cos^4 3x \sin^2 3x dx$

b) $\int \sin^3 4x dx$

d) $\int \frac{dx}{4+3cosx-4\sin x}$

2.5. a) $\int \cos 4x \sin x dx$

b) $\int tg^3(4-x)dx$

d) $\int \frac{dx}{3+5\sin x+3cosx}$

2.6. a) $\int \sqrt[3]{\cos^4 x}\sin^3 x dx$

b) $\int tg^2(5x+1)dx$

d) $\int \frac{6\sin x+\cos x}{1+cosx}dx$

2.7. a) $\int \sqrt[3]{\sin^4 x}\cos^3 x dx$

b) $\int tg^5 4x dx$

d) $\int \frac{dx}{5-3cosx}$

2.8. a) $\int cos^3 2x\sin^3 2x dx$

b) $\int ctg^3 \frac{x}{2}dx$

d) $\int \frac{dx}{5+4\sin x}$

2.9. a) $\int cos^3 2x\sin^5 2x dx$

b) $\int \cos^4 3x dx$

d) $\int \frac{dx}{8+4cosx}$

2.10. a) $\int \cos x\sin 9x dx$

b) $\int \cos^3 4x dx$

d) $\int \frac{dx}{4\sin^2 x-5cos^2 x}$

2.11. a) $\int \cos 2x\cos 5x dx$

b) $\int x tg^2 x^2 dx$

d) $\int \frac{dx}{8-4\sin x+7\cos x}$

2.12. a) $\int \cos^4 x\sin x dx$

b) $\int (1-tg 2x)^2 dx$

d) $\int \frac{dx}{3+2cosx-\sin x}$

2.13. a) $\int \sin 5x\sin 7x dx$

b) $\int (1+ctg 2x)^2 dx$

d) $\int \frac{dx}{2\sin^2 x+7\cos^2 x}$

2.14. a) $\int \sin^4 5x\cos 5x dx$

b) $\int (tg 2x+ctg 2x)^2 dx$

d) $\int \frac{dx}{8+7cosx-4\sin x}$

2.15. a) $\int \frac{\cos^3 x}{\sqrt[3]{\sin^5 x}}dx$

b) $\int (1+\cos 3x)^2 dx$

d) $\int \frac{dx}{4\sin^2 x+8\cos x\sin x}$

2.16. a) $\int \cos^4 x\sin 2x dx$

b) $\int ctg^2 \frac{x}{3}dx$

d) $\int \frac{dx}{3+3cosx+2\sin x}$

2.17. a) $\int \cos^3 x\sin 2x dx$

b) $\int tg^4 \frac{x}{3}dx$

d) $\int \frac{dx}{5\sin^2 x-3\cos^2 x}$

2.18. a) $\int \sin 5x cos 7x dx$

b) $\int tg^3 \frac{x}{2}dx$

d) $\int \frac{dx}{5+3cosx+\sin x}$

2.19. a) $\int \cos 5x \sin 7x dx$

b) $\int ctg^4 \frac{x}{2} dx$

d) $\int \frac{dx}{3 + cosx + \sin x}$

2.20. a) $\int \cos^5 x \sin^3 x dx$

b) $\int tg^4 3x dx$

d) $\int \frac{dx}{16\sin^2 x + 7\cos^2 x}$

2.21. a) $\int \cos^2 x \sin^4 x dx$

b) $\int ctg^3 (x+2) dx$

d) $\int \frac{dx}{7\sin x - 3cosx}$

2.22. a) $\int \sqrt[5]{\cos^4 x} \sin 2x dx$

b) $\int \cos^4 (x+3) dx$

d) $\int \frac{dx}{4cosx - 6\sin x}$

2.23. a) $\int \cos^2 3x \sin^2 3x dx$

b) $\int (1 - tg 3x)^2 dx$

d) $\int \frac{dx}{3 - 2\sin^2 x}$

2.24. a) $\int \cos^2 3x \sin^3 3x dx$

b) $\int (1 - \sin 3x)^2 dx$

d) $\int \frac{2 - \sin x + 3\cos x}{1 + cosx} dx$

2.25. a) $\int \cos 5x \cos 7x dx$

b) $\int tg^3 (2x+3) dx$

d) $\int \frac{dx}{5 + 3\sin^2 x}$

2.26. a) $\int \cos^3 x \sin^7 x dx$

b) $\int (2 + \sin 5x)^2 dx$

d) $\int \frac{7 + 6\sin x - 5\cos x}{1 + cosx} dx$

2.27. a) $\int \cos 2x \sin^2 x dx$

b) $\int (tg 3x - ctg 3x)^2 dx$

d) $\int \frac{dx}{6 - 3\cos^2 x}$

2.28. a) $\int \sqrt[3]{\cos^2 3x} \sin 3x dx$

b) $\int ctg^3 (2x-3) dx$

d) $\int \frac{dx}{2 + 3cosx + 4\sin x}$

2.29. a) $\int \sqrt[3]{\cos^2 x} \sin^3 x dx$

b) $\int tg^2 \frac{2x}{3} dx$

d) $\int \frac{\sin^2 x dx}{3\sin^2 x - \cos^2 x}$

2.30. a) $\int \cos 2x \cos 7x dx$

b) $\int tg^4 (x+3) dx$

d) $\int \frac{dx}{8 - 3\sin^2 x}$

3

3.1. a) $\int \frac{\sqrt{x+1} dx}{\sqrt[3]{x+1} - \sqrt[6]{x+1}}$

b) $\int \frac{(x+3)dx}{\sqrt{x^2 - 2x + 6}}$

3.2. a) $\int \frac{xdx}{2+\sqrt{x+4}}$

b) $\int \frac{dx}{x\sqrt{x^2+x-2}}$

3.3. a) $\int \frac{\sqrt[6]{x+2}dx}{\sqrt[3]{x+2}+\sqrt{x+2}}$

b) $\int \frac{dx}{\sqrt{(4+x^2)^3}}$

3.4. a) $\int \frac{dx}{\sqrt[3]{(2x+3)^2}-\sqrt{2x+3}}$

b) $\int \frac{dx}{x^2\sqrt{x^2+25}}$

3.5. a) $\int \frac{(x-1)dx}{x\sqrt{x-2}}$

b) $\int \frac{dx}{x\sqrt{1+x-x^2}}$

3.6. a) $\int \frac{\sqrt{x+3}-\sqrt[3]{x+3}}{\sqrt[3]{(x+3)^2}+\sqrt[6]{x+3}}dx$

b) $\int \frac{(x-3)dx}{\sqrt{2x^2-4x+1}}$

3.7. a) $\int \frac{\sqrt[3]{(x+1)^2}+\sqrt[6]{x+1}}{\sqrt{x+1}+\sqrt[3]{x+1}}dx$

b) $\int \frac{(2x+1)dx}{\sqrt{1+x-3x^2}}$

3.8. a) $\int \frac{\sqrt[4]{x}+\sqrt{x}}{\sqrt{x}+1}dx$

b) $\int \frac{(2x-10)dx}{\sqrt{1+x-x^2}}$

3.9. a) $\int \frac{dx}{3+\sqrt{x+5}}$

b) $\int \frac{(2x+5)dx}{\sqrt{9+8x+4x^2}}$

3.10. a) $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$

b) $\int \frac{(3x+4)dx}{\sqrt{13+6x+x^2}}$

3.11. a) $\int \frac{\sqrt{x+1}-2\sqrt[3]{x-1}}{\sqrt{x+1}+2\sqrt[3]{x+1}}dx$

b) $\int \frac{(3x-1)dx}{\sqrt{2x^2-5x+1}}$

3.12. a) $\int \frac{(x+1)dx}{x\sqrt{x-1}}$

b) $\int \frac{(5x+2)dx}{\sqrt{x^2+3x-4}}$

3.13. a) $\int \frac{\sqrt{x+2}dx}{\sqrt[3]{x+2}+\sqrt[6]{x+2}}$

b) $\int \frac{(x-4)dx}{\sqrt{2x^2-x+7}}$

3.14. a) $\int \frac{\sqrt{x+3}dx}{1+\sqrt[3]{x+2}}$

b) $\int \frac{(4x+1)dx}{\sqrt{2+x-x^2}}$

3.15. a) $\int \frac{(x+3)dx}{x\sqrt{x-4}}$

b) $\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$

3.16. a) $\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt{x}+\sqrt[6]{x}}dx$

b) $\int \frac{(5x-3)dx}{\sqrt{2x^2+4x-5}}$

3.17. a) $\int \frac{\sqrt{3x+1}-2}{\sqrt{3x+1}+2\sqrt[3]{3x+1}}dx$

b) $\int \frac{(3x+2)dx}{\sqrt{4+2x-x^2}}$

3.18. a) $\int \frac{(x^3-1)dx}{\sqrt{x+2}}$

b) $\int \frac{dx}{x\sqrt{x^2+x-3}}$

3.19. a) $\int \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[3]{x}-\sqrt[6]{x}-1}dx$

b) $\int \frac{(x+5)dx}{\sqrt{3-6x-x^2}}$

3.20. a) $\int \frac{\sqrt[6]{3x+1}+1}{\sqrt{3x+1}-\sqrt[3]{3x+1}}dx$

b) $\int \frac{(x-9)dx}{\sqrt{4+2x-x^2}}$

3.21. a) $\int \frac{x^3dx}{\sqrt{x-2}}$

b) $\int \frac{(3x-4)dx}{\sqrt{2x^2-6x+1}}$

3.22. a) $\int \frac{\sqrt{x}}{1-\sqrt[4]{x}}dx$

b) $\int \frac{(7x-1)dx}{\sqrt{2-3x-x^2}}$

3.23. a) $\int \frac{\sqrt{x}}{x-4\sqrt[3]{x^2}}dx$

b) $\int \frac{dx}{(x-1)\sqrt{1+x-x^2}}$

3.24. a) $\int \frac{x^2dx}{\sqrt{x-3}}$

b) $\int \frac{dx}{x\sqrt{1-x-x^2}}$

3.25. a) $\int \frac{x-\sqrt[3]{x^2}}{x(1+\sqrt[6]{x})}dx$

b) $\int \frac{dx}{x\sqrt{x^2-3x+2}}$

3.26. a) $\int \frac{\sqrt{x-2}dx}{3+\sqrt{x-2}}$

b) $\int \frac{(7x+1)dx}{\sqrt{2-4x-x^2}}$

3.27. a) $\int \frac{\sqrt{x}}{3x+\sqrt[3]{x^2}}dx$

b) $\int \frac{dx}{(x+1)\sqrt{x^2-3x+2}}$

3.28. a) $\int \frac{dx}{3+\sqrt{x-5}}$

b) $\int \frac{dx}{x\sqrt{1-3x-2x^2}}$

3.29. a) $\int \frac{dx}{2+\sqrt{x-8}}$

b) $\int \frac{dx}{(x+1)\sqrt{2-x-x^2}}$

3.30. a) $\int \frac{\sqrt{x}}{1-\sqrt[3]{x}}dx$

b) $\int \frac{(5x+1)dx}{\sqrt{3-6x-x^2}}$

6.5. Aniq integral va uni hisoblash

Aytaylik, $y=f(x)$ funksiya $[a;b]$ kesmada aniqlangan bo‘lsin. Bu kesmani ixtiuoriy $a=x_0<x_1<x_2<...<x_n=b$ nuqtalar bilan n ta uzunliklari $\Delta x_i=x_i-x_{i-1}\left(i=\overline{1,n}\right)$ bo‘lgan qismiy bo‘laklarga bo‘lamiz va har bir bolakdan bittadan ixtiyoriy ξ_i nuqtalarni tanlab olamiz, bu yerda $x_{i-1}<\xi_i<x_i$ $i=\overline{1,n}$. Quyidagi ko‘rinishdagi yig‘indini tuzamiz:

$$\sigma_n=\sum_{i=1}^{n} f(\xi_i)\Delta x_i. \tag{6.11}$$

Bu yig‘indi $y=f(x)$ funksiyaning $[a;b]$ kesmadagi ***integral yig‘indisi*** deyiladi. Integral yig‘indi asosi Δx_i balandligi $f(\xi_i)$ bo‘lgan to‘g‘ri to‘rtburchaklarning yuzalarining algebraik yig‘indisini beradi.

Integral yig‘indi σ_n ning qismiy bo‘laklar uzunliklarining eng kattasi nolga intilgandagi limiti $f(x)$ funksiyadan a dan b gacha olingan ***aniq integral*** deyiladi va $\int_a^b f(x)dx$ kabi belgilanadi, ya’ni ta’rif bo‘yicha

$$\lim_{\max\Delta x_i\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i=\int_a^b f(x)dx \tag{6.12}$$

6.2-Teorema *Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo‘lsa, u shu kesmada integrallanuvchidir, ya’ni bunday funksiya uchun (5.1) integral yig‘indining limiti mavjud va bu limit $[a;b]$ kesmani qismiy bo‘laklarga bo‘lish va ulardan ξ_i nuqtalarni tanlash usuliga bog‘liq emas.*

Agar $[a;b]$ kesmada $f(x)\geq 0$ bo‘lsa, $\int_a^b f(x)dx$ aniq integral $f(x)$ funksiya gafigi, Ox o‘qi va $x=a$, $x=b$ to‘g‘ri chiziqlar bilan chegaralangan ***egri chiziqli trapetsiya yuzini*** ifodalaydi.

1-Izoh. *Aniq integral integrallash o‘zgaruvchisiga bog‘liq emas:*

$$\int_a^b f(x)dx=\int_a^b f(t)dt=\int_a^b f(z)dz.$$

2-Izoh. *Aniq integralning chegaralari almashtirilsa, integralning ishorasi o‘zgaradi:*

$$\int_a^b f(x)dx=-\int_b^a f(x)dx.$$

3-Izoh. *Agar aniq integralning integrallash chegaralari teng bo'lsa, uning qiymati nolga teng:*

$$\int_a^a f(x)dx = 0.$$

Aniq integralning asosiy xossalari ($f(x), \varphi(x)$ funksiyalarni mos kesmalarda integrallanuvchi deb faraz qilamiz):

1. *Bir nechta funksiyaning algebraik yig'indisining aniq integrali qo'shiluvchilar integrallarining yig'indisiga teng. Ikki qo'shiluvchi bo'lgan hol bilan cheklanamiz:*

$$\int_a^b [f(x) \pm \varphi(x)]dx = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx.$$

2. *O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, agar* $k = const$ *bo'lsa, u holda*

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$

3. *Agar* $[a;b]$ *kesmada funksiya o'z ishorasini o'zgartirmasa, u holda bu funksiya aniq integralining ishorasi funksiya ishorasi bilan bir xil bo'ladi, ya'ni:*

a) *agar* $[a;b]$ *kesmada* $f(x) \geq 0$ *bo'lsa, u holda*

$$\int_a^b f(x)dx \geq 0$$

b) *agar* $[a;b]$ *kesmada* $f(x) \leq 0$ *bo'lsa, u holda*

$$\int_a^b f(x)dx \leq 0$$

4. *Agar* $[a;b]$ *kesmada ikki* $f(x)$ *va* $\varphi(x)$ *funksiya* $f(x) \geq \varphi(x)$ *shartni qanoatlantirsa, u holda*

$$\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx$$

5. *Agar* $[a;b]$ *kesma bir necha qismlarga bo'linsa, u holda* $[a;b]$ *kesma bo'yicha aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng.* $[a;b]$ *kesma ikki qismga bo'lingan hol bilangina cheklanamiz, ya'ni agar* $a<c<b$ *bo'lsa, u holda*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

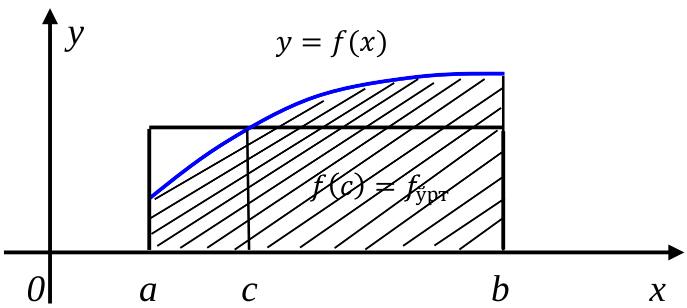
6. *Agar* m *va* M *sonlar* $f(x)$ *funksiyaning* $[a;b]$ *kesmadagi eng kichik va eng katta qiymatlari bo'lsa, u holda*

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

6.3-Teorema *(o'rta qiymat haqida).* *Agar* $f(x)$ *funksiya* $[a;b]$ *kesmada uzluksiz bo'lsa, bu kesmaning ichida shunday* $x=c$ *nuqta topiladiki,*

$$\int_a^b f(x)dx = f(c)(b-a)$$

Funksiyaning bu nuqtadagi qiymati uning shu kesmadagi o'rta qiymati bo'ladi.



6.1-chizma.

O'rta qiymat haqidagi teoremaning geometrik ma'nosi quyidagicha(1-chizma): yuqoridan integral osti funksiyasi $f(x)$ ning grafigi bilan chegaralangan, $(b-a)$ asosli egri chiziqli trapesiyaning yuzi o'shanday asosli va balandligi funksiyaning $f(c)$ o'rta qiymatiga teng to'g'ri to'rtburchakning yuziga tengdosh.

9. *Agar* $f(x)$ *funksiya kesmada uzluksiz va* $\Phi(x)=\int_a^x f(x)dx$ *bo'lsa, quyidagi tenglik o'rinli*

$$\Phi'(x) = \left(\int_a^x f(x)dx \right)' = f(x)$$

10. *Agar $F(x)$ funksiya $f(x)$ funksiyaning qandaydir boshlang'ich funksiyasi bo'lsa,*

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \qquad (6.13)$$

tenglik o'rinli. Bu formula *Nyuton-Leybnits formulasi* deyiladi.

6.35-misol

Aniq integralni hisoblang: $\int_1^4 \frac{x^2-2}{\sqrt{x}}dx$.

►Integral ostidagi funksiyani hadlab bo'lamiz va yuqoridagi xossalardan foydalanib integralni hisoblaymiz.

$$\int_1^4 \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}} \right) dx = \int_1^4 x^{3/2}dx - 2\int_1^4 \frac{1}{\sqrt{x}}dx = \frac{2}{5}x^{5/2}\Big|_1^4 - \sqrt{x}\Big|_1^4 = \frac{2}{5}(32-1) - (2-1) = 11\frac{2}{5}. \blacktriangleleft$$

6.36-misol

Aniq integralni hisoblang: $\int_1^2 \frac{3x-4}{x^3+4x}dx$.

►Integral ostidagi funksiyani sodda kasrlarga ajratamiz:

$$\frac{3x-4}{x^3+4x} = \frac{3x-4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}, \quad A(x^2+4) + Bx^2 + Cx \equiv 3x - 4,$$

$$\begin{array}{l|l} x^2 & A + B = 0 \\ x^1 & C = 3 \\ x^0 & 4A = -4 \end{array}$$

Bundan, $A = -1, B = 1, C = 3$. Natijada,

$$\int_1^2 \frac{2x-3}{x^3+4x}dx = \int_1^2 \left(-\frac{1}{x} + \frac{x+3}{x^2+4} \right) dx = -\int_1^2 \frac{dx}{x} + \int_1^2 \frac{xdx}{x^2+4} + 3\int_1^2 \frac{dx}{x^2+4} =$$

$$= -\ln x\Big|_1^2 + \frac{1}{2}\ln(x^2+4)\Big|_1^2 + \frac{3}{2}arctg\,\frac{x}{2}\Big|_1^2 = -\ln 2 + \frac{1}{2}\ln\frac{8}{5} + \frac{3}{2}\left(arctg\,1 - arctg\,\frac{1}{2} \right) =$$

$$= \ln\frac{\sqrt{10}}{5} + \frac{3}{2}arctg\frac{1}{3}. \blacktriangleleft$$

$y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz, $x=\varphi(t)$ funksiya hosilasi bilan $[\alpha;\beta]$ kesmada uzluksiz va monoton, $\varphi(\alpha)=a$, $\varphi(\beta)=b$ va $y=f(\varphi(t))$ murakkab funksiya $[\alpha;\beta]$da uzluksiz bo‘lsa, u holda quyidagi

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \tag{6.4}$$

aniq integralda o‘zgaruvchini almashtirish formulasi o‘rinli.

6.37-misol

Aniq integralni hisoblang: $\int_0^2\sqrt{4-x^2}dx$.

► $x=2\sin t$ deb almashtirish bajaramiz. U holda $dx=2\cos tdt$, $x=0$ da $\alpha=0$ va $x=2$ da $\beta=\pi/2$ ni hosil qilamiz. Natijada, (5.4) formulaga ko‘ra,

$$\int_0^2\sqrt{4-x^2}dx=\int_0^{\pi/2}\sqrt{4-4\sin^2x}\cdot 2\cos tdt=\int_0^{\pi/2}4\cos^2tdt=2\int_0^{\pi/2}(1+\cos 2t)dt=$$

$$=2\left(t+\frac{1}{2}\sin 2t\right)\Bigg|_0^{\pi/2}=\pi\,.◄$$

Agar $u(x)$ va $v(x)$ funksiyalar $[a;b]$ kesmada uzluksiz hosilalarga ega bo‘lsa, quyidagi

$$\int_a^b u(x)dv(x)=u(x)v(x)\Big|_a^b-\int_a^b v(x)du(x)$$

yoki qisqacha

$$\int_a^b udv=uv\Big|_a^b-\int_a^b vdu \tag{6.5}$$

aniq integralda bo‘laklab integrallash formulasi o‘rinli.

6.38-misol

Aniq integralni hisoblang: $\int_1^e x^2\ln xdx$.

$$►\int_1^e x^2\ln xdx=\left|\begin{matrix}u=\ln x, & du=\dfrac{dx}{x}\\ dv=x^2dx, & v=x^3/3\end{matrix}\right|=\frac{x^3}{3}\ln x\Big|_1^e-\frac{1}{3}\int_1^e x^2dx=\frac{e^3}{3}-\frac{x^3}{9}\Big|_1^e=\frac{2e^3+1}{9}\,.◄$$

26-Auditoriya topshiriqlari

Berilgan aniq integrallarni hisoblang.

1. $\int_1^2 \frac{2x^6+2}{x^4}dx$.

2. $\int_{-1}^2 \frac{dx}{x^2+4x+5}$.

3. $\int_1^{e^3} \frac{dx}{x\sqrt{1+\ln x}}$.

4. $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x-\cos^3 x}dx$.

5. $\int_1^2 \frac{dx}{x\sqrt{x^2+5x+1}}$

6. $\int_{-2}^2 \frac{dx}{1+\sqrt{x+2}}$

7. $\int_{\pi/3}^{\pi/2} \frac{dx}{\sin^3 x}$

8. $\int_0^3 \frac{dx}{(9+x^2)\sqrt{9+x^2}}$

9. $\int_1^{\sqrt{3}} arctg\frac{1}{x}dx$

10. $\int_0^{\pi/4} xtg^2xdx$.

26-Mustaqil yechish uchun testlar

1. Aniq integralni hisoblang: $\int_0^1 \frac{x^3}{x^2+1}dx$

A) $\frac{1}{2}+\ln 2$ B) $\frac{1}{2}+\ln\sqrt{2}$ C) $\frac{1}{2}-\ln\sqrt{2}$ D) 1

2. Aniq integralni hisoblang: $\int_0^{\pi/2} x\cos xdx$

A) $\frac{\pi}{2}-1$ B) $\frac{\pi}{2}+1$ C) $\frac{\pi}{2}$ D) 1

3. Aniq integralni hisoblang: $\int_e^5 \frac{3}{x\ln x}dx$.

A) $3\ln(\ln 5)$ B) $5\ln(\ln 3)$ C) $3\ln(5e)$ D) $5\ln(3e)$

4. Aniq integralni hisoblang: $\int_0^{\pi/4} \sin^3 2xdx$.

A) $-\frac{1}{3}$ B) $\frac{1}{3}$ C) $-\frac{2}{3}$ D) $\frac{2}{3}$

5. Aniq integralni hisoblang: $\int_3^4 \frac{dx}{x^2-3x+2}$

A) $\ln\frac{2}{3}$ B) $\ln\frac{4}{3}$ C) $\ln\frac{3}{2}$ D) $\ln 2$

6.6. Aniq integralning tatbiqlari

6.6.1. Yassi shakllar yuzlarini hisoblash.

Yuqorida berilganidek, $[a;\,b]$ kesmada $f(x)\geq 0$ bo‘lsa aniq integral geometrik nuqtai nazardan egri chiziqli trapetsiyaning yuzini ifodalaydi. Ixtiyoriy yassi shaklni esa bir nechta egri chiziqli trapetsiyalar yuzlarining yig‘indisi yoki ayirmasi deb qarash mumkin. Bundan har qanday yassi shakllarning yuzini aniq integral yordamida hisoblash mumkinligi kelib chiqadi.

6.39-misol

Berilgan $y = x^2 - 2x$ funksiya grafigi, Ox koordinata chizig‘i va $x=-1,\ x=1$ to‘g‘ri chiziqlar bilan chegaralangan shaklning yuzini hisoblang.

►Avval berilgan chiziqlar bilan chegaralangan yassi shaklni yasaymiz (6.1-chizma). Izlanayotgan yuza $S = |S_1| + |S_2|$ yoki $S = S_1 - S_2$ dan iborat,

$$S = \int_{-1}^{0} (x^2 - 2x)dx - \int_{0}^{1} (x^2 - 2x)dx =$$

$$= \left(\frac{x^3}{3} - x^2\right)\Bigg|_{-1}^{0} - \left(\frac{x^3}{3} - x^2\right)\Bigg|_{0}^{1} = -\left(-\frac{1}{3} - 1\right) - \left(\frac{1}{3} - 1\right) = 2.$$ ◄

Umumiy holda, agar yassi shakl ikkita $y = f_1(x)$, $y = f_2(x)$ funksiyalar grafiklari va $x = a,\ x = b$ vertikal chiziqlar bilan chegaralangan bo‘lib, $f_1(x) \leq f_2(x), x \in [a;\,b]$ bo‘lsa, bu yassi figura yuzi

$$S = \int_{a}^{b} (f_2(x) - f_1(x))dx \qquad (6.16)$$

formula bilan hisoblanadi.

Agar egri chiziqli trapetsiyaning chegarasidagi egri chiziq $x = x(t),\ y = y(t)$ parametrik tenglamalar bilan berilsa, u holda bu yassi shakl yuzi

$$S = \int_{\alpha}^{\beta} y(t)x'(t)dt \qquad (6.17)$$

formula bilan hisoblanadi. Bu yerda α va β chegara $a = x(\alpha),\ b = x(\beta)$ $(y(t) \geq 0, t \in [\alpha;\,\beta])$ tenglamalardan aniqlanadi.

6.40-misol

Ellips $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$ chizig‘i bilan chegaralangan shakl yuzuni toping.

► Avval ellipsning parametrik tenglamasini yozamiz: $x = a\cos t$, $y = b\sin t$. Shaklning simmetrikligini va (6.17) formulani e'tiborga olib quyidagini hosil qilamiz:

$$S = 4\int_0^a ydx = 4\int_{\pi/2}^0 b\sin t\cdot(-a\sin t)dt = 4ab\int_0^{\pi/2}\sin^2 tdt = 2ab(t - \frac{1}{2}\sin 2t)\Big|_0^{\pi/2} = \pi ab$$

.◄

Egri chiziq qutb koordinatalar sistemasidagi $r = r(\varphi)$ tenglama bilan berilgan bo'lsin. Agar OM_1M_2 yassi shakl $r = r(\varphi)$ tenglama bilan berilgan M_1M_2 egri chiziq va φ_1, φ_2 qutb burchaklariga mos keluvchi OM_1, OM_2 qutb radiuslari bilan chegaralangan egri chiziqli sektor bo'lsa, uning yuzi

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi \qquad (6.18)$$

formula bilan hisoblanadi.

6.41-misol

Qutb koordinatasida berilgan chiziq bilan chegaralangan shakl yuzini hisoblang: $r = 4\cos 3\varphi$.

► $S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$,

Qutb radiusi $r \geq 0$, ya'ni $4\cos 3\varphi \geq 0$, $\cos 3\varphi \geq 0$ bo'ladi.

Bundan,

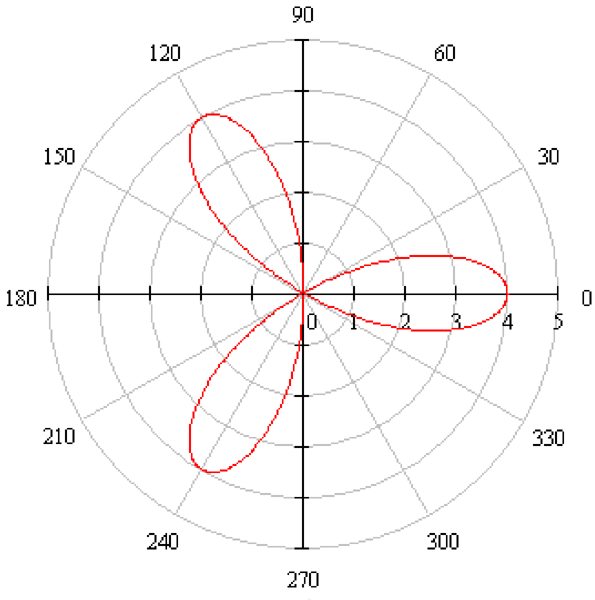
$$-\frac{\pi}{2} + 2\pi n \leq 3\varphi \leq \frac{\pi}{2} + 2\pi n, n \in Z,$$

$$-\frac{\pi}{6} + \frac{2\pi n}{3} \leq \varphi \leq \frac{\pi}{6} + \frac{2\pi n}{3}, n \in Z$$

Topilgan oraliqda $r = 4\cos\varphi$ shaklni yasaymiz(2-chizma):

$$S = 6\cdot\frac{1}{2}\int_{-\pi/6}^0 16\cos^2 3\varphi d\varphi = 24\int_{-\pi/6}^0 (1+\cos 6\varphi)d\varphi = 24(\varphi + \frac{1}{6}\sin 6\varphi)\Big|_{-\frac{\pi}{6}}^0 =$$

$$= 24(0 + 0 + \frac{\pi}{6} + \frac{1}{6}\cdot 0) = 4\pi. ◄$$



6.2-chizma

6.6.2. Egri chiziq yoyi uzunligini hisoblash.

AB egri chiziq yoyi $y = f(x)$ tenglama bilan berilgan bo‘lsin, bu yerda $f(x)$ uzluksiz differensiyalanuvchi funksiya. U holda uning $l = AB$ uzunligi quyidagi formula bilan hisoblanadi:

$$l = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx \qquad (6.19)$$

Bu yerda $A(a;\ f(a))$ va $B(b;\ f(b))$ yoy uchlari bo‘ladi.

Agar silliq egri chiziq $x = x(t),\ y = y(t)$ tenglamalar bilan berilgan bo‘lib, $x(t)$, $y(t)$ -uzluksiz differensiallanuvchi funksiyalar bo‘lsa, AB egri chiziq yoyi uzunligi l quyidagi formula bilan hisoblanadi:

$$l = \int_\alpha^\beta \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt \qquad (6.20)$$

Bu yerda α va β chegara t parametrning yoyning A va B chegaralariga mos keluvchi qiymatlaridir.

Agar silliq egri chiziq yoyi qutb koordinatalar sistemasidagi $r = r(\varphi)$ tenglama bilan berilgan bo‘lsa, u holda yoy uzunligi

$$l = \int_{\varphi_1}^{\varphi_2} \sqrt{r^2(\varphi) + \left(r'(\varphi)\right)^2}\, d\varphi \qquad (6.21)$$

formula bilan hisoblanadi, bu yerda φ_1 va φ_2 yoyning A va B chegaralariga mos qutb burchaklaridir.

6.42-misol

Ushbu $y=\frac{2}{3}\sqrt{x^3}$ egri chiziqning $x_1=3,\ x_2=8$ absissali uchlari orasidagi yoyi uzunligini toping.

►(6.4)dan foydalanamiz.

$$l=\int_3^8\sqrt{1+\left(\sqrt{x}\right)^2}dx=\int_3^8\sqrt{1+x}dx=\frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_3^8=\frac{2}{3}\left(9^{\frac{3}{2}}-4^{\frac{3}{2}}\right)=\frac{38}{3}=12\frac{2}{3}.$$ ◄

6.43-misol

Ushbu $x=a(t-\sin t),\ y=a(1-\cos t)$ sikloidaning 1-arkasi uzunligini toping.

► $x'_t=a(1-\cos t),\ y'_t=a\sin t$. Sikloidaning 1-arkasida $0\le t\le 2\pi$ ekanligidan va (6.5)dan foydalanib hisoblaymiz.

$$l=\int_0^{2\pi}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}dt=a\int_0^{2\pi}\sqrt{2-2\cos t}dt=2a\int_0^{2\pi}\sin\frac{t}{2}dt=$$

$$=-4a\cos\frac{t}{2}\Big|_0^{2\pi}=-4a(-1-1)=8a.$$ ◄

6.6.3. Jism hajmini hisoblash.

Fazoda Ox o‘qiga proyeksiyasi $[a;b]$ kesma bo‘lgan qandaydir jism berilgan bo‘lsin. $x\in[a;b]$ nuqtadan o‘tuvchi Ox o‘qiga perpendikulyar har qanday tekislikning kesm bilan kesishmasi yuzi $S(x)$ ga teng bo‘lgan shaklni hosil qiladi. U holda bu jismning hajmi quyidagi formula bilan hisoblanadi:

$$V=\int_a^b S(x)dx \qquad (6.22)$$

Xususiy holda, $y=f(x)$ funksiya gafigi bilan berilgan AB egri chiziq, Ox o‘qi va $x=a,\ x=b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyani Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jismning ko‘ndalang kesimi yuzi $S(x)=\pi f^2(x)$ bo‘ladi. Shuning uchun bu ***aylanma jismning hajmi***

$$V = \pi \int_a^b f^2(x)dx \qquad (6.23)$$

formula bilan hisoblanadi. Qisqacha, $V = \pi \int_a^b y^2 dx$.

Xuddi shu kabi, yassi shaklni Oy o‘qi atrofida aylanishidan hosil bo‘lgan jism hajmini topish uchun $V = \pi \int_c^d x^2 dy$ formula qo‘llanadi.

Eslatma. *Qutb koordinatalar sistemasining $(r;\varphi)$ o‘zgaruvchilari o‘rniga $(\rho;\varphi)$ o‘zgaruvchilarini ishlatish ham mumkin. U holda yuqoridagi (6.18) va (6.21) formulalar quyidagi ko‘rinishni oladi:*

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} \rho^2(\varphi)d\varphi \qquad va \qquad l = \sqrt{\rho^2(\varphi) + (\rho'(\varphi))^2}\,d\varphi$$

27-Auditoriya topshiriqlari

1. Berilgan egri chiziqlar bilan chegaralangan shakl yuzini toping:

$$a) \quad y^2 = x + 5, \quad y^2 = -x + 4$$

$$b) \quad y = (x-4)^2, \quad y = 16 - x^2$$

2. $x = a(t - \sin t), \quad y = a(1 - \cos 3t)$ sikloidaning 1-arkasi va Ox o‘qi bilan chegaralangan shakl yuzini hisoblang.

3. $r = a(1 - \cos\varphi)$ kardioida chizig‘i bilan chegaralangan shakl yuzini toping.

4. $y = \frac{1}{3}\sqrt{(2x-1)^3}$ tenglama bilan berilgan egri chiziqning $x_1 = 2,\ x_2 = 8$ absissali uchlari orasidagi yoyi uzunligini hisoblang.

5. $x = a\cos t,\ y = b\sin t$ ellips chizig‘ining yoy uzunligini toping.

6. $r = a(1 - \cos\varphi)$ kardioida chizig‘i yoy uzunligini hisoblang.

7. $z = \frac{x^2}{4} + \frac{y^2}{2}$, $z = 1$ sirtlar bilan chegaralangan jism hajmini toping.

8. $x = a(t - \sin t), \quad y = a(1 - \cos 3t)$ sikloidaning 1-arkasi va Ox o‘qi bilan chegaralangan shaklni absissa o‘qi atrofida aylanishidan hosil bo‘lgan jism hajmini toping.

27-Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri yuzani topish formulasi emas?

$A)\ \int_{\alpha}^{\beta} y(t)x'(t)dt$ $B)\ \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$ $C)\ \int_{\alpha}^{\beta} f^2(x)dx$ $D)\ \int_{\alpha}^{\beta} x(t)y'(t)dt$

2. Berilgan $f(x)=\sqrt{x^3}$ funksiyani $x=0$ va $x=5$ chiziqlar orasidagi yoyi uzunligini toping

$A)\ 12$ $B)\ 12\frac{5}{27}$ $C)\ 12\frac{7}{27}$ $D)\ 12\frac{11}{27}$

3. Quyidagi $r^2=4\cos 2\varphi$ chiziq bilan chegaralangan figuraning yuzini hisoblang

$A)\ 12$ $B)\ 8$ $C)\ 4$ $D)\ 16$

4. $x=4(t-\sin t),\quad y=4(1-\cos t),\quad 0\le t\le\pi$ parametrik tenglama bilan berilgan egri chiziq yoy uzunligini hisoblang

$A)\ 12$ $B)\ 8$ $C)\ 4$ $D)\ 16$

5. Aniq integral yordamida hajm hisoblash formulasi berilgan variantni aniqlang

$A)\ \int_{\alpha}^{\beta} y(t)x'(t)dt$ $B)\ \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$ $C)\ \int_{\alpha}^{\beta} f^2(x)dx$ $D)\ \int_{\alpha}^{\beta} x(t)y'(t)dt$

6.7. Birinchi va ikkinchi tur xosmas integrallar, ularni hisoblash va yaqinlashishga tekshirish

6.7.1. Chegarasi cheksiz xosmas integrallar.

6.1-Ta'rif. *Yarim $[a,+\infty)$ intervalda uzluksiz bo'lgan funksiyaning* ***xosmas integrali*** *quyidagicha belgilanadi:*

$$\int_a^{+\infty} f(x)dx$$

va ushbu tenglik bilan aniqlanadi:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx \qquad (6.24)$$

Agar (6.24) formulada o'ngda turgan limit mavjud bo'lsa, u holda xosmas integral ***yaqinlashuvchi*** deyiladi. Bu limit integralning qiymati sifatida qabul qilinadi. Agar ko'rsatilgan limit mavjud bo'lmasa, xosmas integral ***uzoqlashuvchi*** deb ataladi.

Agar integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang'ich funksiya ma'lum bo'lsa, u holda xosmas integralning yaqinlashuvchimi yoki yo'qmi ekanini aniqlash mumkin. N'yuton-Leybnis formulalari yordamida quyidagiga ega bo'lamiz:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx = \lim_{b\to+\infty} F(x)\Big|_a^b = \lim_{b\to+\infty}[F(b)-F(a)] = F(+\infty)-F(a).$$

Shunday qilib, agar $x\to+\infty$ da $F(x)$ boshlang'ich funksiya ma'lum bo'lsa (biz uni $F(+\infty)$ bilan belgiladik), u holda xosmas integral yaqinlashuvchi, agar bu limit mavjud bo'lmasa, u holda xosmas integral uzoqlashuvchi bo'ladi.

6.44-misol

► Berilgan $f(x)=e^{-kx}$ funksiya uchun $F(x)=-\frac{1}{k}e^{-kx}$ funksiya boshlang'ich funksiya bo'ladi.

N'yuton-Leybnis formulasini qo'llaymiz:

$$I=\int_0^{+\infty} e^{-kx}dx = \lim_{b\to+\infty}\left(-\frac{1}{k}e^{-kx}\Big|_0^b\right) = -\frac{1}{k}\lim_{b\to+\infty}\left(e^{-kb}-1\right).$$

Agar $k>0$ bo'lsa, $I=\frac{1}{k}$ integral yaqinlashuvchi.

Agar $k\le 0$ bo'lsa, $I=\infty$ integral uzoq lashuvchi. ◄

Xosmas integral $(-\infty, b]$ yarim cheksiz integralda ham shunga o‘xshash aniqlanadi:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx = \lim_{a\to-\infty} F(x)\Big|_{a}^{b} = F(b) - F(-\infty).$$

bu yerda $F(-\infty)$ $F(x)$ boshlang‘ich funksiyaning $x\to-\infty$ dagi limiti.

Agar $f(x)$ funksiya butun sonlar o‘qida uzluksiz bo‘lsa, u holda umumlashgan xosmas integral quyidagi formula bilan aniqlanadi:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{s} f(x)\,dx + \int_{s}^{+\infty} f(x)\,dx \qquad (6.25)$$

bu yerda $s-$ ixtiyoriy tayinlangan nuqta.

Agar (6.25) formulada o‘ng tomonda turgan ikkala integral yaqinlashuvchi bo‘lsa, u holda chap tomondagi xosmas integral ham yaqinlashuvchi bo‘ladi.

6.45-misol

Ushbu

$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}.$$

integralni yaqinlashuvchiligini tekshiring.

► (6.25) formulada $s=0$ deb faraz qilib, quyidagini hosil qilamiz:

$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{+\infty}\frac{dx}{1+x^2}.$$

Tenglikning o‘ng qismidagi xosmas integrallar yaqinlashuvchi bo‘ladi, chunki

$$\int_{-\infty}^{0}\frac{dx}{1+x^2} = arctgx\Big|_{-\infty}^{0} = arctg0 - arctg(-\infty) = \frac{\pi}{2},$$

$$\int_{0}^{+\infty}\frac{dx}{1+x^2} = arctgx\Big|_{0}^{+\infty} = arctg(+\infty) - arctg0 = \frac{\pi}{2}.$$

Shuning uchun ushbuga ega bo‘lamiz:

$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Integarl yaqinlashuvchi va uning qiymati π ga teng. ◄

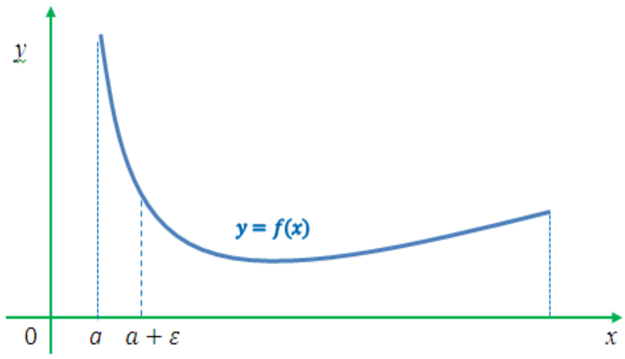
6.7.2 Cheksiz funksiyalarning xosmas integrallari.

6.2-Ta'rif. $(-\infty, b]$ *intervalda uzluksiz va* $x = a$ *da aniqlanmagan yoki uzilishga ega bo'lgan* $f(x)$ *funksiyaning (1-shakl) xosmas integrali quyidagicha belgilanadi:*

$$\int_a^b f(x)\,dx$$

va ushbu tenglik bilan aniqlanadi:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to +\infty} \int_{a+\varepsilon}^{b} f(x)\,dx \qquad (6.26)$$



6.3-chizma

Agar (6.26) formulada o'ngda turgan limit mavjud bo'lsa, u holda xosmas integral ***yaqinlashuvchi*** deyiladi.

Agar ko'rsatilgan limit mavjud bo'lmasa, u holda xosmas integral ***uzoqlashuvchi*** deyiladi.

Agar integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang'ich funksiya ma'lum bo'lsa, u holda N'yuton-Leybnis formulasini qo'llash mumkin:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} F(x)\Big|_{a+\varepsilon}^{b} = \lim_{\varepsilon \to 0} \left[F(b) - F(a+\varepsilon)\right] = F(b) - F(a)$$

Spunday qilib, agar $x \to a$ da $F(x)$ boshlang'ich funksiyaning limiti mavjud bo'lsa (biz uni $F(a)$ bilan belgiladik), u holda xosmas integral yaqinlashuvchi, agarda bu limit mavjud bo'lmasa, u holda xosmas integral uzoqlashuvchi bo'ladi.

$[a, b)$ intervalda uzluksiz va $x = b$ da aniqlanmagan yoki **II** tur uzilishga ega bo'lgan $f(x)$ funksiyaning xosmas integrali ham shunga o'xshash aniqlanadi:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx = \lim_{\varepsilon \to 0} F(x)\Big|_a^{b-\varepsilon} = \lim_{\varepsilon \to 0} \left[F(b-\varepsilon) - F(a)\right] = F(b) - F(a),$$

bu yerda $F(b) - F(x)$ boshlang'ich funksiyaning $x \to b$ dagi limiti.

Agarda $f(x)$ funksiya $[a,b]$ kesmaning biror-bir $x=s$ oraliq nuqtasida cheksiz uzilishga ega yoki aniqlanmagan bo‘lsa, u holda xosmas integral quyidagi integral bilan aniqlanadi:

$$\int_a^b f(x)dx=\int_a^s f(x)dx+\int_s^b f(x)dx \tag{6.27}$$

Agar (6.27) formulaning o‘ng tomonida turgan intervalardan aqalli bittasi uzoqlashuvchi bo‘lsa, u holda xosmas integral uzoqlashuvchi bo‘ladiyu

Agar (6.27) ning o‘ng tomonidagi ikkala integral yaqinlashuvchi bo‘lsa, u holda tenglikning chap tomonidagi xosmas integral ham yaqinlashuvchi bo‘ladi.

6.46-misol

Ushbu

$$\int_0^4 \frac{dx}{\sqrt{x}}$$

integral ning yaqinlashuvchanligini tekshiring.

► $x\to 0$ da $f(x)=\frac{1}{\sqrt{x}}\to\infty$. $x=0$ nuqta $[0,4]$ kesmaning chap oxirida yotadi. Shuning uchun quyidagiga ega bo‘lamiz:

$$\int_0^4 \frac{dx}{\sqrt{x}}=2\sqrt{x}=4-0=4.$$

Integral yaqinlashuvchi. ◄

6.7.3 Absolyut va shartli yaqinlashuvchanlik.

Ishorasini saqlamaydigan funksiyalarning xosmas integrallarini izlashni ba’zida nomanfiy funksiya bo‘lgan holga olib kelishga imkon beradigan alomatni keltiramiz.

Agar $\int_a^{+\infty}|f(x)|dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty}f(x)dx$ integral ham yaqinlashuvchi bo‘ladi.

Bunda oxirgi integral ***absolyut*** yaqinlashuvchi interval deb ataladi.

Agarda $\int_a^{+\infty}f(x)dx$ integral yaqinlashuvchi, $\int_a^{+\infty}|f(x)|dx$ integral esa uzoqlashuvchi bo‘lsa, u holda $\int_a^{+\infty}f(x)dx$ integral ***shartli*** yaqinlashuvchi integral deb ataladi.

6.47-misol

Ushbu

$$\int_0^{+\infty}\frac{\cos x}{1+x^2}dx,\qquad \int_0^{+\infty}\frac{\sin x}{1+x^2}dx.$$

integrallarning yaqinlashuvchanligini tekshiring.

►Integral ostidagi funksiyalar ushbu shartlarni qanoatlantiradi:

$$\left|\frac{\cos x}{1+x^2}\right|\le\frac{1}{1+x^2},\qquad \left|\frac{\sin x}{1+x^2}\right|\le\frac{1}{1+x^2}.$$

$$\int_0^{+\infty}\frac{dx}{1+x^2}=arctgx\Big|_0^{+\infty}=arctg(+\infty)-arctg0=\frac{\pi}{2}$$

integral yaqinlashuvchi, shuning uchun

$$\int_0^{+\infty}\left|\frac{\sin x}{1+x^2}\right|dx\qquad \int_0^{\infty}\left|\frac{\cos x}{1+x^2}\right|dx$$

integrallar ham yaqinlashuvchi bo‘ladi. ◄

28-Auditoriya topshiriqlari

Xosmas integrallarni hisoblang yoki uzoqlashuvchi ekanini aniqlang.

1. $\int_e^{\infty}\frac{dx}{x(\ln x)^5}$ (Javob: 0,25).

2. $\int_1^{\infty}\frac{dx}{\sqrt{x}}$ (Javob: Uzoqlashuvchi).

3. $\int_0^{\infty}x^5e^{-x^2}dx$ (Javob: 1).

4. $\int_{-\infty}^{\infty}\frac{2xdx}{x^2+1}$ (Javob: Uzoqlashuvchi).

5. $\int_{\sqrt{2}}^{\infty}\frac{dx}{x\sqrt{x^2-1}}$ (Javob: $\frac{\pi}{4}$).

6. $\int_0^{1}\frac{dx}{x^3}$ (Javob: Uzoqlashuvchi).

7. $\int_0^{1/e}\frac{dx}{x(\ln x)^2}$ (Javob: 1).

8. $\int_0^{2}\frac{dx}{x^2-4x+3}$ (Javob: Uzoqlashuvchi).

Xosmas integrallarni yaqinlashishga tekshiring.

9. $\int_{e^2}^{\infty}\frac{dx}{x\ln\ln x}$ (Javob: Uzoqlashuvchi).

10. $\int_{0}^{\pi/2}\frac{1-\cos x}{x^k}$ (Javob: $k\le 2$ da yaqinlashuvchi, $k>2$ da uzoqlashuvchi).

28-Mustaqil yechish uchun testlar

1. Quyidagi $\int_{1}^{\infty}\frac{2}{x^2}dx$ xosmas integralni hisoblang

$A)\ 2$ $B)\ 1$ $C)\ 0$ $D)\ 3$

2. Quyidagi $\int_{0}^{1}\frac{dx}{\sqrt[3]{x^2}}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

$A)\ 1$ $B)\ 2$ $C)\ 3$ $D)$ *uzoqlashuvchi*

3. Quyidagi $\int_{1}^{\infty}\frac{dx}{\sqrt[3]{x^2}}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

$A)\ 1$ $B)\ 2$ $C)\ 3$ $D)$ *uzoqlashuvchi*

4. Quyidagi $\int_{0}^{2}\frac{dx}{x^2-3x+2}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

$A)\ \ln 2$ $B)\ -\ln 4$ $C)\ -\ln 2$ $D)$ *uzoqlashuvchi*

5. Quyidagi $\int_{0}^{4}\frac{dx}{\sqrt{4-x}}$ xosmas integralni hisoblang yoki uzoqlashuvchi ekanini aniqlang

$A)\ 1$ $B)\ 2$ $C)\ 3$ $D)$ *uzoqlashuvchi*

9-Shaxsiy topshiriqlar

1-topshiriq. Aniq integrallarni hisoblang.

1.1 a) $\int_{1}^{\sqrt{2}}\frac{\sqrt{4-x^2}}{x^2}dx$ b) $\int_{0}^{\pi/4}\frac{xdx}{\cos^2 x}$

1.2 a) $\int_{1}^{2}\frac{\sqrt{x^2-1}}{x}dx$ b) $\int_{0}^{2}arctg\sqrt{x+1}dx$

1.3 a) $\int_{0}^{\pi/4}\frac{dx}{\cos^2 x+3\sin^2 x}$ b) $\int_{1}^{3}\ln(3x+2)dx$

1.4 a) $\int_{0}^{\ln 3}\frac{dx}{e^x(1+3e^{-2x})}$ b) $\int_{0}^{1}\frac{\arcsin x}{\sqrt{x+1}}dx$

1.5 a) $\int_{0}^{\pi/2}\frac{dx}{\cos x-3\sin x}$ b) $\int_{1}^{2}x^2\ln xdx$

1.6 a) $\int_{e^2}^{e^3}\frac{\ln xdx}{x(1+\ln^2 x)}$ b) $\int_{0}^{1}\frac{\arccos x}{\sqrt{x+1}}dx$

1.7 a) $\int_{0}^{4}x^2\sqrt{16-x^2}dx$ b) $\int_{0}^{\pi/4}xtg^2 xdx$

1.8 a) $\int_{\pi/6}^{\pi/2}\frac{3\cos^3 x}{\sin^4 x}dx$ b) $\int_{0}^{1}\left(x^2+x\right)e^x dx$

1.9 a) $\int_{0}^{\pi/2}\sin^4 x\cos^5 xdx$ b) $\int_{1}^{2}(x-1)\ln xdx$

1.10 a) $\int_{0}^{\sqrt{5}}\sqrt{5-x^2}dx$ b) $\int_{1}^{\sqrt{3}}arctg\frac{1}{x}dx$

1.11 a) $\int_{-1}^{1}\frac{xdx}{\sqrt{5x-1}}$ b) $\int_{0}^{\pi/2}x\sin x\cos xdx$

1.12 a) $\int_{1/\sqrt{3}}^{1}\frac{dx}{x^2\sqrt{1+x^2}}$ b) $\int_{0}^{\pi/4}x\sin^2 xdx$

1.13 a) $\int_{4}^{9}\frac{\sqrt{x}dx}{\sqrt{x}-1}$ b) $\int_{0}^{\pi}x^2\sin xdx$

1.14 a) $\int_{0}^{4}\frac{dx}{1+\sqrt{2x+1}}$ b) $\int_{-2}^{0}x^2e^{-\frac{x}{2}}dx$

1.15 a) $\int_{\ln 2}^{\ln\sqrt{2}}\frac{dx}{e^x\sqrt{1-e^{-2x}}}$ b) $\int_{2/3}^{1}arctg(3x-2)dx$

1.16 a) $\int_0^{\pi/3} \frac{dx}{2+\cos x}$ b) $\int_1^{e^2} \sqrt{x} \ln x dx$

1.17 a) $\int_3^6 \frac{\sqrt{x^2-9}}{x^4} dx$ b) $\int_0^1 \frac{\ln(x+2)}{(x+2)^2} dx$

1.18 a) $\int_2^4 \frac{dx}{x\sqrt{x-1}}$ b) $\int_0^1 \frac{\arcsin(x/2)}{\sqrt{2-x}} dx$

1.19 a) $\int_{-2}^2 x^2\sqrt{4-x^2} dx$ b) $\int_0^{\pi/6} \frac{xdx}{\cos^2 2x}$

1.20 a) $\int_1^{13} \frac{x+1}{\sqrt[3]{2x+1}} dx$ b) $\int_{1/2}^1 arc\sin(1-x)dx$

1.21 a) $\int_0^5 \frac{dx}{2x+\sqrt{3x+1}}$ b) $\int_{\pi/6}^{\pi/4} xctg^2 xdx$

1.22 a) $\int_3^6 \frac{\sqrt{x^2-9}}{x^4} dx$ b) $\int_1^e (x+3)\ln^2 xdx$

1.23 a) $\int_1^2 \frac{\sqrt{x^2-1}}{x} dx$ b) $\int_1^e x\ln^2 xdx$

1.24 a) $\int_0^{\sqrt{6}} \sqrt{6-x^2} dx$ b) $\int_1^3 \ln(2x+3)dx$

1.25

2-topshiriq. Quyidagi tenglama bilan berilgan chiziqlar bilan chegaralangan figura yuzini hisoblang.

2.1. $r = 4\sin^2\varphi$

2.2. $x = 3(\cos t + t\sin t),\ y = 3(\sin t - t\cos t), y = 0\ (0 \le t \le \pi)$

2.3. $r = 3\sin 4\varphi$

2.4. $r = 4\cos 3\varphi,\ r = 2\ (r \ge 2)$

2.5. $x = 4(t - \sin t),\ y = 4(1-\cos t)$

2.6. $y = x+1, y = \cos x,\ y = 0$

2.7. $r = 6\sin 3\varphi,\ r = 3\ (r \ge 3)$

2.8. $r = \cos\varphi + \sin\varphi$

2.9. $r = 1/2 + \sin\varphi$

2.10. $r = 6\cos 3\varphi,\ r = 3\ (r \ge 3)$

2.11. $r = \cos 3\varphi$

2.12. $r = \sin\varphi,\ r = 2\sin\varphi$

2.13. $r = 1/2 + \cos\varphi$

2.14. $x = 7\cos^3 t,\ y = 7\sin^3 t$

2.15. $y^2 = x^3, x = 4, y = 0$

2.16. $r = \cos\varphi - \sin\varphi$

2.17. $r = 1 + \sqrt{2}\sin\varphi$

2.18. $r = 5(1 - \cos\varphi)$

2.19. $r = \sqrt{3}\cos\varphi,\ r = \sin\varphi, 0 \le \varphi \le \pi/2$

2.20. $x = 5\cos^3 t,\ y = 5\sin^3 t$

2.21. $r = 3(1 - \cos\varphi)$

2.22. $r = 6\cos 3\varphi,\ r = 3\ (r \le 3)$

2.23. $r = \sin 3\varphi$

2.24. $r = 2(1 - \cos\varphi), r = 2(r \ge 2)$

2.25. $r = 5(1 + \cos\varphi)$

3-topshiriq. Quyidagi tenglamalar orqali berilgan chiziqning yoyi uzunligini hisoblang.

3.1. $r = 5(1 + \cos\varphi)$

3.2. $\begin{cases} x = 3(2\cos t - \cos 2t), \\ y = 3(2\sin t - \sin 2t), \end{cases}$

$0 \le t \le 2\pi.$

3.3. $y = 1 + \ln(\cos x),\ (0 \le x \le \pi/6)$

3.4. $\begin{cases} x = 4(\cos t + t\sin t), \\ y = 4(\sin t - t\cos t), \end{cases}$

$0 \le t \le 2\pi.$

3.5. $\begin{cases} x = (t^2 - 2)\sin t + 2t\cos t, \\ y = (2 - t^2)\cos t + 2t\sin t, \end{cases}$

$0 \le t \le \pi.$

3.6. $\begin{cases} x = 2\sin^3 t \\ y = 2\cos^3 t \end{cases} \quad 0 \le t \le \pi$

3.7. $\begin{cases} x = e^t(\cos t + \sin t), \\ y = e^t(\cos t - \sin t), \end{cases}$

$0 \le t \le \pi.$

3.8. $\begin{cases} x = 4\sin^3 t \\ y = 4\cos^3 t \end{cases}$

3.9. $\begin{cases} x = 3{,}5\left(2\cos t - \cos 2t\right), \\ y = 3{,}5\left(2\sin t - \sin 2t\right), \end{cases}$

$0 \le t \le \pi/2.$

3.10. $\begin{cases} x = 6\cos^3 t, \\ y = 6\sin^3 t, \end{cases}$

$0 \le t \le \pi/3.$

3.11. $\begin{cases} x = \left(t^2 - 2\right)\sin t + 2t\cos t, \\ y = \left(2 - t^2\right)\cos t + 2t\sin t, \end{cases}$

$0 \le t \le \pi/3.$

3.12. $\begin{cases} x = 8\left(\cos t + t\sin t\right), \\ y = 8\left(\sin t - t\cos t\right), \end{cases}$

$0 \le t \le \pi/4.$

3.13. $r = 2\sin^3(\varphi/3),\ (0 \le \varphi \le \pi/6)$

3.14. $\begin{cases} x = 3\left(\cos t + t\sin t\right), \\ y = 3\left(\sin t - t\cos t\right), \end{cases}$

$0 \le t \le \pi/3.$

3.15. $\begin{cases} x = 3\left(t - \sin t\right), \\ y = 3\left(1 - \cos t\right), \end{cases}$

$\pi \le t \le 2\pi.$

3.16. $\begin{cases} x = \mathrm{e}^t\left(\cos t + \sin t\right), \\ y = \mathrm{e}^t\left(\cos t - \sin t\right), \end{cases}$

$\pi/2 \le t \le \pi.$

3.17. $\begin{cases} x = 2{,}5\left(t - \sin t\right), \\ y = 2{,}5\left(1 - \cos t\right), \end{cases}$

$\pi/2 \le t \le \pi.$

3.18. $\begin{cases} x = 3{,}5\left(2\cos t - \cos 2t\right), \\ y = 3{,}5\left(2\sin t - \sin 2t\right), \end{cases}$

$0 \le t \le \pi/2.$

3.19. $\begin{cases} x = 6(\cos t + t\sin t), \\ y = 6(\sin t - t\cos t), \end{cases}$ $0 \le t \le \pi.$

3.20. $\begin{cases} x = (t^2 - 2)\sin t + 2t\cos t, \\ y = (2 - t^2)\cos t + 2t\sin t, \end{cases}$ $0 \le t \le \pi/2.$

3.21. $\begin{cases} x = 8\cos^3 t, \\ y = 8\sin^3 t, \end{cases}$ $0 \le t \le \pi/6.$

3.22. $\begin{cases} x = (t^2 - 2)\sin t + 2t\cos t, \\ y = (2 - t^2)\cos t + 2t\sin t, \end{cases}$ $0 \le t \le 2\pi.$

3.23. $\begin{cases} x = 4(t - \sin t), \\ y = 4(1 - \cos t), \end{cases}$ $\pi/2 \le t \le 2\pi/3.$

3.24. $y^2 = x^3$ ning $x = 4$ bilan kesilgan qismi.

3.25. $\begin{cases} x = 5\sin^3 t \\ y = 5\cos^3 t \end{cases}$ $0 \le t \le \pi$

4-topshiriq. Quyidagi chiziqlar bilan chegaralangan figuraning Ox o‘qi (1-12 variantlar uchun), Oy o‘qi (13-25 variantlar uchun) atrofida aylanishidan hosil bo‘lgan jism hajmini toping.

4.1. $y^2 = x^3, x = 0, y = 4$

4.2. $x = 4(t - \sin t),\ y = 4(1 - \cos t), y = 0$

4.3. $y = 5\cos x, y = \cos x, x = 0, x \ge 0$

4.4. $y = 2x - x^2, y = 4x - 2x^2$

4.5. $y = \sin^2 x, x = \pi/2, y = 0$

4.6. $x = \sqrt[3]{y - 2}\ x = 1, y = 1$

4.7. $y = xe^x, y = 0,\ x = 1$

4.8. $y = 2x - x^2, y = -x + 2, x = 0$

4.9. $y = 3\sin x, y = \sin x, 0 \le x \le \pi$

4.10. $x = 3\cos^2 t,\ y = 2\sin^2 t$

4.11. $y = \arccos x,\ y = \arcsin x, x = 0$

4.12. $(y-1)^2 = x,\ y = x-1$

4.13. $x = 2\cos^3 t,\ y = 2\sin^3 t$

4.14. $y = \arccos x,\ y = \arcsin x, y = 0$

4.15. $y = (x-1)^2,\ y = 1$ $y=(x-1)^2,\quad y=1.$

4.16. $y^2 = x-2,\ y = x^3,\ y = 0, y = 1$

4.17. $y = x^3, y = x^2$

4.18. $y = \arccos(x/5),\ y = \arcsin(x/3), y = 0$

4.19. $(y-1)^2 = x,\ y = x-1$

4.20. $y = (x-2)^2,\ y = 4-x$

4.21. $y = \arccos x,\ y = \arcsin x, x = 0$

4.22. $y = (x-1)^2,\ x = 0,\ x = 3,\ y = 0$

4.23. $x = 2\cos^2 t,\ y = 5\sin^2 t$

4.24. $y = x^3, y = x$

4.25. $y = (x-1)^2,\ x = 3,\ y = 0$

5-topshiriq. Xosmas integrallarni hisoblang yoki uzoqlashuvchi ekanini isbotlang.

5.1.a) $\int_0^\infty \frac{xdx}{16x^2+1}$ b) $\int_0^1 \frac{dx}{\sqrt{2-4x}}$

5.2. $\int_1^\infty \frac{16xdx}{16x^4-1}$ b) $\int_{-1}^3 \frac{2x-3}{\sqrt[3]{x^2}}dx$

5.3.a) $\int_0^\infty \frac{x^3dx}{\sqrt{16x^4+1}}$ b) $\int_0^1 \frac{e^{1/x}}{x^2}dx$

5.4.a) $\int_1^\infty \frac{xdx}{\sqrt{16x^4-1}}$ b) $\int_1^3 \frac{dx}{\sqrt[3]{2-4x}}$

5.5.a) $\int_{-\infty}^0 \frac{xdx}{\sqrt{(x^2+1)^3}}$ b) $\int_{1/2}^2 \frac{\ln(2x-1)}{2x-1}dx$

5.6.a) $\int_0^\infty \frac{x^2dx}{\sqrt[3]{(x^3+1)^4}}$ b) $\int_{1/4}^1 \frac{dx}{20x^2-9x+1}$

5.7.a) $\int_0^\infty \frac{xdx}{\sqrt[4]{(x^2+16)^5}}$ b) $\int_{1/2}^1 \frac{dx}{(1-x)\ln^2(1-x)}$

5.8.a) $\int_4^\infty \frac{xdx}{\sqrt{x^2-4x+1}}$ b) $\int_0^{2/3} \frac{\sqrt[3]{\ln(2-3x)}}{2-3x}dx$

5.9. a) $\int_{1}^{\infty}\frac{xdx}{x^2-4x+5}$ b) $\int_{0}^{1}\frac{xdx}{1-x^4}$

5.10. a) $\int_{-1}^{\infty}\frac{dx}{x^2+4x+5}$ b) $\int_{0}^{\pi/6}\frac{\cos 3x}{\sqrt[3]{(1-\sin 3x)^2}}dx$

5.11. a) $\int_{1}^{\infty}\frac{arctg2x}{4x^2+5}dx$ b) $\int_{0}^{1}\frac{2xdx}{\sqrt{1-x^4}}$

5.12. a) $\int_{0}^{\infty}\frac{dx}{4x^2+4x+5}$ b) $\int_{0}^{1/3}\frac{\cos xdx}{\sqrt[7]{\sin^2 x}}$

5.13. a) $\int_{0}^{\infty}\frac{xdx}{9x^2+6x+5}$ b) $\int_{4/5}^{1}\frac{5dx}{\sqrt[3]{4-5x}}$

5.14. a) $\int_{0}^{\infty}\frac{(x+3)dx}{\sqrt[3]{x^2+6x+5}}$ b) $\int_{0}^{\pi/2}\frac{e^{tgx}dx}{\cos^2 x}$

5.15. a) $\int_{0}^{\infty}\frac{3-x^2}{x^2+4}dx$ b) $\int_{0}^{1}\frac{e^{\pi-\arcsin x}dx}{\pi\sqrt{1-x^2}}$

5.16. a) $\int_{0}^{\infty}\frac{3+\sqrt{arctg3x}dx}{9x^2+1}$ b) $\int_{1}^{2}\frac{3dx}{\sqrt[3]{4x-x^2-4}}$

5.17. a) $\int_{1}^{\infty}\frac{4dx}{x(1+\ln^2 x)}$ b) $\int_{\pi/2}^{\pi}\frac{\sin 2xdx}{\sqrt[3]{1-\cos^2 x}}$

5.18. a) $\int_{0}^{\infty}x\sin xdx$ b) $\int_{0}^{1/3}\frac{5dx}{\sqrt[3]{1-3x}}$

5.19. a) $\int_{-\infty}^{-1}\frac{5dx}{(x^2-4x)\ln 3}$ b) $\int_{1}^{3}\frac{xdx}{\sqrt[3]{(x^2-1)^4}}$

5.20. a) $\int_{0}^{\infty}\frac{\pi dx}{(1+4x^2)arctg^2 2x}$ b) $\int_{0}^{1/3}\frac{dx}{9x^2-9x+2}$

5.21. a) $\int_{1}^{\infty}\frac{(2x+1)dx}{\sqrt{4x^2+4x+5}}$ b) $\int_{0}^{\pi/2}\frac{3\sin^3 xdx}{\sqrt{\cos x}}$

5.22. a) $\int_{0}^{\infty}\frac{dx}{(x^2+2x)\ln 5}$ b) $\int_{0}^{1}\frac{x^4dx}{\sqrt[3]{1-x^5}}$

5.23. a) $\int_{0}^{\infty}2xe^{-3x}dx$ b) $\int_{0}^{\sqrt{5}}\frac{\sqrt[3]{5}xdx}{\sqrt[3]{5-x^2}}$

5.24. a) $\int_{-\infty}^{0}3xe^{2x}dx$ b) $\int_{0}^{3}\frac{2x^2dx}{\sqrt{9-x^6}}$

5.25. a) $\int_{0}^{\infty}x^3e^{-x^2}dx$ b) $\int_{0}^{1/5}\frac{3xdx}{\sqrt[5]{1-25x^2}}$

VII BOB. KO'P O'ZGARUVCHILI FUNKSIYALARNING DIFFERENSIAL HISOBI

7.1. Ko'p o'zgaruvchili funksiyalar. Xususiy hosilalar va to'la differensial

Ko'p o'zgaruvchili funksiyalar. R^2 fazoda biror D to'plamning bir-biriga bog'liq bo'lmagan ixtiyoriy (x,y) haqiqiy sonlari juftligiga biror qoidaga ko'ra E to'plamdagi yagona z haqiqiy son mos qo'yilgan bo'lsa, D to'plamda ***ikki*** x ***va*** y ***o'zgaruvchili*** z ***funksiya aniqlangan*** deyiladi. Ikki o'zgaruvchili funksiya simvolik tarzda quyidagicha belgilanadi: $z = f(x,y)$, $z = z(x,y)$ va h.k. Bunda x, y erkli o'zgaruvchilar yoki argumentlar, z esa erksiz o'zgaruvchi yoki ***funksiya*** deb ataladi.

D to'plam ***funksiyaning aniqlanish*** sohasi, E to'plam ***o'zgarish sohasi*** yoki ***qiymatlar to'plami*** deyiladi. Har bir juft haqiqiy songa Oxy koordinatalar sistemasida bitta M nuqta va bitta nuqtaga bir juft haqiqiy son mos kelganligi uchun ikki o'zgaruvchili funksiyani M nuqtaning funksiyasi deb ham qaraladi, hamda $z = f(x,y)$ o'rniga $z = f(M)$ deb ham yozish mumkin. Funksiyaning D aniqlanish sohasi sohaga tegishli yoki tegishli bo'lmagan chiziqlar bilan chegaralangan Oxy tekislikning bir qismini ifodalaydi. Birinchi holda D soha yopiq soha deyiladi va $\overline{D}$ kabi belgilanadi, ikkinchi holda esa ochiq soha deyiladi. Funksiya butun Oxy tekislikda aniqlangan bo'lisi ham mumkin. Ikki o'zgaruvchili funksiya berilish usullari ham, bir o'zgaruvchili funksiyaga o'xshash bo'ladi. Ko'proq funksiyaning ***analitik usulda*** berilishini qaraymiz. Masalan,

1) $z = x^2 + y^2$ funksiya analitik usulda berilgan bo'lib, Oxy tekislikning hamma nuqtalari uchun aniqlangan, o'zgarish sohasi $[0,+\infty)$ dan iborat bo'ladi;

2) $z = \sqrt{4 - x^2 - y^2}$ funksiya aniqlangan bo'lishi uchun $4 - x^2 - y^2 \geq 0$ yoki $x^2 + y^2 \leq 4$ bo'lishi kerak, bunday nuqtalar to'plami markazi koordinatlar boshida radiusi 2 ga teng bo'lgan doiradan iborat, qiymatlar to'plami $[0,2)$ bo'ladi.

Ikki o'zgaruvchili funksiyaning geometrik tasviri fazoda tenglamasi $z = f(x,y)$ bo'lgan sirtni ifodalaydi. Masalan: 1) $z = 2x + 3y - 6$ ikki o'zgaruvchili funksiya fazoda $2x + 3y - z - 6 = 0$ tekislikni tasvirlaydi. 2) $x^2 + y^2 + z^2 = R^2$ sfera tenglamasi bo'lib, $z = \pm\sqrt{R^2 - x^2 - y^2}$ ikki o'zgaruvchili funksiyalar grafiklari fazoda sferani ifodalaydi.

D to‘plamning har bir (x, y, z) haqiqiy sonlar uchligiga biror qoida bo‘yicha E to‘plamdagi yagona u haqiqiy son mos qo‘yilgan bo‘lsa, D to‘plamda uch o‘zgaruvchining funksiyasi aniqlangan deyiladi.

Bunda x, y, z erkli o‘zgaruvchilar yoki argumentlar, u esa erksiz o‘zgaruvchi yoki funksiya deb ataladi. Uch o‘zgaruvchining funksiyasi $u = f(x, y, z)$ kabi belgilanadi.

Uch o‘zgaruvchili funksiya aniqlanish sohasi R^3 fazoning biror nuqtalar to‘plami yoki butun fazo bo‘lishi mumkin. Masalan: $u = \sqrt{25 - x^2 - y^2 - z^2}$ funksiya $25 - x^2 - y^2 - z^2 \geq 0$ yoki $x^2 + y^2 + z^2 \leq 25$ shartda aniqlanganligi uchun $x^2 + y^2 + z^2 = 25$ sfera va uning ichida aniqlangan.

To‘rt o‘zgaruvchili va umuman n o‘zgaruvchili funksiyaga ham yuqoridagidek ta’rif berish mumkin. Bunday funksiyalar mos ravishda $u = f(x, y, z, t)$ yoki $y = f(x_1, x_2, x_3, x_4)$, $y = f(x_1, x_2, ..., x_n)$ kabi belgilanadi.

To‘rt va undan ortiq o‘zgaruvchiga bog‘liq funksiyalarning aniqlanish sohasini chizmalarda ko‘rgazmali namoyish etish mumkin emas. Ammo, uni tasvirlash mumkin bo‘lmasa y yo‘q deyish mumkin emas. Masalan, to‘rtinchi o‘zgaruvchi fazodagi temperatura, beshinchisi zichlik va h.k bo‘lishi mumkin.

Ikki o‘zgaruvchili funksiya limiti, **uzluksizligi va uzilishi.** Ikki o‘zgaruvchili funksiyaning limiti tushunchasini berishdan oldin, berilgan nuqtaning δ- atrofi tushunchasini kiritamiz. $P_0(x_0, y_0)$ ***nuqtaning*** δ ***atrofi*** deb koordinatalari quyidagi shartni qanoatlantiruvchi $P(x, y)$ nuqtalar to‘plamiga aytiladi: $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ yoki qisqacha $\rho(P, P_0) < \delta$, bu yerda $\rho(P, P_0)$ belgi P va P_0 nuqtalar orasidagi masofani bildiradi.

Demak, P_0 nuqtaning δ atrofi deganda P_0 markazli δ radiusli doiraning ichida yotuvchi barcha P nuqtalar tushuniladi. $n(n \geq 3)$ o‘lchovli fazoda $P_0(x_1, x_2, ..., x_n)$ nuqtaning δ atrofi ham shunga o‘xshash aniqlanadi.

Ikki o‘zgaruvchili $z = f(x, y) = f(P)$ funksiya P_0 nuqtaning biror atrofida aniqlangan bo‘lsa (P_0 nuqtada aniqlanmagan bo‘lishi mumkin) va ixtiyoriy $\varepsilon > 0$ uchun shunday $\delta > 0$ topilsaki, $\rho(P, P_0) = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ tengsizlikni qanoatlantiruvchi barcha $P(x, y)$ nuqtalar uchun

$$|f(x, y) - A| < \varepsilon \quad \text{yoki} \quad |f(P) - A| < \varepsilon$$

tengsizlik bajarilsa, A o‘zgarmas son $z = f(x, y)$ ***funksiyaning*** P_0 ***nuqta*** $(P \to P_0)$***dagi limiti*** deyiladi, va

$$\lim_{P \to P_0} f(P) = A \text{ yoki } \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A$$

kabi belgilanadi.

7.1-misol.

$\lim\limits_{\substack{x \to 2 \\ y \to 0}} \frac{\sin xy}{y}$ limitni hisoblang.

► $P_0\ (2;0)$ nuqtada $\frac{\sin xy}{y}$ funksiya aniqlanmagan. Limitning xossalaridan

$\lim\limits_{\substack{x \to 2 \\ y \to 0}} \frac{\sin xy}{y} = \lim\limits_{\substack{x \to 2 \\ y \to 0}} \left(x \cdot \frac{\sin xy}{xy} \right) = \lim\limits_{\substack{x \to 2 \\ y \to 0}} x \cdot \lim\limits_{\substack{x \to 2 \\ y \to 0}} \frac{\sin xy}{xy} = 2 \cdot 1 = 2$, chunki $\lim\limits_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$.◄

$z = f(x, y)\ (z = f(P))$ funksiya $P_0(x_0, y_0)$ nuqtada hamda uning biror atrofida aniqlangan va

$$\lim_{P \to P_0} f(P) = f(P_0) \text{ yoki } \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0)$$

bo‘lsa, ya’ni funksiyaning $P_0(x_0, y_0)$ nuqtadagi limiti funksiyaning shu nuqtadagi qiymatiga teng bo‘lsa, ***funksiya*** $P_0(x_0, y_0)$ ***nuqtada uzluksiz*** deyiladi. Uzluksizlik shartlari bajarilmagan nuqtalari funksiyaning ***uzilish nuqtalari*** deyiladi.

$z = f(x, y)$ funksiya biror D sohaning har bir nuqtasida uzluksiz bo‘lsa, u holda funksiya D sohada uzluksiz deyiladi.

7.2-misol.

Ushbu $z = \frac{1}{x^2 - y^2}$ funksiyaning uzilish nuqtalarini toping.

►Funksiya koordinatalari $x^2 - y^2 = 0$ tenglamani qanoatlantiruvchi nuqtalarda uzilishga ega. Bu $y = x$ va $y = -x$ to‘g‘ri chiziqlar bo‘lib, bu to‘g‘ri chiziqlarga tegishli har bir nuqtada funksiya uzilishga ega bo‘ladi.◄

Ikki o‘zgaruvchili uzluksiz funksiya ham bir o‘zgaruvchili uzluksiz funksiya ega bo‘lgan asosiy xossalarga ega bo‘ladi. (Bu xossalarni takrorlash o‘quvchiga tavsiya etiladi).

Ikki o‘zgaruvchili funksiya xususiy, to‘liq orttirmalari va xususiy hosilalari. $z = f(x, y)$ funksiyada x o‘zgaruvchiga biror Δx orttirma berib, y ni o‘zgarishsiz qoldirsak, funksiya $\Delta_x z$ orttirma olib, bu orttirmaga z funksiyaning x ***o‘zgaruvchi bo‘yicha xususiy orttirmasi*** deyiladi va quyidagicha yoziladi:

$$\Delta_x z = f(x + \Delta x, y) - f(x, y).$$

Xuddi shunday, y o'zgaruvchiga Δy orttirma berib x o'zgarishsiz qolsa, unga z funksiyaning y ***o'zgaruvchi bo'yicha xususiy orttirmasi*** deyiladi va quyidagicha yoziladi:

$$\Delta_y z = f(x, y+\Delta y) - f(x,y).$$

x va y o'zgaruvchilar mos ravishda Δx va Δy orttirmalar olsa, $z = f(x,y)$ funksiya $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$ to'liq orttirma oladi

Agar $\lim\limits_{\Delta x\to 0}\dfrac{\Delta_x z}{\Delta x}$ chekli limit mavjud bo'lsa, u holda bu limitga $z = f(x,y)$ funksiyaning x ***o'zgaruvchi bo'yicha xususiy hosilasi*** deyiladi va $\dfrac{\partial z}{\partial x}$ yoki $z'_x = f'_x(x,y)$ bilan belgilanadi. Agar $\lim\limits_{\Delta y\to 0}\dfrac{\Delta_y z}{\Delta y}$ chekli limit mavjud bo'lsa, u holda bu limitga $z = f(x,y)$ funksiyaning y ***o'zgaruvchi bo'yicha xususiy hosilasi*** deyiladi va $\dfrac{\partial z}{\partial y}$ yoki $z'_y = f'_y(x,y)$ bilan belgilanadi.

Xususiy hosilalar ta'riflaridan ko'rinadiki bir argumentli funksiyani differensiallashning hamma qoida va formulalari o'z kuchida qoladi.

Istalgan chekli sondagi o'zgaruvchili funksiyaning xususiy hosilalari ham yuqoridagidek aniqlanadi.

7.3-misol.

Ushbu $z = x^2 + 2xy + 3y^2$ funksiyaning birinchi tartibli xususiy hosilalarini toping.

►Oldin y ni o'zgarmas deb z'_x ni topamiz:

$$z'_x = (x^2 + 2xy + 3y^2)'_x = (x^2)'_x + (2xy)'_x + (3y^2)'_x = 2x + 2y,$$

endi x ni o'zgarmas deb z'_y ni topamiz:

$$z'_y = (x^2 + 2xy + 3y^2)'_y = (x^2)'_y + (2xy)'_y + (3y^2)'_y = 2x + 6y. ◄$$

7.4-misol.

Ushbu $z = x^{x^y}$ funksiyaning xususiy hosilalarini toping.

► $z = x^{x^y} = e^{x^y \ln x}$ bo'lgani uchun

$$z'_x = e^{x^y \ln x}\left(yx^{y-1}\ln x + x^{y-1}\right) = x^{x^y + y - 1}(y\ln x + 1),$$

$$z'_y = e^{x^y \ln x} x^y \ln^2 x = x^{x^y + y}\ln^2 x. ◄$$

29-Auditoriya topshiriqlari

1. Quyidagi funksiyalarning aniqlanish sohasini toping va uning qandayligini izohlang.

1) $z=\sqrt{1-x^2-9y^2}$; 2) $z=\dfrac{1}{\sqrt{x+y}}+\sqrt{y-x}$;

3) $z=\ln(x+y^2-2)$; 4) $u=\sqrt{4-x^2+y^2-z^2}$;

5) $z=\sqrt{x-\sqrt{y}}$; 6) $z=\arcsin\dfrac{x}{y-1}$.

2. Quyidagi limitlarni hisoblang.

1) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{2-\sqrt{xy+4.}}{xy}$; 2) $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$.

3. Quyidagi funksiyalarning uzilish nuqtalarini toping.

1) $z=\dfrac{6}{x^2-4y^2-4}$; 2) $z=\dfrac{y^2+2x}{y^2-2x}$; 3) $z=\dfrac{1}{\sin^2\pi x+\sin^2\pi y}$.

4. Quyidagi funksiyalarni $x\to 0,\ y\to 0$ da uzluksizlikka tekshiring.

1) $z=\dfrac{x^2y^2}{x^2+y^2},\ f(0,0)=0$; 2) $z=\dfrac{xy}{x^2+y^2},\ f(0,0)=0$;

3) $z=\dfrac{x^2y^2}{x^4+y^4},\ f(0,0)=0$; 4) $z=\dfrac{x^4-y^4}{x^4+y^4},\ f(0,0)=0$.

5. Quyidagi funksiyalarning xususiy hosilalarini toping:

1) $z=x^3+3x^2y-y^3$; 2) $z=\ln\left(x+\sqrt{x^2+y^2}\right)$; 3) $u=\dfrac{y}{x}+\dfrac{z}{y}-\dfrac{x}{z}$;

4) $z=\sqrt{1-\left(\dfrac{x+y}{xy}\right)^2}+\arcsin\dfrac{x+y}{y}$.

6. $z=\dfrac{y}{\cos(x^2-y^2)}$ bo‘lsa, $\dfrac{1}{x}\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\dfrac{\partial z}{\partial y}=\dfrac{z}{y^2}$ tenglik bajarilishini tekshiring.

29-Mustaqil yechish uchun testlar

1. Ikki o‘zgaruvchili funksiyaning xususiy orttirmasi to‘g‘ri berilgan variantni aniqlang.

A) $f(x+\Delta x,\ y)-f(x,\ y+\Delta y)$

B) $f(x+\Delta x,\ y+\Delta y)-f(x,\ y)$

D) $f(x,\ y+\Delta y)-f(x,\ y)$

E) $f(x,\ y+\Delta y)-f(x+\Delta x,\ y)$

2. Ikki o‘zgaruvchili funksiyaning to‘liq orttirmasi to‘g‘ri berilgan variantni aniqlang.

A) $f(x+\Delta x,\ y)-f(x,\ y+\Delta y)$

B) $f(x+\Delta x,\ y+\Delta y)-f(x,\ y)$

D) $f(x,\ y+\Delta y)-f(x,\ y)$

E) $f(x,\ y+\Delta y)-f(x+\Delta x,\ y)$

3. Quyidagi funksiyalardan qaysi biri $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$ tenglikni qanoatlantiradi?

A) $z=tg(x^2+y^2)$ B) $z=ye^{y^2-x^2}$

D) $z=\frac{2x-y}{x+3y}$ E) To‘g‘ri javob yo‘q.

4. Quyidagi funksiyalardan qaysi biri $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0$ tenglikni qanoatlantiradi?

A) $z=tg(x^2+y^2)$ B) $z=ye^{y^2-x^2}$

D) $z=\frac{2x-y}{x+3y}$ E) To‘g‘ri javob yo‘q.

5. Quyidagi funksiyalardan qaysi biri $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$ tenglikni qanoatlantiradi?

A) $z=tg(x^2+y^2)$ B) $z=ye^{y^2-x^2}$

D) $z=\frac{2x-y}{x+3y}$ E) To‘g‘ri javob yo‘q.

7.2. To‘la differensial. Yuqori tartibli xususiy hosila va differensiallar. Murakkab va oshkormas funksiya hosilasi

To‘la differensial va taqribiy hisoblash formulasi. $z=f(x,y)$ funksiya va ixtiyoriy $P(x,y)$ nuqtani qaraymiz. Ma’lumki, x *va* y o‘zgaruvchilar mos ravishda Δx va Δy orttirmalar olsa, $P_1(x+\Delta x, y+\Delta y)$ nuqtaga ega bo‘lamiz. Bu nuqtalar orasidagi masofa $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$ bo‘lishi ravshan. $z=f(x,y)$ funksiya $\Delta z=f(x+\Delta x, y+\Delta y)-f(x,y)$ to‘liq orttirma oladi.

Agar $z = f(x, y)$ funksiyaning $P(x, y)$ nuqtadagi to‘liq orttirmasini

$$\Delta z = A \cdot \Delta x + B \cdot \Delta y + o(\rho) \tag{7.1}$$

ko‘rinishda ifodalash mumkin bo‘lsa, bu funksiya $P(x, y)$ nuqtada ***differensiallanuvchi*** deyiladi, bu yerda A, B lar Δx va Δy ga bog‘liq bo‘lmagan sonlar, oxirgi qo‘shiluvchi $\Delta x \to 0,\ \Delta y \to 0\ (\rho \to 0)$ da yuqori tartibli cheksiz kichik funksiya.

Differensiallanuvchi $z = f(x, y)$ funksiya orttirmasining argumentlarning $\Delta x, \Delta y$ orttirmalariga nisbatan chiziqli ifodasi bo‘lgan bosh bo‘lagi funksiyaning ***to‘la differensiali*** deyiladi va dz bilan belgilanadi. $z = f(x, y)$ funksiyaning to‘la differensiali

$$dz = A \cdot \Delta x + B \cdot \Delta y \tag{7.2}$$

formula bilan aniqlanadi.

7.1-Teorema*(differensiallanuvchanlikning zaruriy sharti).* *Agar x funksiya $P(x, y)$ nuqtada differensiallanuvchi bo‘lsa, u holda u shu nuqtada $f'_x(x, y)$ va $f'_y(x, y)$ hususiy hosilalarga ega bo‘ladi, bunda* $A = f'_x(x, y),\ B = f'_y(x, y)$.

Yuqoridagi (7.1) va (7.2) formulalarda A va B kattaliklarni xususiy hosilalarga almashtirib, quyidagilarga ega bo‘lamiz:

$$\Delta z = f'_x(x, y) \cdot \Delta x + f'_y(x, y) \cdot \Delta y + o(\rho) \tag{7.3}$$

$$dz = f'_x(x, y) \cdot \Delta x + f'_y(x, y) \cdot \Delta y \tag{7.4}$$

Erkli o‘zgaruvchilarning orttirmalari ularning differensiallariga bevosita teng, ya’ni $dx = \Delta x,\ dy = \Delta y,$ shuning uchun to‘la differensial

$$dz = f'_x(x, y)dx + f'_y(x, y)dy \text{ yoki } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{7.5}$$

formula bilan hisoblanadi. $f'_x(x, y)dx$ va $f'_y(x, y)dy$ qo‘shiluvchilar ***xususiy differensiallar*** deb ataladi, ular mos ravishda $d_x z$ va $d_y z$ bilan belgilanadi. Demak, $dz = d_x z + d_y z.$

To‘la differensialdan funksiyaning taqribiy qiymatlarini hisoblashda foydalanish mumkin, ya’ni (7.3), (7.4) formulalardan $\Delta z = dz + o(\rho)$ va cheksiz kichik ρ larda $\Delta z \approx dz$. Demak,

$$f(x + \Delta x,\ y + \Delta y) \approx f(x, y) + f'_x(x, y)dx + f'_y(x, y)dy. \tag{7.6}$$

Bu esa ***taqribiy hisoblash*** formulasidir. Uch o‘zgaruvchili $u = F(x, y, z)$ funksiyaning to‘la differensiali

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \qquad (7.7)$$

formula bilan hisoblanadi. (1.6) formulani uch o‘zgaruvchili funksiya uchun ham umumlashtirish mumkin.

7.2-**Teorema*(differensiallanuvchanlikning yetarli sharti).*** *Agar z = f(x, y) funksiya **P(x, y)** nuqtaning biror* δ *atrofida xususiy hosilalarga ega bo‘lib, bu hosilalar nuqtaning o‘zida uzluksiz bo‘lsa, u holda funksiya shu nuqtada differensiallanuvchi bo‘ladi.*

7.5-misol.

Ushbu $z = \ln(x^2 + y^2)$ funksiyaning to‘la differensialini toping.

►Xususiy hosilalarni topamiz;

$$z'_x = \frac{(x^2+y^2)'_x}{x^2+y^2} = \frac{2x}{x^2+y^2}, \quad z'_y = \frac{(x^2+y^2)'_y}{x^2+y^2} = \frac{2y}{x^2+y^2},$$

(7.5) formulaga asosan, $dz = \frac{2x}{x^2+y^2}dx + \frac{2y}{x^2+y^2}dy$ bo‘ladi. ◄

7.6-misol.

O‘lchovlari $a = 8m$, $b = 6m$, $c = 3m$ bo‘lgan parallelepipedning uzunligi va eni mos ravishda 10 sm va 5 sm ga ko‘paytirilsa, balandligi esa 15 sm kamaysa uning hajmi qanday o‘zgaradi.

►Parallelepipedning hajmi $v = xyz$; x, y, z uning o‘lchamlari. Hajm orttirmasini taqriban $\Delta V \approx dV$ formuladan hisoblash mumkin.

$$dV = yxdx + xzdy + xydz.$$

Shartga ko‘ra

$x = 8$, $y = 6$, $z = 3$, $dx = 0.1$, $dy = 0.05$, $dz = -0.15$ bo‘lganligi uchun

$$\Delta V \approx dV = 66\cdot 3\cdot 0.1 + 8\cdot 3\cdot 0.05 + 8\cdot 6(-0.15) = -4{,}2.$$

Shunday qilib, hajm taxminan $4.2m^3$ ga kamayadi.◄

7.7-misol.

To‘la differensial formulasidan foydalanib:

$arcctg\left(\frac{1{,}97}{1{,}02} - 1\right)$ ni taqribiy hisoblang.

► To‘la differensial formulasidan taqribiy hisoblashda foydalanish uchun, oldin qiymati taqribiy hisoblanadigan funksiyaning analitik ifodasini tanlash zarur, keyin boshlang‘ich nuqtani shunday tanlash kerakki funksiyaning va xususiy hosilalarning bu nuqtadagi qiymatlarini jadvalsiz hisoblash mumkin bo‘lsin. Shundan keyin (7.6) formuladan foydalanish kerak.

$arcctg\left(\frac{1{,}97}{1{,}02} - 1\right)$ ifoda $f(x,y) = arcctg(\frac{x}{y} - 1)$ funksiyaning $P_1(1{,}97;1{,}02)$ nuqtadagi qiymati deyish mumkin. Boshlang‘ich nuqta uchun $P_0 = (2;1)$ ni olsak, $\Delta x = 1.97 - 2 = -0.03$, $\Delta y = 1.02 - 1 = 0.02$ bo‘ladi. Endi xususiy hosilalarni topib, ularning P_0 nuqtadagi qiymatlarini hisoblaymiz:

$$f'_x(x,y) = \left[arcctg\left(\frac{x}{y} - 1\right)'_x\right] = -\frac{y}{y^2 + (x-y)^2};$$

$$f'_y(x,y) = \left[arcctg\left(\frac{x}{y} - 1\right)'_y\right] = \frac{x}{y^2 + (x-y)^2};$$

$$f'_x(2;1) = -\frac{1}{1 + (2-1)^2} = -0.5; \qquad f'_y(2;1) = \frac{2}{1 + (2-1)^2} = 1.$$

(7.6) dan foydalansak,

$$arcctg\left(\frac{1.97}{1.02} - 1\right) \approx arcctg\left(\frac{2}{1} - 1\right) + (-0.5)(-0.03) + 1\cdot 0.02 =$$

$$= \frac{\pi}{4} + 0.015 + 0.02 = 0.82$$

bo‘ladi.◄

Yuqori tartibli xususiy hosilalar va differensiallar. $z = f(x,y)$ funksiyaning ***ikkinchi tartibli xususiy hosilalari*** deb birinchi tartibli xususiy hosilalardan olingan xususiy hosilalarga aytiladi. Ikkinchi tartibli xususiy hosilalar quyidagicha belgilanadi:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx}'' = f_{xx}''(x,y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = z_{xy}'' = f_{xy}''(x,y);$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = z_{yx}'' = f_{yx}''(x,y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z_{yy}'' = f_{yy}''(x,y).$$

$f_{xy}''(x,y)$ va $f_{yx}''(x,y)$ xususiy hosilalar aralash xususiy hosilalar deyiladi. Aralash xususiy hosilalar uzluksiz bo‘lgan nuqtalarda ular o‘zaro teng bo‘ladi.

Uchinchi va undan yuqori tartibli xususiy hosilalar ham yuqoridagidek aniqlanadi.

Ushbu $\frac{\partial^n z}{\partial x^m \partial y^{n-m}}$ yozuv z funksiyani m marta x o‘zgaruvchi bo‘yicha va $(n-m)$ marta y o‘zgaruvchi bo‘yicha differensiallashni bildiradi.

Birinchi tartibli to‘la differensialdan olingan to‘la differensial ***ikkinchi tartibli to‘la differensial*** deyiladi. $d(dz) = d^2 z$ kabi aniqlanib, xususiy hosilalar orqali quyidagicha topiladi.

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dxdy + \frac{\partial^2 z}{\partial y^2} dy^2 \qquad (7.8)$$

7.8-misol.

Ushbu $z = x^2 y^3$ funksiyaning ikkinchi tartibli to‘la differensialini toping.

►Xususiy hosilalarni topamiz:

$$z_x' = \left(x^2 y^3\right)_x' = 2xy^3; \; z_y' = 3x^2 y^2; \; z_{xx}'' = 2y^3, \; z_{xy}'' = 6xy^2, \; z_{yx}'' = 6xy^2, \; z_{yy}'' = 6xy^2,$$

(7.8) formulaga asosan ikkinchi tartibli to‘la differensial

$$d^2 z = 2y^3 dx^2 + 12xy^2 dxdy + 6x^2 y dy^2. ◄$$

Murakkab va oshkormas funksiyaning hosilasi. Ikki o‘zgaruvchining $z = z(u,v)$ differensiallanuvchi funksiyasi berilgan bo‘lsin. u va v argumentlar ham $\boldsymbol{x}$ erkli o‘zgaruvchining differensiallanuvchi funksiyalari bo‘lsin, ya’ni $u = u(x)$, $v = v(x)$. ***Murakkab funksiyaning hosilasi*** quyidagicha hisoblanadi:

$$\frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx} . \tag{7.9}$$

Ikki o‘zgaruvchili $z = z(u,v)$ $u = u(x,y)$, $v = v(x,y)$ ***murakkab funksiyaning xususiy hosilalari*** quyidagi formulalar bilan hisoblanadi:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} , \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}. \tag{7.10}$$

Oshkormas $F(x, y(x)) = 0$ ***funksiyaning hosilasi***

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0, \quad y_x' = -\frac{F_x'(x,y)}{F_y'(x,y)} \tag{7.11}$$

formula orqali hisoblanadi.

Uchta o‘zgaruvchini bog‘laydigan $F(x, y, z) = 0$, $z = z(x, y)$ ***oshkormas funksiyaning xususiy hosilalari*** quyidagi formulalardan topiladi:

$$z_x' = -\frac{F_x'(x,y,z)}{F_z'(x,y,z)}, \quad z_y' = -\frac{F_y'(x,y,z)}{F_z'(x,y,z)}. \tag{7.12}$$

7.9-misol.

Quyidagi funksiyaning xususiy hosilalarini toping:

$z = u^v$, bu yerda $u = y\sin x$, $v = y\cos x$.

►Berilgan funksiya ikki o‘zgaruvchili murakkab funksiyadir. Avval z dan u va v o‘zgaruvchilar bo‘yicha xususiy hosila hisoblaymiz

$$\frac{\partial z}{\partial u} = v \cdot u^{v-1}, \quad \frac{\partial z}{\partial v} = u^v \cdot \ln u,$$

so‘ngra u va v funksiyalardan x va y o‘zgaruvchilar bo‘yicha xususiy hosila hisoblaymiz

$$\frac{\partial u}{\partial x} = y\cos x, \quad \frac{\partial u}{\partial y} = \sin x;$$
$$\frac{\partial v}{\partial x} = -y\sin x, \quad \frac{\partial v}{\partial y} = \cos x.$$

Endi (7.10) formulalardan foydalansak,

$$\frac{\partial z}{\partial x} = v \cdot u^{v-1} \cdot y\cos x + u^v \cdot \ln u \cdot (-y\sin x) =$$
$$(y\sin x)^{y\cos x}\left[\frac{y\cos^2 x}{\sin x} - y\sin x \ln(y\sin x)\right],$$

$$\frac{\partial z}{\partial y} = v \cdot u^{v-1} \cdot \sin x + u^v \cdot \ln u \cdot \cos x = (y\sin x)^{y\cos x}\left[\cos x + \cos x \ln(y\sin x)\right]. \blacktriangleleft$$

7.10-misol.

Ushbu $x^2 - 2y^2 + 3z^2 - yz + y = 0$ oshkormas funksiya uchun $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar topilsin.

►Tenglamaning chap tomonini *F(x,y,z)* deb belgilab, xususiy hosilalarini topamiz

$$F_x'(x,y,z) = 2x, \quad F_y'(x,y,z) = -4y - z + 1, \quad F_z'(x,y,z) = 6z - y.$$

Oshkormas funksiyaning xususiy hosilalari uchun (7.12) formulalardan foydalanib quyidagi yechimlarga ega bo‘lamiz:

$$\frac{\partial z}{\partial x} = -\frac{F_x'(x,y,z)}{F_z'(x,y,z)} = -\frac{2x}{6z-y}; \quad \frac{\partial z}{\partial y} = -\frac{F_y'(x,y,z)}{F_z'(x,y,z)} = -\frac{1-4y-z}{6z-y}. \blacktriangleleft$$

30- Auditoriya topshiriqlari

1. Quyidagi funksiyalarning to‘la differensiallarini toping:

1) $z = arctg\frac{x+y}{1-xy}$; 2) $u = x^{y^2 z}$; 3) $u = zarctg\,\frac{y}{x}$.

2. $z = x^2 y$ funksiya uchun $P_0(4;5)$ nuqtada $\Delta x = -0{,}1$, $\Delta y = 0{,}2$ bo‘lganda dz va Δz larni hisoblang.

3. Taqribiy hisoblang:

1) $(1{,}02)^3(0{,}97)^3$; 2) $1{,}98^3\sqrt{3{,}01^2+3{,}97^2}$.

4. $z=\ln\left(x+\sqrt{x^2+y^2}\right)$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

5. Ko‘rsatilgan tartibli xususiy hosilalarini toping:

1) $f(x,y)=2x^3y+3xy^2$, f_{xyx},

2) $f(x,y)=x\sin y$, $\dfrac{\partial^3 z}{\partial x\partial y^2}$.

6. $z=e^{-x}\cos y-e^{-y}\sin x$ funksiya $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$ Laplas tenglamasini qanoatlantirishini tekshiring.

7. Berilgan funksiyalarning ikkinchi tartibli to‘la differensialini toping.

1) $z=e^{3x+\cos y}$; 2) $z=x^2\ln\dfrac{y}{x}$.

8. $z=\ln\left(e^x+e^y\right)$ funksiya uchun $\dfrac{\partial z}{\partial x}$ -? Agar $y=\sin x$ bo‘lsa, $\dfrac{dz}{dx}$ ni toping.

9. $z=\sin u\cos v$, $u=2e^x$ va $v=\ln(x^2+1)$ bo‘lsa, $\dfrac{dz}{dx}$ ni toping.

10. $z=arctg\dfrac{u}{v}$, $u=x\sin y$ va $v=y\cos x$ bo‘lsa, $\dfrac{\partial z}{\partial x}$ va $\dfrac{\partial z}{\partial y}$ larni toping.

11. $\sin(xy)-x^2-y^2=0$ bo‘lsa, $\dfrac{dy}{dx}$ ni toping.

12. $x\cos y+y\cos z+z\cos x=1$ bo‘lsa, $\dfrac{\partial z}{\partial x}$ va $\dfrac{\partial z}{\partial y}$ larni toping.

30-Mustaqil yechish uchun testlar

1. $z=f(x,y)$ funksiyaning to‘la differensiali formulasini ko‘rsating.

A) $dz=f'_x(x,y)dy+f'_y(x,y)dx$ B) $dz=\dfrac{dz}{dx}\partial x+\dfrac{dz}{dy}\partial y$

D) $dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy$ E) A va D.

2. To‘la differensial yordamida taqribiy hisoblash formulasini toping.

A) $f(x+\Delta x,\ y)\approx f(x,\ y)+df(x,\ y)$

B) $f(x+\Delta x,\ y+\Delta y)-f(x,\ y)\approx df(x,\ y)$

D) $f(x+\Delta x,\ y)\approx f(x,\ y)-df(x,\ y)$,

E) $f(x+\Delta x,\ y+\Delta y)+f(x,\ y)\approx df(x,\ y)$

3. Quyidagilardan qaysi biri $z=x\sin(x+2y)$ funksiyaning ikkinchi tartibli xususiy hosilasi bo‘ladi?

A) $2\cos(x+2y)-2\sin(x+2y)$ B) $-4\sin(x+2y)$

D) $\cos(x+2y)-x\sin(x+2y)$ E) To‘g‘ri javob yo‘q.

4. $x^3-3y^2+3z^2-2yz+3y=0$ oshkormas funksiya uchun $\frac{\partial z}{\partial x}$ topilsin.

A) $\frac{3(x^2-2)}{6z-2y}$ B) $\frac{3(x^2-2)}{2y-3z}$

D) $\frac{x^2}{3z-2y}$ E) $\frac{3x^2}{2y-6z}$ u

5. $z=\ln(u^2+v^2)$, $u=x\sin y$ va $v=y\cos x$ bo‘lsa, $\frac{\partial z}{\partial y}$ ning $M_0(\pi;\ \pi)$ nuqtadagi qiymatini toping.

A) $\frac{1}{\pi^2}$ B) $\frac{\pi}{2}$ D) $\frac{2}{\pi}$ E) $\frac{2}{\pi^2}$

7.3. Ikki o‘zgaruvchili funksiyaning ekstremumlari. Yopiq sohada funksiyaning eng katta va eng kichik qiymatlari

Ikki o‘zgaruvchili funksiya ekstremumi. $z=f(x,y)$ funksiyaning $P_0(x_0,y_0)$ nuqtadagi qiymati uning bu nuqtaning biror atrofining istalgan $P(x,y)$ nuqtasidagi qiymatlaridan katta (kichik) bo‘lsa, ya’ni $f(x_0,y_0)>f(x,y)$ $\left(f(x_1,y_1)<f(x,y)\right)$ bo‘lsa, $z=f(x,y)$ ***funksiya*** $P_0(x_0,y_0)$ ***nuqtada maksimum(minimum)ga ega*** deyiladi.

Funksiyaning maksimum yoki minimumi uning ***ekstremumi*** deyiladi. Funksiya ekstremumga ega bo‘lgan nuqta uning ***ekstremum nuqtasi*** deyiladi.

Ekstremumning zaruriy shartlari. $P_0(x_0;y_0)$ nuqta uzluksiz $z=f(x,y)$ funksiyaning ekstremum nuqtasi bo‘lsa, bu nuqtada

$$\begin{cases} f'_x(x_0, y_0) = 0 \\ f'_y(x_0, y_0) = 0 \end{cases}$$

bo‘ladi, yoki ulardan hech bo‘lmaganda bittasi mavjud bo‘lmaydi.

Bunday nuqtalarga ***kritik nuqtalar*** deyiladi. Shuni ta’kidlaymizki hamma kritik nuqtalar ham ekstremum nuqtalar bo‘lavermaydi. Kritik nuqtada ekstremum bo‘lmasligi ham mumkin.

Ekstremumning yetarli shartlari.

Ikkinchi tartibli xususiy hosilalarning kritik nuqtadagi qiymatlarini

$$A = f''_{xy}(x_0, y_0), \quad B = f''_{xy}(x_0, y_0), \quad C = f''_{yy}(x_0, y_0)$$

bilan belgilaymiz va quyidagi determinantni tuzamiz

$$\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$$

$z = f(x, y)$ funksiya $P_0(x_0, y_0)$ nuqtani o‘z ichiga oluvchi biror sohada uchinchi tartibligacha uzluksiz hosilaga ega bo‘lsin. U holda

1. $\Delta = AC - B^2 > 0$ bo‘lsa, $z = f(x, y)$ funksiya $P_0(x_0, y_0)$ nuqtada ekstremumga ega bo‘lib,

1) $A < 0 (C < 0)$ bo‘lganda $P_0(x_0, y_0)$ nuqtada maksimumga,

2) $A > 0 (C > 0)$ bo‘lganda $P_0(x_0, y_0)$ nuqtada minimumga erishadi.

2. $\Delta = AC - B^2 < 0$ bo‘lsa, $P_0(x_0, y_0)$ nuqtada ekstremum mavjud bo‘lmaydi.

3. $\Delta = AC - B^2 = 0$ bo‘lsa, ekstremum bo‘lishi ham, bo‘lmasligi ham mumkin.

7.11-misol.

Ushbu $z = f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ funksiyani ekstremumga tekshiring.

►Bu funksiya butun Oxy tekislikda aniqlangan. Birinchi tartibli xususiy hosilalarini topamiz:

$$f'_x = 4x^3 - 4x + 4y; \; f'_y = 4y^3 + 4x - 4y$$

Ekstremumga ega bo‘lishning zaruriy shartidan:

$$\begin{cases} 4x^3 - 4x + 4y = 0 \\ 4y^3 + 4x - 4y = 0 \end{cases} \Rightarrow, \qquad \begin{cases} y = -x^3 + x \\ y^3 + x - y = 0 \end{cases}$$

$$\begin{cases} (x - x^3)^3 + x^3 = 0 \\ y = -x^3 + x \end{cases} \Rightarrow \begin{cases} 2x - x^2 = 0 \\ y = -x^3 + x \end{cases}$$

Demak, uchta $O(0;0)$, $P_1\left(-\sqrt{2};\sqrt{2}\right)$ va $P_2\left(\sqrt{2};-\sqrt{2}\right)$ kritik nuqtalarga ega bo‘lamiz, boshqa kritik nuqtalar yo‘q, chunki $f_x'(x,y), f_y'(x,y)$ xususiy hosilalar O*xy* tekislikning boshqa hamma nuqtalarida mavjud va noldan farqli.

Ikkinchi tartibli xususiy hosilalarni topamiz:

$$f_{xx}''(x,y) = 12x^2 - 4;\ f_{xy}''(x,y) = 4;\ f_{yy}''(x,y) = 12y^2 - 4.$$

$O(0;0)$ nuqtada ekstremumning yetarli shartini tekshiramiz: $A = -4,\ \ B = 4,\ \ C = -4$; $\Delta = AC - B^2 = -4\cdot(-4) - 4^2 = 0$ bo‘lib, yuqoridagi yetarli shart javob bermaydi. Bu nuqta atrofida berilgan funksiya musbat ham, manfiy ham bo‘lishini ko‘ramiz, masalan, O*x* o‘qi bo‘yicha ($y = 0$)

$$f(x,y)\big|_{y=0} = f(x,0) = x^4 - 2x^2 = -x^2\left(2 - x^2\right) < 0.$$

$y = x$ bissektrisa bo‘yicha, $f(x,y)\big|_{y=x} = f(x,x) = 2x^4 > 0$ bo‘ladi. Shunday qilib, $O(0,0)$ ning atrofida $\Delta f(x,y)$ orttirma ishorasini bir xil saqlamaydi, demak, ekstremum yo‘q.

$P_1\left(-\sqrt{2};\sqrt{2}\right)$ va $P_2\left(\sqrt{2};-\sqrt{2}\right)$ nuqtalarda yetarli shartni tekshiramiz, bu nuqtalar uchun $A = 20$, $A = 4$, $C = 20$ bo‘lib, $\Delta = AC - B^2 = 400 - 16 > 0$ va $A = 20 > 0$, demak, $P_1\left(-\sqrt{2};\sqrt{2}\right)$ va $P_2\left(\sqrt{2};-\sqrt{2}\right)$ nuqtalarda funksiya minimumga ega, $f_{\min}(P_1) = f_{\min}(P_2) = -8$. ◄

7.12-misol.

Ushbu $z = \sqrt{(x-1)^2 + (y-1)^2}$ funksiyaning ekstremumini toping.

► $\dfrac{\partial z}{\partial x} = \dfrac{x-1}{2\sqrt{(x-1)^2 + (y-1)^2}}$, $\dfrac{\partial z}{\partial y} = \dfrac{y-1}{\sqrt{(x-1)^2 + (y-1)^2}}$.

$P_0(1;1)$ nuqtada xususiy hosilalar mavjud emas. Demak, P_0 (1;1) nuqta kritik nuqta bo‘ladi. Bu nuqtada ekstremumni tekshirish uchun Δz orttirmaning P_0 nuqta atrofida ishorasini tekshiramiz:

$$\Delta z=\sqrt{(1+\Delta x-1)^2+(1+\Delta y-1)^2}=\sqrt{\Delta x^2+\Delta y^2}>0,$$

bu ishora $P_0(1;1)$ nuqtaning istalgan atrofida saqlanadi ya’ni $P_0(1;1)$ nuqtada funksiya minimumga ega $z_{\min}=f(1;1)=0$.◄

Shartli ekstremum. Ushbu $z=f(x,y)$ funksiyaning shartli ekstremumi deb bu funksiyaning x va y o‘zgaruvchilarni $\varphi(x,y)=0$ tenglama(bog‘lash tenglamasi) bilan bog‘langanlik shartida erishadigan ekstremumiga aytiladi.

Agar bog‘lash tenglamasi $\varphi(x,y)=0$ dan $y=y(x)$ ni topish mumkin bo‘lsa, uni $z=f(x,y)$ funksiyaga qo‘yib shartli ekstremum topish masalasini bir o‘zgaruvchili $z=f(x,y(x))$ funksiya ekstremumini topishga keltiriladi.

7.13-misol.

Ushbu $z=\sqrt{(x-1)^2+(y-1)^2}$ funksiyaning $x+y-4=0$ shartdagi ekstremumini toping.

► $y=4-x$ ni berilgan tenglamaga qo‘yib bir o‘zgaruvchi x ning funksiyasini hosil qilamiz:

$$z=\sqrt{2x^2-8x+10}.$$

Bu yerda funksiyani $z=\sqrt{2(x-2)^2+2}$ shaklda yozish orqali uning ekstremimini elementar usulda topish mumkin. $x=2$ nuqtada funksiya o‘zining eng kichik qiymatiga erishadi. Demak, $P_0(2,\ 2)$ nuqta berilgan funksiyaning minimum nuqtasi va $z_{\min}=\sqrt{2}$ ekan.◄

Bog‘lash tenglamasini parametrik tenglamalar orqali ifodalanganda ham shartli ekstremum topish masalasi bir o‘zgaruvchining ekstremumini topishga keltiriladi.

Ko‘p hollarda, $z=f(x,y)$ funksiyaning $\varphi(x,y)=0$ shartdagi ekstremumini Lagranj funksiyasi deb ataluvchi $\Phi(x,y,\lambda)=f(x,\ y)+\lambda\varphi(x,y)$ funksiyani oddiy ekstremumga tekshirish yordamida topiladi, bu yerda λ - noma’lum o‘zgarmas.

Lagranj funksiyasi ekstremumining zaruriy sharti quyidagicha:

$$\begin{cases} \Phi_x(x,y,\lambda)=0 \\ \Phi_y(x,y,\lambda)=0 \\ \Phi_\lambda(x,y,\lambda)=0 \end{cases} \Rightarrow \begin{cases} f_x(x,\ y)+\lambda\varphi_x(x,y)=0 \\ f_y(x,\ y)+\lambda\varphi_y(x,y)=0 \\ \varphi(x,y)=0. \end{cases}$$

7.14-misol.

Ushbu $z=2x+y$ funksiyaning $x^2+y^2-5=0$ shartdagi ekstremumini toping.

► *1-usul.* $x^2+y^2-5=0$ ni $x=\sqrt{5}\cos t$, $y=\sqrt{5}\sin t$ parametrik tenglamalar bilan ifodalab, bir o'zgaruvchi t ning funksiyasini hosil qilamiz va kritik nuqtasini topamiz:

$$z(t)=2\sqrt{5}\cos t+\sqrt{5}\sin t.$$

$$z'(t)=-2\sqrt{5}\sin t+\sqrt{5}\cos t,\ z'(t)=0\Rightarrow t_0=arctg2+n\pi,\ n\in Z.$$

t_0 nuqtalarda

$$\begin{cases} \cos t_0=2\sin t_0 \\ \cos^2 t_0+\sin^2 t_0=1 \end{cases},$$

ya'ni, 1) $\sin t_{01}=\dfrac{1}{\sqrt{5}}$; $\cos t_{01}=\dfrac{2}{\sqrt{5}}$ 2) $\sin t_{02}=-\dfrac{1}{\sqrt{5}}$; $\cos t_{02}=-\dfrac{2}{\sqrt{5}}$.

$z''(t)=-2\sqrt{5}\cos t-\sqrt{5}\sin t$, $z''(t_{01})<0$, $z''(t_{02})>0$ bo'lgani uchun t_{01} ga mos $P_1(2,\ 1)$ nuqta maksimum va t_{02} ga mos $P_1(-2,-1)$ nuqta minimum nuqta boladi va $z_{\max}(2,\ 1)=5$, $z_{\min}(-2,-1)=-5$.

2-usul. Lagranj funksiyasini tuzamiz:
$\Phi(x,y,\lambda)=2x+y+\lambda(x^2+y^2-5)$. Xususiy hosillarini nolga tenglab yechamiz:

$$\begin{cases} 2+2\lambda x=0 \\ 1+2\lambda y=0 \\ x^2+y^2-5=0 \end{cases} \Rightarrow \begin{cases} x=-\dfrac{1}{\lambda} \\ y=-\dfrac{1}{2\lambda} \\ \lambda^2=\dfrac{1}{4} \end{cases}$$

Bundan $\lambda_1=-1/2,\ x_1=2, y_1=1$ va $\lambda_2=1/2,\ x_2=-2, y_2=-1$ larni topamiz.

$P_1(2,\ 1)$ nuqtada $z_{max}(2,\ 1)=5$ va $P_1(-2,-1)$ nuqtada $z_{min}(-2,-1)=-5$. ◄

Ikki o‘zgaruvchili funksiyaning yopiq sohadagi eng katta va eng kichik qiymatlarini topish. Chegaralangan yopiq sohada differensiallanuvchi funksiya o‘zining ***eng katta va eng kichik qiymatlariga*** yo sohada yotuvchi kritik nuqtalarda, yo bu soha chegarasida erishadi.

7.15-misol.

Ushbu $z=x^2+y^2-xy+x+y$ funksiyaning $x\le 0, y\le 0, x+y\ge -3$ sohadagi eng katta va eng kichik qiymatlarini toping.

►Soha AOB uchburchakdan iborat. Soha ichidagi kritik nuqtalarni topamiz:
$$\begin{cases}\dfrac{\partial z}{\partial x}=2x-y+1=0\\ \dfrac{\partial z}{\partial y}=2y-x+1=0\end{cases}$$

bundan $x=-1, y=-1$ bo‘lib, P_0 (-1,-1) kritik nuqtaga ega bo‘lamiz. Funksiyani soha chegarasida tekshiramiz: AO chegarada $y=0$ bo‘lib, $z=x^2+x$ funksiya hosil bo‘ladi. Bu funksiyaning ekstremumi:

$z'_x=2x+1=0$, $x=-\dfrac{1}{2}=-0{,}5$

bo‘ladi. Demak, P_1 (-0,5, 0) AO chegaradagi kritik nuqta. Tenglamasi $x=0$, BO chegarada $z=y^2+y$ funksiya hosil bo‘lib, $z'_y=2y+1=0$, $y=-1/2$. Demak, $P_2\left(0,-\dfrac{1}{2}\right)$ BO chegaradagi kritik nuqta bo‘ladi. Tenglamasi $y=-3-x$ bo‘lgan AB chegarada $z=3x^2+9x+6$ funksiya hosil bo‘lib, $z'_x=6x+9=0$ $x=-\dfrac{3}{2}$. AB ning tenglamasidan $y=-3+\dfrac{3}{2}=-\dfrac{3}{2}$, demak, AB chegaradagi kritik nuqta $P_3\left(-\dfrac{3}{2},-\dfrac{3}{2}\right)$ bo‘ladi.

Berilgan funksiyaning P_0, P_1, P_2, P_3 kritik nuqtalardagi, hamda A, B, O nuqtalardagi qiymatlarni hisoblaymiz:

$z_0 = f(P_0) = f(-1,-1) = -1$; $z_1 = f(P_1) = f\left(-\frac{1}{2},-0\right) = -\frac{1}{4}$;

$z_2 = f(P_2) = f\left(0,-\frac{1}{2}\right) = -\frac{1}{4}$; $z_3 = f(P_3) = f\left(-\frac{3}{2},-\frac{1}{2}\right) = -\frac{3}{4}$;

$z_4 = f(O) = f(0,0) = 0$; $z_5 = f(A) = f(-3,0) = 6$;

$z_6 = f(B) = f(0,-3) = 6$.

Funksiyaning topilgan barcha qiymatlarini taqqoslab,

$$z_{eng\,kat.} = f(A) = f(B) = 6 \quad va \quad z_{eng\,kich.} = f(P_0) = -1$$

degan xulosaga kelamiz.◀

31-Auditoriya topshiriqlari

1. Berilgan funksiyalarning kritik nuqtalarini toping.

1) $z = 2x^3 + xy^2 + 5x^2 + y^2$;

2) $z = e^{2x}(x + y^2 + 2y)$;

3) $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$;

4) $5x^2 + 5y^2 + 5z^2 - 2xy - 2yz - 2xz - 72 = 0$.

2. Berilgan funksiyalarni ekstremumga tekshiring.

1) $z = x^3 + 3xy^2 - 15x - 12y$;

2) $z = 4(x - y) - x^2 - y^2$;

3) $z = 2xe^{-(x^2+y^2)}$;

4) $z = x^4 + y^4 - 2x^2 - 4xy - y^2$.

3. Yopiq sohada eng katta va eng kichik qiymatlarini toping.

1) $z = x^2 + y^2 + xy - 4x - 2y + 5$, $\overline{D}: x \ge 0,\ x + y \le 3,\ x - y \le 3$;

2) $z = e^{-x^2-y^2}(2x^2 + 3y^2)$, $\overline{D}:\ x^2 + y^2 \le 4$.

31-Mustaqil yechish uchun testlar

1. $z = x^3 + y^2 + 6xy + 9x - 2y + 5$ funksiyaning kritik nuqtalari berilgan javobni aniqlang.

A) (1,–2), (5,–14)   B) (1,–2), (5,–6)

D) (1,–2)   E) $(5,-14)$.

2. $z = x^3 + y^2 + 6xy + 9x - 2y + 5$ funksiyaning ekstremum nuqtalari berilgan javobni aniqlang.

A) (1,–2), (5,–14)   B) (1,–2), (5,–6)

D) (1,–2)   E) $(5,-14)$.

3. Quyidagilardan qaysi biri $z = \sqrt{x^2 + y^2 + 4y + 4}$ funksiyaning ekstremum nuqtasi bo‘ladi?

A) (1,–2)   B) (0,–2)   D) (0, 0)   E) mavjud emas.

4. $z = 6xy - x^2 - y^2 - 5x + 5y + 4$ funksiyaning ekstremumi berilgan javobni aniqlang.

A) $z_{\min}(-1,-2) = 0$ B) $z_{\max}(-1, 1) = 6$

D) $z_{\min}(1,-1) = -14$ E) mavjud emas.

5. Funksiyaning berilgan yopiq sohadagi eng katta va eng kichik qiymatini toping. $z = x^2 - y^2$, $\overline{D}: x^2 + 2y^2 \le 4$

A) 4 va -4 B) 4 va -1 D) 2 va -4 E) 2 va -1

10-Shaxsiy topshiriqlar

1

Berilgan funksiyalarning xususiy hosilalarini va to‘la differensialini toping.

1.1. $z = ctg\sqrt{\dfrac{x}{x-y}}$

1.2. $z = \ln(3y^2 - x^4)$

1.3. $z = \sin\sqrt{\dfrac{y}{x+y}}$

1.4. $z = \arccos\dfrac{y}{x}$

1.5. $z = \cos\left(x - \sqrt{xy^3}\right)$

1.6. $z = \cos\sqrt{x^2 + y^3}$

1.7. $z = \sin\sqrt{x - y^3}$

1.8. $z = ctg\sqrt{xy^3}$

1.9. $z = \sin\sqrt{y/x^3}$

1.10. $z = \cos\sqrt{\dfrac{x}{x+y}}$

1.11. $z = \ln\left(y^2 + e^{-2x}\right)$

1.12. $z = arctg\left(x\sqrt{y}\right)$

1.13. $z = ye^{-\sqrt{x^2+y^2}}$

1.14. $z = ctg(3x^2 + \sqrt{y})$

1.15. $z = arctg(x/y^3)$

1.16. $z = \arcsin(2x^2\sqrt{y})$

1.17. $z = \cos\dfrac{x-y}{x^2+y^2}$

1.18. $z = \cos(3x^2 - \sqrt{y})$

1.19. $z = e^{-\sqrt{x^2+y^2}}$

1.20. $z = tg\,\dfrac{2x-y^2}{x}$

1.21. $z = \arcsin(xy) + 3x^2y$

1.22. $z = y\ln\left(3x^2 - y\right)$

1.23. $z = \sin\dfrac{2x-y}{x+2y}$

1.24. $z = xctg(x/y^3)$

1.25. $z = arctg\left(x^2 + y^2\right)$

1.26. $z = y\cos(3x^2 - \sqrt{y})$

1.27. $z = xarctg\left(x\sqrt{y}\right)$

1.28. $z = e^{-x^2+y^2}$

1.29. $z = y\ln\left(3x^2 + \sqrt{y}\right)$

1.30. $z = \sqrt{x}e^{-x^2+y^2}$

2

Berilgan $z=z(u,v)$, $u=u(x)$, $v=v(x)$ murakkab funksiyaning $x=x_0$ nuqtadagi hosilasini toping.

2.1. $z=e^{v-2u-1}$, $u=\cos x$, $v=\sin x$, $x_0=\pi/2$.

2.2. $z=\ln\left(e^u+e^{-v}\right)$, $u=x^2$, $v=x^3$, $x_0=-1$.

2.3. $z=u^2e^v$, $u=\cos x$, $v=\sin x$, $x_0=\pi$.

2.4. $z=u^v$, $u=\ln(x-1)$, $v=e^{x/2}$, $x_0=2$.

2.5. $z=\sqrt{u+v^2+3}$, $u=\ln x$, $v=x^2$, $x_0=1$.

2.6. $z=\arccos\dfrac{2u}{v}$, $u=\sin x$, $v=\cos x$, $x_0=\pi$.

2.7. $z=\dfrac{u}{v}$, $u=e^x$, $v=2-e^{2x}$, $x_0=0$.

2.8. $z=u^2e^{-v}$, $u=\sin x$, $v=\sin^2 x$, $x_0=\pi/2$.

2.9. $z=e^{u-2v}$, $u=\sin x$, $v=x^3$, $x_0=0$.

2.10. $z=\ln\left(e^u+e^v\right)$, $u=x^2$, $v=x^3$, $x_0=1$.

2.11. $z=u^v$, $u=e^x$, $v=\ln x$, $x_0=1$.

2.12. $z=\ln\left(e^{-u}+e^{-2v}\right)$, $u=x^2$, $v=x^3/3$, $x_0=-1$.

2.13. $z=\arcsin\dfrac{u}{v}$, $u=\sin x$, $v=\cos x$, $x_0=\pi$.

2.14. $z=\dfrac{u^2}{v+1}$, $u=1-2x$, $v=arctgx$, $x_0=0$.

2.15. $z=e^{v-2u}$, $u=\sin x$, $v=x^3$, $x_0=0$.

2.16. $z=\arcsin\dfrac{u^2}{v}$, $u=\sin x$, $v=\cos x$, $x_0=\pi$.

2.17. $z=\dfrac{u^2}{v}$, $u=1-2x$, $v=1+arctgx$, $x_0=0$.

2.18. $z=\sqrt{u^2+v+3}$, $u=\ln x$, $v=x^2$, $x_0=1$.

2.19. $z=\arcsin\dfrac{u}{2v}$, $u=\sin x$, $v=\cos x$, $x_0=\pi$

2.20. $z=\ln\left(e^{-u}+e^v\right)$, $u=x^2$, $v=x^3$, $x_0=-1$.

2.21. $z=\dfrac{2u}{v}$, $u=e^x$, $v=2-e^{2x}$, $x_0=0$.

2.22. $z=arctg(uv)$, $u=x+3$, $v=e^x$, $x_0=0$

2.23. $z=\frac{v}{u}-\frac{u}{v},\ u=\sin x,\ v=\cos x,\ x_0=\pi/4.$

2.24. $z=ar\cos\frac{u^2}{v},\ u=\sin x,\ v=\cos x,\ x_0=\pi.$

2.25. $z=\sqrt{u+v+3},\ u=\ln x,\ v=x^2,\ x_0=1.$

2.26. $z=\ln\left(e^{2u}+e^{v}\right),\ u=x^2,\ v=x^4,\ x_0=1.$

2.27. $z=arctg(u+v),\ u=x^2+2,\ v=4-x,\ x_0=1.$

2.28. $z=\frac{u}{v}-\frac{v}{u},\ u=\sin 2x,\ v=tg^2x,\ x_0=\pi/4.$

2.29. $z=\sqrt{u^2+v^2+3},\ u=\ln x,\ v=x^3,\ x_0=1.$

2.30. $z=\frac{v}{u},\ u=e^x,\ v=1-e^{2x},\ x_0=0.$

3

Berilgan oshkormas $z=z(x,y)$ funksiyaning $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ xususiy hosilalarini toping.

3.1. $cos^2x+cos^2y+cos^2z=3/2.$

3.2. $z^3+3xyz+3y=7.$

3.3. $e^{z-1}=cosxcosy+z.$

3.4. $x\cos y+yccos+zccos=\pi/2.$

3.5. $3x^2y^2+2xyz^2-2x^3z+4y^3z=4.$

3.6. $\ln z=x+2y-z+\ln 3.$

3.7. $x^3+2y^3+z^3-3xyz-2y-5=0.$

3.8. $e^x-xyz-x+1=0.$

3.9. $\sqrt{x^2+y^2}+z^2-3z=5.$

3.10. $x^2+y^2+z^2-2xz=4.$

3.11. $x^2+y^2+z^2-xy=2.$

3.12. $3x+y^2+z^2=xz+5.$

3.13. $e^z+x+2yz-y-4=0.$

3.14. $x^2+y^2+z^2-z=5.$

3.15. $x^2+y^2+z^2-6x+2z=0.$

3.16. $xyz=z^2-2.$

3.17. $x^2-2y^2+3z^2-yz+y=4$.

3.18. $x^2+y^2+z^2+2xz=5$.

3.19. $x^2-2y^2+z^2-4x+2z+2=0$.

3.20. $x^3+y^3+z^3-3xyz=4$.

3.21. $x^2-2xy-3y^2+6x-2y+z^2-8z+20=0$.

3.22. $x^2+y^2+z^2=y-2z+3$.

3.23. $x^2+y^2+z^2+2xy+yz-4x-3y-z=0$.

3.24. $x^2+y^2+z^2+6z+2x-4y-12=0$.

3.25. $x^2+2y^2+3z^2=59$.

3.26. $z^2=xy-z+x^2-4$.

3.27. $2x^2+2y^2+z^2-8xz-z+6=0$.

3.28. $x^3+3xyz-z^2=27$.

3.29. $x+y+z+2=xyz$.

3.30. $x^2+y^2+z^2-2xy-2xz-2yz=13$.

4

Berilgan funksiyalarni ekstremumga tekshiring.

4.1. $z=x^2y(4-x-y)$.

4.2. $z=2y\log(2-x^2)+y^2$.

4.3. $z=6(x-y)-3x^2-3y^2$.

4.4. $z=y\sqrt{x}-2y^2-x+14y$.

4.5. $z=x^3+8y^3-6xy+5$.

4.6. $z=x+\frac{1}{6}x^6+y^2\left(y^2-1\right)$.

4.7. $z=1+15x-2x^2-2y^2-xy$.

4.8. $z=3+6x-x^2-y^2-xy$.

4.9. $z=2x^3+2y^3-6xy+5$.

4.10. $z=x^2+xy+y^2+\frac{1}{x}+\frac{1}{y}$.

4.11. $z=y\sqrt{x}-y^2-x+6y$.

4.12. $z=x^2+xy+y^2+x-y+5$.

4.13. $z=e^{3x^2-6xy+2y^2}$.

4.14. $z=3x^3+3y^3-9xy+10$.

4.15. $z = xye^{-\frac{x}{5}-\frac{y}{6}}$

4.16. $z = x\sqrt{y} - x^2 - y + 6x + 3.$

4.17. $z = xy(6 - x - y).$

4.18. $z = x^2 + y^2 + xy - 6x - 9y + 2.$

4.19. $z = x^3 + y^3 - 3xy.$

4.20. $z = 2xy - 2x^2 - 4y^2.$

4.21. $z = 2(x + y) - x^2 - y^2.$

4.22. $z = x^3 + 8y^3 - 6xy + 1.$

4.23. $z = 2xy - 3x^2 - 2y^2 + 10.$

4.24. $z = 2xy - x^2 - y^2 + 9.$

4.25. $z = 5x^2 + y^2 - 3xy + 4.$

4.26. $z = xy(12 - x - y).$

4.27. $z = 1 + 15y - 2x^2 - 2y^2 - xy.$

4.28. $z = x^2 - xy + y^2 + x + y + 2.$

4.29. $z = x^2 + xy + y^2 + \frac{3}{x} + \frac{\sqrt[3]{9}}{y}.$

4.30. $z = 6xy - x^2y - xy^2 + 6.$

5

Berilgan funksiyalarning ko‘rsatilgan yopiq sohadagi eng katta va kichik qiymatlarini toping.

5.1. $z = 3xy - x^2y - xy^2 + 6,\ \overline{D}: 0 \le x \le 2,\ 0 \le y \le 3.$

5.2. $z = x^2 - xy + y^2 - 4x,\ \overline{D}: x \ge 0, y \ge 0, 3x + 2y - 12 \le 0.$

5.3. $z = x + 3y + 4,\ \overline{D}: x \ge 0, y \ge 0, x^2 + y^2 - 1 \le 0.$

5.4. $z = x^2(y+1) - 2y,\ \overline{D}: \sqrt{1 + x^2} \le y \le 2.$

5.5. $z = xy,\ \overline{D}:\ x^2 + y^2 \le 1.$

5.6. $z = 3x + y - xy,\ \overline{D}: x \ge 0,\ x \le y \le 4.$

5.7. $z = x^2 + 2xy - 4x + 8y - 2,\ \overline{D}: 0 \le x \le 1,\ 0 \le y \le 2.$

5.8. $z = 5x^2 - 3xy + y^2,\ \overline{D}: 0 \le x \le 1,\ 0 \le y \le 1.$

5.9. $z = x^2 + y^2 - 2x - 2y + 8,\ \overline{D}: x \ge 0,\ y \ge 0,\ x + y \le 1.$

5.10. $z = 3x + 6y - x^2 - xy - y^2,\ \overline{D}: 0 \le x \le 1,\ 0 \le y \le 1.$

5.11. $z = x^2 - 2y^2 + 4xy - 6x + 5,\ \overline{D}: x \ge 0,\ y \ge 0,\ x + y \le 3.$

5.12. $z = xy - 2x - y$, $\overline{D}:\ 0 \le x \le 3,\ 0 \le y \le 4$.

5.13. $z = x^2 + 2xy - 10$, $\overline{D}:\ x^2 - 4 \le y \le 0$.

5.14. $z = x^2 - 2xy + y^2 + 4x$, $\overline{D}:\ x \ge -3,\ 0 \le y \le 1 - x$.

5.15. $z = \frac{1}{2}x^2 - xy$, $\overline{D}:\ 2x^2 \le y \le 8$.

5.16. $z = 3x^2 + 3y^2 - 2x - 2y + 1$, $\overline{D}:\ x \ge 0,\ 0 \le y \le 1 - x$.

5.17. $z = 2x^2 + 3y^2 + 1$, $\overline{D}: 0 \le y \le \sqrt{9 - \frac{9}{4}x^2}$.

5.18. $z = 2x^2 + 2xy - \frac{1}{2}y^2 - 4x$, $\overline{D}:\ x \ge 0,\ 2x \le y \le 2$.

5.19. $z = x^2 + xy - 4$, $\overline{D}:\ 4 - x^2 \le y \le 0$.

5.20. $z = 6xy - 9x^2 - 9y^2 + 4x + 4y$, $\overline{D}:\ 0 \le x \le 1,\ 0 \le y \le 1$.

5.21. $z = 2x^2y - x^3y - x^2y^2$, $\overline{D}:\ x \ge 0,\ 0 \le y \le 6 - x$.

5.22. $z = x^2y(4 - x - y)$, $\overline{D}:\ x \ge 0,\ 0 \le y \le 6 - x$.

5.23. $z = x^3 - 3xy + y^3$, $\overline{D}:\ 0 \le x \le 2,\ -1 \le y \le 2$.

5.24. $z = 4(x - y) - x^2 - y^2$, $\overline{D}:\ x \ge 0, x + 2y \le 4,\ x - 2y \le 4$.

5.25. $z = 4 - 2x^2 - y^2$, $\overline{D}:\ 0 \le y \le \sqrt{1 - x^2}$.

5.26. $z = 1 - x^2 - y^2$, $\overline{D}: (x - 1)^2 + (y - 1)^2 \le 1$.

5.27. $z = x^2 + 2xy - y^2 - 4x$, $\overline{D}:\ x \le 3,\ 0 \le y \le x + 1$.

5.28. $z = \sin x + \sin y + \sin(x + y)$, $\overline{D}:\ 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}$.

5.29. $z = \cos x + \cos y + \cos(x + y)$, $\overline{D}:\ 0 \le x \le \frac{3\pi}{2},\ 0 \le y \le \frac{3\pi}{2}$.

5.30. $z = \cos x \cos y \cos(x + y)$, $\overline{D}:\ 0 \le x \le \pi,\ 0 \le y \le \pi$.

VIII BOB. ODDIY DIFFERENSIAL TENGLAMALAR

8.1. Birinchi tartibli differensial tenglamalar

Asosiy tushunchalar. Erkli o‘zgaruvchilar, ularning noma’lum funksiyasi va bu funksiyaning hosilalari(yoki differensiallari)ni bog‘lovchi munosabatga ***differensial tenglama*** deyiladi. Agar noma’lum funksiya faqat bitta o‘zgaruvchiga bog‘liq bo‘lsa, bunday differensial tenglama ***oddiy differensial tenglama***, ***a***gar noma’lum funksiya ikki yoki undan ortiq o‘zgaruvchilarga bog‘liq bo‘lsa, bunday differensial tenglama ***xususiy hosilali differensial tenglama*** deyiladi.

Differensial tenglamaga kirgan hosilalarning eng yuqori tartibiga ***differensial tenglamaning tartibi*** deyiladi.

$y'' - y'\cos x - x^2 y = 0$, $\quad y''' = \cos x$ tenglamalar mos ravishda ikkinchi va uchinchi tartibli oddiy differensial tenglamalarga misol bo‘ladi.

Bu paragrafda biz faqat birinchi tartibli oddiy differensial tenglamalar bilan tanishamiz, u
umumiy holda

$$F(x, y, y') = 0 \qquad (8.1)$$

yoki y'ga nisbatan yechilgan

$$y' = f(x,\ y) \qquad (8.2)$$

ko‘rinishda belgilanadi. Birinchi tartibli differensial tenglamalarni, ba’zida, *differensial shakl* deb ataluvchi

$$P(x, y)dx + Q(x, y)dy = 0 \qquad (8.3)$$

ko‘rinishda yozish qulay.

Differensial tenglamaning yechimi (yoki *integrali*) deb, tenglamaga qo‘yganda uni ayniyatga aylantiradigan har qanday differensiallanuvchi $y = \varphi(x)$ funksiyaga aytiladi. Differensial tenglamaning yechimini topish jarayoni ***differensial tenglamani integrallash*** deb yuritiladi.

8.1-misol.

Ushbu $y = xe^{3x}$ funksiya $y'' - 6y' + 9y = 0$ tenglamaning yechimi ekanini isbotlang.

► $y = xe^{3x}$ funksiya va uning $y' = (3x+1)e^{3x}$, $y'' = (9x+6)e^{3x}$ hosilalarini berilgan tenglamaga qo‘yamiz va ayniyat hosil qilamiz:

$(9x+6)e^{3x} - 6(3x+1)e^{3x} + 9xe^{3x} = e^{3x}(9x + 6 - 18x - 6 + 9x) \equiv 0$. ◄

Birinchi tartibli $y' = f(x,\ y)$ differensial tenglamaning D sohadagi ***umumiy yechimi*** deb quyidagi shartlarni qanoatlantiruvchi $y = \varphi(x,\ C)$ funksiyaga aytiladi: 1) u biror to‘plamga tegishli ixtiyoriy o‘zgarmas C da berilgan tenglamaning yechimi bo‘ladi; 2) ixtiyoriy $y_0 = y(x_0)$ $((x_0, y_0) \in D)$ sohadagi boshlang‘ich shart uchun o‘zgarmas C ning shunday yagona C_0 qiymatini topish mumkinki, $y = \varphi(x, C_0)$ funksiya berilgan boshlang‘ich shartni qanoatlantiradi.

$y = \varphi(x, C)$ umumiy yechimdan o‘zgarmasning muayyan $C = C_0$ qiymatida hosil qilinadigan har qanday $y = \varphi(x, C_0)$ qiymatiga ***xususiy yechim*** deyiladi.

$y' = f(x, y)$ tenglamaning $y_0 = y(x_0)$ boshlang‘ich shartni qanoatlantiruvchi xususiy yechimini topish masalasi ***Koshi masalasi*** deyiladi.

Differensial tenglama har qanday yechimining Oxy tekisligidagi grafigi *integral chiziq* deyiladi. Shunday qilib, $y = \varphi(x, C)$ umumiy yechimga Oxy tekisligida bir-biridan faqat o‘zgarmas C ga farq qiladigan integral chiziqlarlar oilasi, $y_0 = y(x_0)$ boshlang‘ich shartni qanoatlantiruvchi xususiy yechimiga esa bu oilaning $M(x_0, y_0)$ nuqtadan o‘tuvchi egri chizig‘i mos keladi.

8.1-Teorema(Koshi). *Agar* $f(x, y)$ *funksiya* D *sohada uzluksiz bo‘lsa va uzluksiz* $\frac{\partial f}{\partial y}$ *xususiy hosilaga ega bo‘lsa, u holda* $y' = f(x, y)$ *differensial tenglamaning* $y_0 = y(x_0)$ *boshlang‘ich shartdagi yechimi mavjud va yagona.*

Eslatma. Differensial tenglamaning umumiy yechimidan hosil qilib bo‘lmaydigan yechimlari ham bo‘lishi mumkin. Bunday yechimlar ***maxsus yechimlar*** deb ataladi va uning ixtiyoriy nuqtasida Koshi teoremasining shartlari buziladi.

Masalan, $y' = 3\sqrt[3]{(y-1)^2}$ tenglamaning umumiy yechimi: $y = (x+C)^3 + 1$, C ixtiyoriy o‘zgarmas. $y = 1$ funksiya ham tenglamaning yechimi, lekin uni umumiy yechimdan hech qanday o‘zgarmas C da hosil qilib bo‘lmaydi. Demak, $y = 1$ tenglamaning maxsus yechimi ekan.

8.1.1. O‘zgaruvchilari ajraladigan differensial tenglamalar.

Ushbu

$$P(x)dx + Q(y)dy = 0 \tag{8.4}$$

ko‘rinishdagi tenglamaga ***o‘zgaruvchilari ajralgan differensial tenglama*** deyiladi. Uning umumiy integrali

$$\int P(x)dx + \int Q(y)dy = C \tag{8.5}$$

bo‘ladi, bu yerda C - ixtiyoriy o‘zgarmas.

Ushbu

$$M_1(x)N_1(y)dx + M_2(x)N_2(y)dy = 0 \tag{8.6}$$

yoki

$$y' = f_1(x)f_2(y) \tag{8.7}$$

ko'rinishdagi tenglamalarga ***o'zgaruvchilari ajraladigan differensial tenglamalar*** deiladi. Ularda o'zgaruvchilarni ajratish quyidagicha bajariladi. (8.6) tenglamaning ikkala qismini, $N_1(y) \neq 0$, $M_2(x) \neq 0$ deb faraz qilib, $N_1(x)M_1(x)$ ga bo'lamiz. (8.7) tenglamada $y' = \frac{dy}{dx}$ ekanini e'toborga olib, uning ikkala qismini dx ga ko'paytiramiz va $f_2(y) \neq 0$ ga bo'lamiz. Natijada, (8.4) shakldagi o'zgaruvchilari ajralgan

$$\frac{M_1(x)}{M_2(x)}dx + \frac{N_2(y)}{N_1(y)}dy = 0, \quad f_1(x)dx + \frac{dy}{f_2(y)} = 0$$

tenglamalar hosil bo'ladi va ularni integrallanadi:

$$\int \frac{M_1(x)}{M_2(x)}dx + \int \frac{N_2(y)}{N_1(y)}dy = C, \quad \int f_1(x)dx - \int \frac{dy}{f_2(y)} = C.$$

8.2-misol.

Ushbu $y'y\sqrt{\frac{1-x^2}{1-y^2}} + 1 = 0$ differensial tenglamaning umumiy integralini toping.

►Tenglamada $y' = \frac{dy}{dx}$ ekanini e'toborga olib, $\frac{dy}{dx}y\sqrt{\frac{1-x^2}{1-y^2}} + 1 = 0$ ni hosil qilamiz va o'zgaruvchlarini ajratamiz:

$$\frac{y}{\sqrt{1-y^2}}dy + \frac{dx}{\sqrt{1-x^2}} = C.$$

Integrallab, umumiy yechimini topamiz

$C = \arcsin x - \sqrt{1-y^2}$.◄

8.1.2. Bir jinsli differensial tenglamalar.

Agar $f(x,y)$ funksiya uchun $f(tx,ty) = t^m f(x,y)$ shart bajarilsa(bu yerda t-ixtiyoriy parametr), $f(x,y)$ funksiya ***n o'lchovli bir jinsli funksiya*** deb ataladi, bunda n biror son.

Masalan, $f(x,y) = xy - y^2$ funksiya uchun $f(tx,ty) = tx \cdot ty - (ty)^2 = t^2(xy - y^2)$ bo'lib, bu funksiya $n = 2$ o'lchovli bir jinsli funksiya bo'ladi. $f(x,y) = \frac{x^2 + y^2}{xy}$, $n = 0$ o'lchovli bir jinsli funksiyadir.

$y' = f(x,y)$ differensial tenglamada $f(x,y)$ funksiya nol o'lchovli bir jinsli funksiya bo'lsa, bunday differensial tenglamaga ***birinchi tartibli bir jinsli differensial tenglama*** deyiladi.

$f(tx,ty) = f(x,y)$ shartga bo'ysunadigan nol o'lchovli bir jinsli funksiya $f(x,y) = \varphi\left(\frac{y}{x}\right)$ ko'rinishda yozilishi mumkin. Haqiqatdan ham, t parametrni ixtiyoriy tanlab olish mumkin bo'lgani uchun $t = \frac{1}{x}$ deb olamiz. U holda $f(x,y) = f(tx,ty) = f(1,\frac{y}{x}) = \varphi\left(\frac{y}{x}\right)$.

Shunday qilib, bir jinsli differensial tenglamani $y' = \varphi\left(\frac{y}{x}\right)$ ko'rinishda ifodalash mumkin. Bir jinsli tenglama $y = xu(x)$ almashtirish bilan o'zgaruvchilari ajraladigan

$$xu' = \varphi(u) - u \qquad (8.8)$$

differensial tenglamaga keltiriladi.

Eslatma. *Ushbu* $P(x,y)dx + Q(x,y)dy = 0$ *differensial tenglamada* $P(x,y)$ *va* $Q(x,y)$ *funksiyalar bir xil o'lchovli bir jinsli funksiya bo'lsa, u holda bu tenglama bir jinsli differensial tenglama bo'ladi.*

8.3-misol

Ushbu $y' = \frac{x+2y}{2x-y}$ differensial tenglamani yeching.

►Bir jinsli differensial tenglama bo'lgani uchun o'zgaruvchini quyidagicha almashtirami

$$\frac{y}{x} = u \Rightarrow y = ux \Rightarrow y' = u + u'x,$$

$$u + x\frac{du}{dx} = \frac{1+2u}{2-u}, \qquad x\frac{du}{dx} = \frac{1+u^2}{2-u}.$$

O'zgaruvchilarini ajratamiz va integrallaymiz:

$$\frac{2-u}{1+u^2}du = \frac{dx}{x},$$

$$\int\left(\frac{2}{1+u^2} - \frac{u}{1+u^2}\right)du = \int\frac{dx}{x},$$

$$2arctgu - \frac{1}{2}\ln\left|1+u^2\right| = \ln|x| + \ln C,$$

$$2arctg\,\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)=\ln|x|+\ln C,$$

$$2arctg\,\frac{y}{x}=\ln\left(C\sqrt{x^2+y^2}\right).◄$$

32-Auditoriya topshiriqlari

1. $y=\frac{2+Cx}{1+2x}$ _funksiya $2\left(1+x^2y'\right)=y-xy'$ tenglamaning yechimi bo‘la oladimi?

2. $y=Ce^x-e^{-x}$ funksiya $xy''+2y'-xy=0$ tenglamaning yechimi bo‘la oladimi?

3. $e^{\frac{y}{x}}=Cy$ tenglama bilan berilgan oshkormas funksiya $xyy'-y^2=x^2y'$ tenglamaning yechimi bo‘la oladimi?

Quyidagi o‘zgaruvchilari ajraladigan differensial tenglamalarning umumiy yechimini toping:

4. $\left(1+y^2\right)dy+xydy=0$.

5. $y'-xy^2=2xy.$

6. $y'=\frac{(2+x)y}{x(1+x)}$.

7. $x\ln xy'=y^2-1$.

8. $y'=a^{x+y}$, $a>0$, $a\neq 1$.

9. $y'+\sin\frac{x+y}{2}=\sin\frac{x-y}{2}$.

Quyidagi bir jinsli differensial tenglamalarning umumiy yechimini toping:

10. $y'=\frac{y}{x}+\frac{x}{y}$.

11. $y'=\frac{y^2}{x^2+xy}$

12. $4x^2y'=y^2+6xy-3x^2$.

13. $x^2y'-y^2e^{x/y}=xy$

14. $xy'=y+\sqrt{x^2-y^2}\arcsin\frac{y}{x}$.

15. $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0$.

Quyidagi differensial tenglamalarning Koshi masalasi yechimini toping

16. $(x+xy)dy+(y-xy)dx=0,\ y(1)=1$.

17. $y'\sin x=y\ln y,\ y(\pi/2)=e$.

18. $2(y-xy')=1+x^2y'$, $y(1)=1$.

19. $xy'=y(1+\ln y-\ln x)$, $y(1)=e^2$.

20. $xy'=x\sin\dfrac{y}{x}+y$, $y(2)=\pi$.

32-Mustaqil yechish uchun testlar

1. Erkli o'zgaruvchi va noma'lum funksiya hamda uning hosilalari yoki differensiallarini bog'lovchi munosabatga … deyiladi.
A) oddiy differensial tenglama; B) xususiy hosilali differensial tenglama
D) differensial tenglama; E) differensial tenglamaning tartibi.

2. Differensial tenglamaning umumiy yechimidan C ixtiyoriy o'zgarmasning muayyan qiymatida hosil qilinadigan yechimga … deyiladi.
A) integral chiziq; B) xususiy yechim;
D) umumiy yechim; E) maxsus yechim.

3. Differensial tenglamani yeching: $y'=2xy-x^2y'$.
A) $y=C(1+x^2)$; B) $y=C(1-x^2)$;
D) $y=C(x+x^2)$; E) $y=C(x-x^2)$.

4. Quyidagilardan qaysilari o'zgaruvchilari ajraladigan differensial tenglama:
1) $xy+y^2=(2x^2+xy)y'$; 3) $(\sqrt{xy}-\sqrt{x})dx+(\sqrt{y}+\sqrt{xy})dy=0$;
2) $y'=e^{x+y}+e^{x-y}$; 4) $xdy=2(y-\sqrt{xy})dx$?
A) 1),2); B) 1),4); D) 2),3); E) 2),4).

5. Quyidagilardan qaysilari bir jinsli differensial tenglama:
1) $xy+y^2=(2x^2+xy)y'$; 3) $(\sqrt{xy}-\sqrt{x})dx+(\sqrt{y}+\sqrt{xy})dy=0$;
2) $y'=e^{x+y}+e^{x-y}$; 4) $xdy=2(y-\sqrt{xy})dx$?
A) 1),2); B) 1),4); D) 2),3); E) 2),4).

8.1.3. Birinchi tartibli chiziqli differensial tenglamalar.

Noma`lum funksiya va uning hosilasiga nisbatan chiziqli bo'lgan differensial tenglama ***birinchi tartibli chiziqli differensial tenglama*** deb ataladi.

Chiziqli tenglamaning umumiy ko'rinishi quyidagicha:

$$y'+P(x)y=Q(x), \tag{8.9}$$

bu yerda $P(x)$ va $Q(x)$ lar x ning uzluksiz funksiyalari(yoki o‘zgarmaslar).

Agar $P(x) \equiv 0$ yoki $Q(x) \equiv 0$ bo‘lsa, (8.9) tenglama o‘zgaruvchilari ajraladigan tenglama bo‘ladi. $P(x) \neq 0$ va $Q(x) \neq 0$ deb faraz qilamiz. (8.9) tenglamaning yechimini x ning ikkita funksiyasining ko‘paytmasi shaklida izlaymiz(***Bernulli almashtirishi***):

$$y = u(x)v(x) \Leftrightarrow y = uv \tag{8.10}$$

Bu funksiyalardan birini ixtiyoriy tanlab olish mumkin, ikkinchisini esa (8.9) tenglama asosida aniqlanadi. (8.10) tenglikdan y' ni hisoblaymiz

$$y' = u'v + uv' \ . \tag{8.10}$$

y *va* y' ni (8.9) tenglamaga qo‘yamiz

$$u'v + uv' + P(x)uv = Q(x) \tag{8.11}$$

yoki

$$u'v + u(v' + P(x)v) = Q(x) . \tag{8.12}$$

Funksiyalardan birini ixtiyoriy tanlab olish mumkin bo‘lgani uchun v funksiyani qavs ichida turgan ifoda nolga teng bo‘ladigan qilib tanlaymiz, ya’ni

$$v' + P(x)v = 0 \tag{8.13}$$

bo‘lishini talab qilamiz. U holda u funksiyani topish uchun (8.12) tenglikdan quyidagi tenglamani hosil qilamiz:

$$u'v = Q(x) \tag{8.14}$$

Dastlab, (8.13) tenglamadan v ni topamiz:

$$v = C \cdot e^{-\int P(x)dx} .$$

(8.13) tenglamaning noldan farqli birorta yechimi zarur, shuning uchun $C = 1$ deb olamiz. U holda

$$v = e^{-\int P(x)dx} . \tag{8.15}$$

v ning bu topilgan ifodasini (8.14) tenglamaga qo‘yib, u funksiya uchun o‘zgaruvchilari ajraladigan tenglamani hosil qilamiz:

$$u' \cdot e^{-\int P(xdx} = Q(x) .$$

Bu tenglamaning yechimi

$$u = \int Q(x) \cdot e^{\int P(x)dx} \cdot dx + C . \tag{8.16}$$

(8.15) va (8.16) lar u va v ning x orqali ifodalarini beradi. u va v ni (8.10)ga qo‘yib, berilgan chiziqli tenglamaning umumiy yechimini hosil qilamiz

$$y = e^{-\int P(x)dx} \cdot \left(C + \int Q(x) \cdot e^{\int P(x)dx} \cdot dx \right) . \tag{8.17}$$

Quyida biz (8.9) tenglamani yechishning ***o‘zgarmasni variatsiyalash usuli(Lagranj usuli)*** bilan tanishamiz. Buning uchun berilgan tenglamaning

$$y' + P(x)y = 0$$

bir jinsli qismining umumiy yechimi $y = Ce^{-\int P(x)dx}$ topiladi. Ixtiyoriy o‘zgarmas C ni x ning funksiyasi $C(x)$ deb olsak,

$$y = C(x)e^{-\int P(x)dx} \tag{8.18}$$

hosil bo‘ladi. Uni (8.9) tenglamaga qo‘yamiz va

$$C'(x)e^{-\int P(x)dx} = Q(x) \tag{8.19}$$

tenglamani hosil qilamiz. Bundan

$$C(x) = \int Q(x)e^{\int P(x)dx}dx + C \tag{8.20}$$

bo‘ladi va uni (8.18) ga qo‘ysak, berilgan (8.9) tenglamaning umumiy yechimi (8.17) hosil bo‘ladi.

Eslatma. *Ayrim hollarda differensial tenglama x ni y ning funksiyasi deb qaralganda chiziqli bo‘lishi mumkin. Bu hollarda* $\frac{dx}{dy} + p(y)x = q(y)$ *tenglama yuqoridagi biror usulda yechiladi.*

8.4-misol.

Koshi masalasi yechimini toping.

$$y' - y\cos x = \frac{1}{2}\sin 2x, y(0) = 0.$$

► Birinchi tartibli chiziqli differensial tenglama, chunki

$$P(x) = -\cos x,\ Q(x) = \frac{1}{2}\sin 2x$$

$y = uv$ deb belgilab yechiladi.

$$v = e^{-\int P(x)dx} = e^{\int \cos x dx} = e^{\sin x}$$

$$u = \int Q(x)e^{\int P(x)dx}dx = \frac{1}{2}\int \frac{\sin 2x}{e^{\sin x}}dx = \frac{1}{2}\int e^{-\sin x}\sin 2x dx =$$

$$= \int e^{-\sin x}\sin x\cos x dx = \left| \begin{matrix} \sin x = t \\ \cos x dx = dt \end{matrix} \right| = \int te^{-t}dt =$$

$$= \left| \begin{matrix} u = t, & du = dt \\ dv = e^{-t}dt, & v = -e^{-t} \end{matrix} \right| = -te^{-t} + \int e^{-t}dt = -te^{-t} - e^{-t} + C = -\sin xe^{-\sin x} - e^{-\sin x} + C.$$

U holda umumiy yechim

$$y = e^{\sin x}(-\sin xe^{+\sin x} - e^{-\sin x} + C),$$

yoki

$$y = -\sin x - 1 + Ce^{\sin x}.$$

Boshlang‘ich shartga ko‘ra,

$$y(0) = 0 \Rightarrow 0 = 0 - 1 + C \Rightarrow C = 1.$$

Demak, Koshi masalasining yechimi

$$y = -\sin x - 1 + e^{\sin x}. \blacktriangleleft$$

8.1.4. Bernulli tenglamasi.

Ushbu

$$y' + P(x)y = Q(x)y^n \tag{8.21}$$

ko'rinishdagi differensial tenglamaga ***Bernulli tenglamasi*** deyiladi, bu yerda $P(x)$ va $Q(x)$ lar x ning uzluksiz funksiyalari, hamda $n \neq 0$, $n \neq 1$.

Bernulli tenglamasini y^n ga bo'lamiz:

$$y^{-n} \cdot y' + P(x) \cdot y^{-n+1} = Q(x). \tag{8.22}$$

So'ngra $z = y^{-n+1}$ almashtirish bajarib, $z' = (-n+1) \cdot y^{-n} \cdot y'$ ekanligini hisobga olib (8.21)ga qo'ysak,

$$z' + (-n+1) \cdot P(x) \cdot z = (-n+1) \cdot Q(x) \tag{8.23}$$

birinchi tartibli chiziqli differensial tenglamaga ega bo'lamiz. Chiziqli differensial tenglamaning umumiy yechimi topiladi, hamda z o'rniga y^{-n+1} ni qo'yib, Bernulli tenglamasining umumiy yechimi topiladi.

8.5-misol.

Ushbu $y' + xy = xy^3$ differensial tenglamaning umumiy yechimini toping.

► Berilgan tenglamani y^3 bo'lib,

$$y^{-3}y' + y^{-2}x = x$$

tenglamani hosil qilamiz. $y^{-2} = z$ almashtirish bajarsak, $z' = -2y^{-3}y'$ bo'ladi. Bularni tenglamaga qo'yib,

$$z' - 2xz = -2x$$

chiziqli tenglamaga kelamiz. Bu tenglamaning umumiy yechimini (8.17)ga asosan topish mumkin:

$$z = e^{2\int xdx}\left[C + \int (-2x)e^{-2\int xdx}dx\right] = e^{x^2}\left[C - \int 2xe^{-x^2}dx\right] =$$

$$e^{x^2}\left[C + \int e^{-x^2}d(-x^2)\right] = e^{x^2}\left[C + e^{-x^2}\right] = Ce^{x^2} + 1.$$

Shunday qilib,

$$z = C \cdot e^{x^2} + 1$$

bo'ladi, z ning o'rniga y^{-2} ni qo'yib, berilgan Bernulli tenglamasining umumiy yechimini hosil qilamiz

$$y^{-2} = C \cdot e^{x^2} + 1, \quad y^2 = \frac{1}{Ce^{x^2} + 1}. \blacktriangleleft$$

Eslatma. *Ayrim hollarda differensial tenglama x ni y ning funksiyasi deb qaralganda chiziqli bo'lishi mumkin. Bu hollarda* $\frac{dx}{dy}+p(y)x=q(y)$ *tenglama yechiladi.*

33-Auditoriya topshiriqlari

Birinchi tartibli chiziqli differensial tenglamalarni yeching:

1. $xy'-y=x^2\cos x$.

2. $(1+x^2)y'-2xy=(1+x^2)^2$.

3. $xy'=y+\frac{2x^2}{1+x^2}$.

4. $2ydx+(y^2-6x)dy=0$.

5. $y'=\frac{y}{2y\ln y+y-x}$.

Bernulli tenglamalarini yeching:

6. $y'=\frac{1}{x}y-y^2$.

7. $y'+\frac{2y}{x}=\frac{2\sqrt{y}}{\cos^2 x}$.

8. $y'=\frac{1}{x}y+\frac{x}{y}\ln x$.

9. $y^3(2y'+y)=x$.

10. $xdx=\left(\frac{x^2}{y}-y^3\right)dy$.

Koshi masalasi yechimini toping:

11. $y'\cos x-y\sin x=1,\ y\left(\frac{\pi}{2}\right)=0$.

12. $y'\sqrt{1+x^2}+y=\arcsin x=1,\ y(1)=0$.

13. $y'+y=e^{x/2}\sqrt{y},\ y(0)=9/4$.

14. $y'+\frac{3x^2y}{x^3+1}=y^2(x^2+1)\sin x,\ y(0)=1$.

33-Mustaqil yechish uchun testlar

1. Birinchi tartibli chiziqli differensial tenglamaning umumiy ko'rinishi qanday?

A) $y' = f_1(x) f_2(y)$; B) $P(x,y)dx + Q(x,y)dy = 0$;
D) $y' + P(x)y = Q(x)$; E) $y' + P(x)y = Q(x)y^n$.

2. Qaysi biri chiziqli differensial tenglama va qanday almashtirish yordamida yechiladi:

1) $ye^x dx + (2y + e^x)dy = 0$; 2) $e^{x^2} dy + x(1 + 2xy)dx = 0$?

A) 1), $y = ux$; B) 1), $y = uv$; D) 2), $y = uv$; E) 2), $y = ux$ **3.**

$y' - \frac{2y}{x} = x^2 e^x$ chiziqli differensial tenglamani yeching.

A) $y = x^2(xe^x + C)$; B) $y = e^x(x^2 + C)$; D) $y = x^2(e^x + C)$; E) $y = Cx^2 + xe^x$.

4. Quyidagilardan qaysi birida x ni y ning funksiyasi deb chiziqli differensial tenglama hosil qilinadi:

1) $ydx - (2x + y^3)dy = 0$; 2) $y'(x + y^2) = y$?

A) faqat 1) da; B) 1) va 2) da; D) faqat 2)da ; E) to‘g‘ri javob yo‘q.

5. Quyidagilardan qaysi birida $z = y^{-1}$ almashtirish orqali chiziqli differensial tenglama hosil qilinadi:

1) $(y - y^2)dx - (x - 1)dy = 0$; 2) $xy + y^2 = (2x^2 + xy)y'$; 3) $y' - 2ytgx + y^2 \sin^2 x = 0$?

A) faqat 1) da; B) 1) va 2)da; D) 1) va 3) da ; E) faqat 3)da.

8.2. Yuqori tartibli differensial tenglamalar. Tartibi pasayadigan differensial tenglamalar

Asosiy tushunchalar. Ushbu

$$F\left(x, y, y', y'', ..., y^{(n)}\right) = 0 \qquad (8.24)$$

ko‘rinishdagi tenglama $n-$ ***tartibli differensial tenglama*** tenglama deyiladi.

Berilgan tenglamani ayniyatga aylantiradigan n marta differensiallanuvchi $\varphi(x)$ funksiyaga uning ***yechimi*** deyiladi. Bunday tenglamalar uchun uning $y(x_0) = y_0$, $y'(x_0) = y'_0$, … \ $y^{(n-1)}(x_0) = y_0^{(n-1)}$ boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish masalasi ***Koshi masalasi*** deyiladi. Agar $y = \varphi(x, C_1, C_2, ... C_n)$ funksiya o‘zgarmas $C_1, C_2, ... C_n$ larning mos qiymatlarida tenglamaga qo‘yilgan ixtiyoriy Koshi masalasining yechimi bo‘lsa, bu funksiya (8.24) tenglamaning ***umumiy yechimi*** deyiladi. Umumiy yechimdan $C_1, C_2, ... C_n$ larning muayyan qiymatida hosil qilingan yechimga ***xususiy yechim*** deyiladi. $n-$tartibli differensial tenglamalarni ayrim hollardagina integrallash mumkin. Bulardan biri tartibini pasaytirish mumkin bo‘lgan differensial tenglamalardir.

$y^{(n)} = f(x)$ **ko‘rinishdagi differensial tenglamalar.** Bunday tenglamalarning umumiy yechimi n marta ketma-ket integrallash orqali topiladi.

$$y^{(n)} = f(x)\ y^{(n-1)} = \int f(x)dx = f_1(x) + \overline{C}_1,\ y^{(n-2)} = \int \left(f_1(x) + \overline{C}_1\right)dx = f_2(x) + \overline{C}_1 x + \overline{C}_2,$$

$\ldots,\ y = f_n(x) + C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_{n-1}x + C_n$, bu yerda $C_i = \overline{C}_i/(n-i)!$.

8.6-misol.

Ushbu $y'' = \dfrac{1}{x} - \sin x$ differensial tenglamani yeching.

►Ikki marta integrallaymiz:

$$y' = \int\left(\frac{1}{x} - \sin x\right)dx = \ln x - \cos x + C_1,$$

$$y = \int(\ln x - \cos x + C_1)dx = x\ln x - x + \sin x + C_1 x + C_2.$$

Demak, berilgan tenlamaning yechimi: $y = x\ln x - x + \sin x + C_1 x + C_2$.◄

$F\left(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}\right) = 0$ **ko‘rinishdagi differensial tenglamalar.** Bunday tenglamalar
$y^{(k)} = z$ almashtirish orqali tartibi pasaytiriladi. Bu holda

$$F\left(x, z, z', \ldots, z^{(n-k)}\right) = 0$$

$(n-k)$ – tartibli differensial tenglama hosil bo‘ladi. Xususan, $n = k+1$ uchun birinchi tartibli $F(x, z, z') = 0$ differensial tenglama hosil bo‘ladi. Bu tenglamaning yechimini k marta integrallab berilgan tenglamaning yechimi topiladi.

8.7-misol.

Ushbu $xy'' = y'\ln\dfrac{y'}{x}$ differensial tenglamani yeching.

► $y' = z$ deb belgilab yechamiz.

$$xz' = z\ln\frac{z}{x},\quad z' = \frac{z}{x}\ln\frac{z}{x}.$$

Hosil bo‘lgan bir jinsli differensial tenglamada $z = ux$ almashtirish bajaramiz.

$$u'x + u = u\ln u \quad \text{yoki} \quad \frac{du}{u(\ln u - 1)} = \frac{dx}{x}.$$

Integrallab,

$$\ln(\ln u - 1) = \ln x + \ln C_1 \quad \text{yoki} \quad \ln u - 1 = C_1 x.$$

Bundan $u=e^{1+C_1x}$ yechimni olamiz va u dan y o‘zgaruvchiga qaytamiz, ya’ni $y'=xe^{1+C_1x}$. Natijada, $y=\int xe^{1+C_1x}dx=\frac{1}{C_1}xe^{1+C_1x}-\frac{1}{C_1^{\,2}}e^{1+C_1x}+C_2$ ◄

8.8-misol.

Ushbu $y'''ctgx+y''=2$ differensial tenglamani yeching.

► $y''=z$ deb belgilab yechamiz:

$z'+tgx\cdot z=2tgx$.

Hosil bo‘lgan chiziqli differensial tenglamada $z=uv$ almashtirish bajaramiz.

$v'+tgx\cdot v=0$ va $u'v=2tgx$.

Birinchi tenglamaning $v=\cos x$ yechimini ikkinchi tenglamaga qo‘yamiz:

$u'=\frac{2\sin x}{\cos^2 x}$.

Bundan $u=\frac{2}{\cos x}+C_1$ hosil bo‘ladi. Natijada, birinchi tartibli chiziqli differensial tenglamaning yechimi $z=uv=\cos x\left(\frac{2}{\cos x}+C_1\right)=2+C_1\cos x$ ga ega bo‘lamiz. Bu tenglikni ikki marta integrallab berilgan tenglamaning yechimi topiladi:

$$y'=\int(2+C_1\cos x)dx=2x+C_1\sin x+C_2;$$

$$y=\int(2x+C_1\sin x+C_2)dx=x^2-C_1\cos x+C_2x+C_3.$$ ◄

$F\left(y,y',y'',...,y^{(n)}\right)=0$ **ko‘rinishdagi differensial tenglamalar.** Erkli x o‘zgaruvchi oshkor qatnshmagan bunday tenglamalar $y'=z(y)$ almashtirish orqali tartibi pasaytiriladi. Bu holda, $y''=z'(y)\cdot y'=z'z$, $y'''=z''z^2+(z')^2z$, ... va hakozo almashtirishlardan foydalanib, tenglamaning tartibi bittaga pasaytiriladi.

8.9-misol.

Koshi masalasini yeching: $2yy''=1+\left(y'\right)^2$, $y(0)=2,\ y'(0)=1$.

► $y'=z,\ y''=z\cdot\frac{dz}{dy}$ deb belgilab yechamiz:

$$2yz\cdot\frac{dz}{dy}=1+z^2.$$

Hosil bo‘lgan birinchi tartibli o‘zgaruvchilari ajraladigan differensial tenglamani yechamiz.

$$\frac{2zdz}{1+z^2}=\frac{dy}{y} \text{ yoki } \ln\left(1+z^2\right)=\ln y+\ln C_1.$$

Bundan $z=\pm\sqrt{C_1y-1}$, ya'ni $y'=\pm\sqrt{C_1y-1}$ hosil bo'ladi. $y(0)=2,\ y'(0)=1$ boshlang'ich shartlarga ko'ra, $C_1=1$. Natijada,

$$y'=\sqrt{y-1} \text{ yoki } \frac{dy}{\sqrt{y-1}}=dx$$

tenglikga ega bo'lamiz. Bu tenglikni integrallab berilgan tenglamaning umumiy integralini topamiz va $y(0)=2$ boshlang'ich shartdan foydalanamiz:

$$2\sqrt{y-1}=x+C_2 \text{ va } C_2=2 \Rightarrow 2\sqrt{y-1}=x+2 .$$

Demak, Koshi masalasining yechimi: $y=\left(\frac{x+2}{2}\right)^2+1$.◄

34-Auditoriya topshiriqlari

Quyidagi tartibi pasayadigan differensial tenglamalarni yeching:

1. $y''=2\sin x\cos^2 x-\sin^3 x$.

2. $y'''\sin^4 x=\sin 2x$.

3. $(1-x^2)y''-xy'=2$.

4. $x^2y'''=(y'')^2$.

5. $xy''+y'=(y')^2$

6. $y''(2y+3)-2(y')^2=0$

7. $yy''-(y')^2=y^2\ln y$.

8. $y(1-\ln y)y''+(1+\ln y)(y')^2=0$.

Koshi masalasi yechimini toping.

1. $y''=\frac{\ln x}{x^2},\ y(1)=3,\ y'(1)=1$.

2. $yy''-(y')^2=y^2\ln y,\ y(0)=1,\ y'(0)=1$.

3. $xy'''-y''=x^2+1,\ y(-1)=0,\ y'(-1)=1,\ y''(-1)=0$.

4. $y''=2-y,\ y(0)=2,\ y'(0)=2$.

34-Mustaqil yechish uchun testlar

1. Quyidagi differensial tenglamalardan qaysi birini tartibini pasaytirib hisoblash mumkin va bu qanday amalga oshiriladi:

1) $y''=\frac{1}{4\sqrt{y}}$; 2) $y''=y'+\frac{\ln y}{x^2}$?

A) faqat 1) ni, 2 marta integrallab;

B) 1) va 2) ni, $y'=z(y)$ va $y'=z(x)$ almashtirishlar yordamida;

D) faqat 2) ni $y' = z(x)$ almashtirish yordamida;
E) faqat 1) ni, $y' = z(y)$ almashtirish yordamida.

2. Quyidagi differensial tenglamalardan qaysi birini tartibini pasaytirish mumkin emas?
A) $xy'' = y'(\ln y' - \ln x)$; B) $yy'' = (y')^2 + x$; D) $x^3 y''' = 4y''$;
E) $x^3 y''' = 4\ln x$.

3. Quyidagi differensial tenglamalardan qaysi birini tartibini pasaytirish mumkin emas?
A) $y^{(n)} = f(x)$; B) $F(x, y', y'') = 0$; D) $F(x, y, y'') = 0$;
E) $F(y, y', y'') = 0$.

4. $xy'' = y'$ differensial tenglamani yeching.
A) $y = C_1 x^2 + C_2$; B) $y = x^2 + C_1 x + C_2$; D) $y = \frac{C_1}{2} x^2 + C_2 x$; E) to‘g‘ri javob yo‘q.

5. Quyidagi differensial tenglamalarning qaysi birini $y' = z$, $y'' = z \cdot z'$ almashtirish orqali tartibi pasaytiriladi?
A) $y'' = (y')^2 + y'$; B) $yy'' = (y')^2 + x$; D) $x^3 y'' = 4y'$;
E) $x^3 y''' = 4\ln x$.

8.3.Yuqori tartibli chiziqli differensial tenglamalar

Asosiy tushunchalar. Agar n - tartibli differensial tenglamada izlanayotgan funksiya va uning hosilalari birinchi darajada qatnashsa, bunday tenglama ***n - tartibli chiziqli differensial tenglama*** deyiladi.
U quyidagi ko‘rinishga ega:

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) y = f(x).$$

Bu yerda $a_0(x), a_1(x), \ldots, a_n(x)$ lar va $f(x)$ biror D sohada berilgan x ning ma’lum uzluksiz funksiyalari (o‘zgarmas bo‘lishi ham mumkin). $a_i(x)\left(i = \overline{1,n}\right)$ funksiyalar ***tenglamaning koeffitsientlari*** deyiladi, shu bilan birga $a_0(x) = 1$ (agar u 1 ga teng bo‘lmasa tenglamaning hamma hadlarini unga bo‘lishimiz mumkin), $f(x)$ funksiya esa ***ozod hadi*** deyiladi.
Agar $f(x) \neq 0$ bo‘lsa, ushbu

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) y = f(x) \qquad (8.25)$$

tenglama ***n - tartibli chiziqli bir jinsli bo‘lmagan differensial tenglama*** deyiladi.

Agar $f(x) = 0$ bo‘lsa, ushbu

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) y = 0 \qquad (8.26)$$

tenglama ***n - tartibli chiziqli bir jinsli differensial tenglama*** deyiladi.

8.3.1. Chiziqli bir jinsli differensial tenglamalar yechimining tuzilishi.

(8.26) tenglamalarning umumiy yechimlarini topishda ularning $y_1(x), y_2(x),..., y_n(x)$ xususiy yechimlarining chiziqli bog'liq va chiziqli erkliligi asosiy rol o'ynaydi.

Agar bir vaqtda nol bo'lmagan shunday $\alpha_1, \alpha_2, ..., \alpha_n$ o'zgarmas sonlar mavjud bo'lsaki, $\sum_{i=1}^{n} \alpha_i y_i(x) \equiv 0$, $x \in (a,b)$ ayniyat o'rinli bo'lsa, $y_1(x), y_2(x),..., y_n(x)$ funksiyalar sistemasi $x \in (a,b)$ da ***chiziqli bog'liq sistema*** deyiladi. Agar bu ayniyat barcha $\alpha_i = 0$ bo'lgandagina bajarilsa, $y_1(x), y_2(x),..., y_n(x)$ funksiyalar sistemasi $x \in (a,b)$da ***chiziqli erkli sistema*** deyiladi.

Agar $x \in (a,b)$da (n-1) –tartibligacha hosilalari uzluksiz bo'lgan $y_1(x), y_2(x),..., y_n(x)$ funksiyalar sistemasining ***Vronskiy determinanti*** $W(x) = W[y_1, y_2, ..., y_n]$ $x \in (a,b)$da aynan nolga teng bo'lsa, ya'ni

$$W(x) = \begin{vmatrix} y_1 & y_2 & ... & y_n \\ y_1' & y_2' & ... & y_n' \\ . & . & . & . \\ y_1^{(n-1)} & y_2^{(n-1)} & ... & y_n^{(n-1)} \end{vmatrix} \equiv 0$$

bo'lsa, $y_1(x), y_2(x),..., y_n(x)$ funksiyalar sistemasi $x \in (a,b)$ da chiziqli bog'liq sistema bo'ladi. Agar (a,b) oraliqning hech bir nuqtasida $W(x) = W[y_1, y_2, ..., y_n] \neq 0$ bo'lsa, $y_1(x), y_2(x),..., y_n(x)$ funksiyalar sistemasi $x \in (a,b)$da chiziqli erkli sistema bo'ladi.

n - tartibli chiziqli bir jinsli differensial tenglamaning n ta chiziqli erkli yechimlari sistemasi uning ***fundamental yechimlari sistemasi*** deyiladi.

8.2-Teorema. *Agar $y_1(x), y_2(x),..., y_n(x)$ funksiyalar chiziqli bir jinsli differensial tenglama yechimlarining fundamental sistemasi bo'lsa, u holda bu tenglamaning umumiy yechimi $y_1(x), y_2(x),..., y_n(x)$ yechimlarining*

$$y = C_1 y_1(x) + C_2 y_2(x) + ... + C_n y_n(x),$$

chiziqli kombinatsiyasidan iborat bo'ladi, bu yerda $C_1, C_2, ..., C_n$ lar ixtiyoriy o'zgarmaslar.

8.3.2. O'zgarmas koeffitsientli chiziqli bir jinsli differensial tenglamalar.

Ushbu

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y = 0 \tag{8.28}$$

tenglama ***o'zgarmas koeffitsientli n - tartibli chiziqli bir jinsli differensial tenglama*** deyiladi. Bu yerda $a_0, a_1, ..., a_n$ koeffitsientlar - biror haqiqiy sonlar. (8.28) tenglamaning fundamental yechimlari sistemasini topish uchun uning

$$k^n + a_1 k^{n-1} + ... + a_{n-1} k + a_n = 0 \qquad (8.29)$$

xarakteristik tenglamasi tuziladi. (8.29) tenglama n - tartibli bo'lgani uchun uning n ta ildizi mavjud, ular haqiqiy yoki kompleks, orasida karralilari ham bo'lishi mumkin.

(8.28) tenglamaning umumiy yechimi (8.29) tenglama yechimlarining xarakteriga bog'liq quyidagicha tuziladi:

1) har bir k sodda haqiqiy yechimga umumiy yechimda Ce^{kx} ko'rinishdagi qo'shiluvchi mos keladi;
2) har bir m karrali k haqiqiy yechimga umumiy yechimda $\left(C_1 + C_2 x + ... + C_m x^{m-1}\right)e^{kx}$ ko'rinishdagi qo'shiluvchi mos keladi;
3) har bir juft $k_{1,2} = \alpha \pm \beta i$ qo'shma kompleks sodda yechimga umumiy yechimda $e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$ ko'rinishdagi qo'shiluvchi mos keladi;
4) har bir juft m karrali $k_{1,2} = \alpha \pm \beta i$ qo'shma kompleks yechimga umumiy yechimda $e^{\alpha x}\left[\left(C_1 + C_2 x + ... + C_m x^{m-1}\right)\cos\beta x + \left(C_1 + C_2 x + ... + C_m x^{m-1}\right)\sin\beta x\right]$ ko'rinishdagi qo'shiluvchi mos keladi.

Xususan, ushbu

$$y'' + py' + qy = 0 \qquad (8.30)$$

o'zgarmas koeffitsientli ikkinchi tartibli chiziqli bir jinsli differensial tenglamaning

$$k^2 + pk + q = 0 \qquad (8.31)$$

xarakteristik tenglamasi ildizlari uchun uch hol:

1) haqiqiy va har xil $k_1 \neq k_2$;
2) haqiqiy va teng $k_1 = k_2 = k$;
3) qo'shma-kompleks $k_{1,2} = \alpha \pm \beta i$ bo'lishi mumkin.

Bu hollarga (8.28) tenglamaning quyidagi fundamental yechimlari va umumiy yechimi mos keladi:

1) $y_1 = e^{k_1 x}, y_2 = e^{k_2 x}, y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$.
2) $y_1 = e^{k_1 x}, y_2 = xe^{k_2 x}, y = (C_1 + C_2 x)e^{k_1 x}$.
3) $y_1 = e^{\alpha x}\cos\beta x, y_2 = e^{\alpha x}\sin\beta x, y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

8.10-misol.

Ushbu $y''-5y'+6y=0$ differensial tenglamaning umumiy yechimini toping.

►Berilgan tenglamaga mos xarakteristik tenglamani tuzamiz:

$$k^2-5k+6=0.$$

Xarakteristik tenglamaning ildizlari $k_1=2,\quad k_2=3$ bo‘lgani uchun umumiy yechim:

$$y=C_1e^{2x}+C_2e^{3x}.◄$$

8.11-misol.

Ushbu $y'''+6y''+9y'=0$ differensial tenglamaning umumiy yechimini toping.

► Berilgan tenglamaning

$$k^3+6k^2+9k=0$$

xarakteristik tenglamasi $k_1=0$ sodda va $k_2=k_3=-3$ ikki karrali haqiqiy ildizlarga ega. Shuning uchun berilgan tenglamaning umumiy yechimi:

$$y=C_1+(C_2+C_3x)e^{-3x}. ◄$$

8.12-misol.

Ushbu $y''-4y'+13y=0$ differensial tenglamaning umumiy yechimini toping.

► Berilgan tenglamaning xarakteristik tenglamasi

$$k^2-4k+13=0$$

$k_{1,2}=2\pm 3i$ qo‘shma-kompleks ildizga ega. Shuning uchun berilgan tenglamaning umumiy yechimi:

$$y=e^{2x}(C_1\cos x+C_2\sin x). ◄$$

8.13-misol.

Ushbu $y''-y'-2y=0$ differensial tenglamaning $x=0$ bo‘lganda $y=8,\quad y'=7$ bo‘ladigan xususiy yechimini toping.

►Berilgan tenglamaga mos xarakteristik tenglama

$$k^2-k-2=0$$

bo‘lib, ildizlari - $k_1=-1,\quad k_2=2$. Demak, tenglamaning umumiy yechimi

$$y=C_1e^{-x}+C_2e^{2x}$$

bo‘ladi. Oxirgi tenglikdan hosila olamiz

$$y'=-C_1e^{-x}+2C_2e^{2x}.$$

$x=0$ bo‘lganda $y=8,\quad y'=7$ boshlang‘ich shartlarga asosan,

$$\begin{cases} C_1+C_2=8 \\ -C_1+2C_2=7 \end{cases}$$

tenglamalar sistemasi hosil bo‘ladi. Oxirgi tenglamalar sistemasidan $C_1=3,\quad C_2=5$ larni aniqlaymiz. Shunday qilib, izlanayotgan xususiy yechim:

$$y = 3e^{-x} + 5e^{2x}. ◄$$

8.3.3. O‘zgarmas koeffitsientli chiziqli bir jinsli bo‘lmagan differensial tenglamalar.

Ushbu

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y = f(x) \qquad (8.32)$$

tenglama ***o‘zgarmas koeffitsientli n - tartibli chiziqli bir jinsli bo‘lmagan differensial tenglama*** deyiladi.

8.3-Teorema. *Chiziqli bir jinsli bo‘lmagan differensial tenglamaning umumiy yechimi tenglamaning biror xususiy yechimi va bu tenglamaga mos chiziqli bir jinsli differensial tenglamaning umumiy yechimi yig‘indusiga teng.*

Chiziqli bir jinsli bo‘lmagan differensial tenglama umumiy yechimini topish uchun, agar unga mos chiziqli bir jinsli differensial tenglamaning umumiy yechimi ma’lum deb hisoblasak, bitta xususiy yechimini aniqlash kifoya.

Quyida o‘zgarmas koeffitsientli ikkinchi tartibli chiziqli bir jinsli bo‘lmagan

$$y'' + py' + qy = f(x) \qquad (8.33)$$

tenglama uchun ***ixtiyoriy o‘zgarmasni variatsiyalash usulini*** ko‘rib chiqamiz. y_1, y_2 funksiyalar mos chiziqli bir jinsli tenglama (8.30) ning fundamental yechimlari bo‘lsin. U holda (8.33)ning umumiy yechimni

$$y(x) = C_1(x)y_1 + C_2(x)y_2$$

ko‘rinishda qidiriladi, bu yerda $C_1(x), C_2(x)$ funksiyalar

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0 \\ C_1'(x)y_1 + C_2'(x)y_2 = f(x) \end{cases}$$

tenglamalar sistemasidan aniqlanadi. Sistemaning yechimi esa

$$C_1(x) = -\int \frac{y_2 f(x)dx}{W[y_1, y_2]}; \quad C_2(x) = -\int \frac{y_1 f(x)dx}{W[y_1, y_2]}$$

formulalardan topiladi.

8.14-misol.

Ushbu $y'' + y = tgx$ differensial tenglamani yeching.

►Berilgan tenglamaga mos bir jinsli tenglamaning xarakteristik tenglamasi

$$k^2 + 1 = 0$$

$k_{1,2} = \pm i$ ildizlarga ega, $\bar{y} = C_1 \cos x + C_2 \sin x$ uningi umumiy yechimi.

Berilgan tenglamaning umumiy yechimini

$$y = C_1(x)\cos x + C_2(x)\sin x$$

ko‘rinishda qidiramiz. $C_1(x), C_2(x)$ funksiyalarni

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0, \\ -C_1'(x)\sin x + C_2'(x)\cos x = tgx \end{cases}$$

tenglamalar sistemasidan aniqlanadi. Bu sistemani yechib, so‘ng uniintegrallab quyidagilarni topamiz:

$$C_1'(x) = -\frac{\sin^2 x}{\cos x}, \quad C_2'(x) = \sin x,$$

$$C_1(x) = -\int \frac{\sin^2 x}{\cos x} dx = \sin x - \ln tg\left(\frac{x}{2} + \frac{\pi}{4}\right) + C_1,$$

$$C_2(x) = -\cos x + C_2.$$

Demak, berilgan differensial tenglamaning umumiy yechimi:

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln tg\left(\frac{x}{2} + \frac{\pi}{4}\right). \blacktriangleleft$$

35-Auditoriya topshiriqlari

Differensial tenglamalarning umumiy yechimini toping:

1. $y'' - 2y' - 4y = 0$.

2. $y'' - 4y' + 4y = 0$.

3. $y'' + 8y' + 25y = 0$.

4. $y''' + 9y' = 0$.

5. $y^{IV} + 5y'' + 4y = 0$.

6. $y^{IV} - 6y''' + 9y'' = 0$.

7. $y^{IV} + 16y = 0$.

8. $y^{VI} + 3y^{IV} + 3y'' + y = 0$.

Tenglamalarning berilgan boshlang‘ch shartni qanoatlantiradigan yechimini toping:

9. $y'' + 5y' + 6y = 0;\ y(0) = 1, y'(0) = -6$.

10. $y'' - 10y' + 25y = 0;\ y(0) = 0, y'(0) = 1$.

11. $y'' - 6y' + 18y = 0;\ y(0) = 2, y'(0) = 3$.

12. $9y'' + y = 0;\ y\left(\frac{3\pi}{2}\right) = 2, y'\left(\frac{3\pi}{2}\right) = 0$.

Tenglamalarni ixtiyoriy o‘zgarmasni variatsiyalash usulida yeching:

13. $y'' + y = \frac{1}{\sqrt{\cos 2x}}$.

14. $y'' + y = ctg 2x$.

15. $y'' + 5y' + 6y = \frac{1}{1 + e^{2x}}$.

16. $y''\cos\frac{x}{2}+\frac{1}{4}y\cos\frac{x}{2}=1$.

35-Mustaqil yechish uchun testlar

1. Quyidagi funksiyalar sistemasidan qaysi biri chiziqli erkli bo‘ladi:
1) $e^{-x},\ e^{x},\ chx$; 2) $\ln 2x,\ \ln 3x,\ \ln 4x$?
A) 2) - chiziqli erkli; D) ikkalasi ham chiziqli erkli;
B) 1) - chiziqli erkli; E) ikkalasi ham chiziqli bog‘liq.

2. $y''-8y'+7y=0$ differensial tenglamaning umumiy yechimini aniqlang.
A) $y=C_1e^{-x}+C_2e^{-7x}$; B) $y=C_1e^{3x}+C_2e^{4x}$; D) $y=C_1e^{-3x}+C_2e^{-4x}$;
E) $y=C_1e^{x}+C_2e^{7x}$

3. $y^{IV}-y=0$ differensial tenglamaning fundamental yechimlari sistemasi nechta funksiyadan iborat?
A) 2 ta; B) 4 ta; D) 3 ta ; E) 1 ta.

4. $y''+2y'=0$ differensial tenglamaning fundamental yechimlarini aniqlang.
A) $y_1=e^{-2x},\ y_2=2e^{-2x}$; B) $y_1=e^{-2x},\ y_2=xe^{-2x}$; D) $y_1=x,\ y_2=e^{-2x}$;
E) $y_1=1,\ y_2=e^{-2x}$.

5. $y''+py'+qy=f(x)$ tenglamani o‘zgarmasni variatsiyalash usulida yechishda $C_1(x)$ va $C_2(x)$ funksiyalarni aniqlovchi tenglamalar sistemasini tuzing.

A) $\begin{cases}C_1y_1+C_2y_2=0\\ C_1y_1'+C_2y_2'=f(x)\end{cases}$; B) $\begin{cases}C_1'y_1+C_2'y_2=0\\ C_1'y_1'+C_2'y_2'=f(x)\end{cases}$;

D) $\begin{cases}C_1y_1'+C_2y_2'=0\\ C_1y_1'+C_2y_2'=f(x)\end{cases}$; E) $\begin{cases}C_1'y_1+C_2'y_2=0\\ C_1y_1'+C_2y_2'=f(x)\end{cases}$.

8.3.4. O‘ng tomoni maxsus ko‘rinishdagi o‘zgarmas koffitsientli bir jinsli bo‘lmagan chiziqli differensial tenglamalar.

Ushbu

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y'+a_ny=f(x) \qquad (8.34)$$

tenglamada $f(x)=e^{\alpha x}[P_n(x)\cos\beta x+Q_m(x)\sin\beta x]$ ko‘rinishga ega bo‘lsa, o‘ng tomoni maxsus ko‘rinishdagi o‘zgarmas koeffitsientli bir jinsli bo‘lmagan chiziqli differensial tenglama deb ataymiz. Bu yerda $P_n(x)$ va $Q_m(x)$ lar mos ravishda n – va m – darajali ko‘phadlar, α va β - o‘zgarmaslar. (8.34) tenglama ***xususiy yechimni tanlash usuli(noma’lum koeffitsientlar usuli)***da yechiladi. Buning uchun xususiy yechim

$$y_* = x^r e^{\alpha x}\left[\overline{P}_l(x)\cos\beta x + \overline{Q}_l(x)\sin\beta x\right] \qquad (8.35)$$

ko‘rinishda qidiriladi. Bu yerda r xarakteristik tenglama $k^n + a_1 k^{n-1} + ... + a_{n-1}k + a_n = 0$ ning $\alpha + \beta i$ ildizining karraligi ko‘rsatkichi(agar bunday ildiz bo‘lmasa, $r = 0$), $\overline{P}_l(x)$ va $\overline{Q}_m(x)$ lar noma’lum koeffitsientli l – darajali ko‘phadlar, bunda $l = \max(n,m)$.

Agar $f(x)$ funksiyada $\cos\beta x$ va $\sin\beta x$ lardan faqat bittasi ishtirok etsa ham y_* da ularning ikkalasi olinadi.

y_* xususiy yechimni (8.34) tenglamaga qo‘yib, ayniyatdan noma’lum o‘zgarmas koeffitsientlar topiladi.

Agar (8.34) da $f(x) = f_1(x) + f_2(x)$ bo‘lib, y_{1*}, y_{2*} lar mos ravishda $y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1}y' + a_n y = f_1(x)$ va $y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1}y' + a_n y = f_2(x)$ tenglamalarning xususiy yechimlari bo‘lsa, (8.34) tenglamaning xususiy yechimi $y_* = y_{1*} + y_{2*}$ ko‘rinishda bo‘ladi.

Eslatma. $f(x)$ funksiya ko‘rinishining quyidagi xususiy hollari mavjud:

1) $f(x) = Ae^{\alpha x}$, $A -$ o`zgarmas $(\alpha + \beta i \equiv \alpha)$.
2) $f(x) = A\cos\beta x + B\sin\beta x$, A va $B -$ o`zgarmas $(\alpha + \beta i \equiv \beta i)$.
3) $f(x) = P_n(x)$ $(\alpha + \beta i \equiv 0)$.
4) $f(x) = P_n(x)e^{\alpha x}$ $(\alpha + \beta i \equiv \alpha)$.
5) $f(x) = P_n(x)\cos\beta x + Q_m(x)\sin\beta x$ $(\alpha + \beta i \equiv \beta i)$.
6) $f(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$, A va $B -$ o`zgarmas.

8.15-misol.

Ushbu $y'' - 3y' + 2y = e^{3x}(x^2 + x)$ differensial tenglamani yeching

► $f(x) = e^{3x}(x^2 + x)$; $P_n(x) = x^2 + x; \alpha = 3$

$y'' - 3y' + 2y = 0$ bir jinsli qismining xarakteristik tenglamasi va uning ildizlari: $k^2 - 3k + 2 = 0 \Rightarrow k_1 = 2$, $k_2 = 1$ .Demak, $\overline{y} = C_1 e^{2x} + C_2 e^x$ - bir jinsli tenglamaning umumiy yechimi.

$\alpha = 3$ xarakteristik tenglamaning ildizi emas, chunki xarakteristik tenglamaning ildizlari $k_1 = 2$, $k_2 = 1$. Bundan, $r = 0$.

Endi berilgan tenglamaning y_* xususiy yechimini $y_* = (Ax^2 + Bx + C)\cdot e^{3x}$ ko‘rinishda izlaymiz. Bunda $\overline{P}_n(x) = Ax^2 + Bx + C$, $A,\ B,\ C$ lar noma’lum.

$y'_* = (3Ax^2 + 3Bx + 3C + 2Ax + B)e^{3x}$, $y''_* = (9Ax^2 + 9Bx + 9C + 12Ax + 6B + 2A)e^{3x}$

y_*, y'_*, y''_* larni berilgan tenglamaga qo‘yib ixchamlasak,

$$2Ax^2 + (6A + 2B)x + 2A + 3B + 2C = x^2 + x$$

$$\left.\begin{aligned} x^2&:\ 2A = 1 \\ x&:\ 6A + 2B = 1 \\ x^0&:\ 2A + 3B + 2C = 0 \end{aligned}\right\} \Rightarrow A = \frac{1}{2}; B = -1; C = 1.$$

$y_* = \left(\frac{x^2}{2} - x + 1\right)e^{3x}$ - xususiy yechim topildi. Demak, umumiy yechim:

$$y = \bar{y} + y_* = C_1 e^{2x} + C_2 e^x + \left(\frac{x^2}{2} - x + 1\right)e^{3x}.$$ ◀

8.16-misol.

Ushbu $y'' + y' - 2y = e^x(\cos x - 7\sin x)$ differensial tenglamani yeching

▶ Bir jinsli qismini yechamiz:

$y'' + y' - 2y = 0,\ y = e^{\alpha x} \Rightarrow k^2 + k - 2 = 0 \Rightarrow k_1 = 1, k_2 = -2$, demak, $\bar{y} = C_1 e^x + C_2 e^{-2x}$

$\alpha = 1,\ \beta = 1;\quad \alpha + \beta i = 1 + i$ - xarakteristik tenglamaning ildizi emas.

$y_* = e^x(A\cos x + B\sin x)$,

$y'_* = ((A+B)\cos x + (B-A)\sin x)e^x$,

$y''_* = (-2A\sin x + 2B\cos x)e^x$.

Topilganlarni tenglamaga qo‘yamiz:

$-(3A+B)\sin x + (3B-A)\cos x = \cos x - 7\sin x$;

$\left.\begin{matrix} 3A+B=7 \\ 3B-A=1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} A=2 \\ B=1 \end{matrix}\right\}$, bundan, $y_* = e^x(2\cos x + \sin x)$. Demak,

$y = \bar{y} + y_* = C_1 e^x + C_2 e^{-2x} + e^x(2\cos x + \sin x)$ - umumiy yechim.◀

8.17-misol.

Ushbu $y'' + y = 2\sin x$ differensial tenglamani yeching

▶ $y'' + y = 0,\quad k^2 + 1 = 0 \Rightarrow k_{1,2} = \pm i$.

$\bar{y} = C_1\cos x + C_2\sin x$ - bir jinsli qismining umumiy yechimi.

$\beta i = i$ - demak, βi xarakteristik tenglamaning ildizi, $r = 1$.

$y_* = x(A\cos x + B\sin x)$ ko‘rinishda izlaymiz. A, B -?

$y'_* = A\cos x + B\sin x + x(-A\sin x + B\cos x)$

$y''_* = -2A\sin x + 2B\cos x - x(A\cos x + B\sin x)$.

Topilganlarni tenglamaga qo‘yamiz:

$-2A\sin x + 2B\cos x - Ax\cos x - Bx\sin x + Ax\cos x + Bx\sin x = 2\sin x$,

$-2A\sin x + 2B\cos x = 2\sin x$.

$\left.\begin{matrix} -2A=2 \\ 2B=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} A=-1 \\ B=0 \end{matrix}\right\}$, bundan, $y_* = -x\cos x$. Demak, umumiy yechim:

$y = \bar{y} + y_* = C_1\cos x + C_2\sin x - x\cos x$.◀

36-Auditoriya topshiriqlari

Quyidagi differensial tenglamalarning umumiy yechimlarini toping.

1. $y'' + y = 2x^2 + 1$
2. $y'' + y' = \sin x$
3. $y'' + 2y' + y = e^{-x}$
4. $y'' - 2y' = 2e^{-2x} + 3x$
5. $2y'' + y' = x$
6. $y'' + 2y' + 2y = 3\sin x$
7. $y'' - y' = e^x \cos 2x$
8. $y'' - 5y' + 6y = 3e^{2x} + \cos 2x.$
9. $y'' + 2y' + 5y = 2e^x \sin 2x$
10. $y'' - 2y' = e^x \cos 2x + 3x$

36-Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y'' + 2y' = 2x + 1$ differensial tenglama xususiy yechimining umumiy ko'rinishi bo'ladi?

A) $y_* = Ax + B,$ B) $y_* = Ax^2 + Bx + C,$ D) $y_* = Ax^2 + Bx,$ E) $y_* = Ax^2.$

2. $y'' - 2y' + y = 3e^x$ differensial tenglama xususiy yechimining umumiy ko'rinishini toping.

A) $y_* = Ae^x$ B) $y_* = Axe^x$ D) $y_* = Ax^2e^x$ E) $y_* = (Ax^2 + Bx + C)e^x.$

3. Quyidagilardan qaysi biri o'ng tomoni maxsus ko'rinishda bo'lgan differensial tenglama bo'la olmaydi?

A) $y'' + y = x\cos x$ B) $y'' + 2y' + 2y = x + \sin x$

D) $y'' - 2y' + y = xe^{x^2}$ E) $y'' + y = \dfrac{\sin x}{e^x}.$

4. $y'' - 2y' = e^x - 4x$ differensial tenglamaning umumiy yechimini toping.

A) $y = C_1 + C_2e^{2x} + x^2 + x - e^x$ B) $y = C_1 + C_2e^{2x} + 2x^2 + x - e^x$

D) $y = 2x^2 + x - e^x$ E) $y = x^2 + x - e^x$

5. $y'' - 2y' = e^x - 4x$ differensial tenglamaning xususiy yechimini toping.

A) $y = C_1 + C_2e^{2x} + x^2 + x - e^x$ B) $y = C_1 + C_2e^{2x} + 2x^2 + x - e^x$

D) $y = 2x^2 + x - e^x$ E) $y = x^2 + x - e^x$

11-Shaxsiy uy topshiriqlari

1

Birinchi tartibli differensial tenglamalarning umumiy yechimini toping.

1.1. $\cos^3 y \cdot y' - \cos(x + 2y) = \cos(x - 2y)$

1.2. $e^x tgy dx = (1 - e^x)\sec^2 y dy$

1.3. $x\sqrt{1 + y^2} + yy'\sqrt{1 + x^2} = 0.$

1.4. $(e^{2x} + 5)dy + ye^{2x}dx = 0.$

1.5. $y'y\sqrt{\dfrac{1 - x^2}{1 - y^2}} + 1 = 0.$

1.6. $4xdx - 3ydy = 3x^2ydy - 2xy^2dx.$

1.7. $\sqrt{4+y^2}dx - ydy = x^2ydy.$

1.8. $\left(\mathrm{e}^x + 8\right)dy - y\,\mathrm{e}^x\,dx = 0.$

1.9. $y' = 3^{x^2}x\left(1+y^2\right).$

1.10. $3^{x^2+y}dy + xdx = 0.$

1.11. $3e^x\sin ydx + (1-e^x)\cos ydy =$
.

1.12. $y\ln y + xy' = 0.$

1.13. $y'\sin x = y\ln y.$

1.14. $y' = \dfrac{1+y^2}{xy\left(1+x^2\right)}.$

1.15. $\left(3+y^2\right)dx - \dfrac{e^x}{x}ydy = 0.$

1.16. $\sqrt{5+y^2} + y'y\sqrt{1-x^2} = 0.$

1.17. $y\left(1+\ln y\right) + xy' = 0.$

1.18. $\sqrt{3+y^2}dx - ydy = x^2ydy.$

1.19. $6xdx - 6ydy = 3x^2ydy - 2xy^2dx.$

1.20. $\sqrt{4-x^2}y' + xy^2 + x = 0.$

1.21. $\sqrt{5+y^2}dx + 4\left(x^2y + y\right)dy = 0.$

1.22. $\dfrac{e^{-x^2}dy}{x} + \dfrac{dx}{\cos^2 y} = 0.$

1.23. $6xdx - 6ydy = 2x^2ydy - 3xy^2dx.$

1.24. $x\sqrt{3+y^2}dx + y\sqrt{2+x^2}dy = 0.$

1.25. $\left(1+y^2\right)dx - xydy = 0$

1.26. $\dfrac{dy}{dx} = \dfrac{2x}{3y^2+1}.$

1.27. $\dfrac{dy}{x} + \dfrac{dx}{y}e^{x-y} = 0$

1.28. $y\left(1+\ln y\right) + xy' = 0.$

1.29. $x\left(1+y^{\backslash 6}\right)dx - y^2\left(1+x^3\right)dy = 0.$

1.30. $e^{1+x^2}tgydx - \dfrac{e^{2x}}{x-1}dy = 0.$

2

Birinchi tartibli differensial tenglamalarning umumiy integralini toping.

2.1. $3y\sin\dfrac{3x}{y}dx + \left(y - 3x\sin\dfrac{3x}{y}\right)dy = 0.$

2.2. $xy' - y = \dfrac{x}{arctg\dfrac{y}{x}}.$

2.3. $y - xy' = x\sec\dfrac{y}{x}.$

2.4. $xy + y^2 = \left(2x^2 + xy\right)y'.$

2.5. $xy' - y = xtg\dfrac{y}{x}.$

2.6. $xy' = y\cos\ln\dfrac{y}{x}.$

2.7. $xy' = \dfrac{3y^3 + 4yx^2}{2y^2 + 2x^2}.$

2.8. $x - y\cos\dfrac{y}{x}dx + x\cos\dfrac{y}{x}dy = 0$

2.9. $xy' = \sqrt{x^2+y^2} + y.$

2.10. $xy' = \dfrac{3y^3 + 2yx^2}{2y^2 + x^2}.$

2.11. $y' = \dfrac{y^2}{x^2} + 4\dfrac{y}{x} + 2$

2.12. $xy' = 2\sqrt{x^2+y^2} + y$

2.13. $2y' = \dfrac{y^2}{x^2} + 6\dfrac{y}{x} + 3.$

2.14. $xy' - y = (x+y)\ln\frac{x+y}{x}$

2.15. $y' = \frac{x+2y}{2x-y}$.

2.16. $3y' = \frac{y^2}{x^2} + 8\frac{y}{x} + 4$.

2.17. $xy' = \frac{3y^3 + 6yx^2}{2y^2 + 3x^2}$.

2.18. $y' = \frac{x^2 + 3xy - y^2}{3x^2 - 2xy}$.

2.19. $xy' = \sqrt{2x^2 + y^2} + y$.

2.20. $y' = \frac{x^2 + xy - y^2}{x^2 - 2xy}$.

2.21. $x^2dy = (6x^2 + 6xy + y^2)dx$.

2.22. $xy' = 3\sqrt{x^2 + y^2} + y$.

2.23. $y^2 + x^2y' = xyy'$.

2.24. $ydx + (2\sqrt{xy} - x)dy = 0$.

2.25. $2x^2dy = (8x^2 + 8xy + y^2)dx$.

2.26. $y' = \frac{x^2 + xy - 3y^2}{x^2 - 4xy}$.

2.27. $xy' = 4\sqrt{2x^2 + y^2} + y$.

2.28. $(x^2 - xy - y^2)dy = y^2dx$.

2.29. $(8y + 10x)dx + (5y + 7x)dy = 0$

2.30. $xy' = \frac{3y^3 + 8yx^2}{2y^2 + 4x^2}$.

3

Koshi masalasi yechimini toping

3.1. $y' - y/x = x^2, \quad y(1) = 0.$

3.2. $y' - y\operatorname{ctg} x = 2x\sin x, \quad y(\pi/2) = 0.$

3.3. $y' + y\cos x = \frac{1}{2}\sin 2x, \quad y(0) = 0.$

3.4. $y' + y\operatorname{tg} x = \cos^2 x, \quad y(\pi/4) = 1/2.$

3.5. $y' - \frac{y}{x+2} = x^2 + 2x, \quad y(-1) = 3/2.$

3.6. $y' - \frac{1}{x+1}y = e^x(x+1), \quad y(0) = 1.$

3.7. $y' - \frac{y}{x} = x\sin x, \quad y\left(\frac{\pi}{2}\right) = 1.$

3.8. $y' + \frac{y}{x} = \sin x, \quad y(\pi) = \frac{1}{\pi}.$

3.9. $y' + \frac{y}{2x} = x^2, \quad y(1) = 1.$

3.10. $y' + \frac{2x}{1+x^2}y = \frac{2x^2}{1+x^2}, \quad y(0) = \frac{2}{3}.$

3.11. $y' - \frac{2x-5}{x^2}y = 5, \quad y(2) = 4.$

3.12. $y' + \frac{y}{x} = \frac{x+1}{x}e^x, \quad y(1) = e.$

3.13. $y' - \frac{y}{x} = -\frac{12}{x^3}, \quad y(1) = 4.$

3.14. $y' - \frac{y}{x} = -2\frac{\ln x}{x}, \quad y(1) = 1.$

3.15. $y' + 2xy = -2x^3, \quad y(1) = e^{-1}.$

3.16. $y' + \frac{2}{x}y = x^3, \quad y(1) = -5/6.$

3.17. $y' + \frac{y}{x} = 3x, \quad y(1) = 1.$

3.18. $y' - \frac{2xy}{1+x^2} = 1 + x^2, \quad y(1) = 3.$

3.19. $y' + \frac{1-2x}{x^2}y = 1, \quad y(1) = 1.$

3.20. $y' + \frac{3y}{x} = \frac{2}{x^3}, \quad y(1) = 1.$

3.21. $y' + \frac{xy}{2(1-x^2)} = \frac{x}{2}, \quad y(0) = \frac{2}{3}.$

3.22. $y' + xy = -x^3, \quad y(0) = 3.$

3.23. $y' - \frac{2}{x+1}y = e^x (x+1)^2, \quad y(0) = 1.$

3.24. $y' + 2xy = x e^{-x^2} \sin x, \quad y(0) = 1.$

3.25. $y' - 2y/(x+1) = (x+1)^3, \; y(0) = 1/2.$

3.26. $y' - y\cos x = -\sin 2x, \quad y(0) = 3.$

3.27. $y' - 4xy = -4x^3, \quad y(0) = -1/2.$

3.28. $y' - y\cos x = \sin 2x, \quad y(0) = -1.$

3.29. $y' - 3x^2 y = x^2(1+x^3)/3, \quad y(0) = 0.$

3.30. $y' - \frac{y}{x} = -\frac{\ln x}{x}, \quad y(1) = 1.$

4

Differensial tenglamaning umumiy yechimini toping

4.1. $y'''x\ln x = y''$.

4.2. $xy''' + y'' = 1$.

4.3. $2xy''' = y''$.

4.4. $xy''' + y'' = x + 1$.

4.5. $y''' \operatorname{ctg} 2x + 2y'' = 0$.

4.6. $x^3 y''' + x^2 y'' = 1$.

4.7. $\operatorname{tg} x \cdot y'' - y' + \dfrac{1}{\sin x} = 0$.

4.8. $x^2 y'' + xy' = 1$.

4.9. $\operatorname{tg} x \cdot y''' = 2y''$.

4.10. $y''' \operatorname{cth} 2x = 2y''$.

4.11. $x^4 y'' + x^3 y' = 1$.

4.12. $xy''' + 2y'' = 0$.

4.13. $\left(1 + x^2\right) y'' + 2xy' = x^3$.

4.14. $x^5 y''' + x^4 y'' = 1$.

4.15. $xy''' - y'' + \dfrac{1}{x} = 0$.

4.16. $xy''' + y'' + x = 0$.

4.17. $\operatorname{th} x \cdot y^{IV} = y'''$.

4.18. $y''' \operatorname{tg} x = y'' + 1$.

4.19. $xy''' + y'' = \sqrt{x}$.

4.20. $y''' \operatorname{tg} 5x = 5y''$.

4.21. $y''' \operatorname{th} 7x = 7y''$.

4.22. $x^3 y''' + x^2 y'' = \sqrt{x}$.

4.23. $\operatorname{cth} x \cdot y'' - y' + \dfrac{1}{\operatorname{ch} x} = 0$.

4.24. $\left(x + 1\right) y''' + y'' = \left(x + 1\right)$.

4.25. $\left(1 + \sin x\right) y''' = \cos x \cdot y''$.

4.26. $\operatorname{cth} xy'' + y' = \operatorname{ch} x$.

4.27. $x^4 y'' + x^3 y' = 4$.

4.28. $-xy''' + 2y'' = \frac{2}{x^2}.$

4.29. $y'' + \frac{2x}{x^2+1} y' = 2x.$

4.30. $\left(1+x^2\right)y'' + 2xy' = 12x^3.$

5

Differensial tenglamaning umumiy yechimini toping

5.1. $y''' - 4y'' + 5y' - 2y = (16-12x)e^{-x}.$

5.2. $y''' - y' = 2e^x + \cos x.$

5.3. $y''' - 3y'' + 2y' = (1-2x)e^x.$

5.4. $y'' + 2y' = 4e^x(\sin x + \cos x).$

5.5. $y''' - y'' - y' + y = (3x+7)e^{2x}.$

5.6. $y'' - 4y' + 4y = -e^{2x}\sin 6x.$

5.7. $y''' - 2y'' + y' = (2x+5)e^{2x}.$

5.8. $y'' + 9y = -18\sin 3x - 18e^{3x}.$

5.9. $y''' + y'' - y' - y = (8x+4)e^x.$

5.10. $y'' + 2y' + 5y = -\sin 2x.$

5.11. $y'' - 4y' + 4y = e^{2x}\sin 3x.$

5.12. $y''' - 3y' + 2y = (4x+9)e^{2x}.$

5.13. $y''' - 9y' = -9e^{3x} + 18\sin 3x - 9\cos 3x.$

5.14. $y''' - 3y' + 2y = (4x+9)e^{2x}.$

5.15. $y''' + 4y'' + 5y' + 2y = (12x+16)e^x.$

5.16. $y'' + 6y' + 13y = e^{-3x}\cos 4x.$

5.17. $y''' - 3y'' - y' + 3y = (4-8x)e^x.$

5.18. $y''' + y'' - 2y' = (6x+5)e^x.$

5.19. $y'' + y = 2\cos 7x - 3\sin 7x.$

5.20. $y''' - y'' - 2y' = (6x-11)e^{-x}.$

5.21. $y''' - y'' - 4y' + 4y = (7-6x)e^x.$

5.22. $y''' + 4y'' + 4y' = (9x+15)e^x.$

5.23. $y''' - 3y'' - y' + 3y = (4-8x)e^x.$

5.24. $y'' + y = 2\cos 4x + 3\sin 4x.$

5.25. $y''' - 5y'' + 6y' = 6x^2 + 2x - 5.$

5.26. $y^{IV} - 3y''' + 3y'' - y' = x - 3.$

5.27. $y''' + 5y'' + 7y' + 3y = (16x + 20)e^x.$

5.28. $y'' + 4y' = 16 \operatorname{sh} 4x.$

5.29. $y'' + 2y' + 5y = 10 \cos x.$

5.30. $y'' - y' = 2 \operatorname{ch} x.$

IX BOB. OPERATSION HISOB. ASLDAN TASVIRNI VA TASVIRDAN ASLNI TOPISH

Quyidagi shartlarni qanoatlantiruvchi $f(t)$ funksiyaga ***asl(original)*** deb ataladi:

a) $f(t)$ funksiya uzluksiz yoki chekli oraliqda chekli sondagi I tur uzilish nuqtalariga ega;
b) $t<0$ da $f(t)=0$;
c) $M>0$ va $s_0>0$ o‘zgarmas sonlar mavjud bo‘lib, $f(t)\le Me^{s_0 t}$ o‘rinli (s_0- funksiyaning o‘sish ko‘rsatkichi).

Eng sodda asl funksiyalardan biri **Xevisayd** birlik funksiyasidir:

$$\sigma(t)=\begin{cases}1 & agar \quad t\ge 0\\ 0 & agar \quad t<0\end{cases}$$

Agar $f(t)$ funksiya a) va d) shartlarni qanoatlantirsa, $f(t)\sigma(t)$ asl funksiya bo‘ladi. Qulaylik uchun shunchaki $f(t)$ deb yoziladi, lekin $t<0$ da $f(t)=0$ hisoblanadi.

$$F(p)=\int_0^{+\infty} e^{-pt}f(t)dt \tag{9.1}$$

Laplas almashtirishi, $F(p)$ funksiya esa $f(t)$ funksiyaning ***Laplas tasviri***, $L-$ tasviri, yoki qisqacha, ***tasviri*** deb ataladi.
Agar $F(p)$ funksiya $f(t)$ funksiyaning tasviri bo‘lsa, u quyidagicha yoziladi:

$$F(p)\risingdotseq f(t) \quad \text{yoki} \quad L\{f(t)\}=F(p) \tag{9.2}$$

9.1-misol.

Xevisayd birlik funksiyaning tasvirini toping.

► $L\{\sigma(t)\}=\int_0^{+\infty} e^{-pt}\cdot 1dt=-\frac{e^{-pt}}{p}\Big|_0^{+\infty}=\frac{1}{p}$.◄

9.2-misol.

Ushbu $f(t)=e^t$ funksiyaning tasvirini toping.

► $L\{e^t\}=\int_0^{+\infty} e^{-pt}\cdot e^t dt=\int_0^{+\infty} e^{-(p-1)t}dt=-\frac{e^{-(p-1)t}}{p-1}\Big|_0^{+\infty}=\frac{1}{p-1}$.◄

9.3-misol.

Ushbu $f(t)=\sin t$ funksiyaning tasvirini toping.

► $L\{\sin t\}=\int_0^{+\infty} e^{-pt}\sin tdt=-\frac{e^{-pt}}{p}\sin t\Big|_0^{+\infty}+\frac{1}{p}\int_0^{+\infty} e^{-pt}\cos tdt=\frac{1}{p^2}\int_0^{+\infty} e^{-pt}\cos tdt$

$$L\{\sin t\}=\int_0^{+\infty} e^{-pt}\sin t\ dt=\frac{e^{-pt}}{-p}\sin t\Big|_0^{+\infty}+$$

$$+\frac{1}{p}\int_{0}^{+\infty} e^{-pt}\cos t\ dt = -\frac{1}{p^2}e^{-pt}\cos t\Big|_0^{+\infty} - \frac{1}{p^2}\int_{0}^{+\infty} e^{-pt}\sin t\ dt =$$

$$= \frac{1}{p^2} - \frac{1}{p^2}L\{\sin t\}$$

Bundan, $L\{\sin t\} = \frac{1}{1+p^2}$. ◀

9.1. Laplas almashtirishining xossalari

Chiziqlilik xossalari. Agar $F_1(p) \xrightarrow{\cdot\cdot} f_1(t), F_2(p) \xrightarrow{\cdot\cdot} f_2(t)$ bo'lsa, ixtiyoriy C_1 va C_2 lar uchun

$$C_1F_1(p) + C_2F_2(p) \xrightarrow{\cdot\cdot} C_1f_1(t) + C_2 \cdot f_2(t)\ . \tag{9.3}$$

O'xshashlik teoremasi. Agar $F(p) \xrightarrow{\cdot\cdot} f(t)$ bo'lsa, ixtiyoriy $a > 0$ uchun

$$\frac{1}{a} \cdot F\left(\frac{p}{a}\right) \xrightarrow{\cdot\cdot} f(at). \tag{9.4}$$

9.4-misol.

Ushbu $f(t) = 3e^{2t} + \sin 3t$ funksiyaning tasvirini toping.

▶ $e^t \xleftarrow{\cdot\cdot} \frac{1}{p-1}$, $\sin t \xleftarrow{\cdot\cdot} \frac{1}{p^2+1}$ bo'lgani uchun $e^{2t} \xleftarrow{\cdot\cdot} \frac{1}{2}\cdot\frac{1}{\frac{p}{2}-1} = \frac{1}{p-2}$,

$\sin 3t \xleftarrow{\cdot\cdot} \frac{1}{3}\cdot\frac{1}{(p/3)^2+1} = \frac{3}{p^2+9}$. $F(p) = \frac{3}{p-2} + \frac{3}{p^2+9}$. ◀

Siljish teoremasi. Agar $F(p) \xrightarrow{\cdot\cdot} f(t)$ bo'lsa,

$$F(p+\alpha) \xrightarrow{\cdot\cdot} e^{-\alpha t}f(t) \tag{9.5}$$

9.5-misol.

Ushbu $f(t) = e^{-2t}\sin 3t$ funksiyaning $F(p)$ tasvirini toping.

▶ $\sin 3t \xleftarrow{\cdot\cdot} \frac{3}{p^2+9}$ bo'lgani uchun $e^{-2t}\sin 3t \xleftarrow{\cdot\cdot} \frac{3}{(p+2)^2+9}$,

$F(p) = \frac{3}{p^2+4p+13}$. ◀

Kechikish teoremasi. Agar $F(p) \xrightarrow{\cdot\cdot} f(t)$ bo'lsa, ixtiyoriy $\tau > 0$ uchun

$$e^{-\tau p}F(p) \xrightarrow{\cdot\cdot} f(t-\tau)\ . \tag{9.6}$$

Aslni differensiallash teoremasi. Agar $F(p) \xrightarrow{\cdot\cdot} f(t)$ bo'lsa,

$$f'(t) \leftarrow\!\!\!\div\!\!\!- pF(p) - f(0)\ . \qquad \textbf{(9.7)}$$

9.6-misol.

Ushbu $f(t) = sin^2 t$ funksiyaning $F(p)$ tasvirini toping.

▶ $f(t) = 0,\quad f'(t) = \sin 2t \leftarrow\!\!\!\div\!\!\!- \dfrac{2}{p^2+4}$, (7) formulaga ko‘ra, $pF(p) = \dfrac{2}{p^2+4}$.

$F(p) = \dfrac{2}{p\left(p^2+4\right)}$. ◀

Tasvirni differensiallash teoremasi.

Agar $F(p) \xrightarrow{\ \div\ } f(t)$ bo‘lsa, $F'(p) \xrightarrow{\ \div\ } tf(t)$. (9.8)

Aslni integrallash teoremasi.

Agar $F(p) \xrightarrow{\ \div\ } f(t)$ bo‘lsa, $\int\limits_0^t f(\tau)d\tau \leftarrow\!\!\!\div\!\!\!- \dfrac{F(p)}{p}$. (9.9)

Tasvirni integrallash teoremasi.

Agar $F(p) \xrightarrow{\ \div\ } f(t)$ bo‘lsa, $\dfrac{f(t)}{t} \leftarrow\!\!\!\div\!\!\!- \int\limits_0^p F(z)dz$. (9.10)

Asllar o‘ramasining tasviri haqida teorema.

Agar $F(p) \xrightarrow{\ \div\ } f(t)$, $\Phi(p) \xrightarrow{\ \div\ } \varphi(t)$ bo‘lsa, u holda

$$F(p)\Phi(p) \xrightarrow{\ \div\ } \int\limits_0^t f(\tau)\varphi(t-\tau)\,\mathrm{d}\tau\ . \qquad (9.11)$$

Dyuamel integrali.

$$pF(p)\Phi(p) \xrightarrow{\ \div\ } f(t)\varphi(0) + \int\limits_0^t f(\tau)\varphi_t'(t-\tau)d\tau\ . \qquad (9.12)$$

9.7-misol.

Ushbu $f(t) = \dfrac{\sin^2 t}{t}$ funksiyaning $F(p)$ tasvirini toping.

▶ $\sin^2 t \leftarrow\!\!\!\div\!\!\!- \dfrac{2}{p\left(p^2+4\right)}$, (9.10) formulaga ko‘ra,

$$\frac{\sin^2 t}{t} \leftarrow\!\!\!\div\!\!\!- \int\limits_0^p \frac{2dq}{q\left(q^2+4\right)} = \frac{1}{2}\int\limits_0^p \left(\frac{1}{q} - \frac{q}{q^2+4}\right)dq = \frac{1}{2}\ln\frac{q}{\sqrt{q^2+4}}\Bigg|_0^p = \frac{1}{2}\ln\frac{p}{\sqrt{p^2+4}}.$$ ◀

9.8-misol.

Ushbu $F(p)=\dfrac{1}{p^2+4p+3}$ funksiyaning $f(t)$ aslini toping.

► $F(p)=\dfrac{1}{p^2+4p+3}=\dfrac{1}{2}\left(\dfrac{1}{p+1}-\dfrac{1}{p+3}\right)\xrightarrow{\cdot}\dfrac{1}{2}\left(e^{-t}-e^{-3t}\right).$ ◄

9.9-misol.

Ushbu $F(p)=\dfrac{p}{p^2-4p+9}$ funksiyaning $f(t)$ aslini toping.

► $F(p)=\dfrac{p}{p^2-4p+9}=\dfrac{p-2}{(p-2)^2+5}+\dfrac{2}{\sqrt{5}}\cdot\dfrac{\sqrt{5}}{(p-2)^2+5},$

$f(t)=e^{2t}\cos\sqrt{5}t+\dfrac{2}{\sqrt{5}}\sin\sqrt{5}t.$ ◄

9.10-misol.

Ushbu $F(p)=\dfrac{e^{-2p}}{p^2-9}$ funksiyaning $f(t)$ aslini toping.

► $\dfrac{1}{p^2-9}=\dfrac{1}{3}\cdot\dfrac{3}{p^2-9}\xrightarrow{\cdot}\dfrac{1}{3}sh3t$, $\dfrac{e^{-2p}}{p^2-9}\xrightarrow{\cdot}\dfrac{1}{3}sh3(t-2)$, $f(t)=\dfrac{1}{3}sh(3t-6).$ ◄

37-Auditoriya topshiriqlari

Quyidagi funksiyalar asl funksiya bo‘la oladimi?

1. $f(t)=1/(t-2)$

2. $f(t)=cht$

3. $f(t)=tgt$

4. $f(t)=3^t$

5. $f(t)=e^{-t}\cos t$

6. $f(t)=t^t$.

Quyidagi asl funksiyalarning Laplas tasvirini toping.

1. $f(t)=4+3e^{-t}$

2. $f(t)=4t^2e^{-t}$

3. $f(t)=cht\sin 2t$

4. $f(t)=e^{-3t}\cos^2 t$

5. $f(t)=\sin^3 t$

6. $f(t)=4t+3e^{-t}\sin 3t$

7. $f(t)=te^{-t}\sin 3t$

8. $f(t)=\int_0^t \tau sh2\tau d\tau$

9. $f(t)=\int_0^t \tau^2e^{-\tau}d\tau$

10. $f(t)=\dfrac{\sin^2 t}{t}$.

Quyidagi tasvir funksiyalarning asl funksiyalarini toping.

1. $F(p)=\dfrac{1}{p-2}$

2. $F(p)=\dfrac{1}{p^2-2p+3}$

3. $F(p)=\dfrac{1}{(p-2)^2(p^2+1)}$

4. $F(p)=\dfrac{p-2}{p(p^2+4p+5)}$

5. $F(p)=\dfrac{e^{-2p}}{(p-2)(p+1)}$

6. $F(p)=\dfrac{e^{-3p}}{(p^2-2)(p^2+1)}$

37-Mustaqil yechish uchun testlar

1. Quyidagi funksiyalarning qaysi biri asl funksiya bo‘ladi?

A) $f(t)=1/(t-2)$ B) $f(t)=e^{-t}\sin t$ D) $f(t)=e^{t^2}$ E) B va D.

2. Kechikish teoremasi keltirilgan javobni toping.

A) $F(p+\alpha)\xrightarrow{\cdot}e^{-\alpha t}f(t)$ B) $e^{-\tau p}F(p)\xrightarrow{\cdot}f(t-\tau)$

D) $f(at)\xleftarrow{\cdot}\frac{1}{a}F\left(\frac{p}{a}\right)$ E) $e^{\tau p}F(p)\xrightarrow{\cdot}f(t-\tau)$.

3. O‘xshashlik teoremasi keltirilgan javobni toping.

A) $F(p+\alpha)\xrightarrow{\cdot}e^{-\alpha t}f(t)$ B) $e^{-\tau p}F(p)\xrightarrow{\cdot}f(t-\tau)$

D) $f(at)\xleftarrow{\cdot}\frac{1}{a}F\left(\frac{p}{a}\right)$ E) $e^{\tau p}F(p)\xrightarrow{\cdot}f(t-\tau)$.

4. $f(t)=3-2e^{-5t}$ funksiyaning tasvirini aniqlang.

A) $\frac{3}{p}-\frac{2}{p+5}$ B) $\frac{3}{p^2}-\frac{2}{p+5}$ D) $\frac{3}{p}-\frac{2}{p-5}$ E) $\frac{3}{p^2}-\frac{2}{p-5}$.

5. $F(p)=\frac{3}{(p-1)^3}$ funksiyaning tasvirini aniqlang.

A) $f(t)=te^t$ B) $f(t)=t^2e^t$ D) $f(t)=\frac{3}{2}t^2e^t$ E) Tog‘ri javob yo‘q.

9.2 O‘zgarmas koeffitsientli chiziqli differensial tenglama va tenglamalar sistemasini yechishning operatsion hisob usuli

Ushbu

$$x^{(n)}(t)+a_1x^{(n-1)}(t)+....+a_{n-1}x'(t)+a_nx(t)=f(t) \qquad (9.13)$$

differensial tenglamada $a_1,a_2,...,a_{n-1},a_n$ o‘zgarmas sonlar, $x(t),x'(t),...,x^{(n-1)}(t),x^{(n)}(t),f(t)$ lar asl funksiyalar bo‘lsin. Quyidagi Koshi masalasi yechimini topishning operatsion usulini qaraymiz:

$$x(0)=x_0,\ x'(0)=x_0',...,x^{(n-1)}(0)=x_0^{(n-1)}. \qquad (9.14)$$

$x(t)\leftarrow\!\!\!\div X(p),\ f(t)\leftarrow\!\!\!\div F(p)$ bo‘lsin. U holda aslni differensiallash formulasidan foydalanamiz:

$$x'(t)\leftarrow\!\!\!\div pX(p)-x(0),\ x''(t)\leftarrow\!\!\!\div p^2F(p)-px(0)-x'(0),...$$

$$x^{(n)}(t)\leftarrow\!\!\!\div p^nX(p)-p^{n-1}x(0)-...-px^{(n-2)}(0)-x^{(n-1)}(0)..$$

(1) tenglamaga tasvirlarni qo‘yib, $X(p)$ noma’lumga nisbatan chiziqli tenglamani hosil qilamiz. Uni ixcham holda yozilishi

$$Q_n(p)X(p)=F(p)+R_{n-1}(p). \qquad (9.15)$$

(3) tenglamadan $X(p)$ topiladi va uning asli (9.13) tenglamaning (9.14) shartlarni qanoatlantiruvchi yechimi bo‘ladi.

9.11-misol.

Ushbu $x'+x=e^{-t},\quad x(0)=1$ tenglamani yeching.

► $x(t)\leftarrow\!\!\!\div X(p),\ e^{-t}\leftarrow\!\!\!\div\dfrac{1}{p+1},\ x'(t)\leftarrow\!\!\!\div pX(p)-1.$

$pX(p)-1+X(p)=\dfrac{1}{p+1},\ X(p)=\dfrac{1}{(p+1)^2}+\dfrac{1}{p+1}\div\!\!\!\rightarrow te^{-t}+e^{-t},\ x(t)=(t+1)e^{-t}.$ ◄

Agar (9.13) tenglama

$$x(0)=x'(0)=...=x^{(n-1)}(0)=0 \qquad (9.16)$$

boshlang‘ich shartlar bilan berilgan bo‘lsa, Dyuamel integrali yordamida quyidagicha yechiladi. Qo‘shimcha

$$z^{(n)}(t)+a_1z^{(n-1)}(t)+....+a_{n-1}z'(t)+a_nz(t)=1 \qquad (9.17)$$

$$z(0)=z'(0)=...=z^{(n-1)}(0)=0 \qquad (9.18)$$

(9.16) boshlang‘ich shartlar bilan berilgan (9.17) bo‘lsin. Quyidagi

$$X(p)=\frac{F(p)}{Q_n(p)},\ Z(p)=\frac{1}{pQ_n(p)} \qquad (9.19)$$

tenglamani hosil qilamiz. Bundan $X(p)=pF(p)Z(p)$ ekani ma’lum. Dyuamel integralidan foydalanib,

$$x(t)=f(t)z(0)+\int_0^t f(\tau)z_t'(t-\tau)d\tau \qquad (9.20)$$

yechimni topamiz.

9.12-misol.

Ushbu $x''' + x' = \sin t, \quad x(0) = x'(0) = x''(0) = 0$ tenglamani yeching.

► Qo‘himcha tenglama tuzamiz: $z''' + z' = 1, \quad z(0) = z'(0) = z''(0) = 0.$

$$z(t) \leftarrow\!\!\div\!\!- Z(p),\ z'(t) \leftarrow\!\!\div\!\!- pZ(p),\ z''(t) \leftarrow\!\!\div\!\!- p^2 Z(p),\ 1 \leftarrow\!\!\div\!\!- \frac{1}{p}\ .$$

$Z(p) = \dfrac{1}{p^2(p^2+1)} = \dfrac{1}{p^2} - \dfrac{1}{p^2+1}, \quad z(t) = t - \sin t$. (9.20) formuladan va

$z(0) = 0,\ z'_t = 1 - \cos t$ ekanidan foydalanib quyidagini topamiz:

$$x(t) = \int_0^t \sin\tau(1 - \cos(t-\tau))d\tau = \int_0^t \sin\tau d\tau - \int_0^t \sin\tau(\cos t\cos\tau + \sin t\sin\tau)d\tau =$$

$$= -\cos\tau\Big|_0^t - \frac{\cos t}{2}\int_0^t \sin 2\tau d\tau - \frac{\sin t}{2}\int_0^t (1 - \cos 2\tau)d\tau = -\cos t + 1 + \frac{1}{4}\cos t\cos 2\tau\Big|_0^t -$$

$$-\frac{1}{2}t\sin t + \frac{1}{4}\sin t\sin 2\tau\Big|_0^t = -\cos t + 1 - \frac{1}{2}t\sin t + \frac{1}{4}(\cos t\cos 2t + \sin t\sin 2t) - \frac{1}{4}\cos t =$$

$$= -\cos t - \frac{1}{2}t\sin t + 1.$$ ◄

O‘zgarmas koeffitsientli oddiy chiziqli differensial tenglamalar sistemasi ham xuddi yuqoridagi kabi operatsion hisob yordamida ikki noma’lumli algebraik tenglamalar sistemasiga keltirib yechiladi.

9.13-misol.

Koshi masalasini yeching: $\begin{cases} x' = -y + 2 \\ y' = x + 1, \end{cases} \qquad x(0) = -1,\ y(0) = 0.$

► $x(t) \leftarrow\!\!\div\!\!- X(p),\ y(t) \leftarrow\!\!\div\!\!- Y(p)$ bo‘lsin.

$x'(t) \leftarrow\!\!\div\!\!- pX(p) + 1,\ y'(t) \leftarrow\!\!\div\!\!- pY(p),\ 1 \leftarrow\!\!\div\!\!- \dfrac{1}{p}$ dan foydalanib, sistemani qayta yozamiz:

$$\begin{cases} pX(p) + 1 = -Y(p) + \dfrac{2}{p} \\ pY(p) = X(p) + \dfrac{1}{p} \end{cases}.$$

Sistemani yechib,

$$\begin{cases} X(p) = \dfrac{2}{p^2+1} - \dfrac{1}{p} \\ Y(p) = \dfrac{2p}{p^2+1} + \dfrac{2}{p} \end{cases}$$

yechimni hosil qilamiz va asl funksiyalarini topamiz. Bu esa Koshi masalasining yechimi bo‘ladi:

$$\begin{cases} x(t) = 2\sin t - 1 \\ y(t) = 2\cos t + 2 \end{cases} . \blacktriangleleft$$

38-Auditoriya topshiriqlari

Quyidagi differensial tenglamalarni operatsion hisob usulida yeching.

1. $x'' + 2x' + x = t, \quad x(0) = 0, \ x'(0) = 1$

2. $x'' - 3x' = e^t, \quad x(0) = x'(0) = 0$

3. $x'' + 2x' - 3x = e^{-2t}, \quad x(0) = -1, \ x'(0) = 0$

4. $x'' + 2x' = \cos t, \quad x(0) = 0, \ x'(0) = -1$

5. $x'' + x' = t\sin t, \quad x(0) = x'(0) = 0$

Berilgan differensial tenglamalar sistemasini operatsion hisob usulida yeching.

1. $\begin{cases} x' = y - 1, & x(0) = -1 \\ y' = -x + 2, & y(0) = 0. \end{cases}$

2. $\begin{cases} x' = 3x + 4y, & x(0) = 1 \\ y' = 4x - 3y, & y(0) = 1. \end{cases}$

3. $\begin{cases} x' = -2x + y, & x(0) = 0 \\ y' = 3x, & y(0) = 1. \end{cases}$

4. $\begin{cases} x' = 3y, & x(0) = 1 \\ y' = 3x + 1, & y(0) = 1. \end{cases}$

38-Mustaqil yechish uchun testlar

1. Quyidagi Koshi masalalaridan qaysi birini Dyuamel integralidan foydalanib yechish mumkin?

A) $x'' + 2x' = \cos t, \quad x(0) = 0, \ x'(0) = -1$

B) $x'' - 2x' = ch t, \quad x(0) = 0, \ x'(0) = 0$

D) $x'' - 2x' + x = \cos t, \quad x(0) = 1, \ x'(0) = 0$

E) $x'' + x' = \sin t, \quad x(0) = 1, \ x'(0) = 1$.

2. $x'' + 2x' = \cos t, \quad x(0) = 0, \ x'(0) = 0$ differensial tenglama uchun tuzilgan algebraik tenglamaning javobini aniqlang.

A) $X(p) = \dfrac{1}{(p+2)(p^2+1)}$ B) $X(p) = \dfrac{1}{(p-2)(p^2+1)}$

D) $X(p) = \dfrac{p}{(p-2)(p^2+1)}$ E) $X(p) = \dfrac{p}{(p+2)(p^2+1)}$

3. $x' - 2x = e^{3t}, \quad x(0) = 0$ differensial tenglamani operatsion hisob yordamida yeching.

A) $x(t) = t - \sin t$ B) $x(t) = t + e^{3t}$ D) $x(t) = e^{3t} - e^{2t}$ E) $x(t) = e^{2t} - e^{3t}$.

4. $\begin{cases} x' = 3y, & x(0) = 0 \\ y' = 3x, & y(0) = 1 \end{cases}$ differensial tenglamalar sistemasini operatsion hisob yordamida yeching.

A) $\begin{cases} x(t) = \sin 3t \\ y(t) = -\cos 3t \end{cases}$ B) $\begin{cases} x(t) = sh3t \\ y(t) = ch3t \end{cases}$ D) $\begin{cases} x(t) = \sin 3t \\ y(t) = \cos 3t \end{cases}$ E) $\begin{cases} x(t) = sh3t \\ y(t) = -ch3t \end{cases}$.

5. $\begin{cases} x' = 3y, & x(0)=0 \\ y' = 3x, & y(0) = -1 \end{cases}$ differensial tenglamalar sistemasini operatsion hisob yordamida yeching.

A) $\begin{cases} x(t) = \sin 3t \\ y(t) = -\cos 3t \end{cases}$ B) $\begin{cases} x(t) = sh3t \\ y(t) = ch3t \end{cases}$ D) $\begin{cases} x(t) = \sin 3t \\ y(t) = \cos 3t \end{cases}$ E) $\begin{cases} x(t) = sh3t \\ y(t) = -ch3t \end{cases}$.

12-Shaxsiy uy topshiriqlari

1

Berilgan asl funksiyalarning tasvirlarini toping.

1.1. $f(t) = 2e^{-3t} + t\sin 2t$

1.2. $f(t) = 3t^2 e^{-t} + sh3t$

1.3. $f(t) = t^2 + e^{-t}\cos 2t$

1.4. $f(t) = e^{2t} + tch3t$

1.5. $f(t) = t\cos tsh3t$

1.6. $f(t) = \sin(3t-2) + 2$

1.7. $f(t) = \dfrac{e^{2t} - e^{-t}}{t}$

1.8. $f(t) = tsh3t\sin 3t$

1.9. $f(t) = \sin^2(2t-3)$

1.10. $f(t) = \sin 5t \sin 3t$

1.11. $f(t) = \dfrac{1-e^{-t}}{t}$

1.12. $f(t) = \cos^3 t$

1.13. $f(t) = t\cos 3t + e^{-t}$

1.14. $f(t) = \int\limits_0^t \sin 2\tau d\tau$

1.15. $f(t) = t^2\cos 3t$

1.16. $f(t) = t(e^t ch3t)$

1.17. $f(t) = t^2 \sin 2t$

1.18. $f(t) = \sin 5t \cos 3t$

1.19. $f(t) = (t-1)^2 \cos 2(t-1)$

1.20. $f(t) = \dfrac{e^t - 1 - t}{t}$

1.21. $f(t) = \int\limits_0^t \tau sh2\tau d\tau$

1.22. $f(t) = \int\limits_0^t \tau e^{-2\tau} d\tau$

1.23. $f(t) = \dfrac{1-\cos t}{t}$

1.24. $f(t) = e^{-2t}\sin^2 t$

1.25. $f(t) = \cos 3t \cos 2t$

1.26. $f(t) = e^{2t}\sin 3t - 3t^2$

1.27. $f(t) = e^{-2t}\cos^2 t$

1.28. $f(t) = te^t \sin 3t + 2t^3$

1.29. $f(t) = e^t(t\sin 3t + 2t^2)$

1.30. $f(t) = \dfrac{\cos t - \cos 2t}{t}$

2

Berilgan tasvir funksiyalarning asllarini toping.

2.1. $F(p) = \dfrac{p}{(p^2+4)^2}$

2.2. $F(p) = \dfrac{e^{-2p}}{p^2 - 6p + 5}$

2.3. $F(p) = \dfrac{p}{p^2 + 6p + 13}$

2.4. $F(p) = \dfrac{pe^{-2p}}{p^3 - 3p - 2}$

2.5. $F(p) = \dfrac{2p}{(p^2+4)(p+1)}$

2.6. $F(p) = \dfrac{1}{p^2(p^2+2p+2)}$

2.7. $F(p) = \dfrac{1}{(p^2-4)(p+1)}$

2.8. $F(p) = \dfrac{e^{-p}}{p^2+6p+9}$

2.9. $F(p)=\dfrac{1}{(p^2+2)^2}$

2.10. $F(p)=\dfrac{2p+3}{p^3+4p^2+5p}$

2.11. $F(p)=\dfrac{1}{(p^2+4)(p+1)}$

2.12. $F(p)=\dfrac{1}{p^2+1}(e^{-2p}+2e^{-3p})$

2.13. $F(p)=\dfrac{e^{-2p}}{p^2-4}+\dfrac{pe^{-2p}}{p^2-4}$

2.14. $F(p)=\dfrac{e^{-2p}}{p^2+3p+2}$

2.15. $F(p)=\dfrac{2}{p(p^2+3p+2)}$

2.16. $F(p)=\dfrac{1}{(p^2+4)(p^2+1)}$

2.17. $F(p)=\dfrac{1}{p(p^2+4)}+\dfrac{2}{p^3}$

2.18. $F(p)=\dfrac{e^{-2p}}{(p+2)(p-1)}$

2.19. $F(p)=\dfrac{e^{-2p}}{p^2+4}+\dfrac{pe^{-2p}}{p^2+4}$

2.20. $F(p)=\dfrac{1}{p^2(p^2-2p+5)}$

2.21. $F(p)=\dfrac{e^{-p}}{p^2-3p+2}$

2.22. $F(p)=\dfrac{e^{-2p}}{p^2-6p+9}$

2.23. $F(p)=e^{-2p}\cdot\dfrac{2p+3}{p^3-4p^2+5p}$

2.24. $F(p)=\dfrac{p}{p^3+1}$

2.25. $F(p)=\dfrac{e^{-p}}{p^2-1}+\dfrac{pe^{-2p}}{p^2-4}$

2.26. $F(p)=e^{-p}\dfrac{p}{(p+2)^2}$

2.27. $F(p)=\dfrac{1}{p^2+1}(e^{-p}+2e^{-2p})$

2.28. $F(p)=\dfrac{1}{p(p^2-4)}+\dfrac{3}{p^3}$

2.29. $F(p)=\dfrac{p}{(p+4)^2}$

2.30. $F(p)=\dfrac{1}{p^2(p^2+9)}+\dfrac{1}{p^4}$

3

Quyidagi differensial tenglamalarni operatsion hisob usulida yeching.

3.1. $x''-2x'=\cos t,\quad x(0)=0,\ x'(0)=1$

3.2. $x''+2x'-3x=e^{-2t},\quad x(0)=x'(0)=0$

3.3. $x''+3x'-4x=e^{t},\quad x(0)=-1,\ x'(0)=0$

3.4. $x''+2x'+5x=3,\quad x(0)=-1,\ x'(0)=0$

3.5. $x''+2x'+x=t^2,\quad x(0)=1,\ x'(0)=0$

3.6. $x''+2x'+x=2\cos^2 t,\quad x(0)=x'(0)=0$

3.7. $x''-2x'+x=t-\sin t,\quad x(0)=x'(0)=0$

3.8. $x'''+x'=t,\quad x(0)=-3,\ x'(0)=1,\ x''(0)=0$

3.9. $x'''+x'=e^{t},\quad x(0)=0,\ x'(0)=1,\ x''(0)=0$

3.10. $x'''+x''=\cos t,\quad x(0)=-2,\ x'(0)=x''(0)=0$

3.11. $x''-x'=te^{t},\quad x(0)=x'(0)=0$

3.12. $x''+4x=2\cos t\cos 3t,\quad x(0)=x'(0)=0$

3.13. $x''+x=t\cos 2t,\quad x(0)=x'(0)=0$

3.14. $x''+x'+x=te^{t},\quad x(0)=x'(0)=0$

3.15. $x'''+x''=\sin t,\quad x(0)=x'(0)=1,\ \ x''(0)=0$

3.16. $x''+x=\cos t,\quad x(0)=-1,\ x'(0)=1$

3.17. $x'' + 2x' + 5x = t, \quad x(0) = 1, \quad x'(0) = 0$

3.18. $x'' - 4x' + 4x = t^2, \quad x(0) = 1, \quad x'(0) = 0$

3.19. $x'' - 2x' + 2x = 1, \quad x(0) = x'(0) = 0$

3.20. $x^{IV} - x'' = 1, \quad x(0) = x'(0) = x''(0) = x'''(0) = 0$

3.21. $x''' + 3x'' + 3x' + x = 1, \quad x(0) = x'(0) = x''(0) = 0$

3.22. $x'' + x' = te^t, \quad x(0) = 1, \quad x'(0) = 0$

3.23. $x'' - 2x = t\sin t, \quad x(0) = x'(0) = 0$

3.24. $x''' + 3x' = e^{-t}, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1$

3.25. $x'' - 2x' + 2x = e^{-2t}, \quad x(0) = x'(0) = 0$

3.26. $x'' + 4x = 2\cos t, \quad x(0) = x'(0) = 0$

3.27. $x'' + 2x' + x = 2\sin^2 t, \quad x(0) = x'(0) = 0$

3.28. $x''' - 3x'' + 3x' - x = 1, \quad x(0) = x'(0) = x''(0) = 0$

3.29. $x'' - 4x' + 4x = te^t, \quad x(0) = 0, \quad x'(0) = 1$

3.30. $x^{IV} - x'' = \cos t, \quad x(0) = x'(0) = 1, \quad x''(0) = x'''(0) = 0$

X BOB. QATORLAR

10.1. Musbat hadli qatorlar. Yaqinlashish alomatlari

Agar $a_1, a_2, a_3, ..., a_n, ...$ cheksiz haqiqiy sonlar ketma-ketligi berilgan bo‘lsa, ulardan tuzilgan ushbu

$$a_1 + a_2 + a_3 + ... + a_n + ... \qquad (10.1)$$

ifodaga ***cheksiz qator*** (qisqacha, ***qator***) deyiladi.

Qator, qisqacha, $\sum_{n=1}^{\infty} a_n$ ko‘rinishda ham yoziladi. $a_1, a_2, a_3, ..., a_n, ...$ - qatorning ***hadlari***, a_n ga qatorning ***umumiy hadi*** yoki $n - hadi$ deyiladi. $a_n > 0$ bo‘lsa, ***musbat hadli qator*** deyiladi.

$S_1 = a_1, S_2 = a_1 + a_2, \ S_3 = a_1 + a_2 + a_3, ..., \ S_n = a_1 + a_2 + a_3 + ... + a_n, \ ...$

yig‘indilarga qatorning ***xususiy*** (yoki ***qismiy) yig‘indilari*** deyiladi.

$S = \lim_{n\to\infty} S_n$ chekli limit mavjud bo‘lsa, (10.1) qator ***yaqinlashuvchi*** deyiladi va $S = \sum_{n=1}^{\infty} a_n$ bo‘ladi. $\lim_{n\to\infty} a_n = 0$ (10.1) qator yaqinlashuvchiligining zaruriy shartidir.

10.1-misol.

Ushbu $\frac{3}{4} + \frac{7}{8} + \frac{11}{16} + ...$ qatorni qisqa yig‘indi shaklida yozing va qator yaqinlashishining zaruriy shartini tekshiring.

► $\frac{3}{4} + \frac{7}{8} + \frac{11}{16} + ... = \sum_{n=1}^{\infty} \frac{4n-1}{2^{n+1}}$ bo‘lgani uchun umumiy had $a_n = \frac{4n-1}{2^{n+1}}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n-1}{2^{n+1}} = 0$, zaruriy shart bajariladi.◄

10.2-misol.

Ushbu $\frac{7}{10} + \frac{29}{100} + ... + \frac{2^n + 5^n}{2^n \cdot 5^n} + ...$ qator yig‘indisini toping.

► $\frac{7}{10} + \frac{29}{100} + ... + \frac{2^n + 5^n}{2^n \cdot 5^n} + ... = \sum_{n=1}^{\infty} \frac{2^n + 5^n}{2^n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{5^n} + \sum_{n=1}^{\infty} \frac{1}{2^n}$,

$$S_n = \frac{1}{5} + \frac{1}{5^2} + ... + \frac{1}{5^n} + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n} = S_n' + S_n'', \quad S_n' = \frac{\frac{1}{5}\left(1 - \frac{1}{5^n}\right)}{1 - \frac{1}{5}} = \frac{1}{4}\left(1 - \frac{1}{5^n}\right),$$

$$S_n'' = \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}, \ \lim_{n\to\infty} S_n = \lim_{n\to\infty}(S_n' + S'') = \lim_{n\to\infty}\left(\frac{1}{4}\left(1 - \frac{1}{5^n}\right) + 1 - \frac{1}{2^n}\right) = \frac{1}{4} + 1 = \frac{5}{4}.$$

$\frac{7}{10} + \frac{29}{100} + ... + \frac{2^n + 5^n}{2^n \cdot 5^n} + ... = 1{,}25.$ ◄

Taqqoslash alomati. *Agar* $\sum_{n=1}^{\infty} a_n$ *(1),* $\sum_{n=1}^{\infty} b_n$ *(2),* $a_n > 0$, $b_n > 0$ *qatorlar uchun* $a_n \le b_n$ *bo'lib,*

a) (2) qator yaqinlashuvchi bo'lsa, (1) qator ham yaqinlashuvchi;
b) (1) qator uzoqlashuvchi bo'lsa, (2) qator ham uzoqlashuvchi bo'ladi.

Umumlashgan taqqoslash alomati. $\sum_{n=1}^{\infty} a_n$ (1), $\sum_{n=1}^{\infty} b_n$ (2), $a_n > 0$, $b_n > 0$ qatorlar uchun $\lim_{n\to\infty}\frac{a_n}{b_n} = l \ (0 < l < \infty)$ bo'lsa, bu ikkala qator bir vaqtda yoki yaqinlashuvchi, yoki uzoqlashuvchi bo'ladi.

Taqqoslash alomatidan foydalanishdan avval quyidagi ikkita sodda qatorlar bilan tanishamiz. Bular geometrik va garmonik qatorlardir:

$$\sum_{n=1}^{\infty} aq^n \qquad (10.3),$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad (10.4).$$

Bu yerda (10.3) qator $q < 1$ da yaqinlashuchi, (10.4) esa uzoqlashuchi qatordir.

10.3-misol.

Ushbu $1+\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 3^3}+...+\frac{1}{n\cdot 3^n}+...$ qatorni yaqinlashishga tekshiring.

► $a_n = \frac{1}{n\cdot 3^n} \le \frac{1}{3^n}$, $\sum_{n=1}^{\infty}\frac{1}{3^n}$ geometrik qator yaqinlashuvchi bo'lgani uchun taqqoslash alomatiga ko'ra, berilgan qator ham yaqinlashuvchi.◄

Dalamber alomati. $\sum_{n=1}^{\infty} a_n$, $a_n > 0$ qator uchun $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = l$ chekli limit mavjud bo'lib,

a) $l < 1$ bo'lsa, qator yaqinlashuvchi;
b) $l > 1$ bo'lsa, qator uzoqlashuvchi bo'ladi.

Eslatma. 1) $l = \infty$ bo'lsa, qator uzoqlashuvchi.
2) $l = 1$ bo'lsa, Dalamber alomati javob bera olmaydi, boshqa alomatlardan foydalaniladi.

10.4-misol.

Ushbu $\sum_{n=1}^{\infty}\frac{1}{2^n\cdot n!}$ qatorni yaqinlashishga tekshiring.

► $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{\frac{1}{(n+1)!\cdot 2^{n+1}}}{\frac{1}{n!\cdot 2^{n1}}}=\lim\limits_{n\to\infty}\frac{n!\cdot 2^n}{(n+1)!\cdot 2^{n+1}}=\lim\limits_{n\to\infty}\frac{1}{2(n+1)}=0<1,$

Dalamber alomatiga ko‘ra, yaqinlashuvchi.◄

Koshi alomati. $\sum\limits_{n=1}^{\infty}a_n$, $a_n>0$ *qator uchun* $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=l$ *chekli limit mavjud bo‘lib,* *a)* $l<1$ *bo‘lsa, qator yaqinlashuvchi;*

b) $l>1$ *bo‘lsa, qator uzoqlashuvchi bo‘ladi.*

10.5-misol.

Ushbu $\sum\limits_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n=\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\ldots+\left(\frac{n}{2n+1}\right)^n+\ldots$ qatorning yaqinlashishini tekshiring.

►Koshi alomatidan $l=\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n}=\lim\limits_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}<1.$

Shunday qilib, berilgan qator Koshi alomatiga asosan yaqinlashuvchi bo‘ladi.◄

Integral alomati. $\sum\limits_{n=1}^{\infty}a_n$, $a_n>0$ *qator uchun* $a_1>a_2>\ldots>a_n>\ldots$ *bo‘lib,* $a_1=f(1), a_2=f(2),\ldots,a_n=f(n),\ldots$ *bo‘lsa,*

a) $\int\limits_1^{\infty}f(x)dx$ *yaqinlashuvchi bo‘lsa, qator yaqinlashuvchi;*

b) $\int\limits_1^{\infty}f(x)dx$ *uzoqlashuvchi bo‘lsa, qator uzoqlashuvchi bo‘ladi*

10.6-misol.

Umumlashgan garmonik qator $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ ni yaqinlashishga tekshiring.

► $1>\frac{1}{2}>\ldots>\frac{1}{n}>\ldots$ va $a_n=f(n)=\frac{1}{n^p}$, $\int\limits_1^{\infty}\frac{1}{x^p}dx$ integralni qaraymiz.

1) $p=1$ da garmonik qator hosil bo‘ladi, $\int\limits_1^{\infty}\frac{1}{x}dx=\lim\limits_{A\to\infty}(\ln A-\ln 1)=\infty$, qator uzoqlashuvchi;

2) $p<1$ da $\int\limits_1^{\infty}\frac{1}{x^p}dx=\lim\limits_{A\to\infty}\left(\frac{A^{1-p}}{1-p}-\frac{1}{1-p}\right)=\infty$, qator uzoqlashuvchi;

3) $p>1$ da $\int\limits_1^{\infty}\frac{1}{x^p}dx=\lim\limits_{A\to\infty}\left(-\frac{1}{(p-1)A^{p-1}}+\frac{1}{p-1}\right)=\frac{1}{p-1}$, qator yaqinlashuvchi.◄

39-Auditoriya topshiriqlari

1. Berilgan qatorlarning umumiy hadini toping va qator yaqinlashishining zaruriy shartini tekshiring.

a) $\frac{3}{4}+\frac{7}{16}+\frac{11}{64}+...$ b) $\frac{1}{2}+\frac{3}{5}+\frac{5}{8}+\frac{7}{11}+...$

2. Quyidagi qatorlarning yig‘indisini hisoblang.

a) $\sum_{n=1}^{\infty}\frac{1}{4n^2+4n-3}$ b) $\sum_{n=1}^{\infty}\frac{3^n+5^n}{15^n}$

d) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$ e) $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$

3. Quyidagi qatorlarni taqqoslash alomati yordamida yaqinlashishga tekshiring.

a) $\sum_{n=1}^{\infty}\frac{1}{2^{2n-1}\cdot(2n-1)}$

b) $\sum_{n=1}^{\infty}\sin\frac{\pi}{2^n}$

d) $\frac{1}{\ln 2}+\frac{1}{\ln 3}+\frac{1}{\ln 4}+...$

e) $1+\frac{1+2}{1+2^2}+\frac{1+3}{1+3^2}+...+\frac{1+n}{1+n^2}+...$

4. Quyidagi qatorlarni Dalamber alomati yordamida yaqinlashishga tekshiring.

a) $\sum_{n=1}^{\infty}\frac{3^n n!}{1\cdot 3\cdot ...\cdot(2n-1)}$ b) $\sum_{n=1}^{\infty}ntg\frac{\pi}{2^{n+1}}$

d) $\sum_{n=1}^{\infty}n^2\sin\frac{\pi}{2^n}$ e) $\sum_{n=1}^{\infty}\frac{2n}{(n+1)!}$

5. Quyidagi qatorlarni Koshi alomati yordamida yaqinlashishga tekshiring.

a) $\sum_{n=1}^{\infty}\frac{1}{\ln^n(n+1)}$ b) $\sum_{n=1}^{\infty}\arcsin^n\frac{1}{n}$

d) $\sum_{n=1}^{\infty}\frac{3^n\cdot n^{n^2}}{(n+1)^{n^2}}$

6. Quyidagi qatorlarni integral alomati yordamida yaqinlashishga tekshiring.

a) $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^2(n+1)}$ b) $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln(n+1)}$

d) $\sum_{n=1}^{\infty}\left(\frac{1+n}{1+n^2}\right)^2$

39- Mustaqil yechish uchun testlar

1. Quyidagi qatorlardan qaysi biri uchun qator yaqinlashishining zaruriy sharti bajarilmaydi?

A) $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$ B) $\sum_{n=1}^{\infty}\frac{1+n}{1+n^2}$ D) $\sum_{n=1}^{\infty}\frac{1+3n^2}{n+3n^2}$ E) $\sum_{n=1}^{\infty}\sin\frac{\pi}{2^n}$.

2. Geometrik qator berilgan javobni aniqlang.

A) $\sum_{n=1}^{\infty}aq^n$ B) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ D) $\sum_{n=1}^{\infty}\frac{1}{n}$ E) $\sum_{n=1}^{\infty}\frac{1}{\ln^n(n+1)}$.

3. Musbat hadli $\sum_{n=1}^{\infty}a_n$ sonli qator yaqinlashishining Dalamber alomati ifodalangan javobni toping.

A) $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=p$ bo‘lib, $p>1$ bo‘lsa, qator yaqinlashuvchi, $p<1$ bo‘lsa, qator uzoqlashuvchi;

B) $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=p$ bo‘lib, $p<1$ bo‘lsa, qator yaqinlashuvchi, $p>1$ bo‘lsa, qator uzoqlashuvchi;

D) $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=p$ bo‘lib, $p>1$ bo‘lsa, qator yaqinlashuvchi, $p<1$ bo‘lsa, qator uzoqlashuvchi;

E) $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=p$ bo‘lib, $p<1$ bo‘lsa, qator yaqinlashuvchi, $p>1$ bo‘lsa, qator uzoqlashuvchi.

4. Umumiy hadi $a_n=\frac{2n}{n^2+3}$ bo‘lgan qatorni toping.

A) $\frac{1}{2}+\frac{4}{7}+\frac{6}{12}+\frac{8}{19}+...$ B) $\frac{2}{4}+\frac{4}{7}+\frac{8}{12}+\frac{16}{19}+...$

D) $\frac{1}{2}+\frac{4}{7}+\frac{6}{11}+\frac{8}{15}+...$ E) To‘g‘ri javob yo‘q.

5. Musbat hadli $\sum_{n=1}^{\infty}a_n$ sonli qator yaqinlashishining Koshi alomati ifodalangan javobni toping.

A) $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=p$ bo‘lib, $p>1$ bo‘lsa, qator yaqinlashuvchi, $p<1$ bo‘lsa, qator uzoqlashuvchi;

B) $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=p$ bo‘lib, $p<1$ bo‘lsa, qator yaqinlashuvchi, $p>1$ bo‘lsa, qator uzoqlashuvchi;

D) $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=p$ bo‘lib, $p>1$ bo‘lsa, qator yaqinlashuvchi, $p<1$ bo‘lsa, qator uzoqlashuvchi;

E) $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=p$ bo‘lib, $p<1$ bo‘lsa, qator yaqinlashuvchi, $p>1$ bo‘lsa, qator uzoqlashuvchi.

10.2. Ishorasi almashinuvchi qatorlar. Absolyut va shartli yaqinlashish

$$u_1-u_2+u_3-u_4+...+(-1)^{n-1}u_n+...=\sum_{n=1}^{\infty}(-1)^{n-1}u_n \qquad (10.5)$$

ko‘rinishdagi qatorga ***ishoralari navbat bilan almashib keladigan (ishoralari almashinuvchi)*** qatorlar deyiladi. Bu yerda $u_1, u_2, u_3, ..., u_n, ...$ musbat sonlar.

Leybnits teoremasi (alomati). *Agar ishorasi almashinuvchi*

$$u_1-u_2+u_3-u_4+...+(-1)^{n-1}u_n+...$$

qatorda

a) qator hadlarining absolyut qiymatlari kamayuvchi, ya’ni

$$u_1>u_2>u_3>u_4>...>u_n>... \qquad (10.6)$$

bo‘lsa,

b) qator umumiy hadi u_n $n\to\infty$ *da nolga intilsa:*

$$\lim_{n\to\infty}u_n=0, \qquad (10.7)$$

u holda bu qator yaqinlashuvchi bo‘ladi.

10.7-misol.

Ushbu $\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-...+(-1)^{n+1}\frac{1}{(n+1)^2}+...$ qatorning yaqinlashuvchanligini tekshiring.

► $\frac{1}{2^2}>\frac{1}{3^2}>\frac{1}{4^2}>...>\frac{1}{(n+1)^2}>...$ va $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{1}{(n+1)^2}=0$.

Demak, qator yaqinlashuvchi.◄

Endi ixtiyoriy ishorali qatorlarni ko‘raylik.

$$u_1+u_2+u_3+u_4+...+u_n+... \qquad (10.8)$$

qatorning cheksiz ko‘p musbat va cheksiz ko‘p manfiy hadlari bo‘lsa, u holda bu qatorga ***o‘zgaruvchan ishorali qator*** yoki ixtiyoriy hadli qator deyiladi.
(10.8) qator hadlarining absolyut qiymatlaridan

$$|u_1|+|u_2|+|u_3|+|u_4|+...+|u_n|+... \qquad (10.9)$$

qatorni tuzaylik.
(10.8) va (10.9) qatorlar bir paytda yaqinlashuvchi bo‘lsa, (10.8) qatorga ***absolyut yaqinlashuvchi*** qator deyiladi.
Agar (10.8) qator yaqinlashuvchi bo‘lib (10.9) qator uzoqlashuvchi bo‘lsa, u holda berilgan (10.8) qatorga ***shartli yaqinlashuvchi*** qator deyiladi.

10.8-misol.

Ushbu $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^{n}\frac{1}{n}+...$ qatorni absolyut va shartli yaqinlashishga tekshiring.

►Leybnits, alomatiga ko‘ra bu qator yaqinlashuvchi, lekin qator hadlarining absolyut qiymatlaridan tuzilgan $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}+...$ garmonik qator esa uzoqlashuvchi. Demak, qator shartli yaqinlashuvchi.◄

10.1-Teorema. *Agar (10.9) qator yaqinlashuvchi bo‘lsa, (10.8) qator ham yaqinlashuvchi bo‘ladi.*

10.9-misol.

O‘zgaruvchan ishorali

$$\frac{\sin\alpha}{1^2}+\frac{\sin 2\alpha}{2^2}+...+\frac{\sin n\alpha}{n^2}+...$$

qatorning yaqinlashishini tekshiring, bu yerda α -ixtiyoriy haqiqiy son.

► Berilgan qator bilan birga

$$\left|\frac{\sin\alpha}{1^2}\right|+\left|\frac{\sin 2\alpha}{2^2}\right|+...+\left|\frac{\sin n\alpha}{n^2}\right|+...$$

qatorni qaraymiz. Bu qatorni yaqinlashuvchi(p>1)

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...+\frac{1}{n^2}+...$$

qator bilan taqqoslaymiz.

Ravshanki, $\left|\frac{\sin n\alpha}{n^2}\right|\le\frac{1}{n^2}$, $n=1,2,...$

Shu sababli, taqqoslash alomatiga ko‘ra absolyut hadli qator yaqinlashuvchi. U holda yuqorida teoremaga ko‘ra, berilgan qator yaqinlashuvchi.◄

40-Auditoriya topshiriqlari

Berilgan qatorlarni absolyut va shartli yaqinlashishga tekshiring.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{1+n^2}$
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n-1}$
3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)^3}$
4. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^3}{2^n}$
5. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\ln(n+1)}$
6. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{5^n+n}$
7. $\sum_{n=2}^{\infty}\frac{(-1)^{n}}{n-\ln n}$
8. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$

40-Mustaqil yechish uchun testlar

1. Quyidagi qatorlardan qaysi biri absolyut yaqinlashuvchi?

A) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{1+n^2}$, B) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^3}{2^n}$, D) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$, E) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n-1}$.

2. Agar o‘zgaruvchan ishorali $\sum_{n=1}^{\infty}a_n$ qator uzoqlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty}|a_n|$ qator …

A) shartli yaqinlashuvchi B) uzoqlashuvchi
D) absolyut yaqinlashuvchi
E) yaqinlashuvchi ham, uzoqlashuvchi ham bo‘lishi mumkin.

3. Agar o‘zgaruvchan ishorali $\sum_{n=1}^{\infty}a_n$ qator uzoqlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty}|a_n|$ qator …

A) shartli yaqinlashuvchi B) uzoqlashuvchi
D) absolyut yaqinlashuvchi
E) yaqinlashuvchi ham, uzoqlashuvchi ham bo‘lishi mumkin.

4. Agar $\sum_{n=1}^{\infty}|a_n|$ qator yaqinlashuvchi bo‘lsa, u holda o‘zgaruvchan ishorali $\sum_{n=1}^{\infty}a_n$ qator … .

A) shartli yaqinlashuvchi B) uzoqlashuvchi
D) absolyut yaqinlashuvchi
E) yaqinlashuvchi ham, uzoqlashuvchi ham bo‘lishi mumkin.

5. Quyidagi qatorlardan qaysi biri shartli yaqinlashuvchi?

A) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{1+n^2}$, B) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^3}{2^n}$, D) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$, E) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n-1}$.

10.3 Darajali qatorlar. Yaqinlashish radiusi va sohasi

Hadlari x o‘zgaruvchining funksiyalardan iborat bo‘lgan

$$u_1(x)+u_2(x)+...+u_n(x)+... \qquad (10.10)$$

ko‘rinishdagi qatorga ***funksional qator*** deyiladi.

Agar (10.10) qator x ning $x_0,\ x_1,\ x_2,...,\ x_n$ aniq son qiymatlarida yaqinlashuvchi bo‘lsa u holda x ning bu $x_0,\ x_1,\ x_2,...,\ x_n$ son qiymatlar to‘plamiga (10.10) ning ***yaqinlashish sohasi*** deyiladi.

Qatorning dastlabki n ta hadi yig‘indisini $S_n(x)$ bilan belgilaylik:

$$S_n(x)=u_1(x)+u_2(x)+...+u_n(x) \qquad (10.11)$$

Agar

$$\lim_{n\to\infty} S_n(x) = S(x)$$

chekli limit mavjud bo‘lsa, (10.10) funksional qator ***yaqinlashuvchi qator***, $S(x)$ esa uning yig‘indisi deyiladi.

Darajali qator deb,

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ... + a_n(x-x_0)^n + ... \qquad (10.12)$$

ko‘rinishdagi funksional qatorga aytiladi.

$x_0 = 0$ da

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \qquad (10.13)$$

ko‘rinishdagi $\boldsymbol{x}$ ning darajalari bo‘yicha yoyilgan darajali qatorga ega bo‘lamiz. Bu yerda $a_0,\ a_1,\ a_2, ..., a_n, ...$ lar o‘zgarmas sonlar bo‘lib, ularga darajali qatorning koeffitsiyentlari deyiladi.

Demak, darajali qatorlar funksional qatorning xususiy holidan iborat.

Har qanday (10.13) darajali qator $x = 0$ nuqtada yaqinlashuvchi bo‘ladi, chunki bu holda qator $a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + ... + a_n \cdot 0^n + ...$ ko‘rinishda sonli qatorga aylanadi va $\lim_{n\to\infty} S_n(0) = \lim_{n\to\infty} a_0 = a_0$ bo‘ladi.

Bu (10.13) darajali qatorning yaqinlashish radiusini Dalamber alomatidan foydalanib topilgan formulasi

$$\boxed{R = \frac{1}{l} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|} \quad . \qquad (10.14)$$

Bundan, (10.13) ning yaqinlashish radiusi $(-R; R)$ ni hosil qilamiz. Odatda, intervalning chegaralari $x = \pm R$ da qator yaqinlashishga alohida tekshiriladi.

Eslatma. 1) Agar $R = 0$ bo‘lsa, qator faqat $x = 0$ nuqtada yaqinlashuvchi.
2) Agar $R = \infty$ bo‘lsa, qator $(-\infty;\ \infty)$ da yaqinlashuvchi.

10.10-misol.

Darajali qatorning yaqinlashish sohasi topilsin:

$$\frac{2x}{1} - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - ... + (-1)^{n+1} \cdot \frac{(2x)^n}{n} + ...$$

► Bu yerda $a_n = (-1)^{n+1} \cdot \frac{2^n}{n}$, $a_{n+1} = (-1)^{n+2} \cdot \frac{2^{n+1}}{n+1}$. Shu sababli

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{2^n \cdot (n+1)}{n \cdot 2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{2}.$$

$x = \frac{1}{2}$ da $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ qatorga ega bo‘lamiz, bu qator Leybnits alomatiga ko‘ra, yaqinlashuvchi:

a) $1>\frac{1}{2}>\frac{1}{3}>\frac{1}{4}>...>\frac{1}{n}>...$ ***b)*** $\lim\limits_{n\to\infty}|u_n|=\lim\limits_{n\to\infty}\frac{1}{n}=\frac{1}{\infty}=0$.

$x=-\frac{1}{2}$ da $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-...$ qatorga ega bo'lamiz, bu qator garmonik qator sifatida uzoqlashuvchi. Demak, $\left(-\frac{1}{2};\frac{1}{2}\right]$ interval yaqinlashish sohasi bo'ladi. ◄

Yaqinlashish intervalini aniqlash uchun, shuningdek, Koshi alomatidan ham foydalanish mumkin, bu holda

$$R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}. \tag{10.15}$$

10.11-misol.

Darajali qatorning yaqinlashish sohasi topilsin:

$$2x+\left(\frac{9}{4}x\right)^2+\left(\frac{64}{27}x\right)^3+...+\left(\left(\frac{n+1}{n}\right)^n x\right)^n+...$$

► Bu yerda $a_n=\left(\frac{n+1}{n}\right)^{n^2}$, $R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}=\frac{1}{\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^n}=\frac{1}{e}$.

$x=\pm\frac{1}{e}$ da qator uzoqlashuvchi, chunki zaruriy shart bajarilmaydi, $\lim\limits_{n\to\infty}\left(\pm\frac{1}{e}\left(\frac{n+1}{n}\right)^n\right)^n\neq 0$. Demak, $\left(-\frac{1}{e};\frac{1}{e}\right)$ interval yaqinlashish sohasi bo'ladi.◄

(10.13) qatorning yaqinlashish sohasi $(x_0-R;x_0+R)$ dan iborat bo'ladi.

10.12-misol.

Qatorning yaqinlashish sohasini toping: $\sum\limits_{n=1}^{\infty}\frac{(x+1)^n}{(2n+1)3^n}$.

► Bu yerda $a_n=\frac{1}{(2n+1)3^n}$, $a_{n+1}=\frac{1}{(2n+3)3^{n+1}}$, $R=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim\limits_{n\to\infty}\frac{3(2n+3)}{2n+1}=3$.

Shu sababli yaqinlashish oralig'i: $|x+1|<3$, $-3<x+1<3$, $-4<x<2$.

$x=2$ da $\sum\limits_{n=1}^{\infty}\frac{1}{2n+1}$ qator uzoqlashuvchi (garmonik qator bilan taqqoslab aniqlanadi);

$x=-4$ da $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$ qator Leybnits alomatiga ko'ra, yaqinlashuvchi.

Demak, qatorning yaqinlashish sohasi: $[-4;\,2)$. ◄

41-Auditoriya topshiriqlari

Berilgan qatorlarning yaqinlashish sohasini toping.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{1+n^2}$
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+2)^n}{10^n}$
3. $\sum_{n=1}^{\infty}\frac{n!\,x^n}{(2n-1)^3}$
4. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{2n}}{2^n n^3}$
5. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{(n+1)^n}$
6. $\sum_{n=1}^{\infty}\frac{(-x)^{n-1}}{5^n+n}$
7. $\sum_{n=2}^{\infty}n!\,x^n$
8. $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(n+2)\cdot 9^n}$

41-Mustaqil yechish uchun testlar

1. Qatorning yaqinlashish sohasini toping: $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(x+4)^n}{\sqrt{n}}$?

A) $[-5;-3]$ B) $(-1;1)$ D) $[-5;-3)$ E) $(-5;-3]$.

2. Qatorning yaqinlashish sohasini toping: $\sum_{n=1}^{\infty}\frac{(-1)^n(x-3)^n}{(n+1)5^n}$

A) $(-1;1)$ B) $[-2;\ 8]$ D) $[-2;\ 8)$ E) $(-2;\ 8]$.

3. Qatorning yaqinlashish sohasini toping: $\sum_{n=1}^{\infty}\frac{x^n}{1+n^2}$.

A) $(-1;1)$ B) $[-1;\ 1]$ D) $[-2;\ 8)$ E) $(-5;\ 3]$.

4. Agar $\sum_{n=2}^{\infty}a_n(x-a)^n$ uchun $R=0$ bo‘lsa, u holda qator … da yaqinlashuvchi.

A) $x=0$ B) $x=a$ D) $(-\infty;\ \infty)$ E) $(-\infty;\ a]$.

5. Agar $\sum_{n=2}^{\infty}a_n(x-a)^n$ uchun $R=\infty$ bo‘lsa, u holda qator … da yaqinlashuvchi.

A) $x=0$ B) $x=a$ D) $(-\infty;\ \infty)$ E) $(-\infty;\ a]$.

13-Shaxsiy uy topshiriqlari

1

Berilgan qatorlarning yig‘indisini hisoblang.

1.1. $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$.

1.2. $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$.

1.3. $\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}$.

1.4. $\sum_{n=1}^{\infty}\frac{3^n-2^n}{4^n}$.

1.5. $\sum_{n=1}^{\infty}\frac{1}{(3n-1)(3n+2)}$.

1.6. $\sum_{n=1}^{\infty}\frac{5^n-2^n}{10^n}$.

1.7. $\sum_{n=1}^{\infty}\frac{1}{(5n-1)(5n+4)}$.

1.8. $\sum_{n=1}^{\infty}\frac{4^n-3^n}{12^n}$.

1.9. $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+3)}$.

1.10. $\sum_{n=1}^{\infty}\frac{4^n+3^n}{12^n}$.

1.11. $\sum_{n=1}^{\infty}\frac{1}{(3n-1)(3n+5)}$.

1.12. $\sum_{n=1}^{\infty}\frac{5^n+2^n}{10^n}$.

1.13. $\sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n+5)}$.

1.14. $\sum_{n=1}^{\infty}\frac{5^n+4^n}{20^n}$.

1.15. $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+4)}$.

1.16. $\sum_{n=1}^{\infty}\frac{5^n-4^n}{20^n}$.

1.17. $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)}$

1.18. $\sum_{n=1}^{\infty}\frac{7^n+3^n}{21^n}$.

1.19. $\sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n+5)}$.

1.20. $\sum_{n=1}^{\infty}\frac{7^n-3^n}{21^n}$.

1.21. $\sum_{n=1}^{\infty}\frac{1}{(n+3)(n+4)}$.

1.22. $\sum_{n=1}^{\infty}\frac{8^n-3^n}{24^n}$.

1.23. $\sum_{n=1}^{\infty}\frac{1}{(3n-1)(3n+2)}$.

1.24. $\sum_{n=1}^{\infty}\frac{8^n+3^n}{24^n}$.

1.25. $\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}$.

1.26. $\sum_{n=1}^{\infty}\frac{9^n+2^n}{18^n}$.

1.27. $\sum_{n=1}^{\infty}\frac{1}{(2n+3)(2n+5)}$

1.28. $\sum_{n=1}^{\infty}\frac{9^n-2^n}{18^n}$.

1.29. $\sum_{n=1}^{\infty}\frac{1}{(n+5)(n+7)}$.

1.30. $\sum_{n=1}^{\infty}\frac{4^n-3^n}{5^n}$

2

Berilgan qatorlarni yaqinlashishga tekshiring.

2.1. . $\sum_{n=1}^{\infty}\frac{3^n(n+2)!}{n^5}$.

2.2. $\sum_{n=1}^{\infty}\frac{3n-1}{7^n(2n+1)!}$

2.3. $\sum_{n=1}^{\infty}\frac{3^{n-1}}{7^n n^7}$.

2.4. $\sum_{n=1}^{\infty}\frac{(2n-1)!}{3^n(2n+1)}$.

2.5. $\sum_{n=1}^{\infty}\frac{n^n}{2^n(n+1)!}$.

2.6. $\sum_{n=1}^{\infty}n\sin\frac{2\pi}{3^n}$.

2.7. $\sum_{n=1}^{\infty}(3n+1)tg\frac{\pi}{3^n}$.

2.8. $\sum_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdot...\cdot(3n-2)}{2\cdot3\cdot4\cdot...\cdot(n+1)}$.

2.9. $\sum_{n=1}^{\infty}n^3 tg\frac{2\pi}{5^n}$.

2.10. $\sum_{n=1}^{\infty}\frac{7^n(3n-1)}{(2n+1)!}$

2.11. $\sum_{n=1}^{\infty}\frac{n^n}{(n+1)!}$

2.12. $\sum_{n=1}^{\infty}\frac{(n+2)!}{n^n}$

2.13. $\sum_{n=1}^{\infty}\frac{5^n}{4(n+1)!}$

2.14. $\sum_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdot\ldots\cdot(3n-2)}{2\cdot7\cdot12\cdot\ldots\cdot(5n-3)}$.

2.15. $\sum_{n=1}^{\infty}\frac{(2n+1)!}{2^n(n+1)}$.

2.16. $\sum_{n=1}^{\infty}\frac{(2n-1)^3}{(2n)!}$.

2.17. $\sum_{n=1}^{\infty}\frac{(2n^2-1)}{(n+2)!}$.

2.18. $\sum_{n=1}^{\infty}\frac{3n-1}{\sqrt{5^n(2n+1)}}$.

2.19. $\sum_{n=1}^{\infty}\frac{4n+1}{\sqrt{n\cdot5^n}}$.

2.20. $\sum_{n=1}^{\infty}(3n-1)\sin\frac{\pi}{4^n}$.

2.21. $\sum_{n=1}^{\infty}\frac{2n+1}{\sqrt{n\cdot2^n}}$.

2.22. $\sum_{n=1}^{\infty}\frac{1\cdot5\cdot9\cdot\ldots\cdot(4n-3)}{1\cdot4\cdot7\cdot\ldots\cdot(3n-2)}$.

2.23. $\sum_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdot\ldots\cdot(3n-2)}{n^2(n+2)!}$.

2.24. $\sum_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdot\ldots\cdot(3n-2)}{2^n(n+3)!}$.

2.25. $\sum_{n=1}^{\infty}\frac{3^n}{5^n(3n+1)}$.

2.26. $\sum_{n=1}^{\infty}\frac{(n+1)!}{2(2n+1)!}$.

2.27. $\sum_{n=1}^{\infty}\frac{5^n(4n-3)}{1\cdot4\cdot7\cdot\ldots\cdot(3n-2)}$

2.28. $\sum_{n=1}^{\infty}\frac{(2n-1)!}{3^n(2n+1)}$.

2.29. $\sum_{n=1}^{\infty}\frac{(n^2+1)}{(n+2)!}$

2.30. $\sum_{n=1}^{\infty}\frac{3^n}{2^n(2n+1)}$

3

Berilgan qatorlarni yaqinlashishga tekshiring.

3.1. $\sum_{n=1}^{\infty}\frac{2^n}{((n+1)/n)^n}$.

3.2. $\sum_{n=1}^{\infty}\left(\frac{5n-1}{5n}\right)^{n^2}$.

3.3. $\sum_{n=1}^{\infty}arctg^n\frac{1}{2n+1}$.

3.4. $\sum_{n=1}^{\infty}\arcsin^{3n}\frac{1}{2^n}$.

3.5. $\sum_{n=1}^{\infty}\frac{1}{\ln^n(n+2)}$.

3.6. $\sum_{n=1}^{\infty}arctg^n\frac{1}{5^n}$.

3.7. $\sum_{n=1}^{\infty}2^n(n/(n+1))^{n^2}$.

3.8. $\sum_{n=1}^{\infty}3^n(n/(n+1))^{n^2}$.

3.9. $\sum_{n=1}^{\infty}\left(\frac{n^2+5n+3}{3n^2-2}\right)^n$.

3.10. $\sum_{n=1}^{\infty}\frac{4^n}{((n+1)/n)^{n^2}}$.

3.11. $\sum_{n=1}^{\infty}arctg^n\frac{\sqrt{3n+2}}{n+1}$.

3.12. $\sum_{n=1}^{\infty}\arcsin^n\frac{n}{2n+1}$.

3.13. $\sum_{n=1}^{\infty}\left(\frac{n^2+2n+3}{2n^2+1}\right)^n$.

3.14. $\sum_{n=1}^{\infty}\left(\frac{3n-1}{3n}\right)^{n^2}$.

3.15. $\sum_{n=1}^{\infty}\left(\frac{2n^2+3}{3n^2+1}\right)^n$.

3.16. $\sum_{n=1}^{\infty}arcctg^n\sqrt{\frac{n+1}{3n-1}}$.

3.17. $\sum_{n=1}^{\infty}\left(\frac{5n+1}{5n}\right)^{n^2}$

3.18. $\sum_{n=1}^{\infty}\frac{3^n}{((n+1)/n)^n}$.

3.19. $\sum_{n=1}^{\infty}\left(\frac{n+1}{5n}\right)^n$.

3.20. $\sum_{n=1}^{\infty}\frac{n^n}{3^n}$.

3.21. $\sum_{n=1}^{\infty}\frac{5^n}{n^n}\cdot$.

3.22. $\sum_{n=1}^{\infty}\frac{1}{n^n}\cdot\left(\frac{2}{3}\right)^n$

3.23. $\sum_{n=1}^{\infty} arcctg^n\,\frac{\sqrt{3}n+2}{n+1}$.

3.24. $\sum_{n=1}^{\infty}\arcsin^n\sqrt{\frac{n+1}{2n+1}}$.

3.25. $\sum_{n=1}^{\infty}\frac{((n+1)/n)^n}{3^n}$.

3.26. $\sum_{n=1}^{\infty}\frac{2^n}{\ln^n(n+2)}$.

3.27. $\sum_{n=1}^{\infty}\frac{1}{n^n}\cdot\left(\frac{3}{2}\right)^n$.

3.28. $\sum_{n=1}^{\infty}\left(\frac{2n+1}{5n-1}\right)^n$

3.29. $\sum_{n=1}^{\infty}\arccos^n\frac{n+1}{2n+1}$

3.30. $\sum_{n=1}^{\infty}\left(\frac{3n+1}{5n+1}\right)^n$

4

Berilgan qatorlarni absolyut va shartli yaqinlashishga tekshiring.

4.1. $\sum_{n=1}^{\infty}\frac{(-1)^n n}{2n^2+1}$.

4.2. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n^2+1}}$.

4.3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{6n-1}$.

4.4. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n^3+1}}$.

4.5. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln(n+1)}$.

4.6. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+1}$.

4.7. $\sum_{n=1}^{\infty}(-1)^{n+1} tg\,\frac{\pi}{4\sqrt{n}}$.

4.8. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$.

4.9. $\sum_{n=1}^{\infty}(-1)^{n-1} n\ln\left(1+\frac{1}{n^2}\right)$.

4.10. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{n^2+4}$.

4.11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2+4}$.

4.12. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{5^n+4}$.

4.13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{5n(n+1)}$.

4.14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3n}{\sqrt{n^2+9}}$.

4.15. $\sum_{n=1}^{\infty}(-1)^{n+1} tg\,\frac{\pi}{3\sqrt[3]{n^2}}$.

4.16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n-1}{n(n+1)}$.

4.17. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n^2+4}}$.

4.18. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)\ln(n+1)}$.

4.19. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)\ln^2(n+1)}$.

4.20. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{(2n+1)^n}$.

4.21. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+5}{3^n}$.

4.22. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{\pi}{2\sqrt{n}}$.

4.23. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{(2n+1)!}$.

4.24. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{4^n}$.

4.25. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3}{(2n+1)!}$.

4.26. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n!}{(2n+1)!}$.

4.27. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{n(n+1)}$.

4.28. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n^3+4}}$.

4.29. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{2^n}$.

4.30. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(3n+1)!}$.

5

Berilgan qatorlarning yaqinlashish sohasini toping.

5.1. $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n^2+9}}$.

5.2. $\sum_{n=1}^{\infty}\frac{(x-5)^n}{n\cdot 3^n}$.

5.3. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{(n+1)\ln(n+1)}$.

5.4. $\sum_{n=1}^{\infty}(-1)^n\frac{(x-3)^n}{n\cdot 5^n}$.

5.5. $\sum_{n=1}^{\infty}(-1)^n\frac{(x-2)^n}{2n\cdot 4^n}$.

5.6. $\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n-\ln^2 n}$.

5.7. $\sum_{n=1}^{\infty}\frac{x^n}{(n+1)\ln^2(n+1)}$.

5.8. $\sum_{n=1}^{\infty}\frac{(x+2)^{2n}}{n\cdot 4^n}$.

5.9. $\sum_{n=1}^{\infty}\frac{(x+2)^{2n-1}}{(2n-1)\cdot(2n-1)!}$

5.10. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{2^n\ln(n+1)}$.

5.11. $\sum_{n=1}^{\infty}\frac{2^n x^n}{n^2+1}$.

5.12. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n!x^n}{n^n}$.

5.13. $\sum_{n=1}^{\infty}\frac{2^n x^n}{\sqrt{2n+1}}$.

5.14. $\sum_{n=1}^{\infty}\frac{5^n x^n}{(2n+1)\sqrt{3^n}}$.

5.15. $\sum_{n=1}^{\infty}\frac{3^n x^n}{(2n+1)^2\sqrt{3^n}}$.

5.16. $\sum_{n=1}^{\infty}\frac{5^n(x+1)^n}{n^n}$.

5.17. $\sum_{n=1}^{\infty}\frac{3^n(x+2)^n}{n^2}$.

5.18. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}\frac{x^n}{2^n}$.

5.19. $\sum_{n=1}^{\infty}\frac{3^n(x+2)^n}{(2n+1)\sqrt{2^n}}$.

5.20. $\sum_{n=1}^{\infty}\frac{5^n(x+3)^n}{n^2+1}$.

5.21. $\sum_{n=1}^{\infty}\frac{\sqrt{n}x^n}{n!}$.

5.22. $\sum_{n=1}^{\infty}\frac{(5n-2)(x-3)^n}{(n^2+1)\cdot 2^{n+1}}$.

5.23. $\sum_{n=1}^{\infty}\frac{(x+5)^{2n-1}}{2n\cdot 4^n}$.

5.24. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}\frac{x^n}{3^n}$.

5.25. $\sum_{n=1}^{\infty}\frac{3^n x^n}{\sqrt{2n-1}}$.

5.26. $\sum_{n=1}^{\infty}\frac{4^n(x+1)^n}{(n+1)!}$.

5.27. $\sum_{n=1}^{\infty}(-1)^n\frac{(x+1)^{2n-1}}{(2n-1)\cdot(2n-1)!}$.

5.28. $\sum_{n=1}^{\infty}\frac{\sqrt{n^3}(x-2)^n}{n!}$.

5.29. $\sum_{n=1}^{\infty}(nx)^n$

5.30. $\sum_{n=1}^{\infty}n!x^n$

10.4. Teylor va Makloren qatorlari

Agar $f(x)$ funksiya $x = x_0$ nuqtaning biror atrofida aniqlangan va istalgan tartibli hosilaga ega bo‘lsa, u holda bu funksiyani darajali qatorga yoyish mumkin.

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + ... \tag{10.16}$$

(10.16)– ***Teylor qatori (Teylor formulasi)*** deyiladi.
Xususiy holda $x_0 = 0$ bo‘lsa,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + ... \tag{10.17}$$

Makloren qatori hosil bo‘ladi.
Funksiyani Teylor qatoriga yoyish mumkin bo‘lishi uchun $x = x_0$ nuqtaning biror atrofida qatorning qoldiq hadi $n \to \infty$ da cheksiz kichik bo‘lishi zarur va yetarlidir. Shuning uchun har bir holda qatorning $f(x)$ funksiyaga yaqinlashish sohasini topish kerak bo‘ladi.
Teylor qatori qoldiq hadining Lagranj ko‘rinishidagi formulasi

$$R_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}, \quad (0 < \theta < 1) \tag{10.18}$$

(10.18) dan foydalanib, (10.16) ni quyidagicha yozish mumkin:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}$$

10.13-misol.

Ushbu $y = \frac{1}{x}$ funksiyani $x_0 = 1$ nuqtada Teylor qatoriga yoying va yaqinlashish oralig‘ini aniqlang.

► $x = 1$ nuqtada $y = \frac{1}{x}$ funksiyaning hosilalari qiymatlarini topamiz.

$$y(1) = 1, \quad y'(1) = -\frac{1}{x^2}\bigg|_{x=1} = -1, \quad y''(1) = \frac{1 \cdot 2}{x^3}\bigg|_{x=1} = 2!, \quad y'''(1) = -\frac{1 \cdot 2 \cdot 3}{x^4}\bigg|_{x=1} = -3!,$$

$$y^{IV}(1) = \frac{1 \cdot 2 \cdot 3 \cdot 4}{x^5}\bigg|_{x=1} = 4!, \ ... \ y^{(n)}(1) = (-1)^n \frac{n!}{x^{n+1}}\bigg|_{x=1} = (-1)^n \cdot n!.$$

Bundan

$$f(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + ... + (-1)^n (x-1)^n + ... = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

Qatorning yaqinlashish oralig‘ini topamiz

$$a_n = (-1)^n, \ R = \lim_{n \to \infty}\left|\frac{a_{n+1}}{a_n}\right| = 1, \ -1 < x - 1 < 1, \quad 0 < x < 2.$$

$x=0$ va $x=2$ nuqtalarda qator uzoqlashadi. Yaqinlashish sohasi - $0<x<2$. ◀

Quyida bir necha elementar funksiyalarning Makloren qatoriga yoyilmasini keltiramiz:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in (-\infty; \infty). \tag{10.19}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + (-1)^n \frac{x^{2n}}{(2n)!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in (-\infty; \infty). \tag{10.20}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in (-\infty; \infty). \tag{10.21}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1} \cdot \frac{x^n}{n} + ... = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n}, \quad x \in (-1; 1]. \tag{10.22}$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + ... + \frac{m(m-1)\cdot ... \cdot(m-n+1)x^n}{n!} + ... =$$
$$= \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdot ... \cdot(m-n+1)}{n!} x^n, \qquad x \in (-1; 1). \tag{10.23}$$

$$arctgx = x - \frac{x^3}{3} + \frac{x^5}{5} - ... + (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1} + ... = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1}, \quad x \in (-1; 1) \tag{10.24}$$

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ... + \frac{x^{2n-1}}{2n-1} + ... \right) = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}, \quad x \in (-1; 1). \tag{10.25}$$

10.14-misol.

Ushbu $f(x) = \dfrac{3}{(1-x)(1+2x)}$ funksiyani Makloren qatoriga yoying. Hosil bo‘lgan qatorning yaqinlashish sohasini toping.

► Berilgan funksiyani sodda kasrlarga ajratamiz

$$\frac{3}{(1-x)(1+2x)} = \frac{1}{1-x} + \frac{2}{1+2x}.$$

(10.23) yoyilmadan

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \left(|x| < 1\right), \qquad \frac{1}{1+2x} = \sum_{n=0}^{\infty} (-1)^n \cdot (2x)^n, \quad \left(|2x| < 1\right).$$

Bundan

$$\frac{3}{(1-x)(1+2x)}=\sum_{n=0}^{\infty}x^n+2\sum_{n=0}^{\infty}(-1)^n\cdot(2x)^n=\sum_{n=0}^{\infty}\left(1+(-1)^n 2^{n+1}\right)x^n.$$

Yuqoridagi ikkita qator $|x|<1,\ |2x|<1$ da yaqinlashuvchi bo‘lganligi uchun hosil bo‘lgan qator $|x|<1/2$ da yaqinlashuvchi bo‘ladi.◄

Funksiyalarni darajali qatorga yoyish umuman olganda Teylor va Makloren formulalari yordamida amalga oshiriladi. Ammo amaliyotda ko‘p funksiyalarni (10.19)-(10.25) qatorlardan formal ravishda foydalanish orqali darajali qatorga yoyiladi.

10.15-misol.

Funksiyaning darajali qatorga yoyilmasidan foydalanib $\ln 2$ ni $\delta=0{,}0001$ aniqlikda taqribiy hisoblang.

► Bu miqdorni (10.22) formula yordamida $\delta=0{,}0001$ aniqlikda taqribiy hisoblash uchun qatorning 10000 ta hadini olish kerak. Shuning uchun bu yerda (10.25) yoyilmadan foydalanish qulay.

$\dfrac{1+x}{1-x}=2$ deb, $x=\dfrac{1}{3}$ ni aniqlab olamiz va bu qiymatni (10.25) qatorga qo‘yamiz. $\ln 2=2\left(\dfrac{1}{3}+\dfrac{1}{3\cdot 3^3}+\dfrac{1}{5\cdot 3^5}+\ldots+\dfrac{1}{(2n-1)\cdot 3^{2n-1}}+\ldots\right)$

Berilgan aniqlikda hisoblash uchun qoldiq hadini baholaymiz

$$R_n=2\left(\frac{1}{(2n+1)\cdot 3^{2n+1}}+\frac{1}{(2n+3)\cdot 3^{2n+3}}+\ldots\right)<\frac{2}{2n+1}\left(\frac{1}{3^{2n+1}}+\frac{1}{3^{2n+3}}+\ldots\right)=$$

$$=\frac{2}{(2n+1)\cdot 3^{2n+1}}\left(1+\frac{1}{9}+\frac{1}{81}+\ldots\right)=\frac{2}{(2n+1)\cdot 3^{2n+1}}\cdot\frac{1}{1-1/9}=\frac{9}{4(2n+1)\cdot 3^{2n+1}}.$$

$n=3$ bol`lganda $R_n<0{,}00015$. U holda $\ln 2\approx 2\left(\dfrac{1}{3}+\dfrac{1}{3\cdot 3^3}+\dfrac{1}{5\cdot 3^5}\right)\approx 0{,}6931.$ ◄

10.16-misol.

Quyidagi integralni integral ostidagi funksiyaning darajali qatorga yoyilmasidan foydalanib $\delta=0{,}001$ aniqlikda taqribiy hisoblang:

$$\int_0^1\frac{\sin x^2}{x}dx$$

►(10.21) yoyilmada x ning o‘rniga x^2 qo‘yamiz va $1/x$ ga ko‘paytiramiz

$$\sin x^2=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots,$$

$$\frac{\sin x^2}{x}=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots.$$

Hosil bo‘lgan qator butun sonlar o‘qida yaqinlashuvchi bo‘lgani uchun hadlab integrallaymiz

$$\int_0^1 \frac{\sin x^2}{x}dx = \left(\frac{x^3}{3} - \frac{x^7}{3!\cdot 7} + \frac{x^{11}}{5!\cdot 11} - \frac{x^{15}}{7!\cdot 15} + \ldots\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{3!\cdot 7} + \frac{1}{5!\cdot 11} - \frac{1}{7!\cdot 15} + \ldots \ .$$

Leybnits alomatidan kelib chiqadigan natijaga ko‘ra, $|R_n| < u_{n+1}$.

$|R_n| < \frac{1}{5!\cdot 11} < 0{,}00076$, demak, $\int_0^1 \frac{\sin x^2}{x}dx \approx \frac{1}{3} - \frac{1}{3!\cdot 7} \approx 0{,}310.$ ◀

10.17-misol.

Ushbu $y'' - (1+x^2)y = 0$ differensial tenglamaning $y(0) = -2, y'(0) = 2$ boshlang‘ich shartlarni qanoatlantiruvchi yechimining darajali qatorga yoyilmasidagi birinchi beshta hadini yozing.

▶Boshlang‘ich shartlarni tenglamaga qo‘yib, $y''(0) = -2$ ni topamiz. Tenglamani ketma-ket differensiallab quyidagilarni hisoblaymiz

$y''' = 2xy + (1+x^2)y', \ \ y'''(0) = 2;$

$y^{IV} = 2y + 2xy' + 2xy' + (1+x^2)y'', \ \ y^{IV}(0) = -6;$

$y^{V} = 2y' + 4y' + 4xy'' + 2xy'' + (1+x^2)y''', \ \ y^{V}(0) = 14.$

Topilganlarni Makloren qatoriga qo‘yamiz va differensial tenglama yechimining darajali qatorga yoyilmasini hosil qilamiz:

$$y = -2 + 2x - x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{7}{60}x^5 - \ldots \ .$$ ◀

42-Auditoriya topshiriqlari

11. $y = x^5 + 2x^4 - 2x^3 + 3x + 1$ funksiyani $x-1$ ning darajalari bo‘yicha qatorga yoying.

12. Makloren qatoridan bevosita foydalanib, $y = \frac{1}{x+2}$ funksiyani x ning darajalari bo‘yicha qatorga yoying.

13. $y = \sqrt{x^3}$ funksiyani $x = 1$ nuqta atrofida qatorga yoying.

14. $y = \sin\frac{\pi x}{4}$ funksiyani $x = 2$ nuqta atrofida qatorga yoying.

Berilgan funksiyalarni Makloren qatori yoyilmalaridan foydalanib qatorga yoying.

15. $y = e^{-x^2}$.

16. $y = \cos^2 x$.

17. $y = (1 - tgx)\cos x$.

18. $y = \ln(10 + x)$.

19. $y = \sqrt[3]{8 - x^3}$.

20. Funksiyaning darajali qatorga yoyilmasidan foydalanib hisoblang:

a) $y = x^2 \cdot \sqrt[4]{1+x}$, $y^{V}(0) - ?$

b) $y = x^6 e^{x^2}$, $y^{(10)}(0) - ?$

21. Funksiyaning darajali qatorga yoyilmasidan foydalanib 0,001 aniqlikda taqribiy hisoblang:

a) $\sqrt[3]{30}$;

b) e^2;

c) $\sin 1/3$;

d) $\ln 3$.

12. $y' = x^2 + y^2$ differensial tenglamaning $y(1) = 1$ boshlang'ich shartni qanoatlantiruvchi yechimining darajali qatorga yoyilmasidagi birinchi beshta hadini yozing.

42-Mustaqil yechish uchun testlar

1. Quyidagilardan qaysi biri $y = \sin 2x$ funksiyaning Makloren qatoriga yoyilmasi bo'ladi?

A) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{2n!}$, B) $\sum_{n=0}^{\infty} (-1)^n \frac{2^n x^n}{n!}$, D) $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$, E) $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$.

2. $y = x^4 e^x$ funksiyaning darajali qatorga yoyilmasidan foydalanib $y'''(0)$ ni hisoblang.

A) 6; B) $1/6$; D) 0; E) 3.

3. $\sum_{n=4}^{\infty} \frac{x^n}{(n-4)!}$ qator quyidagi qaysi bir funksiyaning Makloren qatoriga yoyilmasi bo'ladi?

A) $\ln(1+4x)$; B) $x^4 e^x$; D) $e^x/4$; E) $\ln(4+x)$.

4. $y = \sin\frac{\pi x}{2}$ funksiyani $x = 2$ nuqta atrofida qatorga yoyilmasidan foydalanib, $y'''(2)$ ni toping.

A) $-\frac{\pi^3}{8}$; B) $\frac{\pi^3}{8}$; D) 0; E) $\frac{\pi^3}{3!\cdot 8}$.

5. $y' = x^2 - y$ differensial tenglamaning $y(1) = 1$ boshlang'ich shartni qanoatlantiruvchi yechimining darajali qatorga yoyilmasidagi birinchi uchta hadini yozing.

A) $y = 1 - x + \frac{x^2}{2} + \ldots$; B) $y = 1 + (x-1)^2 + \ldots$;

D) $y = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{4} + \ldots$; E) $y = 1 + (x-1) + \frac{(x-1)^2}{2} + \ldots$.

10.5. Furye qatorlari

10.5.1. Furye qatori. Quyidagi funksional qator

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+...+a_n\cos nx+b_n\sin nx+...=$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

trigonometrik qator deyiladi, $a_0, a_1, b_1, a_2, b_2, ..., a_n, b_n, ...$ sonlar trigonometrik qatorning ***koeffitsientlari*** deyiladi. Agar qator yaqinlashuvchi bo‘lsa, u holda qatorning yig‘indisi ham davriy funksiya bo‘ladi.
f(x) funksiya bu qatorning yig‘indisi bo‘lsin:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right) \qquad (10.26)$$

Bu qatorni [-π, π] segmentda yaqinlashuvchi deb faraz qilib, uning koeffitsientlarini topiladi.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx. \qquad (10.27)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx. \qquad (10.28)$$

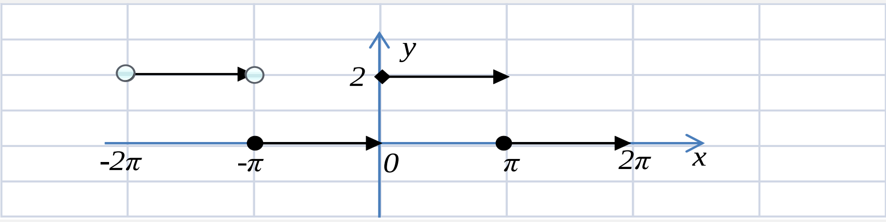
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx. \qquad (10.29)$$

(10.27), (10.28) va (10.29) formulalar bilan aniqlangan (10.26) trigonometrik qator davri 2π bo‘lgan *f(x)* ***funksiyaning Furye qatori*** deb ataladi. a_o, a_n va b_n($n\in N$)sonlar esa ***Furye koeffitsientlari*** deyiladi.

10.18-misol.

Ushbu 2π davrli $f(x)=\begin{cases}0, & agar\ -\pi\le x<0\ \text{bo`lsa},\\ 2, & agar\ 0\le x<\pi\ \text{bo`lsa}.\end{cases}$

funksiyani Furye qatoriga yoying.



►Qatorning koeffitsientlarini (10.27), (10.28) va (10.29) formulalar bo‘yicha topamiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot dx + \frac{1}{\pi}\int_{0}^{\pi} 2dx = 2,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos nx dx = \frac{2}{\pi n}\sin nx\Big|_0^{\pi} = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} 2\sin nx dx = -\frac{2}{\pi n}\cos nx\Big|_0^{\pi} =$$

$$= \frac{2}{\pi n}\left(1-(-1)^n\right).$$

Bundan, $b_{2n} = 0,\ b_{2n-1} = \dfrac{4}{\pi(2n-1)}$ bo‘lganligi uchun berilgan funksiya uchun Furye qatori quyidagicha bo‘ladi:

$$f(x) = 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}. ◄$$

10.5.2.Juft va toq funksiyalarning Furye qatorlari.

f(x) funksiya juft va toq bo‘lgan hollardagi Furye qatoriga yoyilmasi o‘ziga xos formulalar bilan hisoblanadi.

a) $f(-x) = f(x)$ juft funksiya bo‘lsa,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx\ ,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx, \tag{10.30}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0.$$

Shunga ko‘ra juft funksiyaning Furye qatori quyidagicha bo‘ladi:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \tag{10.31}$$

b) $f(-x) = -f(x)$ toq funksiya bo‘lsa,

$a_0 = a_n = 0,$

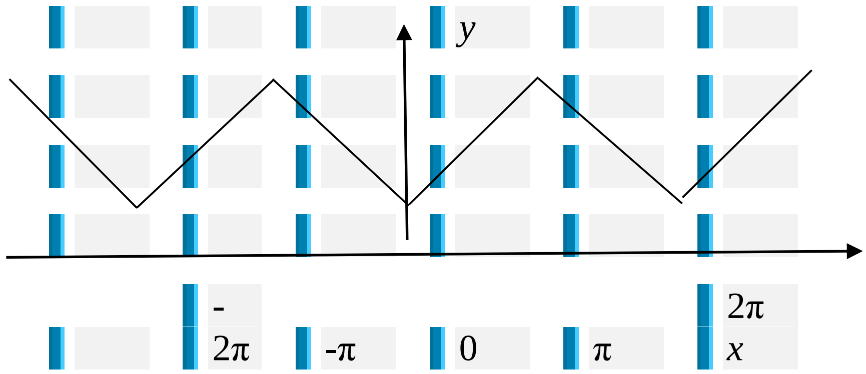
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx. \tag{10.32}$$

Toq funksiyaning Furye qatori quyidagi ko‘rinishda bo‘ladi:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \tag{10.33}$$

10.19-misol.

Ushbu $-\pi < x \le \pi$ intervalda berilgan 2π davrli $f(x)=|x|$ funksiyani Furye qatoriga yoying.



► $f(x)$ juft funksiya bo‘lganligi uchun qatorning koeffitsientlarini (10.30) formulalar bo‘yicha topamiz.

$$b_n = 0; \qquad a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \pi;$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} x\cos kx\,dx = \frac{2}{\pi}\left[\frac{x\sin kx}{k}\Big|_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin kx\,dx\right] = \frac{2}{\pi k^2}\cos kx\Big|_0^{\pi} =$$

$$= \frac{2}{\pi k^2}\left[(-1)^k - 1\right];$$

Demak, $a_{2n} = 0, \qquad a_{2n-1} = -\dfrac{4}{\pi(2n-1)^2};$

Berilgan funksiya uchun Furye qatori quyidagicha bo‘ladi:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}. \blacktriangleleft$$

43-Auditoriya topshiriqlari

Quyidagi 2π davrli funksiyalarni Furye qatoriga yoying.

1. $f(x) = \begin{cases} \pi + 2x, \text{ agar } -\pi < x \le 0 \text{ bo`lsa,} \\ -\pi, \text{ agar } 0 < x \le \pi \text{ bo`lsa.} \end{cases}$

$\left(\text{Javob}: f(x) = -\dfrac{\pi}{2} + \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)x}{(2n-1)^2} - 2\sum_{n=1}^{\infty}\dfrac{\sin nx}{n}.\right)$

2. $f(x) = \begin{cases} x, \text{ agar } -\pi < x \le 0 \text{ bo`lsa,} \\ 2x, \text{ agar } 0 < x \le \pi \text{ bo`lsa.} \end{cases}$

$\left(\text{Javob}: f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}\dfrac{\sin nx}{n}.\right)$

3. $f(x) = x, \; -\pi < x \le \pi.$

$\left(\text{Javob}: f(x) = 2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx.\right)$

4. $(-\pi;\pi]$ oraliqda berilgan 2π davrli $f(x)=x^2$ funksiyani Furye qatoriga yoying. Qator yoyilmasidan foydalanib quyidagi sonli qatorlarning yig‘indisini toping: 1) $\sum_{n=1}^{\infty}\frac{1}{n^2}$; 2) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$; 3) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

43-Mustaqil yechish uchun testlar

1. Quyidagi funksiyalarning qaysi birining Furye qatoriga yoyilmasida $b_n=0$ bo‘ladi?

A) $f(x)=\begin{cases} x, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ 0, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

B) $f(x)=\begin{cases} -x+2, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ x+2, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

D) $f(x)=\begin{cases} x-2, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ x+2, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

E) $f(x)=\begin{cases} x, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ 2, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

2. Quyidagi funksiyalarning qaysi birining Furye qatoriga yoyilmasida $a_0=0,\ a_n=0$ bo‘ladi?

A) $f(x)=\begin{cases} x, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ 0, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

B) $f(x)=\begin{cases} -x+2, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ x+2, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

D) $f(x)=\begin{cases} x-2, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ x+2, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

E) $f(x)=\begin{cases} x, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ 2, & \text{agar } 0<x\le\pi \text{ bo`lsa.} \end{cases}$

3. 2π davrli $f(x)=\begin{cases} -2, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ 1, & \text{agar } 0<x\le\pi \text{ bo`lsa} \end{cases}$ funksiyaning Furye qatoriga yoyilmasidagi a_0 koeffitsientni toping.

A) 1; B) 2; D) -1; E) -2.

4. 2π davrli $f(x)=\begin{cases} -2, & \text{agar } -\pi<x\le 0 \text{ bo`lsa,} \\ 1, & \text{agar } 0<x\le\pi \text{ bo`lsa} \end{cases}$ funksiyaning Furye qatoriga yoyilmasidagi a_n koeffitsientni toping.

A) 0; B) $\frac{2}{\pi n}$; D) $\frac{3(1-(-1)^n)}{\pi n}$; E) $\frac{6}{\pi(2n-1)}$.

5. 2π davrli $f(x)=\begin{cases}-2, \text{ agar } -\pi < x \le 0 \text{ bo`lsa,} \\ 1, \text{ agar } 0 < x \le \pi \text{ bo`lsa}\end{cases}$ funksiyaning Furye qatoriga yoyilmasidagi b_n koeffitsientni toping.

A) 0; B) $\frac{2}{\pi n}$; D) $\frac{3(1-(-1)^n)}{\pi n}$; E) $\frac{6}{\pi(2n-1)}$.

10.5.3. Davri $2l$ bo‘lgan funksiyalarning Furye qatori. Yarim davrda berilgan funksiyalarni Furye qatoriga yoyish

Davri 2l bo‘lgan funksiyalarni Furye qatoriga yoyish. Funksiyaning davri $2l$ bo‘lsa, u $x=\frac{l}{\pi}t$ almashtirish yordamida 2π davrga keltiriladi va hosil bo‘lgan funksiyani Furye qatoriga yoyiladi. So‘ng $t=\frac{\pi}{l}x$ almashtirish bajarib quyidagi formulalarni topamiz:

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)dx; \qquad a_k=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi}{l}x\,dx;$$
$$b_k=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi}{l}x\,dx \tag{10.34}$$

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x. \tag{10.35}$$

Davri $2l$ bo‘lgan juft funksiyalar uchun Furye qatori quyidagicha bo‘ladi:

$$a_0=\frac{2}{l}\int_{0}^{\pi} f(x)dx, \qquad a_k=\frac{2}{l}\int_{0}^{\pi} f(x)\cos\frac{k\pi}{l}x\,dx, \qquad b_k=0; \tag{10.36}$$

$$a_0=\frac{2}{l}\int_{0}^{\pi} f(x)dx, \qquad a_k=\frac{2}{l}\int_{0}^{\pi} f(x)\cos\frac{k\pi}{l}x\,dx, \qquad b_k=0; \tag{10.37}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{l}x. \tag{10.38}$$

Davri $2l$ bo‘lgan toq funksiyalar uchun Furye qatori quyidagicha bo‘ladi:

$$a_0=a_k=0, \qquad b_k=\frac{2}{l}\int_{0}^{\pi} f(x)\sin\frac{k\pi}{l}x\,dx; \tag{10.39}$$

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x. \tag{10.40}$$

10.20-misol.

Ushbu -1 < $x \le 1$ intervalda berilgan $2l = 2$ davrli $f(x) = x - 1$ funksiyani Furye qatoriga yoying va grafigini chizing.

► Berilgan funksiyaning Furye qatorini topishda (10.34), (10.35) formulalardan foydalanamiz, bu yerda $l = 1$.

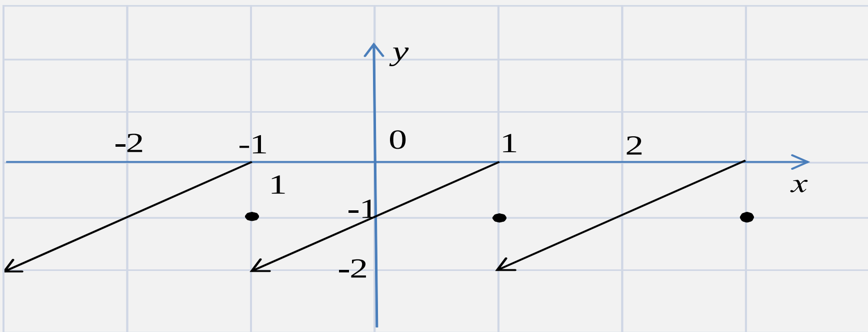
$$a_0 = \int_{-1}^{1}(x-1)dx = \left(\frac{x^2}{2} - x\right)\Bigg|_{-1}^{1} = \frac{1}{2} - 1 - \frac{1}{2} - 1 = -2;$$

$$a_k = \int_{-1}^{1}(x-1)\cos k\pi x\,dx = \left[\frac{(x-1)\sin k\pi x}{k\pi}\Bigg|_{-1}^{1} - \frac{1}{k\pi}\int_{-1}^{1}\sin k\pi x\,dx\right] = \frac{1}{k^2\pi^2}\cos k\pi x\Bigg|_{-1}^{1} = 0$$

$$b_k = \int_{-1}^{1}(x-1)\sin k\pi x\,dx = \left[-\frac{(x-1)\cos k\pi x}{k\pi}\Bigg|_{-1}^{1} + \frac{1}{k\pi}\int_{-1}^{1}\cos k\pi x\,dx\right] = \frac{2\cos k\pi}{k\pi} +$$

$$+\frac{1}{k^2\pi^2}\sin k\pi x\Bigg|_{-1}^{1} = \frac{2(-1)^k}{k^2\pi^2};$$

$$f(x) = -1 + \frac{2}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{\sin n\pi x}{n^2}.$$



◄

Ko‘pincha [0;*l*] kesmada berilgan $f(x)$ funksiyani sinuslar bo‘yicha, yoki kosinuslar bo‘yicha qatorga yoyish masalasi talab etiladi.

$f(x)$ funksiyani kosinuslar bo‘yicha qatorga yoyish uchun funksiyani [0;*l*] kesmadan [-*l*;*l*] kesmaga juft davom ettiriladi. Bu holda Furye qator faqat kosinuslarni o‘z ichiga oladi.

Agar $f(x)$ funksiyani qatorga sinuslar bo‘yicha yoyishni istasak, u holda funksiyani [0; *l*] kesmadan [-*l*;*l*] kesmaga toq davom ettiramiz, bunda $f(0) = 0$ deb olishimiz kerak. Bu holda Furye qator faqat sinuslarni o‘z ichiga oladi.

10.21-misol.

Ushbu $0 < x \le \pi$ intervalda berilgan $f(x) = \sin\frac{x}{2}$ funksiyani kosinuslar bo‘yicha Furye qatoriga yoying.

► Berilgan funksiyani kosinuslar bo'yicha Furye qatoriga yoyish uchun juftga davom ettiramiz:

$$f(x)=\begin{cases}-\sin\frac{x}{2}, & \text{agar } -\pi<x<0 \text{ bo`lsa,}\\ \sin\frac{x}{2}, & \text{agar } 0<x\le\pi \text{ bo`lsa.}\end{cases}$$

$b_n=0$; $\qquad a_0=\frac{2}{\pi}\int_0^{\pi}\sin\frac{x}{2}dx=-\frac{4}{\pi}\cos\frac{x}{2}\Big|_0^{\pi}=\frac{4}{\pi}$;

$$a_k=\frac{2}{\pi}\int_0^{\pi}\sin\frac{x}{2}\cos kx\,dx=\frac{2}{\pi}\int_0^{\pi}\left(\sin\left(\frac{1}{2}+k\right)x+\sin\left(\frac{1}{2}-k\right)x\right)dx=$$

$$=\frac{2}{\pi}\left[-\frac{2\cos kx}{1+2k}\Big|_0^{\pi}-\frac{2\cos kx}{1-2k}\Big|_0^{\pi}\right]=\frac{4}{\pi}\left[-\frac{\cos k\pi}{1+2k}+\frac{1}{1+2k}-\frac{\cos k\pi}{1-2k}+\frac{1}{1-2k}\right]=$$

$$=\frac{8}{\pi}\left[\frac{(-1)^{k+1}}{1-4k^2}+\frac{1}{1-4k^2}\right].$$

Bundan, $a_{2n}=0$, $\qquad a_{2n-1}=\frac{16}{\pi\left(1-4(2n-1)^2\right)}$.

Berilgan funksiya uchun Furye qatori quyidagicha bo'ladi:

$$f(x)=\frac{2}{\pi}-\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{1-4(2n-1)^2}.$$ ◄

44-Auditoriya topshiriqlari

1. Ushbu davri T=2 bo'lgan $f(x)=\begin{cases}0, & \text{agar } -1<x<0 \text{ bo`lsa,}\\ 1, & \text{agar } 0\le x\le 1 \text{ bo`lsa}\end{cases}$ funksiyani Furye qatoriga yoying.

$$\left(\text{Javob}: f(x)=\frac{3}{4}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\pi nx}{n}.\right)$$

2. Ushbu $[-2;2]$ intervalda berilgan T=4 davrli $f(x)=-x$ funksiyani Furye qatoriga yoying.

$$\left(\text{Javob}: f(x)=2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{\pi nx}{2}.\right)$$

3. Ushbu $0<x\le\pi$ intervalda berilgan $f(x)=\sin\frac{x}{2}$ funksiyani sinuslar bo'yicha Furye qatoriga yoying.

$$\left(\text{Javob}: f(x)=\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n\sin nx}{1-4n^2}.\right)$$

4. Ushbu $[0;2]$ intervalda berilgan $f(x)=1-x/2$ funksiyani kosinuslar bo'yicha Furye qatoriga yoying.

$$\left(\text{Javob}: f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi nx}{2}.\right)$$

44-Mustaqil yechish uchun testlar

1. T=4 davrli $f(x)=\begin{cases}0, \text{ agar } -2<x\le 0 \text{ bo`lsa,}\\ 3, \text{ agar } 0<x\le 2 \text{ bo`lsa}\end{cases}$ funksiyaning Furye qatoriga yoyilmasidagi a_0 koeffitsientni toping.

A) $6/\pi$; B) 0; D) $3/\pi$; E) 3.

2. T=4 davrli $f(x)=\begin{cases}0, \text{ agar } -2<x\le 0 \text{ bo`lsa,}\\ 3, \text{ agar } 0<x\le 2 \text{ bo`lsa}\end{cases}$ funksiyaning Furye qatoriga yoyilmasidagi a_n koeffitsientni toping.

A) $6/\pi$; B) 0; D) $3/\pi$; E) 3.

3. T=4 davrli $f(x)=\begin{cases}0, \text{ agar } -2<x\le 0 \text{ bo`lsa,}\\ 3, \text{ agar } 0<x\le 2 \text{ bo`lsa}\end{cases}$ funksiyaning Furye qatoriga yoyilmasidagi b_{2n} koeffitsientni toping.

A) $6/\pi$; B) 0; D) $3/\pi$; E) 3.

4. $(0;\ 2)$ intervalda berilgan $f(x)=1-x$ funksiyani kosinuslar bo‘yicha Furye qatoriga yoyish uchun uni $(-2;\ 0)$ intervalga qanday davom ettiriladi?

A) $f(x)=-1+x,\ x\in(-2;0)$; B) $f(x)=-1-x,\ x\in(-2;0)$;

D) $f(x)=1+x,\ x\in(-2;0)$; E) $f(x)=|1-x|,\ x\in(-2;0)$.

5. $(0;\ 2)$ intervalda berilgan $f(x)=1-x$ funksiyani sinuslar bo‘yicha Furye qatoriga yoyish uchun uni $(-2;\ 0)$ intervalga qanday davom ettiriladi?

A) $f(x)=-1+x,\ x\in(-2;0)$; B) $f(x)=-1-x,\ x\in(-2;0)$;

D) $f(x)=1+x,\ x\in(-2;0)$; E) $f(x)=|1-x|,\ x\in(-2;0)$.

14-Shaxsiy uy topshiriqlari

1

$f(x)$ funksiyani Teylor yoki Makloren qatoriga yoying. Hosil bo‘lgan qatorning yaqinlashish sohasini toping.

1.1. $f(x)=x^3 arctgx,\ x_0=0$ $\left(\text{Javob}: f(x)=\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{2n+2}}{2n-1};\quad |x|\le 1.\right)$

1.2. $f(x)=\cos\dfrac{3x^2}{5},\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\dfrac{(-1)^n 3^{2n}x^{4n}}{5^{2n}(2n)!};\quad -\infty<x<\infty.\right)$

1.3. $f(x)=\dfrac{2}{1-3x^2},\ x_0=0.$ $\left(\text{Javob}: f(x)=2\sum_{n=0}^{\infty}3^n x^{2n};\quad |x|<\dfrac{1}{\sqrt{3}}.\right)$

1.4. $f(x)=x\cos\sqrt{x},\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{n+1}}{(2n)!};\quad 0\le x<\infty.\right)$

1.5. $f(x)=\dfrac{1}{x^2-4x+3},\ x_0=0.$

$\left(\text{Javob}: f(x)=\dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{3^{n+1}-1}{3^{n+1}n!}x^n;\quad -\infty<x<\infty.\right)$

1.6. $f(x)=\ln(5x+3),\ x_0=-\dfrac{2}{5}.$

$\left(\text{Javob}: f(x)=\sum_{n=1}^{\infty}\dfrac{(-1)^n 5^n}{n}\left(x+\dfrac{2}{5}\right)^n;\quad -\dfrac{7}{5}<x\le\dfrac{3}{5}.\right)$

1.7. $f(x)=\sin\dfrac{\pi x}{6},\ x_0=3.$

$\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{\pi}{6}\right)^{2n}\dfrac{(x-3)^{2n}}{(2n)!};\quad -\infty<x<\infty.\right)$

1.8. $f(x)=\dfrac{1}{2x+5},\ x_0=3.$

$\left(\text{Javob}: f(x)=\dfrac{1}{11}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{2}{11}\right)^n(x-3)^n;\quad -\dfrac{5}{2}<x<\dfrac{17}{2}.\right)$

1.9. $f(x)=\dfrac{1}{x^2-4x+3},\ x_0=-2.$

$\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\left(\dfrac{1}{6\cdot 3^n}-\dfrac{1}{10\cdot 5^n}\right)(x+2)^n;\ -5<x<1.\right)$

1.10. $f(x)=\dfrac{1}{(x-3)^2},\ x_0=1.$ $\left(\text{Javob}: f(x)=\dfrac{1}{4}\sum_{n=0}^{\infty}\dfrac{n+1}{2^n}(x-1)^n;\quad -1<x<3.\right)$

1.11. $f(x)=e^{2x},\ x_0=1.$ $\left(\text{Javob}: f(x)=e^2\cdot\sum_{n=0}^{\infty}\dfrac{2^n}{n!}(x-1)^n;\quad -\infty<x<\infty.\right)$

1.12. $f(x)=\dfrac{1}{\sqrt{e^x}},\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^n}{2^n n!};\quad -\infty<x<\infty.\right)$

1.13. $f(x)=2^{-x^2},\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\dfrac{(-1)^n\ln^n 2}{n!}x^{2n};\quad -\infty<x<\infty.\right)$

1.14. $f(x)=shx,\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=1}^{\infty}\dfrac{x^{2n-1}}{(2n-1)!};\quad -\infty<x<\infty.\right)$

1.15. $f(x)=5^x,\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\dfrac{x^n\ln^n 5}{n!};\quad -\infty<x<\infty.\right)$

1.16. $f(x)=\dfrac{1}{x},\ x_0=-2.$ $\left(\text{Javob}: f(x)=-\dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{(x+2)^n}{2^n};\quad -4<x<0.\right)$

1.17. $f(x)=\ln(3x+4),\ x_0=-1.$

$$\left(\text{Javob}: f(x)=\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n}(x+1)^n;\ \ -\frac{4}{3}<x\le-\frac{2}{3}.\right)$$

1.18. $f(x)=\dfrac{1}{\sqrt{4+x}},\ x_0=-3.$

$$\left(\text{Javob}: f(x)=1+\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!}{2^n n!}(x+3)^n;\ \ -4<x\le-2.\right)$$

1.19. $f(x)=\ln\dfrac{1}{x^2-2x+2},\ x_0=1.$ $\left(\text{Javob}: f(x)=\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}x^{2n};\ \ 0\le x\le 2.\right)$

1.20. $f(x)=\sqrt{x},\ x_0=4.$

$$\left(\text{Javob}: f(x)=2+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-2)!(x-4)^n}{2^{4n-2}n!(n-1)!};\ \ 0\le x\le 8.\right)$$

1.21. $f(x)=\sin^2 2x,\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}4^{2n}}{(2n)!}x^{2n};\ \ -\infty<x<\infty.\right)$

1.22. $f(x)=\cos^2 2x,\ x_0=0.$

$$\left(\text{Javob}: f(x)=1+\sum_{n=1}^{\infty}\frac{(-1)^n 6^{2n}}{(2n)!}x^{2n};\ \ -\infty<x<\infty.\right)$$

1.23. $f(x)=\sqrt{1+x^2},\ x_0=0.$

$$\left(\text{Javob}: f(x)=1+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1\cdot3\cdot\ldots\cdot(2n-3)x^{2n}}{2\cdot4\cdot\ldots\cdot(2n-2)\cdot 2n};\ \ |x|<1.\right)$$

1.24. $f(x)=\sqrt[3]{1+x^3},\ x_0=0.$

$$\left(\text{Javob}: f(x)=1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot4\cdot\ldots\cdot(3n-2)x^{3n}}{3^n n!};\ \ |x|<1.\right)$$

1.25. $f(x)=\dfrac{1}{x},\ x_0=3.$ $\left(\text{Javob}: f(x)=\dfrac{1}{3}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{3^n}(x-3)^n;\ \ -\infty<x<\infty.\right)$

1.26. $f(x)=\cos\dfrac{\pi x}{4},\ x_0=2.$

$$\left(\text{Javob}: f(x)=\sum_{n=1}^{\infty}(-1)^n\left(\frac{\pi}{4}\right)^{2n-1}\frac{(x-2)^{2n-1}}{(2n-1)!};\ \ -\infty<x<\infty.\right)$$

1.27. $f(x)=x^2e^{2x},\ x_0=0.$ $\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\dfrac{2^n}{n!}x^{n+2};\ \ -\infty<x<\infty.\right)$

1.28. $f(x)=\dfrac{1}{x+3},\ x_0=-2.$ $\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}(-1)^n(x+2)^n;\ \ -3<x<-1.\right)$

1.29. $f(x)=\cos x,\ x_0=a.$

$$\left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\frac{\cos(a+n\pi/2)}{n!}(x-a)^n;\ \ -\infty<x<\infty.\right)$$

1.30. $f(x)=ch(2x^3),\ x_0=0.\ \left(\text{Javob}: f(x)=\sum_{n=0}^{\infty}\frac{2^n}{n!}x^{6n};\ \ -\infty<x<\infty.\right)$

2

Berilgan miqdorni funksiyaning darajali qatorga yoyilmasidan foydalanib δ aniqlikda taqribiy hisoblang.

2.1. $arctg(1/2),\ \delta=0{,}001.\ (\text{Javob}:0{,}464.)$

2.2. $\cos10°,\ \delta=0{,}0001.\ (\text{Javob}:0{,}9848.)$

2.3. $\arcsin(1/3),\ \delta=0{,}001.\ (\text{Javob}:0{,}340.)$

2.4. $\ln5,\ \delta=0{,}001.\ (\text{Javob}:1{,}609.)$

2.5. $\sin1^0,\ \delta=0{,}0001.\ (\text{Javob}:0{,}0175.)$

2.6. $\sqrt[6]{738},\ \delta=0{,}001.\ (\text{Javob}:3{,}006.)$

2.7. $\ln10,\ \delta=0{,}0001.\ (\text{Javob}:2{,}3026.)$

2.8. $1/\sqrt[7]{136},\ \delta=0{,}001.\ (\text{Javob}:0{,}496.)$

2.9. $\sqrt[4]{90},\ \delta=0{,}001.\ (\text{Javob}:3{,}079.)$

2.10. $\sin1,\ \delta=0{,}00001.\ (\text{Javob}:0{,}00001.)$

2.11. $\sqrt[5]{250},\ \delta=0{,}001.\ (\text{Javob}:,017.)$

2.12. $1/\sqrt[3]{30},\ \delta=0{,}001.\ (\text{Javob}:0{,}302.)$

2.13. $e,\ \delta=0{,}0001.\ (\text{Javob}:2{,}7183.)$

2.14. $1/e,\ \delta=0{,}0001.\ (\text{Javob}:0{,}3679.)$

2.15. $\sqrt[3]{8{,}36},\ \delta=0{,}001.\ (\text{Javob}:2{,}030.)$

2.16. $\sqrt[3]{e},\ \delta=0{,}0001.\ (\text{Javob}:1{,}3956.)$

2.17. $\sqrt{1{,}3},\ \delta=0{,}001.\ (\text{Javob}:1{,}140.)$

2.18. $arctg\frac{\pi}{10},\ \delta=0{,}001.\ (\text{Javob}:0{,}304.)$

2.19. $\ln3,\ \delta=0{,}0001.\ (\text{Javob}:1{,}0986.)$

2.20. $ch2,\ \delta=0{,}0001.\ (\text{Javob}:3{,}7622.)$

2.21. $e^2,\ \delta=0{,}001.\ (\text{Javob}:7{,}389.)$

2.22. $\cos2°,\ \delta=0{,}001.\ (\text{Javob}:0{,}999.)$

2.23. $\sqrt{e},\ \delta=0{,}001.\ (\text{Javob}:1{,}674.)$

2.24. $\sqrt[4]{320},\ \delta=0{,}001.\ (\text{Javob}:4{,}227.)$

2.25. $1/\sqrt[3]{e},\ \delta=0{,}001.\ (\text{Javob}:0{,}716.)$

2.26. $\sqrt[10]{1080},\ \delta=0{,}001.\ (\text{Javob}:2{,}031.)$

2.27. $\lg7,\ \delta=0{,}0001.\ (\text{Javob}:0{,}8451.)$

2.28. $\cos1,\ \delta=0{,}001.\ (\text{Javob}:0{,}541.)$

2.29. $\sin(1/2),\ \delta=0{,}001.\ (\text{Javob}:0{,}479.)$

2.30. $1/\sqrt{e},\ \delta=0{,}0001.\ (\text{Javob}:0{,}6065.)$

3

Berilgan integralni integral ostidagi funksiyaning darajali qatorga yoyilmasidan foydalanib $\delta = 0{,}001$ aniqlikda taqribiy hisoblang.

3.1. $\int_0^{0,25} \ln\left(1+\sqrt{x}\right)dx.$ (Javob: 0,070.)

3.2. $\int_0^{1} arctg\left(x^2/2\right)dx.$ (Javob: 0,162.)

3.3. $\int_0^{0,2} \sqrt{x}e^{-x}\,dx.$ (Javob: 0,054.)

3.4. $\int_0^{0,2} \sqrt{x}\cos x\,dx.$ (Javob: 0,059.)

3.5. $\int_0^{0,5} \frac{arctgx}{x}dx.$ (Javob: 0,487.)

3.6. $\int_0^{0,5} \ln(1+x^3)\,dx.$ (Javob: 0,015.)

3.7. $\int_0^{1} x^2\sin x\,dx.$ (Javob: 0,223.)

3.8. $\int_0^{1} e^{-x^2/2}\,dx.$ (Javob: 0,855.)

3.9. $\int_0^{0,5} \sqrt{1+x^2}\,dx.$ (Javob: 0,480.)

3.10. $\int_0^{0,5} \frac{1}{1+x^5}dx.$ (Javob: 0,484.)

3.11. $\int_0^{0,5} \frac{\sin x^2}{x}dx.$ (Javob: 0,493.)

3.12. $\int_0^{1} \sqrt[3]{1+x^2/2}\,dx.$ (Javob: 1,027.)

3.13. $\int_0^{0,1} \frac{e^x-1}{x}dx.$ (Javob: 0,103.)

3.14. $\int_0^{0,5} x^2\cos 3x\,dx.$ (Javob: 0,018.)

3.15. $\int_0^{0,5} \ln(1+x^2)\,dx.$ (Javob: 0,385.)

3.16. $\int_0^{0,4} \sqrt{x}e^{-x/4}\,dx.$ (Javob: 0,159.)

3.17. $\int_{0,3}^{0,5} \frac{1+\cos x}{x^2}dx.$ (Javob: 2,568.)

3.18. $\int_0^{0,5} \frac{arctgx^2}{x^2}dx.$ (Javob: 0,498.)

3.19. $\int_0^{1} \sin x^2\,dx.$ (Javob: 0,310.)

3.20. $\int_0^{0,1} \frac{\ln(1+x)}{x}dx.$ (Javob: 0,098.)

3.21. $\int_0^{1} \cos\sqrt[3]{x}\,dx.$ (Javob: 0,718.)

3.22. $\int_0^{1} \sqrt{x}\sin x\,dx.$ (Javob: 0,364.)

3.23. $\int_0^{1} arctg\left(\sqrt{x}/2\right)dx.$
(Javob: 0,318.)

3.24. $\int_0^{25} \frac{e^{-2x^2}}{\sqrt{x}}dx.$ (Javob: 0,976.)

3.25. $\int_0^{1} \cos\frac{x^2}{4}dx.$ (Javob: 0,994.)

3.26. $\int_0^{0,5} \frac{x-arctgx}{x^2}dx.$ (Javob: 0,039.)

3.27. $\int_0^{0,4} \sqrt{1-x^3}\,dx.$ (Javob: 0,397.)

3.28. $\int_0^{0,5} e^{-x^2}\,dx.$ (Javob: 0,461.)

3.29. $\int_0^{0,5} \sqrt{1+x^3}\,dx.$ (Javob: 0,508.)

3.30. $\int_0^{0,8} \frac{1-\cos x}{x}dx.$ (Javob: 0,156.)

3

Differensial tenglama yechimining darajali qatorga yoyilmasini noldan farqli birinchi k ta hadini yozing.

4.1. $y'' = 2yy',\ y(0)=0,\ y'(0)=1,\ k=3.\ \left(\text{Javob}: y = x + \dfrac{2x^3}{3!} + \dfrac{12x^5}{5!} + \dots\right)$

4.2. $y'' = \dfrac{y'}{y} - \dfrac{1}{x},\ y(1)=1,\ y'(1)=0,\ k=4.\ \left(\text{Javob}: y = 1 - \dfrac{(x-1)^2}{2!} - \dfrac{(x-1)^4}{4!} + \dfrac{4(x-1)^5}{5!} + \dots\right)$

4.3. $y' = x^2 + xy + e^{-x},\ y(0)=0,\ k=3.\ \left(\text{Javob}: y = x - \dfrac{x^2}{2!} + \dfrac{5x^3}{3!} + \dots\right)$

4.4. $y' = 2x + \cos y,\ y(0)=0,\ k=4.\ \left(\text{Javob}: y = x + x^2 - \dfrac{x^3}{6} - \dfrac{x^4}{4} + \dots\right)$

4.5. $y' = x + \dfrac{1}{y},\ y(0)=1,\ k=4.\ \left(\text{Javob}: y = 1 + x + \dfrac{x^3}{3} - \dfrac{x^4}{3} + \dots\right)$

4.6. $y' = x^2 + e^y,\ y(0)=0,\ k=4.\ \left(\text{Javob}: y = x + \dfrac{x^2}{2} + \dfrac{2x^3}{3} + \dfrac{7x^4}{24} + \dots\right)$

4.7. $y' = xy + e^x,\ y(0)=0,\ k=5.\ \left(\text{Javob}: y = x + \dfrac{x^2}{2} + \dfrac{x^3}{2} + \dfrac{x^4}{6} + \dfrac{x^5}{24} + \dots\right)$

4.8. $y' = 2y^2 + xe^x,\ y(0)=0,\ k=3.\ \left(\text{Javob}: y = \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{8} + \dots\right)$

4.9. $y' = 2\sin x + xy,\ y(0)=0,\ k=3.\ \left(\text{Javob}: y = x^2 + \dfrac{x^4}{6} + \dfrac{11x^6}{360} + \dots\right)$

4.10. $y' = e^{\sin x} + y,\ y(0)=0,\ k=4.\ \left(\text{Javob}: y = x + x^2 + \dfrac{x^3}{2} + \dfrac{5x^4}{24} + \dots\right)$

4.11. $y' = y\cos x + 2\cos y,\ y(0)=0,\ k=3.\ (\text{Javob}: y = 2x + x^2 - x^3 + \dots)$

4.12. $y' = 2y^2 + ye^x,\ y(0)=\dfrac{1}{3},\ k=3.\ \left(\text{Javob}: y = \dfrac{1}{3} + \dfrac{5x^2}{9} + \dfrac{22x^3}{27} + \dots\right)$

4.13. $4x^2y'' + y = 0,\ y(0)=1,\ y'(0)=\dfrac{1}{2},\ k=3.\ \left(\text{Javob}: y = 1 + \dfrac{(x-1)}{2} - \dfrac{(x-1)^2}{8} + \dots\right)$

4.14. $y' = x^2y^2 + y\sin x,\ y(0)=\dfrac{1}{2},\ k=3.\ \left(\text{Javob}: y = \dfrac{1}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{12} + \dots\right)$

4.15. $(1-x)y'' + y = 0,\ y(0)=y'(0)=1,\ k=3.\ \left(\text{Javob}: y = 1 + x - \dfrac{x^2}{2} + \dots\right)$

4.16. $y' = e^{3x} + 2xy^2,\ y(0)=1,\ k=3.\ \left(\text{Javob}: y = 1 + x + \dfrac{5x^2}{2} + \dots\right)$

4.17. $y' = 2x^2 - xy,\ y(0)=0,\ k=3.\ \left(\text{Javob}: y = \dfrac{4x^3}{3!} - \dfrac{16x^5}{5!} + \dfrac{96x^7}{7!} - \dots\right)$

4.18. $y' = e^x - y^2,\ y(0)=0,\ k=4.\ \left(\text{Javob}: y = x + \dfrac{x^2}{2} - \dfrac{x^3}{6} - \dfrac{5x^4}{24} - \dots\right)$

4.19. $y'' = y'^2 + xy,\ y(0)=2,\ y'(0)=-1,\ k=5.\ \left(\text{Javob}: y = 2 - x + \dfrac{x^2}{2} - x^3 + \dfrac{x^4}{2} + \dots\right)$

4.20. $y'=\frac{1-x^2}{y}+1,\ y(0)=1,\ k=5.\left(\text{Javob}: y=1+2x-x^2+\frac{4}{3}x^3-\frac{17}{9}x^4+...\right)$

4.21. $y'=4y-2xy^2+e^{3x},\ y(0)=2,\ k=4.\left(\text{Javob}: y=2+9x+\frac{31x^2}{2}-\frac{11x^3}{6}+...\right)$

4.22. $y''=y\cos y'+x,\ y(0)=1,\ y'(0)=\frac{\pi}{3},\ k=3.\left(\text{Javob}: y=1+\frac{\pi}{3}x+\frac{x^2}{4}+...\right)$

4.23. $y'=x^2+\cos y,\ y(1)=0,\ k=3.\left(\text{Javob}: y=2(x-1)+(x-1)^2-\frac{(x-1)^4}{2}+...\right)$

4.24. $y'=xy+\ln(x+y),\ y(1)=0,\ k=3.\left(\text{Javob}: y=\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{6}+...\right)$

4.25. $yy''=x-(y')^2,\ y(0)=y'(0)=1,\ k=5.\left(\text{Javob}: y=1+x-\frac{x^2}{2}+\frac{2x^3}{3}-\frac{19x^4}{24}+...\right)$

4.26. $y^{IV}=xy+y'x^2,\ y(0)=y'(0)=0,\ y''(0)=y'''(0)=1,\ k=3.\left(\text{Javob}: y=\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^7}{560}+...\right)$

4.27. $y''=y'+2xy,\ y(0)=1,\ y'(0)=0,\ k=5.\left(\text{Javob}: y=1+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{60}+\frac{x^6}{72}+...\right)$

4.28. $y'=y^2+2x,\ y(1)=0,\ k=3.\left(\text{Javob}: y=2(x-1)+(x-1)^2+\frac{4(x-1)^3}{3}+...\right)$

4.29. $y'''=x^2y+y',\ y(0)=y'(0)=0,\ y''(0)=2,\ k=3.\left(\text{Javob}: y=x^2+\frac{x^4}{12}+\frac{x^6}{60}+...\right)$

4.30. $y''=x^2+y^2,\ y(-1)=2,\ y'(-1)=\frac{1}{2},\ k=4.\left(\text{Javob}: y=2+\frac{(x+1)^2}{2}+\frac{5(x+1)^3}{2}+\frac{15(x+1)^4}{16}+...\right)$

5

Quyidagi **(*a*, *b*)** oraliqda berilgan **T** davrli ***f*(*x*)** funksiyalarni Furye qatoriga yoying:

5.1. $f(x)=|x|+1,\quad (-\pi;\pi),\quad T=2\pi.$

5.2. $f(x)=x^2+1,\quad (-2;2)\quad T=4.$

5.3. $f(x)=\begin{cases}0, & agar\ \ -\pi<x<0\ bo'lsa,\\ x+1, & agar\ \ 0\le x<\pi\ bo'lsa.\end{cases}\qquad T=2\pi.$

5.4. $f(x)=x-1,\quad (-2;2),\quad T=4.$

5.5. $f(x)=2+|x|,\quad (-1;1),\quad T=2.$

5.6. $f(x)=\frac{\pi-x}{2},\quad (-\pi;\pi),\quad T=2\pi.$

5.7. $f(x)=|x|-2,\quad (-\pi;\pi),\quad T=2\pi.$

5.8. $f(x)=\begin{cases}-2x, & agar\ -\pi<x<0\ bo'lsa,\\ 1, & agar\ 0\le x\le\pi\ bo'lsa.\end{cases}\qquad T=2\pi.$

5.9. $f(x)=x+1,\quad (-\pi;\pi)\quad T=2\pi.$

5.10. $f(x)=x^2+1,\quad (0;2\pi),\quad T=2\pi.$

5.11. $f(x)=\begin{cases}-x, & agar\ -\pi<x<0\ bo'lsa,\\ 0, & agar\ 0\le x<\pi\ bo'lsa.\end{cases}$ $T=2\pi.$

5.12. $f(x)=\begin{cases}1, & agar\ -1<x<0\ bo'lsa,\\ 3, & agar\ 0\le x\le 1\ bo'lsa.\end{cases}$ $T=2.$

5.13. $f(x)=\sin\frac{x}{2},\quad (-\pi;\pi),\quad T=2\pi.$

5.14. $f(x)=\begin{cases}0, & agar\ -\pi<x\le 0\ bo'lsa,\\ 1+x, & agar\ 0<x\le\pi\ bo'lsa.\end{cases}$ $T=2\pi.$

5.15. $f(x)=\begin{cases}-1, & agar\ -\pi<x<0\ bo'lsa,\\ 2, & agar\ 0\le x\le\pi\ bo'lsa.\end{cases}$ $T=2\pi.$

5.16. $f(x)=\begin{cases}0, & agar\ -2<x<0\ bo'lsa,\\ 3, & agar\ 0\le x\le 2\ bo'lsa.\end{cases}$ $T=4.$

5.17. $f(x)=x^2,\quad (-1;1),\quad T=2.$

5.18. $f(x)=\begin{cases}\cos x, & agar\ -\frac{\pi}{2}<x<\frac{\pi}{2}\ bo'lsa,\\ 0, & agar\ \frac{\pi}{2}\le x\le\frac{3\pi}{2}\ bo'lsa.\end{cases}$ $T=2\pi.$

5.19. $f(x)=|x|+x^2,\quad (-\pi;\pi),\quad T=2\pi.$

5.20. $f(x)=\begin{cases}1, & agar\ -2<x<0\ bo'lsa,\\ -2, & agar\ 0\le x\le 2\ bo'lsa.\end{cases}$ $T=4.$

5.21. $f(x)=\begin{cases}x-2, & agar\ -\pi<x<0\ bo'lsa,\\ 2x, & agar\ 0\le x\le\pi\ bo'lsa.\end{cases}$ $T=2\pi.$

5.22. $f(x)=\cos\frac{x}{2},\quad (-\pi;\pi),\quad T=2\pi.$

5.23. $f(x)=\frac{\pi^2}{12}-\frac{x^2}{4},\quad (-\pi;\pi),\quad T=2\pi.$

5.24. $f(x)=\begin{cases}\pi, & agar\ -\pi<x<0\ bo'lsa,\\ x, & agar\ 0\le x\le\pi\ bo'lsa.\end{cases}$ $T=2\pi.$

5.25. $f(x)=-x|x|,\quad (-1;1),\quad T=2.$

5.26. $f(x)=3-|x|,\quad (-1;1),\quad T=2.$

5.27. $f(x)=\begin{cases}x-1, & agar\ -\pi<x<0\ bo'lsa,\\ 3x, & agar\ 0\le x\le\pi\ bo'lsa.\end{cases}$ $T=2\pi.$

5.28. $f(x)=\cos\frac{3x}{2},\quad (-\frac{\pi}{2};\frac{\pi}{2}),\quad T=\pi.$

5.29. $f(x)=|x|-x^2,\quad (-\pi;\pi),\quad T=2\pi.$

5.30. $f(x)=x^2+1,\quad (-\pi;\pi),\quad T=2\pi.$

XI BOB. KO‘P O‘LCHOVLI INTEGRALLAR

11.1. Ikki o‘lchovli integrallar. Ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish

Chegaralangan $z = f(x,y)$ funksiya Oxy tekislikning qandaydir yopiq D sohasida aniqlanganbo‘lsin. Agar

$$\sum_{k=1}^{n} f(x_k, y_k)\Delta s_k \qquad (11.1)$$

integral yig‘indining D sohaning D_k bo‘laklarga bo‘linishlari va M_k nuqtalarning tanlanish usuliga bog‘liq bo‘lmagan holda $\boldsymbol{d \to 0 (n \to \infty)}$ dagi limiti mavjud bo‘lsa, bu limitga $f(x,y)$ funksiyaning D soha bo‘yicha ***olingan ikki o‘lchovli integrali*** deyiladi

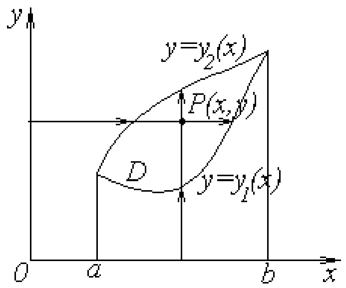
$$\lim_{\substack{d\to 0\\(n\to\infty)}} \sum_{k=1}^{n} f(x_k, y_k)\Delta s_k = \iint_D f(x,y)dxdy,$$

$f(x,y)$ funksiya esa D sohada ***integrallanuvchi*** deyiladi(bu yerda $\Delta S_k - D_k$ bo‘lakning yuzi, $d = \max\limits_{1\le k\le n} d_k$ - bo‘laklar diametrlarining maksimumi).

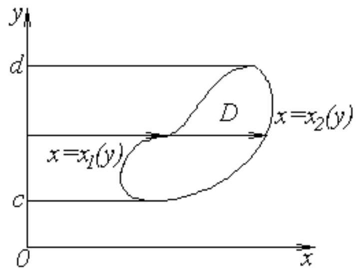
Agar $f(x,y)$ funksiya yopiq D sohada uzluksiz bo‘lsa, u holda bu fuksiya D sohada integrallanuvchi bo‘ladi. Ikki o‘lchovli integrallar ham aniq integrallardagidek chiziqlilik, o‘rta qiymat formulalari, additivlik kabi xossalarga ega.

Hisoblash usullari. Agar D integrallash sohasi Oy o‘qiga nisbatan standart bo‘lsa (11.1-chizma), ikki o‘lchovli integral quyidagicha hisoblanadi:

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy. \qquad (11.2)$$



11.1-chizma



11.2-chizma

Ichki integralda x o‘zgaruvchini ozgarmas kattalik sifatida qabul qilib integrallashni boshlash kerak. (11.2) integralning qiymati qandaydir son bo‘ladi.
Agar D integrallash sohasi Ox o‘qiga nisbatan standart bo‘lsa

(11.2-chizma), ikki o‘lchovli integral quyidagicha hisoblanadi:

$$\iint\limits_D f(x,y)\,dx\,dy = \int\limits_c^d dy \int\limits_{x_1(y)}^{x_2(y)} f(x,y)\,dx. \qquad (11.3)$$

Integrallash chegaralarini tashqi va ichki integrallar uchun almashtirish ikki o‘lchovli integralni karrali integralga keltirish, (11.2) formuladan (11.3) formulaga o‘tish va, aksincha, (11.3) formuladan (11.2) formulaga o‘tish ***integrallash tartibini o‘zgartirish*** deyiladi.

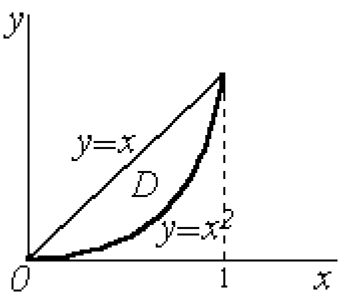
Agar *D* integrallash sohasi *Oy* o‘qiga nisbatan ham, *Ox* o‘qiga nisbatan ham standart bo‘lmasa, u holda uni *Oy* (yoki *Ox*) o‘qiga nisbatan standart bo‘lgan chekli sondagi $D_1, D_2, ..., D_m$ sohalarga bo‘linadi va ikki karrali integralni *D* soha bo‘yicha hisoblashda additivlik xossasidan foydalaniladi.

11.1-misol.

Karrali integralni hisoblang : $\iint\limits_D (x+y^2)\,dx\,dy$.

D - $y = x$ va $y = x^2$ egri chiziqlar bilan chegeralangan soha.

► D soha Oy o‘qiga nisbatan standart hisoblanadi (11.3-chizma).



11.3 - chizma

Ikki o‘lchovli integralni (11.3) formula bo‘yicha takroriy ko‘rinishga olib kelamiz:

$$\iint\limits_D (x+y^2)\,dx\,dy = \int\limits_0^1 dx \int\limits_{x^2}^{x} \left(x+y^2\right)dy.$$

Karrali integralda ichki integralni Nyuton-Leybnis formulasidan foydalanib hisoblaymiz

$$\int\limits_{x^2}^{x} \left(x+y^2\right)dy = \left(xy + \frac{1}{3}y^3\right)\Bigg|_{x^2}^{x} = x^2 - \frac{2}{3}x^3 - \frac{1}{3}x^6.$$

Endi tashqi integralni hisoblaymiz:

$$\int\limits_0^1 \left(x^2 - \frac{2}{3}x^3 - \frac{1}{3}x^6\right)dx = \left(\frac{x^3}{3} - \frac{x^4}{6} - \frac{x^7}{21}\right)\Bigg|_0^1 = \frac{5}{42}. \blacktriangleleft$$

11.1.1. Ikki o‘lchovli integralda o‘zgaruvchilarni almashtirish

$\iint\limits_D f(x,y)\,dx\,dy$ ikki o‘lchovli integralda x, y to‘g‘ri burchakli koordinatalar *x, y* bilan quyidagicha munosabatlar orqali bog‘langan yangi u, v koordinatalarga o‘tkaziladi

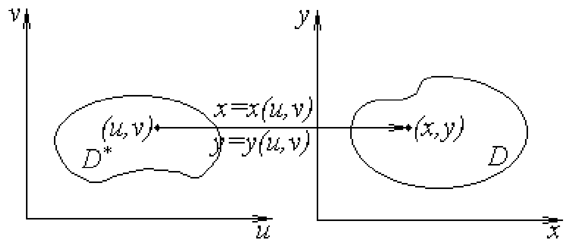
$$x = x(u,v), \quad y = y(u,v). \tag{11.4}$$

Agar (11.4-chizma) D va D^* sohalar o‘rtasida (11.4) munosabatlar orqali o‘zaro bir qiymatli akslantirish o‘rnatilgan bo‘lsa, shu bilan birga akslantirish yakobiani

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

bo‘lsa, quyidagi formula o‘rinlidir:

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D^*} f\big(x(u,v), y(u,v)\big)\left|J(u,v)\right| du\,dv. \tag{11.5}$$



11.4-chizma

Ma’lumki, to‘g‘ri burchakli *x,y* va qutb r, φ koordinatalar o‘zaro

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

munosabatlar bilan bog‘langan. Bu yerda $r \geq 0, \quad 0 \leq \varphi \leq 2\pi$.

Ikki o‘lchovli integralda to‘gri burchakli koordinatalardan qutb koordinatalarga o‘tish quydagi formula orqali amalga oshiriladi:

$$\iint\limits_D f(x,y)dxdy = \iint\limits_{\bar{D}} f(r\cos\varphi, r\sin\varphi)rdrd\varphi\,. \tag{11.6}$$

Integrallash chegaralari *O* qutbning vaziyatiga bog‘liq bo‘ladi.

a) Agar *O* qutb $\varphi = \alpha$ va $\varphi = \beta(\alpha < \beta)$ nurlar hamda $r = r_1(\varphi)$ va $r = r_2(\varphi)(r_1(\varphi) < r_{21}(\varphi))$ chiziqlar bilan chegaralangan *D* soha tashqarisida yotsa, ikki o‘lchovli integral quydagi formula bilan hisoblanadi:

$$\iint_D f(x,y)dxdy = \int_{\alpha}^{\beta} d\varphi \int_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi, r\sin\varphi) r dr. \qquad (11.7)$$

b) Agar O qutb D soha ichida joylashgan bo‘lsa va bu soha chegarasi qutb koordinatalar sistemasida $r = r(\varphi)$ ko‘rinishiga ega bo‘lsa, u holda ikki o‘lchovli integral quydagi formula bilan hisoblanadi:

$$\iint_D f(x,y)dxdy = \int_{0}^{2\pi} d\varphi \int_{0}^{r(\varphi)} f(r\cos\varphi, r\sin\varphi) r dr. \qquad (11.8)$$

c) Agar O qutb $\varphi = \alpha$ va $\varphi = \beta (\alpha < \beta)$ nurlar bilan chegaralangan D soha chegarasida yotsa, shu bilan birga, chegaraning qutb koordinatalar sistemasida tenglamasi $r = r(\varphi)$ ko‘rinishiga ega bo‘lsa, u holda ikki o‘lchovli integral quydagi formula bilan hisoblanadi:

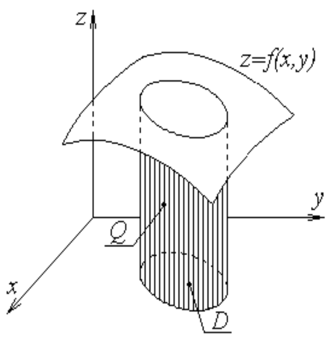
$$\iint_D f(x,y)dxdy = \int_{\alpha}^{\beta} d\varphi \int_{0}^{r(\varphi)} f(r\cos\varphi, r\sin\varphi) r dr. \qquad (11.9)$$

Umumlashgan qutb koordinatalari deb x va y to‘g‘ri burchakli koordinatalar bilan $\boldsymbol{x = ar\cos\varphi, y = br\sin\varphi}$ formulalalar orqali bog‘langan r va φ o‘zgaruvchilarga aytiladi, bunda $r \geq 0,\ 0 \leq \varphi < 2\pi,\ a > 0, b > 0, a \neq b.$ Bu holatda $|J| = abr$ va (11.5) formula quyidagi ko‘rinishga ega bo‘ladi:

$$\iint_D f(x,y)dxdy = ab\iint_{D^*} f(ar\cos\varphi, br\sin\varphi)\, r dr d\varphi. \qquad (11.10)$$

11.1.2. Ikki o‘lchovli integralning tatbiqlari.

Ikki o‘lchovli integralning ***geometrik ma’nosi***: Agar D sohada $f(x,y) \geq 0$ bo‘lsa, u holda ikki karrali integral son jihatidan asosi D bo‘lgan yasovchilari Oz o‘qiga parallel bo‘lgan, yuqoridan $z = f(x,y)$ sirt bilan chegeralangan ***Q silindrik jismning hajmiga*** teng (11.5-chizma).



11.5-chizma

$$V = \iint_D f(x,y)dxdy \qquad (11.11)$$

Xususan, $f(x,y)\equiv 1$, bo‘lganda ikki karrali integral D ***sohaning*** $S(D)$ ***yuziga*** teng, ya’ni

$$S(D)=\iint\limits_{D} dxdy. \tag{11.12}$$

11.2-misol.

Ushbu $z=0, z=x^2+y^2, y=x^2, y=1$ sirtlar bilan chegaralangan $\boldsymbol{Q}$ jismning hajmini hisoblang.

►Berilgan jismni quyidagi ko‘rinishda tasvirlash kerak:

$Q=\{(x,y,z):(x,y)\in D,\quad 0\le z\le x^2+y^2\}$, bunda D — soha Oxy tekislikning $y=x^2$ va $y=1$ egri chiziqlari bilan chegaralangan, ya’ni:

$$D=\{(x,y):\ -1\le x\le 1,\quad x^2\le y\le 1\}.$$

Ikki o‘lchovli integralning geometrik ma’nosiga ko‘ra Q jismning hajmi quyidagicha topiladi:

$$V=\iint\limits_{D}\left(x^2+y^2\right)dxdy=\int\limits_{-1}^{1}dx\int\limits_{x^2}^{1}\left(x^2+y^2\right)dy=\int\limits_{-1}^{1}\left(x^2\left(1-x^2\right)+\frac{1}{3}\left(1-x^6\right)\right)dx=\frac{88}{105}.$$ ◄

Agar $z=f(x,y)$ silliq sirt qismining xOy tekislikdagi proyeksiyasi D_{xy} bo‘lsa, u holda bu ***sirt yuzini*** quyidagi formula bilan hisoblanadi:

$$S=\iint\limits_{D_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy. \tag{11.13}$$

11.3-misol.

Ushbu $y=2\sqrt{x^2+z^2}$ konusning $x^2+z^2=4x$ silindr ichidagi qismi yuzini hisoblang .

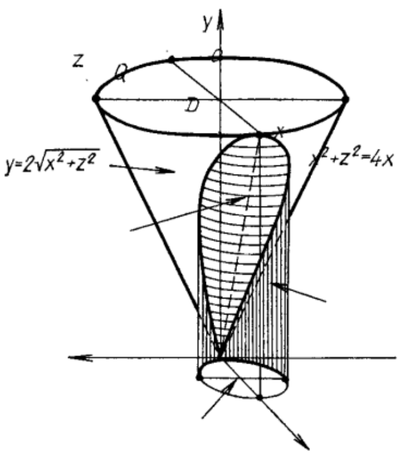
► Berilgan $y=2\sqrt{x^2+z^2}$ konus sirti qismining proyeksiyasi D_{xz} soha silindr asosi bo‘lib, $(x-2)^2+z^2=4$ aylana cizig‘i bilan chegaralangan sohadir (11.6-chizma).

Yuqoridagi (11.13) formulani $y=f(x,z)$ funksiya uchun qo‘llaymiz

$$S=\iint\limits_{D_{xz}}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dxdz.$$

$$\frac{\partial y}{\partial x}=\frac{2x}{\sqrt{x^2+z^2}};\quad \frac{\partial y}{\partial z}=\frac{2z}{\sqrt{x^2+z^2}},$$

u holda izlangan yuza



(11.6-chizma)

$$S=\iint_{D_{xz}}\sqrt{1+\frac{4x^2}{x^2+y^2}+\frac{4z^2}{x^2+z^2}}dxdz=$$

$$=\sqrt{5}\iint_{D_{xz}}dxdz=\left|\begin{matrix} z=r\cos\varphi & dxdz=rdrd\varphi \\ x=r\sin\varphi & r=4\sin\varphi \end{matrix}\right|=$$

$$=\sqrt{5}\int_0^{\pi}d\varphi\int_0^{4\sin\varphi}rdr=8\sqrt{5}\int_0^{\pi}\sin^2\varphi d\varphi=4\sqrt{5}\int_0^{\pi}(1-\cos2\varphi)d\varphi=4\pi\sqrt{5}. \blacktriangleleft$$

Ikki karrali integralning ***fizik ma'nosi***: agar D soha modda taqsimotining $\rho(x,y)$ sirt zichligiga ega, xOy tekislikda yotuvchi qalinligi bir bo'lgan yassi jism bo'lsa, u holda ***jismninig massasini*** quyidagi formula bilan hisoblanadi:

$$m=\iint_D \rho(x,y)dxdy. \qquad (11.14)$$

Jismning Ox va Oy o'qlariga nisbatan **statik momentlari** quyidagi formulalar bo'yicha topiladi:

$$M_x=\int_D y\rho(x,y)dxdy, \qquad M_y=\int_D x\rho(x,y)dxdy. \qquad (11.15)$$

Jism massasi ***og'irlik markazi*** koordinatalari:

$$x_c=\frac{M_y}{m}, \qquad y_c=\frac{M_x}{m}. \qquad (11.16)$$

D yassi jismning koordinata o'qlariga va koordinata boshiga nisbatan ***inersiya momentlari:***

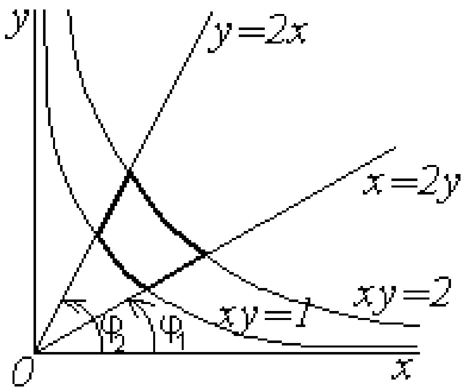
$$I_x=\iint_D y^2\rho(x,y)dxdy, I_y=\iint_D x^2\rho(x,y)dxdy,$$
$$I_0=I_x+I_y=\iint_D (x^2+y^2)\rho(x,y)dxdy \qquad (11.17)$$

formulalar bilan hisoblanadi.

11.4-misol.

Ushbu $\boldsymbol{\rho(x,y)=1}$ zichlikka ega bo'lgan, $xy=1$, $xy=2$, $y=2x$, $x=2y$ egri chiziqlar bilan chegaralangan va I chorakda joylashgan yassi jismning koordinata o'qlariga nisbatan inersiya momentlarini toping.

►Berilgan D yassi jism 11.7-chizmada tasvirlangan. (11.17) formulalarga ko'ra quyidagiga egamiz: $I_x=\iint_D y^2\,dx\,dy,\ I_y=\iint_D x^2\,dx\,dy.$



11.7-chizma

Bu integrallarni qutb koordinatalariga o‘tib hisoblash qulay: $x = r\cos\varphi$, $y = r\sin\varphi$.

$xy = 1$ egri chiziqning qutb koordinatalaridagi tenglamasi:

$r_1(\varphi) = 1/\sqrt{\sin\varphi\cos\varphi}$,

$xy = 2$ ning qutb koordinatalaridagi tenglamasi $r_2(\varphi) = \sqrt{2}/\sqrt{\sin\varphi\cos\varphi}$

U holda φ $\varphi_1 = \operatorname{arctg}\frac{1}{2}$ dan $\varphi_2 = \operatorname{arctg} 2$ gacha o‘zgaradi (11.7-chizma), $[\varphi_1;\varphi_2]$ kesmadan olingan φ ning har bir qiymatida r o‘zgaruvchi $r_1(\varphi)$ dan $r_2(\varphi)$ gacha o‘zgaradi.

Ketma-ket (11.17) formuladan foydalanib, quyidagiga ega bo‘lamiz:

$$I_x = \int_{\varphi_1}^{\varphi_2} d\varphi \int_{r_1(\varphi)}^{r_2(\varphi)} r^3 \sin^2\varphi \, dr = \frac{1}{4}\int_{\varphi_1}^{\varphi_2} \sin^2\varphi \left(r_2^4(\varphi) - r_1^4(\varphi)\right) d\varphi =$$

$$= \frac{3}{4}\int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{\cos^2\varphi} = \frac{3}{4}\operatorname{tg}\varphi \Big|_{\varphi_1}^{\varphi_2} = \frac{3}{4}\left(\operatorname{tg}\varphi_2 - \operatorname{tg}\varphi_1\right) = \frac{3}{4}\left(2 - \frac{1}{2}\right) = \frac{9}{8}.$$

O‘xshash holda quyidagini topamiz: $I_y = \frac{9}{8}$.

45-Auditoriya topshiriqlari

1. Quyidagi integrallarni hisoblang:

a) $\int_0^2 dx \int_{x^2}^{x} \left(x^2 + 2y\right) dy$;

b) $\int_1^2 dx \int_{1/x}^{x} \frac{x^2}{y^2} dy$; d) $\int_0^2 dx \int_x^{3x} (x + 6y)\, dy$.

2. Integrallash sohasi D ma’lum bo‘lsa, $\iint_D f(x,y)\,dx\,dy$ da integrallash chegarasini qo‘ying:

a) $D-$ $x=1,\ x=4,\ 3x-2y+4=0,\ 3x-2y-1=0$ chiziqlar bilan chegaralangan soha.

b) $D-$ $x=0,\ x+y=4,\ y=x^3+1$ chiziqlar bilan chegaralangan soha.

d) $D-$ uchlari $O(0;0),\ A(3;3),\ C(1,\ 5)$ nuqtalarda bo‘lgan ucburchak sohasi.

3. Quyidagi karrali integrallarda chegarani almashtiring:

$a)\ \int_{-2}^{2} dx \int_{0}^{\sqrt{1-x^2}} f(x,y)dy;$

$b)\ \int_{0}^{1} dy \int_{2y}^{5y} f(x,y)dx;$ $\qquad d)\ \int_{2}^{4} dx \int_{-\sqrt{4x-x^2}}^{4-x} f(x,y)dy$

4. $\iint_D (x+y)dxdy$ integralni hisoblang. Bu yerda $D-\ \ x-y=1,\ x-y=2,$ $2x+y-1=0,\ 2x+y-3=0$ chiziqlar bilan chegaralangan soha.

5. $\iint_D (x^2+y^2)dxdy$ integralni qutb koordinatalaridan foydalanib hisoblang. Bu yerda $D-\ \ x^2+y^2=4x$ aylana bilan chegaralangan soha.

6. $\iint_D arctg\frac{y}{x}dxdy$ integralni qutb koordinatalaridan foydalanib hisoblang. Bu yerda $D-\ \ x^2+y^2=1,\ \ x^2+y^2=9,\ \ y=\frac{x}{\sqrt{3}},\ y=\sqrt{3}x$ chiziqlar bilan chegaralangan halqa qismi.

Ikki karrali integral yordamida berilgan sirtlar bilan chegaralangan jismlarning hajmi hisoblang:

7. Koordinata tekisliklari, $x=4,\ y=4$ tekisliklar va $z=x^2+y^2+1$ aylanma paraboloid sirti bilan chegaralangan jism.

8. $z=x^2+y^2$ aylanma paraboloid, koordinata tekisliklari va $x+y=1$ tekislik bilan chegaralangan jism.

9. $y=\sqrt{x},\ y=2\sqrt{x}$ silindrlar va $z=0,\ x+z=6$ tekisliklar bilan chegaralangan jism.

10. $x^2+y^2+z^2=25$ sferaning $x^2+y^2=16$ silindr ichidagi qismi yuzini hisoblang.

45-Mustaqil yechish uchun testlar

1. $\int_{0}^{2} dx \int_{x}^{2} f(x,y)dy$ integrallash chegarasini o‘zgartiring:

A) $\int_{0}^{2} dy \int_{2y}^{3} f(x,y)dx;$ $\qquad$ B) $\int_{0}^{2} dy \int_{2}^{y} f(x,y)dx;$

D) $\int_{0}^{2} dy \int_{0}^{y} f(x,y)dx;$ $\qquad$ E) $\int_{0}^{2} dy \int_{y}^{2} f(x,y)dx.$

2. $\int_{0}^{1} dx \int_{x^2}^{1} (12xy+3)dy$ integralni hisoblang.

A) $16/3$; B) 4; D) $5/6$; E) 6.

3. $x=0,\ y=2, y=x^3+1$ chiziqlar bilan chegaralangan D soha uchun to‘g‘ri tasdiqni toping.
A) Faqat Ox o‘qi bo‘yicha muntazam;
B) Faqat Oy o‘qi bo‘yicha muntazam;
D) Ox va Oy o‘qi bo‘yicha muntazam;
C) Ox va Oy o‘qi bo‘yicha ham muntazam emas.

4. $\iint\limits_D x^2dxdy$ integralni hisoblang. Bu yerda $D-\ \ y=x,\ y=\dfrac{1}{x}, x=2$ chiziqlar bilan chegaralangan soha.
A) 2,25; B) 4,25 ; D) 2,5; E) 4,5 .

5. $\iint\limits_D \sqrt{x^2+y^2}\,dxdy$ integralni hisoblang. Bu yerda $D-\ \ R=3$ radiusli markazi koordinata boshida bo‘lgan doira sohasi.
A) 2π; B) 6π; D) 9π; E) 18π.

11.2. Uch o‘lchovli integrallar va ularning tatbiqlari. Uch o‘lchovli integralda o‘zgaruvchilarni almashtirish

11.2.1.Dekart koordinatalarida uch o‘lchovli integrallarni hisoblash.

Faraz qilaylik $f(x,y,z)=f(P)$ funksiya S sirt bilan chegaralangan yopiq fazoviy V sohada aniqlangan va uzluksiz bo‘lsin. U holda $f(x,y,z)=f(P)$ funksiyaning V soha bo‘yicha ***uch o‘lchovli integrali*** deb integral yig‘indining elementar sohalar diametrlarining eng kattasi nolga intilgandagi limitiga aytiladi:

$$\iiint\limits_V \mathrm{f(x,y,z)dv} = \lim_{\mathrm{maxdiam}\Delta \mathrm{v_i}\to 0}\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta v_i$$

Dekart koordinatalarida uch o‘lchovli integral $\iiint\limits_V \mathrm{f(x,y,z)dxdydz}$ ko‘rinishida yoziladi.
Uch o‘lchovli integralni hisoblash uchta aniq integralni yoki bitta ikki o‘lcovli va bitta aniq integralni ketma-ket hisoblashga keltiriladi.

Agar V soha, ushbu

$$\begin{cases} a\le x\le b \\ y_1(x)\le y\le y_2(x) \\ z_1(x,y)\le z\le z_2(x,y) \end{cases}$$

tengsizliklar sistemasi bilan anliqlangan bo‘lsa u holda uch o‘lchovli integral quyidagi formula bo‘yicha hisoblanadi:

$$\iiint_V f(x,y,z)dxdydz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \qquad (11.18)$$

yoki

$$\iiint_V f(x,y,z)dxdydz = \iint_D dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dx$$

11.5-misol.

Ushbu $\iiint_V (2x+y)dxdydz$ uch o‘lchovli integralni hisoblang. Bu yerda

$V-\ \ y=x,\ y=0, x=1,\ z=1, z=x^2+y^2+1$ sirtlar bilan chegaralangan soha.

► Berilgan sirtlar bo‘yicha quyidagilarni aniqlaymiz:
$0 \le x \le 1,\ 0 \le y \le x,\ \ 1 \le z \le x^2+y^2+1$. Bu holda

$$\int_0^1 dx \int_0^x dy \int_1^{x^2+y^2+1} (2x+y)dz = \int_0^1 dx \int_0^x (2x+y)z\Big|_1^{x^2+y^2+1} dy = \int_0^1 dx \int_0^x (2x+y)(x^2+y^2)dy =$$

$$= \int_0^1 dx \int_0^x \left(2x^3+x^2y+2xy^2+y^3\right)dy = \int_0^1 \left(2x^3y+\frac{1}{2}x^2y^2+\frac{2}{3}xy^3+\frac{1}{4}y^4\right)\Bigg|_0^x dx =$$

$$= \int_0^1 \frac{41}{12}x^4dx = \frac{41}{60}$$ ◄

11.2.2. Uch o‘lchovli integralda o‘zgaruvchilarni almashtirish.

$\iiint_V f(x,y,z)dxdydz$ uch o‘lchovli integralda x,y,z to‘g‘ri burchakli koordinatalardan yangi u,v,w koordinatalarga o‘tiladi:

$$x=x(u,v,w),\quad y=y(u,v,w),\quad z=z(u,v,w), \qquad (11.19)$$

Bu tengliklar u,v,w ga nisbatan bir qiymatli yechiladi:

$$u=u(x,y,z),\quad v=v(x,y,z)\quad w=w(x,y,z). \qquad (11.20)$$

$Oxyz$ fazodagi V soha (11.20) formula orqali akslanadigan $Ouvw$ fazodagi sohani V^* bilan belgilaymiz.

Agar (11.20) funksiyalar V^* sohada birinchi tartibli uzluksiz xususiy hosilalalarga ega bo‘lsa, V^* sohaga akslantirish yakobiani quyidagicha ifodalanadi:

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$$

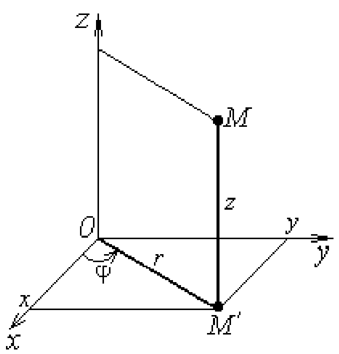
U holda $Oxyz$ fazodagi chegaralangan yopiq V soha $Ouvw$ fazodagi V^* sohaga o‘zaro bir qiymatli akslanadi va uch o‘lchovli integralda o‘zgaruvchilarni almashtirish uchun quyidagi formula o‘rinli bo‘ladi:

$$\iiint_V f(x,y,z)dxdydz=$$
$$=\iiint_{V^*} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|J(u,v,w)\right|dudvdw. \qquad (11.21)$$

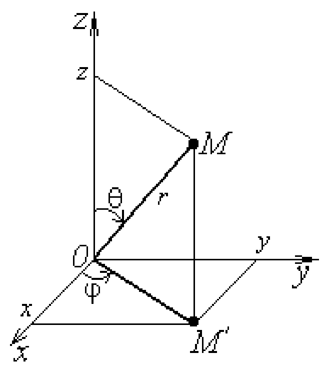
11.2.3. Uch o‘lchovli integralni silindrik koordinatalar sistemasida hisoblash.

Ushbu r,φ,z silindrik koordinatalar x,y,z to‘g‘ri burchakli koordinatalar bilan quyidagi munosabatlar orqali bog‘langan (11.8-chizma):

$$x=r\cos\varphi,\ y=r\sin\varphi,\ z=z. \qquad (11.22)$$



11.8-chizma



11.9-chizma

Bunda $r\geq 0,\quad 0\leq\varphi<2\pi,\quad -\infty<z<+\infty$. x,y,z to‘g‘ri burchakli koordinatalardan r,φ,z (11.22) formulalar bo‘yicha silindrik koordinatalariga o‘tishda $|\boldsymbol{J(r,\varphi,z)}|=\boldsymbol{r}$ bo‘ladi, shuning uchun (11.21) formula quyidagi ko‘rinishga ega bo‘ladi:

$$\iiint_G f(x,y,z)dxdydz=\iiint_{G^*} f(r\cos\varphi,r\sin\varphi,z)rdrd\varphi dz. \qquad (11.23)$$

11.6-misol.

Uch karrali integralni silindrik koordinatalariga o'tib hisoblang.

$$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz.$$

► $0 \le x \le 2,\ 0 \le y \le \sqrt{2x-x^2},\quad 1 \le z \le a$ sohani $x = r\cos\varphi,\ y = r\sin\varphi,\ z = z.$ silindrik almashtirish yordamida $0 \le \varphi \le \pi/2,\ 0 \le r \le 2\cos\varphi,\quad 1 \le z \le a$ sohaga almashtiramiz, berilgan integralni esa (6) formuladan foydalanib hisoblaymiz:

$$\int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} dr \int_0^a zr^2 dz = \int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} \frac{a^2}{2} r^2 dr = \frac{4a^2}{3}\int_0^{\pi/2} \cos^3\varphi d\varphi = .$$

$$= \frac{4a^2}{3}\int_0^{\pi/2}\left(1-\sin^2\varphi\right)d(\sin\varphi) = \frac{4a^2}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\Bigg|_0^{\pi/2} = \frac{8a^2}{9}.$$ ◄

11.2.4. Uch o'lchovli integralni sferik koordinatalar sistemasida hisoblash

Agar M nuqta fazoda x, y, z to'g'ri burchakli koordinatalarga ega bo'lsa, u holda M nuqtaning sferik koordinatalari deb (r, φ, θ) sonlar uchligiga aytiladi, bunda r — M nuqtadan O koordinata boshigacha bo'lgan masofa, φ — OM' (M' — M nuqtaning xOy tekislikdagi proyeksiyasi) nur va Ox o'q orasidagi burchak, θ — Oz o'qining musbat yo'nalishi va OM nur orasidagi burchak (11.9-chizma).

To'g'ri burchakli va sferik koordinatalari orasidagi bog'lanish quyidagi munosabatlar orqali aniqlanadi:

$$x = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta,$$

bunda $r \ge 0, \quad 0 \le \varphi < 2\pi, \quad 0 \le \theta \le \pi$. Shu bilan birga $|J(r,\varphi,\theta)| = r^2\sin\theta$ va (11.21) formula quyidagi ko'rinishga ega bo'ladi

$$\iiint_V f(x,y,z)dxdydz =$$

$$= \iiint_{V^*} f(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta) r^2 \sin\theta dr d\varphi d\theta. \tag{11.24}$$

Umumlashgan sferik koordinatalar deb x, y, z to'g'ri burchakli koordinatalar bilan quyidagi formulalar orqali bog'langan r, φ, θ o'zgaruvchilarga aytiladi:

$$x = ar\cos\varphi\sin\theta, \quad y = br\sin\varphi\sin\theta, \quad z = cr\cos\theta,$$

Bu yerda $r \ge 0,\ 0 \le \varphi < 2\pi,\ 0 \le \theta \le \pi,\ a > 0,\ b > 0,\ c > 0.$ Almashtirish yakobiani $|J(r,\varphi,\theta)| = abcr^2\sin\theta$ va (11.21) formula quyidagi ko'rinishga ega bo'ladi

$$\iiint_V f(x,y,z)dxdydz =$$

$$= abc\iiint\limits_{V^*} f(ar\cos\varphi\sin\theta, br\sin\varphi\sin\theta, cr\cos\theta)r^2\sin\theta drd\varphi d\theta. \quad (11.25)$$

11.2.5. Uch o'lchovli integralning tatbiqlari.

Agar V sohada $\boldsymbol{f(x,y,z)=1}$ bo'lsa, u holda uch o'lchovli integral V sohaning ***hajmiga*** teng, ya'ni

$$V = \iiint\limits_V dxdydz. \quad (11.26)$$

Agar $\rho(x,y,z)$ ni V sohani tashkil etuvchi modda solishtirma zichlik deb hisoblansa, u holda V sohada joylashgan butun ***modda massasi***(uch o'lchovli integralning fizik ma'nosi):

$$m = \iiint\limits_V \rho(x,y,z)dxdydz.$$

Uch o'lchovli integral yordamida, shuningdek, quyidagilarni hisoblash mumkin:

a) Jismning xOy, xOz va yOz koordinata tekisliklariga nisbatan ***statik momentlari:***

$$M_{xy} = \iiint\limits_V z\rho(x,y,z)dxdydz, M_{xz} = \iiint\limits_V y\rho(x,y,z)dxdydz,$$

$$M_{yz} = \iiint\limits_V x\rho(x,y,z)dxdydz, \quad (11.27)$$

bunda $\rho(x,y,z)$ — modda solishtirma zichligi;

b) Jismning ***og'irlik markazi koordinatalari:***

$$x_c = \frac{M_{yz}}{m}, \quad y_c = \frac{M_{xz}}{m}, \quad z_c = \frac{M_{xy}}{m}, \quad (11.28)$$

bunda m — jism massasi;

c) Jismning koordinata tekisliklari, koordinata o'qlari va koordinata boshiga nisbatan ***inersiya momentlari***:

$$I_{xy} = \iiint\limits_V z^2\rho(x,y,z)dxdydz, I_{xz} = \iiint\limits_V y^2\rho(x,y,z)dxdydz,$$

$$(12)$$

$$I_{yz} = \iiint\limits_V x^2\rho(x,y,z)dxdydz,$$

$$I_x = I_{xy} + I_{xz}, \quad I_y = I_{xy} + I_{yz}, \quad I_z = I_{xz} + I_{xz}, \quad I_0 = I_{xy} + I_{xz} + I_{yz}.$$

11.7-misol.

Ushbu $9x^2 + 2y^2 + 18z^2 = 18$ sirt bilan chegaralangan va har bir nuqtasida $\boldsymbol{\rho(x,y,z) = (x^2+y^2)\sqrt{x^2/2+y^2/9+z^2}}$ zichlikka ega bo'lgan jismning massasini hisoblang.

►Jismni chegaralagan sirt ellipsga oid hisoblanadi, unung kanonik tenglamasi esa $x^2/2+y^2/9+z^2=1$ bo‘ladi, yarim o‘qlar $a=\sqrt{2}, \quad b=3, \quad c=1.$

Uch o‘lchovli integralning fizik ma’nosiga ko‘ra V sohani egallagan jism massasi quyidagicha topiladi: $m=\iiint\limits_V (x^2+y^2)\sqrt{x^2/2+y^2/9+z^2}\,dxdydz$

Umumlashgan sferik koordinatalariga o‘tamiz:

$x=\sqrt{2}r\cos\varphi\sin\theta, \quad y=3r\sin\varphi\sin\theta, \quad z=r\cos\theta,$ bundan ellipsoid tenglamasi $r=1$ ekanligi kelib chiqadi. Bunda V soha uchun $0\le r\le 1, 0\le\varphi\le 2\pi, \ 0\le\theta\le\pi$. Bundan kelib chiqadiki,

$$m=3\sqrt{2}\iiint\limits_{G^*} r^2\sin^2\theta(2\cos^2\varphi+9\sin^2\varphi)r\cdot r^2\sin\theta drd\varphi d\theta=$$

$$=3\sqrt{2}\int\limits_0^{2\pi} d\varphi(2\cos^2\varphi+9\sin^2\varphi)\int\limits_0^{\pi} d\theta(1-\cos^2\theta)\sin\theta\int\limits_0^1 r^5dr=\frac{22\sqrt{2}}{3}\pi. ◄$$

46-Auditoriya topshiriqlari

1. Quyidagi integrallarni hisoblang:

a) $\int\limits_0^1 dx\int\limits_0^2 dy\int\limits_0^3 (x+y+z)dz;$ b) $\int\limits_0^a dx\int\limits_0^x dy\int\limits_0^y xyzdz;$

2. Integrallash sohasi V ma’lum bo‘lsa, $\iiint\limits_V f(x,y,z)dxdydz$ da integrallash chegarasini qo‘ying:

a) $V-\ \ y=\sqrt{x},\ y=2\sqrt{x},\ z=0,\ x+z=4$ sirtlar bilan chegaralangan soha.

b) $V-\ \ x=2,\ y=x,\ z=0, z=y^2$ sirtlar bilan chegaralangan soha.

3. $\iiint\limits_V \frac{dxdydz}{(x+y+z+1)^3}$ integralni hisoblang. Bu yerda $V-\ \ x=0, y=0, z=0,\ \ x+y+z=1$ tekisliklar bilan chegaralangan soha.

4. $\iiint\limits_V xydxdydz$ integralni hisoblang. Bu yerda $V-\ \ z=xy$ giperbolik paraboloid va $x+y=1,\ z=0$ tekisliklar bilan chegaralangan soha.

5. Silindrik yoki sferik koordinatalar sistemasiga o‘tib, uch karrali $\iiint\limits_V f(x,y,z)dxdydz$ integralda integrallash chegaralarini qo‘ying:

a) $V-\ \ x^2+y^2=R^2$ silindr va $z=0, z=1, y=x, y=x\sqrt{3}$ tekisliklar bilan chegaralangan coha, 1-oktantdagi qismi.

b) $V-\ \ x^2+y^2=2x$ silindr, $z=0$ tekislik va $z=x^2+y^2$ paraboloid bilan chegaralangan soha.

$d)$ $V-$ $x^2+y^2+z^2 \le R^2$ sharning 1-oktantdagi qismi.

6. $z=4-y^2,\ z=y^2+2$ silindrik sirtlar va $x=-1,\ x=3$ tekisliklar bilan chegaralangan jismning hajmini hisoblang.

7. $z=x^2+y^2,\ z=x^2+2y^2$ paraboloidlar va $y=x,\ y=2x,\ x=1$ tekisliklar bilan chegaralangan jism hajmini hisoblang.

8. $x^2+y^2+z^2=4$ sfera va $x^2+y^2=3z$ paraboloid bilan chegaralangan jism hajmini hisoblang.

9. $y=\frac{1}{2}\sqrt{x^2+y^2},\ y=2$ sirtlar bilan chegaralangan bir jinsli jismning og‘irlik markazini toping.

10. $z=5-x^2-y^2,\ z=1$ sirtlar bilan chegaralangan zichligi $\rho=3$ bo‘lgan bir jinsli jismning Oz o‘qiga nisbatan inersiya momentini toping.

46-Mustaqil yechish uchun testlar

1. $V-$ $y=x^2$ silindrik sirtlar va $y=2x,$ $z=0,\ x+z=4$ tekisliklar bilan chegaralangan soha bo‘lsa, $\iiint\limits_V f(x,y,z)dxdydz$ integralda integrallash chegaralarini qo‘ying.

A) $\int\limits_0^2 dx\int\limits_{2x}^{x^2} dy\int\limits_0^{4-z} f(x,y,z)dz$; B) $\int\limits_0^2 dx\int\limits_{x^2}^{2x} dy\int\limits_0^{4-z} f(x,y,z)dz$;

D) $\int\limits_0^4 dx\int\limits_{2x}^{x^2} dy\int\limits_0^{4-z} f(x,y,z)dz$; E) $\int\limits_0^4 dx\int\limits_{x^2}^{2x} dy\int\limits_0^{4-z} f(x,y,z)dz$.

2. $\int\limits_0^1 dx\int\limits_{-2}^0 dy\int\limits_0^3 2xy^2zdz$ integralni hisoblang.

A) 12; B) 16; D) 9; E) 18.

3. Dekart koordinatalar sistemasidan silindrik koordinatalar sistemasiga o‘tish formulasini aniqlang.

A) $x=r\cos\varphi,\ y=r\sin\varphi,\ z=1$;

B) $x=r\sin\varphi,\ y=r\cos\varphi,\ z=z$;

D) $x=\cos\varphi,\ y=\sin\varphi,\ z=z$;

E) $x=r\cos\varphi,\ y=r\sin\varphi,\ z=1$.

4. Zichligi ρ bo‘lgan jismning Oxy tekisligiga nisbatan statik momentini hisoblash formulasini aniqlang.

A) $\iiint\limits_G z^2\rho(x,y,z)dxdydz$; B) $\iiint\limits_G z\rho(x,y,z)dxdydz$;

D) $\iiint\limits_G xy\rho(x,y,z)dxdydz$; E) $\iiint\limits_G (x^2+y^2)\rho(x,y,z)dxdydz$.

5. Zichligi ρ bo‘lgan jismning Oz o‘qiga nisbatan inersiya momentini hisoblash formulasini aniqlang.

A) $\iiint\limits_G z^2\rho(x,y,z)dxdydz$; B) $\iiint\limits_G z\rho(x,y,z)dxdydz$;

D) $\iiint\limits_G xy\rho(x,y,z)dxdydz$; E) $\iiint\limits_G \left(x^2+y^2\right)\rho(x,y,z)dxdydz$.

XII BOB. EGRI CHIZIQLI INTEGRALLAR

12.1. Birinchi tur egri chiziqli integral

Oxy tekislikda *AB* silliq egri chiziqning har bir nuqtasida aniqlangan *f(x, y)* funksiya berilgan. *AB* silliq egri chiziqning bo‘linish qismlarining eng katta $\Delta\ell_i$ uzunligi nolga intilganda $\sum_{i=1}^{n} f(M_i)\Delta\ell_i = \sum_{i=1}^{n} f(\tilde{x}_i, \tilde{y}_i)\Delta\ell_i$ integral yig‘indinig limiti ***birinchi tur***(yoki *yoy uzunligi bo‘yicha)* ***egri chiziqli integral*** deyiladi va

$$\int_{AB} f(x, y)d\ell = \lim_{d\to 0}\sum_{i=1}^{n} f(\tilde{x}_i, \tilde{y}_i)\Delta\ell_i \tag{12.1}$$

kabi belgilanadi. Bunda $\Delta\ell_i$ kattalik $A_{i-1}A_i$ yoyning uzunligi va $d = \max\Delta\ell_i$ *AB* egri chiziqni kontur yoki integrallash yo‘li deb ataymiz. Agar *f(x,y)* funksiya *AB* konturning hamma nuqtalarida uzluksiz bo‘lsa, bu limit mavjud bo‘ladi. Birinchi tur egri chiziqli integral *AB* integrallash yo‘lining yo‘nalishiga bog‘liq bo‘lmaydi, ya’ni

$$\int_{AB} f(x, y)d\ell = \int_{BA} f(x, y)d\ell.$$

Zichligi $\rho(x,y)$ bo‘lgan moddiy *AB* egri chiziqning *m* massasi $\rho(x,y)$ zichlikdan *AB* egri chiziq bo‘yicha olingan birinchi tur egri chiziqli integralga teng, ya’ni

$$m = \int_{AB} \rho(x, y)dl \tag{12.2}$$

Agar *AB* egri chiziqning har bir nuqtasida $f(x, y)\geq 0$ bo‘lsa u holda birinchi tur egri chiziqli integral son jihatidan yasovchilari *Oz* o‘qiga parellel bo‘lgan yuqoridan z= *f(x, y)* sirt va quyidan *0xy* tekislik bilan chegaralangan sirtning *S* yuziga teng bo‘ladi:

$$S = \int_{AB} f(x, y)dl \tag{12.3}$$

Agar $f(x, y)=1$ bo‘lsa, bu integral son jihatidan *AB* egri chiziqning *L* uzunligiga teng bo‘ladi:

$$L = \int_{AB} dl \tag{12.4}$$

Hisoblash usullari:

1. *AB* egri chiziq parametrik tenglama bilan berilgan bo‘lsa, ya’ni $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

t parametr α dan β gacha o‘zgaradigan bo‘lsin($\alpha < \beta$). Bu holda $d\ell = \sqrt{x_t'^2 + y_t'^2}\,dt$ bo‘lgani uchun birinchi tur egri chiziqli integral

$$\int_{AB} f(x,y)d\ell = \int_{\alpha}^{\beta} f(x(t),y(t))\sqrt{x_t'^2 + y_t'^2}\,dt \qquad (12.5)$$

formula bilan hisoblanadi.

2. *AB* egri chiziq $y=y(x)$ tenglama bilan berilgan bo‘lsa($a \le x \le b$), integral

$$\int_{AB} f(x,y)d\ell = \int_{a}^{b} f(x,y(x))\sqrt{1+y'^2}\,dx \qquad (12.6)$$

formula bilan hisoblanadi.

3. *AB* egri chiziq $x=x(y)$ tenglama bilan berilgan bo‘lsa($c \le y \le d$), integral

$$\int_{AB} f(x,y)d\ell = \int_{a}^{b} f(x(y),y)\sqrt{1+x'^2}\,dy \qquad (12.7)$$

formula bilan hisoblanadi.

4. *AB* egri chiziq $\rho=\rho(\varphi)$ tenglama bilan berilgan bo‘lsa($\varphi_1 \le \varphi \le \varphi_2$), integral

$$\int_{AB} f(x,y)d\ell = \int_{\varphi_1}^{\varphi_2} f(\rho\cos\varphi,\rho\sin\varphi)\sqrt{\rho^2 + \rho'^2}\,d\varphi \qquad (12.8)$$

5. *AB* fazoviy egri chiziq $x=x(t)$, $y=y(t)$, $z=z(t)$ tenglamalar bilan berilgan bo‘lsa($\alpha \le t \le \beta$) , integral

$$\int_{AB} f(x,y,z)d\ell = \int_{\alpha}^{\beta} f(x(t),y(t),z(t))\sqrt{x_t'^2 + y_t'^2 + z_t'^2}\,dt \qquad (12.9)$$

formula bilan hisoblanadi.

12.1-misol.

Agar $y=\ln x$ egri chiziqning $x=1$ va $x=2$ abssissali nuqtalar orasidagi yoyi massasining zichligi $\rho=x^2$ bo‘lsa, bu yoyning massasini toping.

► (12.2) va (12.6) formulalardan foydalanib, quyidagiga ega bo‘lamiz:

$$m = \int_{AB} x^2 d\ell = \int_1^2 x^2\sqrt{1+y'^2}\,dx = \int_1^2 x^2\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx = \int_1^2 x\sqrt{1+x^2}\,dx = \left.\frac{\left(1+x^2\right)^{3/2}}{3}\right|_1^2 =$$

$$= \frac{1}{3}\left(5\sqrt{5} - 2\sqrt{2}\right) \approx 2{,}784 \blacktriangleleft$$

12.2-misol.

Hisoblang: $\int_L (x-y)d\ell$, bu yerda $L: x^2+y^2=4x$ aylana.

►$x^2+y^2=4x$ tenglamani qutb koordinatalar sistemasida ifodalaymiz. Buning uchun $x=\rho cos\varphi$, $y=\rho sin\varphi$ almashtirish bajarsak, L chiziq tenglamasi $\rho=4cos\varphi$ bo‘ladi. $cos\varphi\geq 0$ ekanini hisobga olib, $-\frac{\pi}{2}\leq\varphi\leq\frac{\pi}{2}$ oraliqni topamiz va (12.8) munosabatdan foydalanib, quyidagiga ega bo‘lamiz:

$$\int_L (x-y)d\ell = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\rho\cos\varphi-\rho\sin\varphi)\sqrt{16\cos^2\varphi+16\sin^2\varphi}d\varphi = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho(\cos\varphi-\sin\varphi)d\varphi$$

Endi $\rho=4cos\varphi$ ekanini hisobga olib, aniq integralni hisoblaymiz:

$$4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos^2\varphi-4\sin\varphi\cos\varphi)d\varphi = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2-2\cos2\varphi-2\sin2\varphi)d\varphi =$$

$$=4(2\varphi-\sin2\varphi+\cos2\varphi)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4(\pi-1+\pi+1)=8\pi.$$ ◄

12.3-misol.

Hisoblang: $\int_L (x^{\frac{4}{3}}+y^{\frac{4}{3}})d\ell$,

bu yerda L – $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}, a>0$ tenglama bilan berilgan astroida yoyi.

► $x=a\cos^{\frac{2}{3}}t, y=a\sin^{\frac{2}{3}}t$ astroidaning parametrik tenglamasi bo‘lgani uchun $x^{\frac{4}{3}}+y^{\frac{4}{3}}=a^{\frac{4}{3}}(\cos^4 t+\sin^4 t)$ va

$$d\ell=\sqrt{(-3a\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2}\,dt = 3a|\sin t\cos t|dt.$$

Integral ostidagi funksiya $\frac{\pi}{2}$ davrli, shuning uchun I – chorakda hisoblab 4 ga ko‘paytiramiz. Demak,

$$\int_L (x^{\frac{4}{3}}+y^{\frac{4}{3}})d\ell = 12a^{\frac{7}{3}}\int_0^{\frac{\pi}{2}}(\cos^4 t+\sin^4 t)\sin t\cos t dt = 2a^{\frac{7}{3}}(\cos^6 t+\sin^6 t)\Big|_0^{\frac{\pi}{2}} = 4a^{\frac{7}{3}}.$$ ◄

12.2. Ikkinchi tur egri chiziqli integrallar

Oxy tekilikda har bir nuqtasida *P(x, y)* (Q(x,y)) funksiya berilgan biror *AB* silliq egri chiziqni qarab chiqamiz. Bu chiziqni $A, A_1, A_2, \dots, A_{i-1}, A_i, \dots, A_{n-1}, B$ nuqtalar bilan *n* ta bo‘lakka (yoylarga) ajratamiz va ularning *0x(0y)* koordinata o‘qiga proyeksiyalarini qaraymiz. Har bir yoyda bittadan $M_i(x_i, y_i)$ nuqta tanlab olamiz. $M_i(x_i, y_i)$da berilgan *f(x, y)* funksiya qiymatlarini hisoblab $\Delta x_i = x_i - x_{i-1}$ ($\Delta y_i = y_i - y_{i-1}$) ga ko‘paytiramiz va quyidagi yig‘indini tuzamiz:

$$\sum_{i=1}^{n} P(M_i)\Delta x_i = \sum_{i=1}^{n} P(\tilde{x}_i, \tilde{y}_i)\Delta x_i \left(\sum_{i=1}^{n} Q(M_i)\Delta y_i = \sum_{i=1}^{n} Q(\tilde{x}_i, \tilde{y}_i)\Delta y_i \right) \qquad (12.10)$$

(12.10) ko‘rinishdagi yig‘indilar *P(x,y)(Q(x,y))* funksiya uchun *AB* egri chiziq bo‘ylab olingan ***ikkinchi tur integral yig‘indilar*** deb ataladi.

Bo‘linish qismlari proyeksiyalarining eng katta Δx_i (Δy_i) uzunligi nolga intilganda (12.10) integral yig‘indinig limiti ***ikkinchi tur egri chiziqli integral*** deyiladi va

$$\int_{AB} P(x, y)dx \left(\int_{AB} Q(x, y)dy \right)$$

kabi belgilanadi. Ya’ni

$$\int_{AB} P(x, y)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} P(\tilde{x}_i, \tilde{y}_i)\Delta x_i$$

$$\left(\int_{AB} Q(x, y)dy = \lim_{\max \Delta y_i \to 0} \sum_{i=1}^{n} Q(\tilde{x}_i, \tilde{y}_i)\Delta y_i \right).$$

Agar *P(x,y)(Q(x,y))* funksiya *AB* konturning hamma nuqtalarida uzluksiz bo‘lsa, bu limit mavjud bo‘ladi.

Ikkinchi tur egri chiziqli integral integrallsh yo‘lining yo‘nalishiga bog‘liq bo‘ladi, ya’ni

$$\int_{AB} P(x, y)dx = -\int_{BA} P(x, y)dx.$$

Agar *AB* egri chiziqda ikkita *P(x,y)* va *Q(x,y)* funksiyalar berilgan bo‘lsa,

$$\int_{AB} P(x, y)dx + Q(x, y)dy \qquad (12.11)$$

ikkinchi tur egri chiziqli integralning umumiy ko‘rinishi deb ataladi.

Agar *A* va *B* nuqtalar ustma-ust tushsa, *AB=L* yopiq kontur bo‘lgan holda integral quyidagicha belgilanadi:

$$\oint_{L} P(x, y)dx + Q(x, y)dy.$$

Bu holda yo‘nalish kontur ichidagi yotuvchi soha chapda qoladigan qilib tanlanadi.

Agar *P(x,y)* va *Q(x,y)* funksiyalar $\vec{F}$ - kuchning koordinatalar o‘qidagi proyeksiyalari bo‘lsa $\left(\vec{F}=P\vec{i}+Q\vec{j}\right)$, u holda (2) integral shu kuchning *AB* yo‘lda bajargan ishini ifodalaydi.
L yopiq kontur bo‘yicha hisoblangan quyidagi ikkinchi tur egri chiziqli integral shu kontur bilan chegaralangan sohaning *S* ***yuziga*** teng bo‘ladi:

$$S=\frac{1}{2}\oint_L (xdy-ydx). \tag{12.12}$$

Hisoblash usullari:

1. *AB* egri chiziq parametrik tenglama bilan berilgan bo‘lsa, ya’ni $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$

t parametr α dan β gacha o‘zgaradigan bo‘lsin($\alpha<\beta$).

$$\int_{AB} P(x,y)dx+Q(x,y)dy=\int_{\alpha}^{\beta}[P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)]dt \tag{12.13}$$

formula bilan hisoblanadi.

12.4-misol.

Hisblang: $\int_C (2a-y)dx+xdy,$ bu yerda C – *x=a(t-sint),y=a(1-cost)* sikloida arkasi($0\le t\le 2\pi$).

► *dx=a(1-cost)dt, dy=asintdt* va (12.13) formuladan foydalanamiz:

$$\int_C (2a-y)dx+xdy=\int_0^{2\pi}(a^2(1+\cos t)(1-\cos t)+a^2(t-\sin t)\sin t)dt=a^2\int_0^{2\pi}t\sin t dt=$$

$$=a^2\int_0^{2\pi}td(-\cos t)=-a^2t\cos t\Big|_o^{2\pi}+a^2\int_0^{2\pi}\cos tdt=-a^2 2\pi+a^2\sin t\Big|_o^{2\pi}=-2\pi a^2. ◄$$

2.*AB* egri chiziq *y=y(x)* tenglama bilan berilgan bo‘lsa($a\le x\le b$)

$$\int_{AB} P(x,y)dx+Q(x,y)dy=\int_a^b[P(x,y(x))+Q(x,y(x))y'(x)]dx. \tag{12.14}$$

formula bilan hisoblanadi.

12.5-misol.

Hisoblang: 1. $\int_{AB} xy^2dx+x^2ydy,$ *AB: y=x²*, *A(1,1)* , *B(2,4).*

► (12.14) formuladan foydalanamiz: ($1\le x\le 2$)

$$\int_{AB} xy^2dx+x^2ydy=\int_1^2(x\cdot x^4+x^2\cdot x^2\cdot 2x)dx=3\int_1^2 x^5dx=\frac{1}{2}x^6\Big|_1^2=\frac{1}{2}(2^6-1)=31{,}5 ◄$$

3. *AB* egri chiziq $x=x(y)$ tenglama bilan berilgan bo‘lsa($c \le y \le d$),

$$\int_{AB} P(x,y)dx + Q(x,y)dy = \int_c^d [P(x(y),y)x'(y) + Q(x(y),y)]dy. \qquad (12.15)$$

formula bilan hisoblanadi.

Agar integral ostidagi ifoda qandaydir funksiyaning to‘la differensiali bo‘lsa, ya’ni $du=Pdx+Qdy$ bo‘lsa, u holda integral integrallash yo‘liga bo‘liq bo‘lmaydi, ya’ni $A(x_1,y_2)$, $B(x_2,y_2)$ bo‘yicha integral

$$\int_{AB} P(x,y)dx + Q(x,y)dy = \int_{(x_1,y_1)}^{(x_2,y_2)} P(x,y)dx + Q(x,y)dy = u(x,y)\Big|_{(x_1,y_1)}^{(x_2,y_2)} = u(x_2,y_2) - u(x_1,y_i)$$

(12.16)

Agar u funksiyaning ko‘rinishi bizga ma’lum bo‘lmasa va $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ tenglik o‘rinli bo‘lsa, u holda

$$\int_{(x_1,y_1)}^{(x_2,y_2)} P(x,y)dx + Q(x,y)dy = \int_{x_1}^{x_2} P(x,y_1)dx + \int_{y_1}^{y_2} Q(x_2,y)dy \qquad (12.17)$$

formula bilan hisoblanadi.

12.6-misol.

Hisoblang: $\int_{(2,1)}^{(1,2)} \frac{ydx - xdy}{x^2}$, bu yerda $0y$ o‘qini kesib o‘tmaydigan yo‘l bo‘yicha integral.

► $P(x,y) = \frac{y}{x^2}, Q(x,y) = -\frac{1}{x}, x \neq 0,$ shuning uchun $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{1}{x^2}$.

(12.17) formuladan foydalanamiz:

$$\int_{(2,1)}^{(1,2)} \frac{ydx - xdy}{x^2} = \int_2^1 \frac{dx}{x^2} - \int_1^2 dy = -\frac{1}{x}\Big|_2^1 - y\Big|_1^2 = -1 + \frac{1}{2} - 2 + 1 = -\frac{3}{2}.$$ ◄

Agar $P(x,y)$ va $Q(x,y)$ funksiyalar bo‘lakli silliq L kontur bilan chegaralangan bir bog‘lamli D sohada uzluksiz va uzluksiz xususiy hosilalarga ega bo‘lsa, quyidagi Grin formulasi o‘rinli:

$$\oint_L P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \qquad (12.18)$$

47-Auditoriya topshiriqlari

1. Quyidagi birinchi tur egri chiziqli integrallarni hisoblang:

$a)$ $\int\limits_L \frac{d\ell}{x+2y+5}$, bu yerda L – $y=2x-2$ to‘g‘ri chiziqning AB kesmasi, $A(0;-2)$, $B(1;0)$.

$b)$ $\int\limits_L (x^2+y^2)^3 dl$, bu yerda L – $x=4\cos t,\ \ y=4\sin t$ aylana yoyi.

$d)$ $\int\limits_L arctg\frac{y}{x}dl$, bu yerda L – $r=2\varphi$ Arximed spiralining radiusi R va markazi koordinata boshida bo‘lgan doira ichidagi qismi.

2. $y=\frac{2}{3}x^{3/2},\ 0\le x\le 1$ moddiy yoyning har bir nuqtasidagi zichligi $\rho=y\sqrt{1+x}$ tenglik bilan aniqlansa uning massasini hisoblang.

3. $x^2+y^2=4$ silindrik sirtning Oxy tekislik va $z=2+x^2/2$ sirt bilan chegaralangan qismining yuzini hisoblang.

4. Quyidagi ikkinchi tur egri chiziqli integrallarni hisoblang:

$a)$ $\int\limits_L (x^2+y^2)dy$, bu yerda L – uchlari $A(0;0)$, $B(2;0)$, $C(4;4)$, $D(0;4)$, nuqtalarda bo‘lgan to‘rtburchak(berilgan tartibda aylanishda) konturi.

$b)$ $\int\limits_L xydx+(x^2+y)dy$, bu yerda L – $y=x^2$ parabolaning $A(0;0)$ va $B(2;4)$ nuqtalar orasidagi yoyi.

$d)$ $\oint\limits_L (x^2y-x)dx+(y^2x-2y)dy$, bu yerda L – $\frac{x^2}{9}+\frac{y^2}{4}=1$ ellips konturi, musbat yo‘nalishda.

5. Quyidagi ikkinchi tur egri chiziqli integrallarni berilgan chiziqlar bo‘yicha hisoblang:

$a)$ $\int\limits_{((0;0)}^{(1;1)} xydx+(y-x)dy,\ $ 1) $y=x$, 2) $y=x^2$, 3) $y^2=x$, 4) $y=x^3$.

$b)$ $\int\limits_{((0;0)}^{(1;1)} 2xydx+x^2dy,\ $ 1) $y=x$, 2) $y=x^2$, 3) $y^2=x$, 4) $y=x^3$.

6. Quyidagi ikkinchi tur egri chiziqli integrallarni 1) bevosita va 2) Grin formulasi yordamida hisoblang:

$a)$ $\oint\limits_L (1-x^2)ydx+(1+y^2)xdy$, bu yerda L – $x^2+y^2=4$ aylana konturi, musbat yo‘nalishda;

$b)$ $\oint\limits_L y^2dx+(x+y)^2dy$, bu yerda L – uchlari $A(3;0)$, $B(3;3)$, $C(0;3)$ da bo‘lgan ABC uchburchak konturi, musbat yo‘nalishda.

7. $x^3+x^2-y^2=0$ egri chiziq halqasi bilan chegaralangan shakl yuzini toping.

8. $y = x^3$ egri chiziqning $A(0;0)$ va $B(1;1)$ nuqtalar orasidagi yoyi bo‘ylab $\mathbf{F} = (x^2 + y^2 + 1)\mathbf{i} + 2xy\,\mathbf{j}$ kuch bajargan ishni hisoblang.

47-Mustaqil yechish uchun testlar

1. Hisoblang: $\int_L x^2 dl$, bu yerda $L -$ $y = \ln x$, $\sqrt{3} \le x \le \sqrt{8}$ egri chiziq yoyi.

A) $1\frac{4}{9}$; B) $1\frac{5}{9}$; D) 1; E) 2.

2. Hisoblang: $\int_L x^2 y dl$, bu yerda $L - x^2 + y^2 = 9$ aylananing 1-chorakdagi yoyi.

A) 12; B) 18; D) 9; E) 27.

3. Birinchi tur egri chiziqli integralni hisoblash formulasi noto‘g‘ri berilgan javobni aniqlang.

A) $\int_{AB} f(x,y)d\ell = \int_a^b f(x, y(x))\sqrt{1+y'^2}\,dx$;

B) $\int_{AB} f(x,y)d\ell = \int_{\varphi_1}^{\varphi_2} f(\rho\cos\varphi, \rho\sin\varphi)\sqrt{1+\rho'^2}\,d\varphi$;

D) $\int_{AB} f(x,y)d\ell = \int_a^b f(x(y), y)\sqrt{1+x'^2}\,dy$;

E) $\int_{AB} f(x,y)d\ell = \int_\alpha^\beta f(x(t), y(t))\sqrt{x_t'^2 + y_t'^2}\,dt$.

4. $\int_L ydx$, bu yerda $L - ABC$ siniq chiziq, $A(0;3)$, $B(3;3)$, $C(3;0)$.

5. A) 12; B) 18; D) 9; E) 27.

6. Grin formulasi yordamida hisoblang: $\oint_L -ydx + xdy$, *L:* uchlari *O(0,0), A(1,4), B(4,0)*da bo‘lgan uchburchak konturi(musbat yo‘nalish).

A) 12; B) 8; D) 10; E) 16.

15-Shaxsiy uy topshiriqlari

1

Ikki karrali integralda integrallash tartibini o‘zgartiring.

1.1. $\int_0^{1/2} dx \int_{x^3}^{\sqrt{x}} f(x,y)dy$

1.2. $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dy$

1.3. $\int_0^{2/3} dx \int_x^{2x} f(x,y)dy.$

1.4. $\int_0^1 dy \int_y^{3\sqrt{y}} f(x,y)dx.$

1.5. $\int_{-\sqrt{3}}^{1} dx \int_{-\sqrt{4-x^2}}^{0} f(x,y)dx.$

1.6. $\int_{0}^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)dy$

1.7. $\int_{0}^{1} dy \int_{y}^{2-y} f(x,y)dx.$

1.8. $\int_{0}^{1} dy \int_{\sqrt{y}}^{3-2y} f(x,y)dx.$

1.9. $\int_{0}^{2} dy \int_{y^2/4}^{\sqrt{3-y^2/2}} f(x,y)dx.$

1.10. $\int_{0}^{1} dy \int_{y}^{\sqrt{2-y^2}} f(x,y)dx.$

1.11. $\int_{-6}^{2} dy \int_{(y^2-4)/4}^{2-y} f(x,y)dx.$

1.12. $\int_{0}^{3} dx \int_{x/3}^{2x} f(x,y)dy.$

1.13. $\int_{1}^{2} dx \int_{1/x}^{x} f(x,y)dy.$

1.14. $\int_{-1}^{1} dy \int_{y^2-1}^{1-y^2} f(x,y)dx.$

1.15. $\int_{0}^{2} dx \int_{2x}^{2x+3} f(x,y)dy.$

1.16. $\int_{0}^{1} dy \int_{e^{-y}}^{e^{y}} f(x,y)dx.$

1.17. $\int_{1}^{2} dy \int_{\ln y}^{y} f(x,y)dx.$

1.18. $\int_{0}^{1} dx \int_{x}^{2-x^2} f(x,y)dy.$

1.19. $\int_{1}^{2} dy \int_{1/y}^{y} f(x,y)dx.$

1.20. $\int_{0}^{2} dx \int_{\sqrt{2x-x^2}}^{2} f(x,y)dy.$

1.21. $\int_{0}^{1} dx \int_{\sqrt{x}}^{2-\sqrt{2x-x^2}} f(x,y)dy.$

1.22. $\int_{0}^{\pi/2} dx \int_{\sin x}^{\cos x} f(x,y)dy.$

1.23. $\int_{a/2}^{a} dx \int_{0}^{\sqrt{2ax-x^2}} f(x,y)dy.$

1.24. $\int_{-6a}^{2a} dy \int_{y^2/4a-a}^{2a-y} f(x,y)dx.$

1.25. $\int_{-1}^{2} dy \int_{y^2}^{y+2} f(x,y)dx.$

1.26. $\int_{0}^{4} dy \int_{\sqrt{4y-y^2}}^{2\sqrt{y}} f(x,y)dx$

1.27. $\int_{1}^{2} dy \int_{0}^{\sqrt{4y-y^2}} f(x,y)dx.$

1.28. $\int_{1}^{2} dy \int_{2y}^{6-y} f(x,y)dx.$

1.29. $\int_{0}^{a} dx \int_{(a^2-x^2)/2a}^{\sqrt{a^2-x^2}} f(x,y)dy.$

1.30. $\int_{0}^{1} dy \int_{\sqrt{y}}^{3-2y} f(x,y)dx.$

2

Berilgan chiziqlar bilan chegaralangan bir jinsli figuraning og‘irlik markazi koordinatalarini toping.

2.1. $r = a(1-\cos\varphi).$

2.2. $\sqrt{x} + \sqrt{y} = \sqrt{a}, \quad x = 0, \quad y = 0.$

2.3. $ay = x^2, \quad x + y = 2a \quad (a > 0).$

2.4. $y = a + x, y = a - x, y = 0.$

2.5. $y^2 = ax, \quad y = ax.$

2.6. $y = x^2, \; y = 2x^2, \; x = 1, \; x = 2.$

2.7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x \geq 0, y \geq 0.$

2.8. $y^2 = 2px, \quad x = a \quad (a > 0).$

2.9. $y^2 = ax, \quad x = a, \quad y = 0.$

2.10. $y^2 = 2px, \quad x = 2p.$

2.11. $x^2 + y^2 = R^2, \quad y = 0, \quad y = x \operatorname{tg}\alpha, \quad y \geq 0.$

2.12. $y^2 = 4x + 4, \quad y^2 = -2x + 4.$

2.13. $b^2x^2 + a^2y^2 = a^2b^2, \quad x = 0, \quad y = 0.$

2.14. $y = \sqrt{2x - x^2}, \quad y = 0.$

2.15. $y^2 = 4ax^2 + 4a^2, \quad y = 2a - x, \quad x = 0.$

2.16. $y = 4 - x^2, \quad y = 0.$

2.17. $ay = x^2, \quad y = 2, \quad (a > 0).$

2.18. $y = \sin x, \quad y = 0, \quad x = \pi / 4.$

2.19. $y^2 = 4x + 4, \quad y^2 = -2x + 4.$

2.20. $xy = a^2, \quad y^2 = 8ax, \quad x = 2a \quad (a > 0).$

2.21. $y^2 = x, \quad x^2 = y.$

2.22. $x^2 / 25 + y^2 / 9 = 1, \quad x / 5 + y / 3 = 1.$

2.23. $r = a(1 + \cos\varphi).$

2.24. $ax - y^2 = 0; \quad .ax + y^2 = 0; \quad y = a; \quad a > 0.$

2.25. $y = \sin x, \; y = \cos x; \; x = 0, \; \left(-\frac{\pi}{4};0\right) \notin D$

2.26. $4y = x^2, \; x + y = 8.$

2.27. $y = \sqrt{2x - x^2}, \; x = 1.$

2.28. $y = 1/x, \; y^2 = 8x, \; x = 2.$

2.29. $y^2 = 3x, \; y = 3x.$

2.30. $x^2 / 9 + y^2 / 4 = 1, x / 3 + y / 2 = 1.$

3

H sirt bilan kesilgan L sirt qismi yuzasini toping.

3.1. $L: x^2 + y^2 + z^2 = a^2, \quad H: x^2 + y^2 = ay.$

3.2. $L: z=\sqrt{x^2+y^2}, \quad H: x^2+y^2=2x.$

3.3. $L: x=1-y^2-z^2, \quad H: y^2+z^2=1.$

3.4. $L: x^2+y^2+z^2=a^2,\ H: x^2+y^2=b^2.$

3.5. $L: x^2+y^2=2az; \quad H: (x^2+y^2)^2=2a^2xy.$

3.6. $L: z^2=x^2+y^2, \quad H: z^2=2py.$

3.7. $L: x^2+y^2=2z, \quad H: x^2+y^2=1.$

3.8. $L: x^2+y^2+z^2=R^2, \quad H: (x^2+y^2)^2=R^2(x^2-y^2).$

3.9. $L: x^2+z^2=a^2, \quad H: y^2=a(a-x).$

3.10. $L: x^2+z^2=y^2, \quad H: y^2=2px.$

3.11. $L: x^2+y^2=2ax, \ H: z^2=a(2a-x).$

3.12. $L: x^2+y^2+z^2=2a^2,\ H: x^2+y^2=z^2.$

3.13. $L: x^2+y^2=2z, \quad H: \left(x^2+y^2\right)^2=x^2-y^2.$

3.14. $L: x^2/a+y^2/b=2z,\ H: x^2/a^2+y^2/b^2=c^2.$

3.15. $L: x^2+y^2+z^2=a^2,\ H: x^2/a^2+y^2/b^2=1\ (b<a).$

3.16. $L: 2y=x^2+z^2,\ H: x^2+z^2=1.$

3.17. $L: y^2=x^2+z^2,\ H: x^2+y^2=2z.$

3.18. $L: x^2+y^2+z^2=R^2,\ H: y^2+z^2=Rz.$

3.19. $L: x^2+y^2=2ax,\ H: x^2+y^2=z^2, \quad z=0.$

3.20. $L: x^2+y^2=ax, \quad H: x^2+y^2+z^2=a^2.$

3.21. $L: y^2=4x,\ H: x^2+y^2+z^2=5x.$

3.22. $L: x^2+z^2=4,\ H: x^2+y^2=4.$

3.23. $L: x^2+y^2+z^2=R^2, \quad H: x^2+y^2=z^2\,\mathrm{tg}^2\,\alpha, \quad 0<\alpha<\pi/2.$

3.24. $L: x^2+y^2=2z,\ H: x^2+y^2=3, \quad y\ge 0, \quad x\ge 0, \quad z\ge 0.$

3.25. $L: x^2+z^2=4, \quad H: y^2=4-2x.$

3.26. $L: y^2+z^2=x^2, \quad H: x^2+y^2=R^2.$

3.27. $L: 2z=x^2+y^2, \quad H: x^2+y^2=1.$

3.28. $L: z=\sqrt{x^2+y^2}, \quad H: x^2+y^2=4x.$

3.29. $L: x^2-y^2=2az,\ H: x^2+y^2=3a^2, \quad y\ge 0, \quad x\ge 0, \quad z\ge 0.$

3.30. $L: x^2+y^2=2x, \quad H: x^2+y^2+z^2=4.$

4

Berilgan sirtlar bilan chegaralangan jism hajmini toping.

4.1. $y^2/b^2+z^2/c^2=2x/a,\ x=a$

4.2. $x^2+y^2+z^2-2z=0, \quad x^2+y^2=2-z$

4.3. $x^2+y^2+z^2=16, \quad x^2+y^2+z^2-8z=0.$

4.4. $x^2 + y^2 = z^2, \quad x^2 + y^2 = 6 - z, \quad z \geq 0.$

4.5. $x^2 + y^2 = 4x, \quad x = z, \quad z = 2x.$

4.6. $x^2 + y^2 + z^2 = 3a^2, \quad x^2 + y^2 = 2az.$

4.7. $x^2 + y^2 + z^2 = 2az, \quad x^2 + y^2 = z$, $M(0;0;a)$ nuqtani o‘z ichiga olgan jism.

4.8. $x^2 + y^2 + z^2 = R^2, \quad x^2 + y^2 = Rx.$

4.9. $z = 6 - x^2 - y^2, \quad \sqrt{x^2 + y^2} = z.$

4.10. $az = x^2 + y^2, \quad z = \sqrt{x^2 + y^2}.$

4.11. $az = a^2 - x^2 - y^2, \quad x = 0, \quad y = 0, \quad z = 0.$

4.12. $x^2 + y^2 + z^2 = 4Rz - 3R^2, \quad z^2 = 4\left(x^2 + y^2\right),$ $T(0;0;2R)$ nuqtani o‘z ichiga olgan qismi.

4.12. $z = x^2 + y^2, \quad z = x + y.$

4.13. $x^2 + y^2 + z^2 = 1, \quad z^2 = x^2 + y^2.$

4.14. $z = x^2 + y^2, \quad z^2 = x \cdot y.$

4.15. $x^2 + y^2 + z^2 = a^2, \quad z^2 = x^2 + y^2 \quad (z \geq 0).$

4.16. $x^2 + y^2 + z^2 = R^2, \quad x^2 + y^2 = R(R - 2z).$

4.17. $x^2 + y^2 = hz, \quad z = h.$

4.18. $az = x^2 + y^2, \quad z = \sqrt{x^2 + y^2}, \quad (a > 0).$

4.19. $x = 6 - z^2 - y^2, \quad x^2 = y^2 + z^2.$

4.20. $y^2 + z^2 = 3x, \quad x^2 + y^2 + z^2 = 4.$

4.21. $x^2 + 2y^2 = 2z; \quad z = 2.$

4.22. $z = x^2 - y^2; \quad z = 0, \quad x = 2.$

4.23. $z = x^2 + y^2, \; z = 2\left(x^2 + y^2\right), \; y = x^2, y = x.$

4.24. $z = x^2 + y^2, \; z = x^2 + 2y^2, \; y = x, y = 2x, x = 1.$

4.25. $(x-1)^2 + y^2 = z, \quad 2x + z = 2.$

4.26. $z = 4 - y^2, \; z = y^2 + 2, x = -1, x = 2.$

4.27. $x^2 + y^2 + z^2 = 4, \; z = \frac{1}{3}\left(x^2 + y^2\right)$

4.28. $x^2 + y^2 + z^2 = 4, \; x^2 + y^2 = 4(1 - z).$

4.29. $z = \ln(x + 2), \; z = \ln(6 - x), \; x + y = 2, x - y = 2, y = 0.$

Berilgan sirtlar bilan chegaralangan, zichligi ρ bo‘lgan jism massasini toping.

5.1. $z=4-y^2,\quad z=y^2+2,\quad x=-1,\quad x=2;\quad \rho=1.$

5.2. $x+y+z=a,\quad x=0,\quad y=0,\quad z=0;\quad \rho=z.$

5.3. $z=h,\quad x^2+y^2-z^2=0,\quad \rho=z.$

5.4. $x^2+y^2=a^2,\ x^2-y^2=az,\ z=0\,(z>0),\ \rho=z.$

5.5. $x^2+y^2+z^2=a^2,\ x^2+y^2=z^2,\ \rho=\sqrt{x^2+y^2+z^2}.$

5.6. $x^2+y^2=R^2,\quad z=H,\quad z=0;\quad \rho=x^2+y^2+z^2$

5.7. $x^2+y^2=z,\quad x+y=a,\quad x=0,\quad y=0,\quad z=0,\quad \rho=1.$

5.8. $x^2+z^2=a^2,\quad y^2+z^2=a^2,\quad z\geq 0,\quad \rho=1.$

5.9. $x^2+y^2+z^2\leq 9, x\geq 0, y\geq 0, z\geq 0, 3x+2y=6, \rho=z.$

5.10. $y=\sqrt{x}; y=2\sqrt{x},\ z=0, x+z=6; \rho=x.$

5.11. $x=0,\quad y=0;\quad z=0,\quad y=6,\quad x+z=a,\quad \rho=kx.$

5.12. $x=0,\quad y=0,\quad z=1,\quad z=3,\quad 2x+y=3,\quad \rho=z.$

5.13. $z^2=x^2+y^2,\quad z=h\quad (h>0, z\geq 0);\quad \rho=z^2.$

5.14. $2x+y+z-4=0,\quad x=0\quad z=0,\quad y=0,\quad \rho=y.$

5.15. $x^2+y^2=1,\quad x^2+y^2=2z,\quad x=0,\quad y=0,\quad z=0,\quad \rho=10x.$

5.16. $x^2/a^2+y^2/b^2+z^2/c^2=2, x^2/a^2+y^2/b^2-z^2/c^2=0, (z>0), \rho=1.$

5.17. $x^2/a^2+y^2/b^2+z^2/c^2=1,\ z=0,\ \rho=z.$

5.18. $x^2+y^2+z^2\leq R^2;\quad z\geq 0,\quad \rho=x^2+y^2.$

5.19. $y^2+z^2=2ax;\quad y^2+z^2=2az,\quad x=0;\quad (a>0);\quad \rho=1.$

5.20. $x^2=z^2+y^2,\quad y+z=a,\quad z=0,\quad y=0,\quad x=0,\quad \rho=1.$

5.21. $x^2+y^2+z^2=Rz,\quad \rho=1/\sqrt{x^2+y^2+z^2}.$

5.22. $x^2+y^2+z^2=x,\quad \rho=\sqrt{x^2+y^2+z^2}.$

5.23. $x^2+y^2+z^2=x,\quad \rho=\sqrt{x^2+y^2+z^2}.$

5.24. $x^2+y^2=4y; y+z=4,\ z\geq 0;\ \rho=1/\sqrt{x^2+y^2}.$

5.25. $x^2+y^2=4x; x+z=2,\ z\geq 0;\ \rho=\sqrt{x^2+y^2}.$

5.26. $y=\sqrt{x}; y=2\sqrt{x},\ z=0, x+z=6; \rho=y.$

5.27. $x^2+y^2+z^2=16, x^2+y^2=4x,\ \rho=1.$

5.28. $x=0,\quad y=0,\quad z=1,\quad z=3,\quad x+2y=3,\quad \rho=z.$

5.29. $x^2/a^2+y^2/b^2+z^2/c^2=1,\ z=0,\ \rho=y^2.$

5.30. $x^2/a^2+y^2/b^2+z^2/c^2=1,\ z=0,\ \rho=x^2.$

6

Berilgan masalalarni birinchi tur egri chiziqli integral yordamida yeching.

6.1. Kardioida uzunligini toping: $x = 2a\cos t - a\cos 2t, y = 2a\sin t - a\sin 2t$.

6.2. I kvadrantda joylashgan $x = a\cos^3 t,\ y = a\sin^3 t$ astroida bir jinsli yoyining og‘irlik markazi koordinatalarini toping

6.3. Agar massa taqsimotining har bir nuqtadagi zichligi egri chiziqning ordinata kvadratiga teng bo‘lsa, I chorakda joylashgan $x^2 + y^2 = a^2$ aylana yoyi massasini toping.

6.4. $x = a\cos t,\ y = a\sin t, z = ht, \rho(x, y, z) = 1$ vint chizig‘i bir o‘ramining *Oxy* tekislikka nisbatan statik momentini hisoblang.

6.5. $x = R\cos t,\ y = R\sin t\left(0 \le t \le \dfrac{\pi}{2}\right)$ aylana yoyining O(0, 0) nuqtaga nisbatan inersiya momentini hisoblang

6.6. Nuqtadagi massa taqsimotining zichligi bu nuqtaning ordinatasiga teng bo‘lsa, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ellips tepa yarmining Oy o‘qiga nisbatan statistik momentini hisoblang

6.7. Agar *A*(1;3) va *B*(4;5), bo‘lsa, har bir M nuqtadagi massa taqsimoti zichligi $1/(2x + y)$ bo‘lsa, AB kesma massasini hisoblang

6.8. Agar har bir nuqtadagi massa taqsimoti zichligi $\rho = z^2/(x^2 + y^2)$ bo‘lsa, $x = a\cos t,\ \ y = a\sin t,\ \ z = at$ vint chizig‘I birinchi o‘ramining massasini toping.

6.9. Bir jinsli vint chizig‘ining bir o‘rami massasini toping: $x = a\cos t,\quad y = a\sin t,\quad z = ht,\quad \rho(x,y,z) = 1.$

6.10. Har bir nuqtadagi massa taqsimoti zichligi nuqta ordinatasiga teskari proporsional bo‘lgan, shu bilan birga $(0; a)$ nuqtada zichligi δ. ga teng bo‘lsa, abssisalari $x_1 = 0$ va $x_2 = a$ bo‘lgan nuqtalar orasidagi $y = a/2\ \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ zanjir chizig‘i maydonining massasini hisoblang.

6.11. Zichligi $\rho = \dfrac{1}{\sqrt{x}}$. bo‘lgan $L: y^2 = \dfrac{4}{9}x^3$ egri chiziq yoyining $M_1(3; 2\sqrt{3})$ dan $M_2(8; 32\sqrt{2}/3)$ gacha *Ox* o‘qiga nisbatan qismining statistik momentini hisoblang.

6.12. Agar taqsimot zichligi massasi $\rho = xy$ bo‘lsa, I kvadrantda joylashgan $x = R\cos t, y = R\sin t$ aylana choragining koordinata boshiga nisbatan inersiya momentini toping.

6.13. Absissalari $x = 0, x = a$ nuqtalar orasida bo‘lgan zanjir chizig‘i $y = \dfrac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ uzunligini hisoblang.

6.14. Agar har bir nuqtadagi massa taqsimoti zichligi bu nuqtaning ordinatasiga teng bo‘lsa,

$x = a(2\cos t - \cos 2t), \quad y = a(2\sin t - \sin 2t), \; 0 \le t \le 2\pi$ kardioida massasini toping.

6.15. Agar zichligi $\rho = y$ bo‘lgan $x^2/a^2 + y^2/b^2 = 1$ $M_1(a;0)$ dan $M_2(0;b)$ ellips yoyining nuqtadan nuqtagacha Oy o‘qiga nisbatan statistik momentini toping.

6.16. $x = a\cos^3 t, y = a\sin^3 t, \; 0 \le t \le 2\pi$ astroida uzunligini toping.

6.17. Har bir nuqtadagi massa taqsimoti zichligi $\rho = xy$ bo‘lgan, I chorakda joylashgan $x = a\cos^3 t, y = a\sin^3 t$ astroida yoyining koordinata o‘qlariga nisbatan statistik momentlarini toping.

6.18. Agar egri chiziqning har bir nuqtadasigi massa taqsimoti zichligi bu nuqta ordinatasining kvadratiga teng bo‘lsa, I chorakda joylashgan $x^2 + y^2 = a^2$ aylana qismi massasini toping.

6.19. $y^2 = 2x$ от $M_1(0;0)$ д $M_2(1;\sqrt{2})$ parabola yoyining nuqtadan nuqtagacha qismining Ox o‘qiga nisbatan statik momentini toping.

6.20. Agar zichligi $\rho = \dfrac{1}{\sqrt{x^2 + y^2}}$ bo‘lsa, $y = x/2 - 2$ to‘g‘ri chiziq kesmasining $M_1(0;-2)$ nuqtadan $M_2(4;0)$ nuqtagacha kesmasining massasini toping.

6.21. I kvadrantda joylashgan $x = a\cos^3 t, y = a\sin^3 t$ astroida bir jinsli yoyi og‘irlik markazi koorditalarini toping.

6.22. $x = a\cos t, y = a\sin t, z = ht/2\pi$ vint chizig‘i birinchi o‘ramining koordinata o‘qlariga nisbatan inersiya momentlarini toping.

6.23. Agar zichligi $\rho = \dfrac{1}{x + y}$ bo‘lsa, $y = x + 2$ egri chiziqning от $M_1(2;4)$ nuqtadan $M_2(1;3)$, nuqtagacha massasini toping.

6.24. $x = a(t - \sin t), y = a(1 - \cos t), \; 0 \le t \le 2\pi$ sikloida arkasining Ox o‘qiga nisbatan inersiya momentini toping.

6.25. $x = 4(t - \sin t), y = 4(1 - \cos t), \; 0 \le t \le 2\pi$ sikloida arkasi Ox o‘qiga nisbatan statik momentini toping.

6.26. Agar egri chiziqning har bir nuqtadasigi massa taqsimoti zichligi $\rho = \sqrt{x^2 + y^2}$ bo‘lsa, $x^2 + y^2 = 16y$ aylana yoyining massasini toping.

6.27. Zichligi $\rho = |y|$ bo‘lgan $y^2 = 8x$ $(0 \le x \le 2)$ parabola yoyining m massasini toping.

6.28. Zichligi $\rho = |y|$ bo‘lgan $x = 2cht, y = 2sht (0 \le t \le 2)$ giperbola yoyining m massasini toping.

6.29. Zichligi $\rho = |y|$ bo‘lgan $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, (a \ge b > 0)$ egri chiziq m massasini toping.

6.30. Zichligi $\rho=|y|$ bo‘lgan $\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$ lemniskata yoyi m massasini toping.

7

Quyidagi ikkinchi tur egri chiziqli integrallarni hisoblang.

7.1. $\int\limits_L xdy,$ bu erda L – uchlari $O(0,\ 0)$, $A(4,0)$, $B(4,\ 5)$ bo‘lgan uchburchak konturi, musbat yo‘nalishda.

7.2. $\int\limits_L (x^2-y^2)dy,$ bu erda L – $y=x^2$ parabolaning $O(0,0)$ nuqtadan $A(2,4)$gacha yoyi.

7.3. $\int\limits_L ydx+xdy,$ bu erda L –$x=Rcost$, $y=Rsint$, $t_1=0$ dan $t_2=\pi/2$ gacha chorak aylana.

7.4. $\int\limits_{OA} xydx+(y-x)dy$, bu yerda OA– $y^2=x$ parabolaning $O(0,0)$ nuqtadan $A(1,1)$gacha yoyi.

7.5. $\int\limits_L ydx-xdy,$ bu erda L – $x=4cost$, $y=3sint$ ellips musbat yo‘nalishi bo‘yicha.

7.6. $\int\limits_L -x\cos ydx+y\sin xdy,$ bu yerda L – $(0,0)$ va $(\pi,\ 2\pi)$ nuqtalarni tutashtiruvchi kesma.

7.7. $\int\limits_{AB} \sin ydx+\sin xdy,$ bu yerda AB – $A(0,\pi)$ va $B(\pi,0)$ nuqtalar orasidagi kesma.

7.8. $\int\limits_{(1,0)}^{(6,8)} \frac{xdx+ydy}{\sqrt{x^2+y^2}},$ koordinata boshidan o‘tmaydigan yo‘l bo‘yicha integral.

7.9. $\int\limits_L (x^2-2xy)dx+(y^2-2xy)dy,$ bu yerda L – $y=x^2$ parabola $(-1\le x\le 1)$.

7.10. $\int\limits_L (x^2+y^2)dx+(x^2-y^2)dy,$ bu yerda L - siniq chiziq $y=1-|1-x|,(0\le x\le 2)$.

7.11. $\int\limits_L (x^2+y^2)dy,$ bu yerda L - uchlari $A(0,0)$, $B(3,0)$, $C(3,4)$, $D(0,4)$ nuqtalarda bo‘lgan to‘rtburchakning berilgan yo‘nalishdagi konturi.

7.12. $\int\limits_{AB} (x^2+2y)dx+(2x-y)dy,$ bu yerda AB – $A(-1,2)$ va $B(3,1)$ nuqtalar orasidagi kesma.

7.13. $\int_L (4-y)dx-(2-y)dy,$ bu yerda L - $x=2(t-\sin t), y=2(1-\cos t),$ $0\le t\le 2\pi$.

7.14. $\int_L (xy+x+y)dx+(xy+x-y)dy,$ bu yerda L – $x^2+y^2=2x$ *aylana konturi (musbat yo'nalishda).*

7.15. $\int_{(0,-1)}^{(1,0)} \frac{xdy-ydx}{(x-y)^2},$ $y=x$ to'g'ri chiziqni kesib o'tmaydigan yo'l bo'yicha integral.

7.16. $\int_L (x+y)dx+xdy,$ bu yerda L - $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsning I – chorakdagi yoyi.

7.17. $\int_L (x+y)^2dx-(x^2+y^2)dy,$ bu yerda L - uchlari $A(1,1)$, $B(3,2)$, $C(2,5)$ nuqtalarda bo'lgan uchburchakning musbat yo'nalishdagi konturi.

7.18. $\int_{(-2,-1)}^{(3,0)} (x^4+4xy^3)dx+(6x^2y^2-5y^4)dy.$

7.19. $\int_L xdy,$ bu yerda L - $\frac{x}{2}+\frac{y}{3}=1,$ $x=0, y=0$ uchburchakning musbat yo'nalishda olingan konturi.

7.20. $\int_L \frac{x^2dy-y^2dx}{x^{\frac{5}{3}}+y^{\frac{5}{3}}},$ bu yerda L - $x=a\cos^3 t, y=a\sin^3 t$ astroida yoyi ($(a,0)$ nuqtadan $(0,a)$ nuqtagacha qismi).

7.21. $\int_{AB} xdx+ydy+(x+y-1)dz,$ bu yerda AB – $A(1,1,1)$ va $B(2,3,4)$ nuqtalar orasidagi kesma.

7.22. $\int_{AB} 2xydx+x^2dy,$ bu yerda AB – $y=x^2+1$ parabolaning $A(0,1)$ va $B(2,5)$ nuqtalari orasidagi yoyi.

7.23. $\int_L (y^2-z^2)dx+(z-x)dy-x^2dz,$ bu yerda L – $x=t$, $y=t^2$, $z=t^3$ $(0\le t\le 1)$ fazoviy egri chiziq yoyi.

7.24. $\int_{(0,0)}^{(0,\pi)} e^x\cos ydx-e^x\sin ydy.$

7.25. $\int_L \frac{(x+y)dx-(x-y)dy}{x^2+y^2},$ bu yerda L – $x^2+y^2=9$ aylananing musbat yo'nalishda olingan konturi.

7.26. $\int_L (3+y)dx+(2x-y)dy,$ bu yerda L - $x=2\sin t, y=2\cos t,$ $0\le t\le 2\pi$.

7.27. $\int_{(-2,1)}^{(3,0)} (4x^3+4xy^3)dx+(6x^2y^2-5y^4)dy.$

7.28. $\int_L (x^2+2y)dy,$ bu yerda L - uchlari $A(0,0)$, $B(4,0)$, $C(4,3)$, $D(0,3)$ nuqtalarda bo‘lgan to‘rtburchakning berilgan yo‘nalishdagi konturi.

7.29. $\int_L 2ydx+xdy,$ bu yerda L – $x=3cost,$ $y=4sint$ ellips musbat yo‘nalishi bo‘yicha.

7.30. $\int_L ydx+(3x-2)dy,$ bu yerda L - $3x+2y=6$, $x=0, y=0$ uchburchakning musbat yo‘nalishda olingan konturi.

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