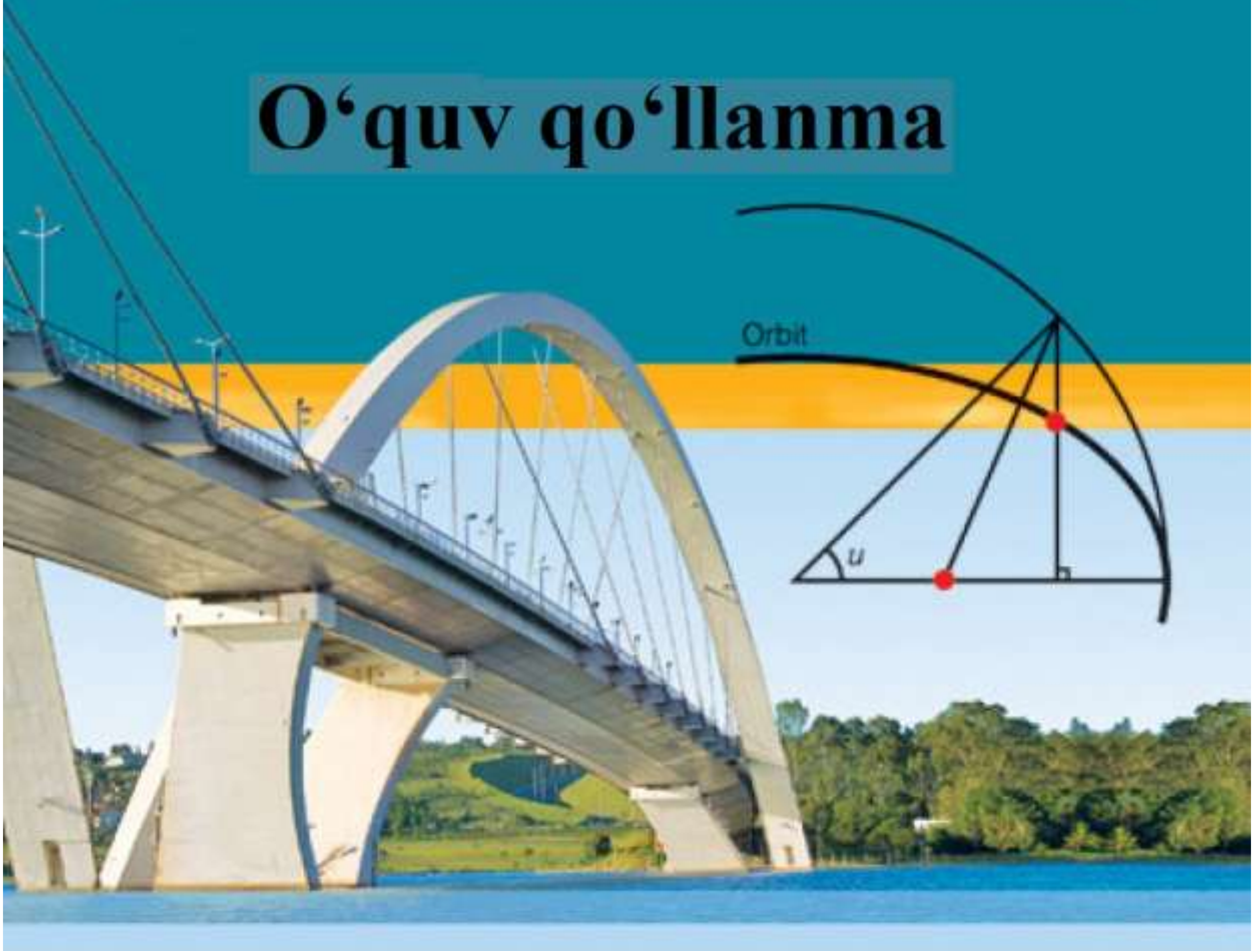
HISOB (CALCULUS)

O‘quv qo‘llanma



S. S. SADADDINOVA

Toshkent – 2023

O‘ZBEKISTON RESPUBLIKASI OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI

MUHAMMAD AL-XORAZMIY NOMIDAGI TOSHKENT AXBOROT TEXNOLOGIYALARI UNIVERSITETI

S. S. SADADDINOVA

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Taqrizchilar:

A.X.Xudoyberdiyev – V.I.Romanovskiy nomidagi Matematika instituti direktor o‘rinbosari, f.-m.f.d., professor.

O‘.N.Qalandarov – Muhammad al – Xorazmiy nomidagi TATU “Oliy matematika” kafedrasi mudiri, f.-m.f.n., dotsent.

Ushbu o‘quv qo‘llanma Muhammad al-Xorazmiy nomidagi Toshkent axborot texnologiyalari universiteti hamda uning bakalavriat bosqichi talabalari uchun mo‘ljallangan bo‘lib, “Hisob (Calculus)” fani materiallarini to‘la qamrab oladi. Jumladan, o‘quv qo‘llanmada kompleks sonlar nazariyasi, sonli va funksional ketma-ketliklar, differensial hisob, integral hisob, sonli va funksional qatorlar, ko‘p o‘zgaruvchili funksiyalar, ikki va uch karrali integrallar, egri chiziqli va sirt integrallari boblari yoritib berilgan. Shuningdek, har bir mavzuning texnikada, iqtisodiyotda, tibbiyotda, ijtimoiy olam va boshqa ko‘plab sohalardagi amaliy tatbiqlari asoslab berilgan.

Kitobda hozirgi zamon hisoblash matematikasi asoslarining yutuqlari o‘z aksini topgan.

O‘quv qo‘llanma barcha muhandis – texnika va iqtisodiyot bakalavriyat ta’lim yo‘nalishlari, shu jumladan Muhammad al – Xorazmiy nomidagi Toshkent axborot texnologiyalari universiteti va uning 5 ta hududiy filiallari hamda pedagogika universitetlari talabalari uchun mo‘ljallangan, shuningdek, litsey va o‘rta maktabning yuqori sinf o‘quvchilari, kichik ilmiy hodimlar va professor-o‘qituvchilar ham foydalanishlari mumkin.

KIRISH

Mazkur o‘quv qo‘llanmani yozishda O‘zbekiston Respublikasi Prezidentining “O‘zbekiston Respublikasi oliy ta’lim tizimini 2030 yilgacha rivojlantirish konsepsiyasini tasdiqlash to‘g‘risida”gi PF-5847 sonli farmonini asos qilib oldik.

O‘quv qo‘llanmani “Hisob (Calculus)” fan dasturi materiallarini to‘liq qamrab oladigan quyidagi 7 ta bobga ajratdik:

I bob. Kompleks sonlar;
II bob. Differensial hisob;
III bob. Integral hisob;
IV bob. Sonli va funksional qatorlar;
V bob. Ko‘p o‘zgaruvchili funksiyalar;
VI bob. Ikki va uch karrali integrallar;
VII bob. Egri chiziqli va sirt integrallari.

Kitobda real hayotda uchraydigan misol va masalalarni olishga harakat qildik, jumladan, iqtisodiy va moliyaviy kattaliklar, ishlab chiqarishdagi o‘sish, sog‘liqni saqlash, ekologik tadqiqotlar, gidrometereologiya ma‘lumotlari asosida, turizm haqida, tarixiy obidalar, kishilar hayoti, psixologiyasi, anatomiyasi, inson miyasining axborotni qabul qilish omillari haqida, ma’lumotlarni uzatish tizimlari, mexanik kattaliklar qatnashgan misol va masalalar tuzdik. Bu bilan matematikaning qo‘llanilish sohalari qanchalik keng ekanligini ko‘rsatishga harakat qildik.

“Hisob (Calculus)” fani tushunchalarini grafik, diagramma va jadvallar yordamida berish bilan talabalarning tasavvurini boyitishga hamda fanga qiziqtirishga urindik. Ushbu fanni o‘rgatish doirasida talabalarning turli bilim va ko‘nikmalarga ega bo‘lishlarini ham inobatga oldik. Ayniqsa, sirtqi, masofaviy hamda onlayn ta’lim oladigan talabalar ham darslikdan foydalana olishlari uchun mavzular ketma-ketligini oddiydan murakkabga, soddadan qiyinga tamoyili asosida yoritdik.

O‘quv qo‘llanmani yozishda fikr mulohazalarini bildirib, kitobning ilmiy va uslubiy jihatdan sifatini oshirishga o‘z hissalarini qo‘shgan, kamchiliklarni to‘g‘rilashga yordamlashgan hamkasblarimdan, dotsent O‘.N.Qalandarov, prof. R.Z.Abdullayev, dots. R.R.Raxmatovlarga, misollar va testlarni tuzishda, chizmalar dizayniga yordamlashgan Sh.E.Tadjibayevaga chuqur minnatdorchiligimni bildiraman.

Muallif

O‘lchov birliklari:

Uzunlik o‘lchovlari				
1 dyuym	0.0833 ft	2.54 sm		
1 fut (ft)	12 dyuym	30.48 sm	0.3048 m	
1 mil	1760 yard	742000 sm	7420 m	7.420 km
1 yard	3 ft	91.44 sm		
Hajm (suyuqlik) o‘lchovlari				
1 fl		0.028 litr		
1 barrel		158.998 litr		
1 gallon		3.785 litr		
1 kvart		0.94625 litr		
Yuza (maydon) o‘lchovlari				
1 kv.mil	640 akr	258.99 ga		
1 akr	4047 m^2	4840 kv. yard	43560 ft^2	
1 kv. yard	0.836 m^2			
Massa (og‘irlik) o‘lchovlari				
1 misqol	4.25 gr	100 ta arpa (bug‘doy) doni og‘irligi		
1 funt	453.6 gr			
Harorat o‘lchovlari				
0^0 C	32^0 F			
5^0 C	41^0 F			
10^0 C	50^0 F			

Selsiy o‘lchov birligidan Farengeyt o‘lchov birligiga o‘tish:

$$F = \frac{9}{5}C + 32$$

Farengeyt o‘lchov birligidan Selsiy o‘lchov birligiga o‘tish:

$$C = \frac{5(F-32)}{9}$$

Belgilashlar:

$\exists$ - mavjudlik belgisi;

$\forall$ - ixtiyoriylik belgisi;

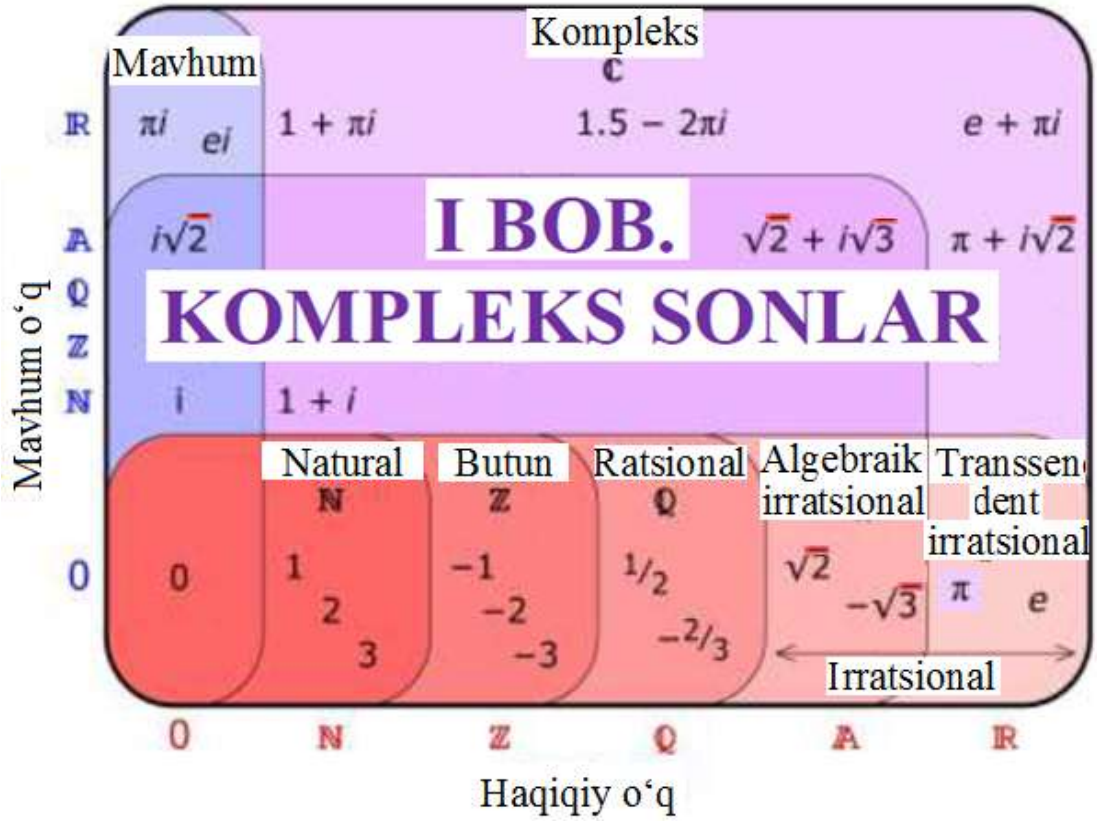
$\in$ - tegishlilik belgisi;

$\cap$ - to‘plamlarning kesishmasi;

$\cup$ - to‘plamlarning birlashmasi;

$\setminus$ - to‘plamlarning ayirmasi.

I BOB. KOMPLEKS SONLAR



1.1-§. Kompleks son haqida tushuncha

Son – matematikaning boshlang‘ich tushunchasi hisonlanadi. Dastlab narsalarni, buyumlarni sanash va qo‘shish amallarini bajarish zaruriyati tufayli natural sonlar paydo bo‘ldi. Natural sonlar to‘plami quyidagicha belgilandi: $N=\{1,2,3,...,n,...\}$.

Hisoblashda ayirish amali kiritilgandan keyin natural sonlar to‘plami yetarli bo‘lmay qoldi, shuning uchun natural sonlarga qarama-qarshi sonlarni va nol sonini kiritish bilan butun sonlar to‘plami hosil qilindi: $Z=\{...,-n,...,-3,-2,-1,0,1,2,3,...,n,...\}$.

Keyinchalik insoniyat ko‘paytirish va bo‘lish amallarini ham o‘ylab topdi va hisob ishlarida natural va butun sonlar bilan kifoyalanib bo‘lmay qoldi. Shu bois ratsional sonlar va keyinroq irratsional sonlar fanga kiritildi. Mana shu barcha sonlarni o‘z ichiga olgan to‘plamni umumiy nom bilan **haqiqiy sonlar to‘plami** deyiladi va ***R*** bilan belgilanadi:

$$N\subset Z\subset R.$$

Fan va texnikaning rivojlanishi bilan turli xil tadqiqotlarda, jumladan tovush to‘lqinlarini tadqiq qilishda haqiqiy sonlar to‘plami uning xususiyatlarini to‘liq ochib bera olmaydi. Shu sababli ham bunday sohalarda faraziy sonlar bo‘lgan kompleks ***C*** sonlarga ehtiyoj sezildi:

$$N\subset Z\subset R\subset C.$$

z **kompleks son** deb, $z = x + iy$ ko‘rinishdagi ifodaga aytiladi, bunda x va y - haqiqiy sonlar, i esa $i^2 = -1$ tenglik bilan aniqlanuvchi mavhum birlik. x va y ni z kompleks sonning **haqiqiy** (inglizcha “real” so‘zining bosh harflari “Re”) va **mavhum** (inglizcha “imajinary” so‘zining bosh harflari) **qismlari** deyiladi va bunday belgilanadi:

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y.$$

Agar $x = 0$ bo‘lsa, u holda $z = 0 + iy = iy$ mavhum son, agar $y = 0$ bo‘lsa, u holda $z = x + i \cdot 0 = x$ haqiqiy son hosil bo‘ladi. Bundan kelib chiqadiki, haqiqiy va mavhum sonlar z kompleks sonning xususiy hollaridir.

Agar ikkita $z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ kompleks sonlarning haqiqiy va mavhum qismlari o‘zaro teng bo‘lsa, ularga **teng kompleks sonlar** deyiladi, ya’ni $\operatorname{Re} z_1 = \operatorname{Re} z_2, \quad \operatorname{Im} z_1 = \operatorname{Im} z_2 \quad \Rightarrow \quad z_1 = z_2.$

Agar $z = x + iy$ kompleks sonning haqiqiy va mavhum qismlari nolga teng bo‘lsa, u nolga teng kompleks son bo‘ladi va aksincha.

Faqat mavhum qismlarining ishorasi bilan farq qiluvchi kompleks sonlarga **qo‘shma kompleks sonlar** deyiladi:

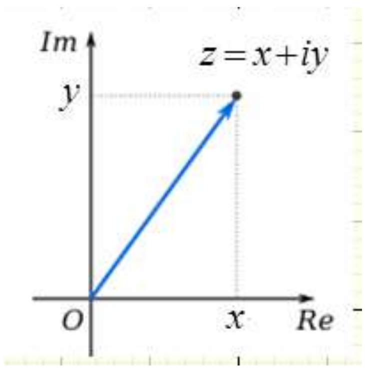
$$z = x + iy \quad \text{va} \quad \bar{z} = x - iy.$$

Ham haqiqiy, ham mavhum qismlarining ishoralari bilan farq qiluvchi kompleks sonlarga **qarama-qarshi kompleks sonlar** deyiladi:

$$z_1 = x + iy \quad \text{va} \quad z_2 = -x - iy.$$

1.1.1. Kompleks sonning geometrik shakli

Har qanday kompleks sonni xOy tekislikda x va y koordinatali $A(x, y)$ nuqta shaklida tasvirlash mumkin va aksincha, tekislikning har bir nuqtasiga bitta kompleks son mos keladi.

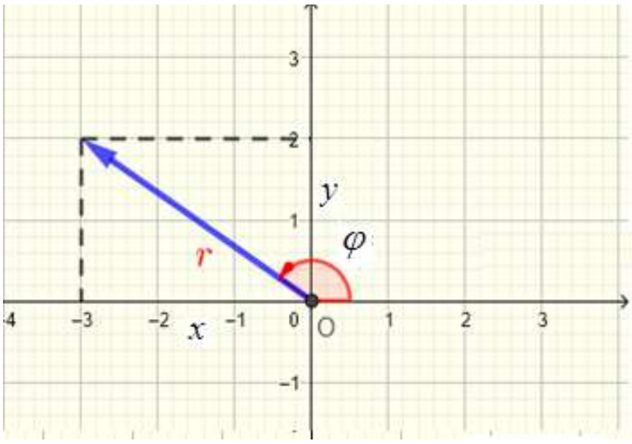


1.1-rasm. Kompleks sonlar tekisligi

Kompleks sonlar tasvirlanadigan tekislik z kompleks o'zgaruvchining tekisligi deyiladi. Kompleks tekislikda z sonni tasvirlovchi nuqtani z nuqta deb ataymiz (1.1-rasm). Ox o'qda yotuvchi nuqtalarga haqiqiy sonlar mos keladi (bunda $y=0$), Oy o'qda yotuvchi nuqtalar sof mavhum sonlarni tasvirlaydi (bunda $x=0$). Shu sababli Ox **haqiqiy o'q**, Oy **mavhum o'q** deyiladi. $A(x,y)$ nuqtani koordinatalar boshi bilan birlashtirib, $\overrightarrow{OA}$ vektorni hosil qilamiz, bu vektorga $z=x+iy$ kompleks sonning **geometrik tasviri** deyiladi.

1.1.1-misol. $z=-3+2i$ kompleks sonni geometrik tasvirlang.

Yechilishi: ► $x=-3$ va $y=2$ larni xOy tekislikda 1.2-rasmda ko'rsatilgandek tasvirlaymiz:



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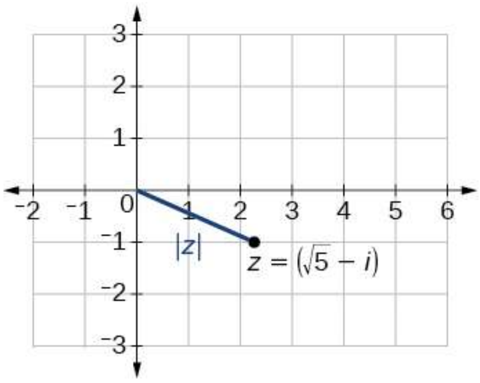
1.2-rasm. $z=-3+2i$ kompleks sonning tasviri

1.1.2-misol. $z=\sqrt{5}-i$ kompleks sonni geometrik tasvirlang.

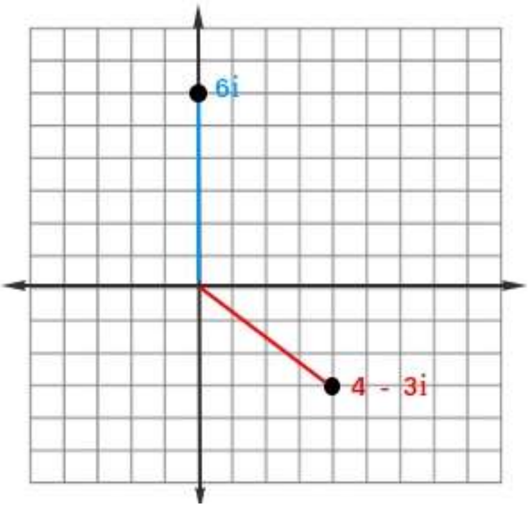
Yechilishi: ► xOy tekislikda $x=\sqrt{5}$ va $y=-1$ larni tasvirlaymiz (1.3-rasm):



1.3-rasm. $z=\sqrt{5}-i$ kompleks sonning tasviri ◄

1.1.3-misol. $z=4-3i$ va $z=6i$ kompleks sonlarni geometrik tasvirlang.

Yechilishi: ► xOy tekislikda $z=4-3i$ komleks son radius vektorni, $z=6i$ esa nuqtani tasvirlaydi (1.4-rasm):



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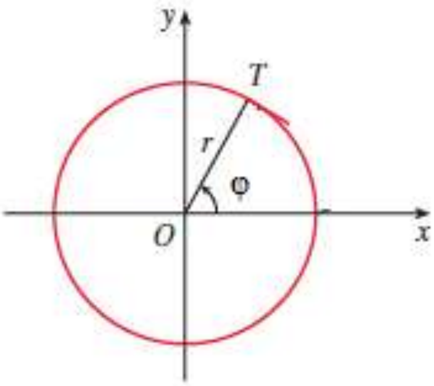
1.4-rasm. $z=4-3i$ va $z=6i$ kompleks sonning tasviri

1.1.2. Kompleks sonning algebraik va trigonometrik shakllari

$z=x+iy$ ifodaga kompleks sonning **algebraik shakli** deyiladi. Endi kompleks sonning trigonometrik shaklini aniqlaymiz.

Koordinatalar boshini – qutb nuqtasi, Ox o'qining musbat yo'nalishini – qutb o'qi deb olib, kompleks tekislikda qutb koordinatalar sistemasini kiritamiz. φ va r larni $T(x,y)$ nuqtaning **qutb koordinatalari** deymiz (1.5-rasm). T nuqtaning qutb radiusi r, ya'ni T nuqtadan qutbgacha bo'lgan masofa z **kompleks sonning moduli** deyiladi va $|z|$ kabi belgilanadi:

$$r=|z|=\sqrt{x^2+y^2} \tag{1.1}$$



1.5-rasm. Qutb koordinatasi bilan kompleks koordinatani bog'lash

T nuqtaning φ qutb burchagini z **kompleks sonning argumenti** deyiladi va $Argz$ kabi belgilanadi. Argument bir qiymatli aniqlanmaydi, uning har bir qiymati $2\pi k$ qo‘shiluvchiga farq qiladi, bunda k–butun son. Argumentning hamma qiymatlari orasidan $0 \le \varphi < 2\pi$ tengsizlikni qanoatlantiruvchi bittasini tanlaymiz. Bu qiymat **bosh qiymat** deyiladi va bunday belgilanadi: $\varphi = Argz$.

Ushbu $\begin{cases} x = r\cdot\cos\varphi, \\ y = r\cdot\sin\varphi \end{cases}$ tengliklarni hisobga olib, z kompleks sonni quyidagicha ifodalash mumkin: $z = x + i\cdot y = r\cdot(\cos\varphi + i\sin\varphi)$, bunda $r = |z| = \sqrt{x^2 + y^2}$ va

$$\varphi = \begin{cases} arctg\,\dfrac{y}{x}, & \text{agar} \quad x > 0,\ y > 0 \ \text{ bo'lsa,} \\ \pi + arctg\,\dfrac{y}{x}, & \text{agar} \quad x < 0 \ \text{ bo'lsa,} \\ 2\pi + arctg\,\dfrac{y}{x}, & \text{agar} \quad x > 0,\ y < 0 \ \text{ bo'lsa} \end{cases} \tag{1.2}$$

formulalar yordamida topiladi.

$$z = r\cdot(\cos\varphi + i\sin\varphi) \tag{1.3}$$

ifodaga kompleks sonning **trigonometrik shakli** deyiladi.

1.1.4-misol. $z = \sqrt{3} - i$ sonni trigonometrik shaklda ifodalang:

Yechilishi: ► Bizga kompleks son algebraik shaklda berilgan. Uni trigonomrtrik shaklga o‘tkazish uchun quyidagi hisob ishlarini bajaramiz:

$$x = \sqrt{3}, \qquad y = -1, \qquad r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2,$$

$$x > 0,\ \ y < 0, \qquad tg\varphi = \frac{y}{x},$$

$$tg\varphi = -\frac{1}{\sqrt{3}}, \qquad \varphi = 2\pi - arctg\frac{1}{\sqrt{3}} = 2\pi - \frac{\pi}{6} = \frac{11}{6}\pi.$$

Topilgan qiymatlarni formulaga qo‘yamiz, natijada ushbu tenglikni hosil qilamiz:

$$z = 2\cdot\left(\cos\frac{11}{6}\pi + i\cdot\sin\frac{11}{6}\pi\right).$$ ◄

1.1.3. Algebraik shakldagi kompleks sonlar ustida amallar

Bizga ikkita algebraik shakldagi kompleks son berilgan bo‘lsin:

$$z_1 = x_1 + iy_1 \text{ va } z_2 = x_2 + iy_2 .$$

I. Qo‘shish. Kompleks sonlarning yig‘indisi deb,

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \quad (1.4)$$

tenglik bilan aniqlanuvchi kompleks songa aytiladi. Bu formuladan vektor ko‘rinishdagi kompleks sonlarni qo‘shish vektorlarni qo‘shish qoidasi bo‘yicha bajarilishi kelib chiqadi. Demak, algebraik shaklda berilgan kompleks sonlarni qo‘shish uchun haqiqiy qismi haqiqiy qismiga, mavhum qismi mavhum qismiga qo‘shilar ekan.

II. Ayirish. Ikkita $z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ kompleks sonlarning ayirmasi deb, shunday songa aytiladiki, u z_2 ga qo‘shilganda yig‘indida z_1 kompleks son hosil bo‘ladi. Demak, algebraik shaklda berilgan kompleks sonlarni ayirish uchun haqiqiy qismi haqiqiy qismidan, mavhum qismi mavhum qismidan ayirilar ekan:

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2) \quad (1.5)$$

Shuni ta’kidlab o‘tamizki, *ikki kompleks son ayirmasining moduli kompleks tekislikda shu sonlarni ifodalovchi nuqtalar orasidagi masofaga teng:*

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad (1.6)$$

1.1.5-misol. $z_1 = 2 + i$ va $z_2 = 2 - 3i$ kompleks sonlarning yig‘indisi va ayirmasini toping.

Yechilishi: ►
$$z_1 + z_2 = (2+i) + (2-3i) = (2+2) + i(1-3) = 4 - 2i$$
$$z_1 - z_2 = (2+i) - (2-3i) = (2-2) + i(1+3) = 4i.$$
◄

III. Ko‘paytirish. Ikkita $z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ kompleks sonning ko‘paytmasi deb, bu sonlarni ikkihad sifatida algebraik qoidalar bo‘yicha ko‘paytirish va $i^2 = -1$ ekanini hisobga olish natijasida hosil bo‘ladigan kompleks songa aytiladi:

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \quad (1.7)$$

1.1.6-misol. Quyidagi kompleks sonlarni ko‘paytiring:

1) $z_1 = \sqrt{3} - 2i$, $z_2 = 7 + 2i\sqrt{3}$;

2) $z_1 = \sqrt{3} - 3i$, $z_2 = 2 + 2\sqrt{3}i$.

Yechilishi: ►

1) $z_1 \cdot z_2 = (\sqrt{3} - 2i) \cdot (7 + 2i\sqrt{3}) = 7\sqrt{3} + 2i \cdot (\sqrt{3})^2 - 14i - 4i^2 \cdot \sqrt{3} = 11\sqrt{3} - 8i$.

2) $z_1 \cdot z_2 = (\sqrt{3} - 3i)(2 + 2\sqrt{3}i) = 2\sqrt{3} + 6i - 6i + 6\sqrt{3} = 8\sqrt{3}$. ◄

IV. Bo'lish. Kompleks sonlarni bo'lish amali ko'paytirishga teskari amal sifatida aniqlanadi. Agar $z \cdot z_2 = z_1$ bo'lsa, z soni $z_1 = x_1 + iy_1$ ning $z_2 = x_2 + iy_2$ kompleks soniga bo'linmasi (ya'ni $z = \frac{z_1}{z_2}$) deyiladi. $z_1 = z \cdot z_2$ tenglikning ikkala qismini $z_2 = x_2 + iy_2$ ga qo'shma bo'lgan $\bar{z}_2 = x_2 - iy_2$ ga ko'paytiramiz: $z_1 \cdot \overline{z_2} = z(z_2 \cdot \overline{z_2})$, bundan quyidagini hosil qilamiz:

$$z = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \cdot \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \qquad (1.8)$$

Demak, z_1 ni z_2 ga bo'lish uchun bo'linuvchi va bo'luvchini bo'luvchiga qo'shma bo'lgan kompleks songa ko'paytirish kerak.

1.1.7-misol. $z_1 = 1 - i$ ni $z_2 = -2 - 2i$ ga bo'ling.

Yechilishi: ►

$$\frac{z_1}{z_2} = \frac{1-i}{-2-2i} = \frac{(1-i)\cdot(-2+2i)}{(-2-2i)\cdot(-2+2i)} = \frac{(-2+2)+i\cdot(2+2)}{4+4} = \frac{4i}{8} = \frac{1}{2}i. \quad ◄$$

V. Darajaga ko'tarish. Mavhum birlik i ning natural darajasi uchun formula topishga harakat qilamiz:

$$i^1 = i,$$
$$i^2 = -1,$$
$$i^3 = i^2 \cdot i = -i,$$
$$i^4 = i^2 \cdot i^2 = 1,$$
$$i^5 = i^4 \cdot i = i.$$

Bulardan umumiy formula hosil qilamiz:

$$i^{4k} = 1, \qquad i^{4k+1} = i, \qquad i^{4k+2} = -1, \qquad i^{4k+3} = -i. \qquad (1.9)$$

1.1.8-misol. $(1+i)^{10}$ ni hisoblang.

Yechilishi: ►

$$(1+i)^{10} = ((1+i)^2)^5 = (1+2i+i^2)^5 = (2i)^5 = 32i^5 = 32i. \quad ◄$$

1.1.4. Trigonometrik shakldagi kompleks sonlar ustida amallar

I. Ko‘paytirish. Bizga z_1 va z_2 kompleks sonlar trigonometrik shaklda berilgan bo‘lsin:

$$z_1 = r_1 \cdot (\cos\varphi_1 + i\sin\varphi_1) \quad \text{va} \quad z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2).$$

Ularning ko‘paytmasini hisoblaymiz:

$$z_1 \cdot z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1)\cdot r_2(\cos\varphi_2 + i\sin\varphi_2) =$$

$$= r_1 \cdot r_2[(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)] =$$

$$= r_1 \cdot r_2\left[(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right].$$

Demak, ikkita kompleks son ko‘paytirilganda ularning modullari ko‘paytiriladi, argumentlari esa qo‘shiladi:

$$z_1 \cdot z_2 = r_1 \cdot r_2\left[(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right] \qquad (1.10)$$

1.1.9-misol. $z_1 = \sqrt{3} - 3i$, $z_2 = 2 + 2\sqrt{3}i$ kompleks sonlarni algebraik shakldan trigonometrik shaklga o‘tkazing va ko‘paytiring.

Yechilishi: ▶ Dastlab kompleks sonlarni (1.1), (1.2) va (1.3) formulalar yordamida trigonometrik shaklga o‘tkazamiz:

$$z_1 = \sqrt{3} - 3i$$

$$x = \sqrt{3}, \quad y = -3, \quad r = \sqrt{x^2 + y^2} = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$$

$$tg\varphi = \frac{y}{x} = \frac{-3}{\sqrt{3}} = -\sqrt{3}, \qquad \varphi = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

Shunda: $z_1 = \sqrt{3} - 3i = 2\sqrt{3}\cdot(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi)$ hosil bo‘ladi,

$$z_2 = 2 + 2i\sqrt{3}$$

$$x = 2, \quad y = 2\sqrt{3}, \quad r = \sqrt{x^2+y^2} = \sqrt{4+12} = 4$$

$$tg\varphi = \frac{y}{x} = \frac{2\sqrt{3}}{2} = \sqrt{3}, \qquad \varphi = \frac{\pi}{3}$$

Shunday qilib, $z_2 = 2 + 2\sqrt{3}i = 4\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ ni topdik.

Endi hosil qilingan trigonometrik shakldagi kompleks sonlarni o‘zaro ko‘paytiramiz:

$$z_1 \cdot z_2 = 2\sqrt{3}\cdot\left(\cos\frac{5}{3}\pi + i\cdot\sin\frac{5}{3}\pi\right)\cdot 4\cdot\left(\cos\frac{\pi}{3} + i\cdot\sin\frac{\pi}{3}\right) =$$

$$= 8\sqrt{3}\cdot\left(\left(\cos(\frac{5}{3}\pi + \frac{\pi}{3}) + i\sin(\frac{5}{3}\pi + \frac{\pi}{3})\right)\right) = 8\sqrt{3}\cdot(\cos 2\pi + i\sin 2\pi) = \blacktriangleleft$$

$$= 8\sqrt{3}\cdot(1 + i\cdot 0) = 8\sqrt{3}.$$

II. Bo‘lish. Agar trigonometrik shakldagi $z_1 = r_1 \cdot (\cos\varphi_1 + i\sin\varphi_1)$ va $z_2 = r_2 \cdot (\cos\varphi_2 + i\sin\varphi_2)$ kompleks sonlar berilgan bo‘lsa, ularning birinchisi ikkinchisiga nisbatini quyidagicha yozamiz:

$$\frac{z_1}{z_2} = \frac{r_1 \cdot (\cos\varphi_1 + i\sin\varphi_1)}{r_2 \cdot (\cos\varphi_2 + i\sin\varphi_2)} = \frac{r_1 \cdot (\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 - i\sin\varphi_2)}{r_2 \cdot (\cos^2\varphi_2 + \sin^2\varphi_2)} =$$

$$= \frac{r_1}{r_2} \cdot \left[(\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2) + i(\sin\varphi_1 \cos\varphi_2 - \cos\varphi_1 \sin\varphi_2)\right] =$$

$$= \frac{r_1}{r_2} \cdot \left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right].$$

Shunday qilib, kompleks sonlarni bo‘lishda bo‘linuvchining moduli bo‘luvchining moduliga bo‘linadi, argumentlari esa ayriladi:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right] \qquad (1.11)$$

III. Darajaga ko‘tarish. Trigonometrik shakldagi (1.3) kompleks sonni darajaga oshirishda kompleks sonlarni ko‘paytirish qoidasidan foydalanamiz.

$n = 2$ darajaga ko‘tarish uchun $z^2 = z \cdot z = r^2(\cos 2\varphi + i\sin 2\varphi)$;

$n = 3$ darajaga ko‘tarish uchun $z^3 = z^2 \cdot z = r^3(\cos 3\varphi + i\sin 3\varphi); \ldots,$

Matematik induksiya tamoyiliga ko‘ra, n darajaga ko‘tarish formulasi

$$z^n = r^n \cdot (\cos n\varphi + i\sin n\varphi) \qquad (1.12)$$

kelib chiqadi. (1.11) formulaga **Muavr formulasi** deyiladi.

1.1.10-misol. $(1+i)^{10}$ ni hisoblang.

Yechilishi:► $(1+i)^{10}$ ni 1.1.8-misolda algebraik shaklda darajaga oshirgan edik. Endi uni trigonometrik shaklda darajaga oshiramiz:

$$z = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

$$z^{10} = (1+i)^{10} = (\sqrt{2})^{10}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} = 32\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) =$$

$$= 32\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = 32\left(\cos\left(2\pi + \frac{\pi}{2}\right) + i\sin\left(2\pi + \frac{\pi}{2}\right)\right) = 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

◄

IV. Ildizdan chiqarish. Bu amal darajaga ko‘tarish amaliga teskari amaldir. Kompleks sonning n darajali ildizi $\sqrt[n]{z}$ deb, shunday W songa aytiladiki, bu sonning n darajasi ildiz ostidagi songa tengdir, ya’ni agar $\sqrt[n]{z}=W$ bo‘lsa, $z=W^n$ o‘rinli.

Agar $z=r\cdot(\cos\varphi+i\sin\varphi)$ va $W=\rho\cdot(\cos\theta+i\sin\theta)$ bo‘lsa, u holda:

$$\sqrt[n]{z}=\sqrt[n]{r\cdot(\cos\varphi+i\sin\varphi)}=\rho\cdot(\cos\theta+i\sin\theta)$$

Muavr formulasiga binoan: $r\cdot(\cos\varphi+i\sin\varphi)=\rho^n\cdot(\cos n\theta+i\sin n\theta)$.

Bundan $\rho^n=r,\ \ n\theta=\varphi+2\pi k$ ekanligini ko‘ramiz. Shunga ko‘ra, ρ va θ ni topamiz: $\rho=\sqrt[n]{r},\quad \theta=\dfrac{\varphi+2\pi k}{n}$. Bunda k - istalgan butun son, $\sqrt[n]{r}$ – arifmetik ildiz. Demak, kompleks sonning n darajali ildizi $\sqrt[n]{z}$ quyidagi formuladan topiladi:

$$\sqrt[n]{z}=\sqrt[n]{r\cdot(\cos\varphi+i\sin\varphi)}=\sqrt[n]{r}\left(\cos\frac{\varphi+2\pi k}{n}+i\sin\frac{\varphi+2\pi k}{n}\right) \tag{1.13}$$

k ga $1,2,3,\dots,n-1$ qiymatlar berib, ildizning n ta har xil qiymatiga ega bo‘lamiz, bu qiymatlarning modullari bir xil. $k>n-1$ da ildizning topilgan qiymatlari bilan bir xil bo‘lgan qiymatlar hosil bo‘ladi. n ta ildizning hammasi markazi koordinatalar boshida bo‘lib, radiusi $\sqrt[n]{r}$ ga teng aylana ichiga chizilgan muntazam n tomonli ko‘pburchak uchlarida yotadi.

1.1.11-misol. $z=\sqrt{3}-i,\ \ n=5,\ \ k=4$ kompleks son uchun z^n va $\sqrt[k]{z}$ ni hisoblang.

Yechilishi: ► $z=\sqrt{3}-i$ kompleks sonni trigonometrik shaklga 1.1.4-misolda keltirganmiz: $z=2\cdot\left(\cos\dfrac{11}{6}\pi+i\cdot\sin\dfrac{11}{6}\pi\right)$.

1) Endi uni $n=5$ darajaga oshirish uchun (1.12) formuladan foydalanamiz:

$$z^5=2^5\cdot\left(\cos\frac{11}{6}\pi+i\cdot\sin\frac{11}{6}\pi\right)^5=32\left(\cos\frac{55}{6}\pi+i\sin\frac{55}{6}\pi\right)=32\left(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi\right)$$

2) Kompleks sonni $k=4$ darajali ildizdan chiqarish uchun (1.13) formuladan foydalanamiz:

$$\sqrt[4]{z}=\sqrt[4]{2}\left(\cos\frac{\frac{11\pi}{6}+2k\pi}{4}+i\sin\frac{\frac{11\pi}{6}+2k\pi}{4}\right)=\sqrt[4]{2}\left(\cos\frac{11\pi+12k\pi}{24}+i\sin\frac{11\pi+12k\pi}{24}\right).$$

4-darajali ildizdan chiqarishda barcha yechimlarni qamrab olish uchun k ga $k=0,1,2,3$ bo‘lgan 4 ta qiymat beramiz:

$$k=0 \quad \text{da} \quad z_1=\sqrt[4]{2}\left(\cos\frac{11\pi+12\cdot 0\pi}{24}+i\sin\frac{11\pi+12\cdot 0\pi}{24}\right)=\sqrt[4]{2}\left(\cos\frac{11\pi}{24}+i\sin\frac{11\pi}{24}\right);$$

$$k=1 \quad \text{da} \quad z_2=\sqrt[4]{2}\left(\cos\frac{11\pi+12\pi}{24}+i\sin\frac{11\pi+12\pi}{24}\right)=\sqrt[4]{2}\left(\cos\frac{23\pi}{24}+i\sin\frac{23\pi}{24}\right);$$

$$k=2 \quad \text{da} \quad z_3=\sqrt[4]{2}\left(\cos\frac{11\pi+12\cdot 2\pi}{24}+i\sin\frac{11\pi+12\cdot 2\pi}{24}\right)=\sqrt[4]{2}\left(\cos\frac{35\pi}{24}+i\sin\frac{35\pi}{24}\right);$$

$$k=3 \quad \text{da} \quad z_4=\sqrt[4]{2}\left(\cos\frac{11\pi+12\cdot 3\pi}{24}+i\sin\frac{11\pi+12\cdot 3\pi}{24}\right)=\sqrt[4]{2}\left(\cos\frac{47\pi}{24}+i\sin\frac{47\pi}{24}\right)=\sqrt[4]{2}\left(\cos\frac{\pi}{24}-i\sin\frac{\pi}{24}\right).$$

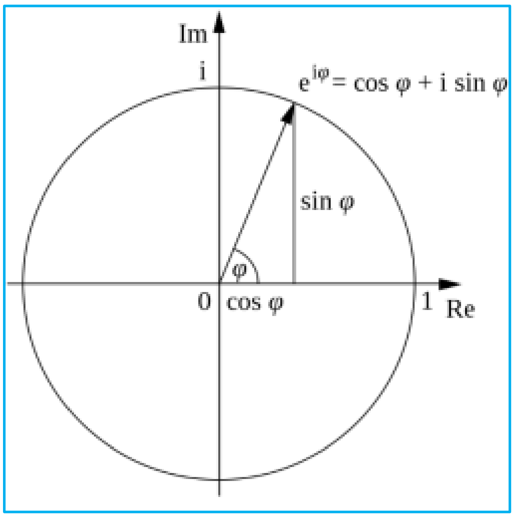
◀

1.1.5. Ko‘rsatkichli shakldagi kompleks sonlar va ular ustida amallar. Eyler formulasi

Kompleks analizda trigonometrik funksiya bilan eksponensial funksiya orasidagi bog‘lanishni ifodalaydigan formulaga **Eyler formulasi** deyiladi va quyidagicha yoziladi:

$$e^{i\varphi}=\cos\varphi+i\sin\varphi \qquad (1.14)$$

bunda $\varphi-$ ixtiyoriy haqiqiy son.



1.6-rasm. Ko‘rsatkichli shakldagi kompleks son tasviri

Kompleks sonni ko‘rsatkichli shaklda ifodalash uchun Eyler formulasini har ikki tomonini r ga ko‘paytiramiz:

$$\cos\varphi+i\sin\varphi=e^{i\varphi} \quad \Rightarrow \quad r\cdot(\cos\varphi+i\sin\varphi)=r\cdot e^{i\varphi}.$$

Oxirgi tenglikda o‘ng tomonda $z=r\cdot(\cos\varphi+i\sin\varphi)$ almashtiramiz bajaramiz. U holda quyidagi tenglik hosil bo‘ladi: $z=r\cdot e^{i\varphi}$.

Shunday qilib, har qanday kompleks sonni **ko‘rsatkichli shaklda** quyidagicha ifodalash mumkin:

$$z = r \cdot e^{i\varphi} \tag{1.15}$$

1.1.12-misol. $1,\ i,\ 1+i,\ -i$ sonlarni ko‘rsatkichli shaklda ifodalang.

Yechilishi:► Algebraik shakldagi kompleks sonlarni ko‘rsatkichli shaklga o‘tkazish uchun (1.1), (1.2) va (1.15) formulalardan foydalanamiz.

1) Agar $z_1 = 1$ bo‘lsa, $r = 1,\ \varphi = 2\pi k$ bo‘ladi, bundan $1 = e^{2k\pi i}$.

2) Agar $z_2 = i$ bo‘lsa, $r = 1,\ \varphi = \frac{\pi}{2}$ bo‘ladi, shunda $i = e^{\frac{\pi}{2}i}$.

3) Agar $z_3 = 1+i$ bo‘lsa, $r = \sqrt{2},\ \varphi = \frac{\pi}{4}$ bo‘ladi, shunda $1+i = \sqrt{2}e^{\frac{\pi}{4}i}$.

4) Agar $z_4 = -i$ bo‘lsa, $r = 1,\ \varphi = \frac{3\pi}{2}$ bo‘ladi, shunda $-i = e^{\frac{3\pi}{2}i}$. ◄

Ko‘rsatkichli shakldagi kompleks sonlar ustida amallar

Bizga 2 ta ko‘rsatkichli shakldagi $z_1 = r_1 \cdot e^{i\varphi_1}$ va $z_2 = r_2 \cdot e^{i\varphi_2}$ kompleks sonlar berilgan bo‘lsin. Ular ustida quyidagi amallarni bajarish qulay:

Ko‘paytirish: $z_1 \cdot z_2 = r_1 \cdot e^{i\varphi_1} \cdot r_2 \cdot e^{i\varphi 2} = r_1 \cdot r_2 \cdot e^{i(\varphi_1+\varphi_2)}$

Darajaga oshirish: $z^n = r^n \cdot e^{in\varphi}$;

Bo‘lish: $\frac{z_1}{z_2} = \frac{r_1 \cdot e^{i\varphi_1}}{r_2 \cdot e^{i\varphi 2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)}$;

Ildiz chiqarish: $\sqrt[n]{z} = \sqrt[n]{re^{i\varphi}} = \sqrt[n]{r}e^{\frac{\varphi+2k\pi}{n}i}$.

Shunday qilib, kompleks sonni 3 xil – algebraik, trigonometrik va ko‘rsatkichli shakllarda yozish mumkin ekan:

$$x + iy = r(\cos\varphi + i\sin\varphi) = r \cdot e^{i\varphi}.$$

1.1.6. Eyler formulasining qo‘llanilishi

Haqiqiy sonni kompleks darajaga oshirish

Siz bilan kompleks sonni haqiqiy darajaga qanday oshirishni o‘rganib chiqdik. Endi haqiqiy sonni kompleks darajaga qanday qilib oshirish mumkin, degan savolga javob izlaymiz. Bizda e sonini kompleks darajaga ko‘taradigan Eyler formulasi bor, keling shu formuladan foydalanamiz. Eyler formulasini quyidagicha yozib olamiz:

$e^{ix} = \cos x + i\sin x$ va uni $x \to x\ln b$, $b > 0$ deb o‘zgartiramiz,

u holda quyidagi tenglikka kelamiz:

$$e^{i(x\ln b)} = \cos(x\ln b) + i\sin(x\ln b) = e^{\ln b^{ix}} = b^{ix}.$$

Shunday qilib, haqiqiy sonni kompleks darajaga oshirish formulasi hosil bo‘ldi:

$$b^{ix} = \cos(x\ln b) + i\sin(x\ln b) \qquad (1.16)$$

Bundan umumiy formulani quyidagicha yozish mumkin:

$$b^{x+iy} = b^x b^{iy} = b^x(\cos(y\ln b) + i\sin(y\ln b)) \qquad (1.17)$$

1.1.13-misol. 5^{3+2i} ni hisoblang.

Yechilishi:▶ 5^{3+2i} ni kompleks darajaga oshirish uchun (1.17) formuladan foydalanamiz:

$$5^{3+2i} = 5^3 \cdot 5^{2i} = 5^3(\cos(2\ln 5) + i\sin(2\ln 5)) \approx -124.63 + 9.65i$$ ◀

Kompleks argumentli trigonometrik funksiyalar

Agar Eyler $e^{ix} = \cos x + i\sin x$ formulasida, $x \to -x$ deb olsak, u holda $e^{-ix} = \cos x - i\sin x$ tenglik hosil bo‘ladi. Bu tengliklarni hadlab ayiramiz:

$$-\begin{cases} e^{ix} = \cos x + i\sin x \\ e^{-ix} = \cos x - i\sin x \end{cases} \Rightarrow e^{ix} - e^{-ix} = (\cos x + i\sin x) - (\cos x - i\sin x)$$

U holda $\sin x$ funksiyaning formulasi hosil bo‘ladi:

$$e^{ix} - e^{-ix} = 2i\sin x \Rightarrow$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad (1.18)$$

Agar bu tengliklarni hadlab qo‘shsak, u holda

$$+\begin{cases} e^{ix} = \cos x + i\sin x \\ e^{-ix} = \cos x - i\sin x \end{cases} \Rightarrow e^{ix} + e^{-ix} = (\cos x + i\sin x) + (\cos x - i\sin x)$$

$\cos x$ funksiyaning formulasini hosil qilamiz:

$$e^{ix} + e^{-ix} = 2\cos x \Rightarrow$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad (1.19)$$

Agar (1.18) tenglikda $x \to ix$ deb o‘zgartirsak-chi, unda quyidagini topamiz:

$$\sin ix = \frac{e^{i^2 x} - e^{-i^2 x}}{2i} = i\frac{e^{i^2 x} - e^{-i^2 x}}{2i^2} = i\frac{e^{-x} - e^{x}}{-2} = i\frac{e^{x} - e^{-x}}{2} = ishx.$$

Shunday qilib, kompleks argumentli sinus funksiya bilan giperbolik sinusning bog‘liqlik formulasini keltirib chiqardik:

$$\sin ix = ishx. \qquad (1.20)$$

Shuningdek, (1.19) tenglikda $x \to ix$ deb o‘zgartirsak, unda quyidagini topamiz:

$$\cos ix = \frac{e^{i^2 x} + e^{-i^2 x}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = chx.$$

Ya’ni kompleks argumentli kosinus funksiya bilan giperbolik kosinusning bog‘liqlik formulasi kelib chiqadi:

$$\cos ix = chx. \qquad (1.21)$$

Agar argumenti to‘liq kompleks sondan iborat bo‘lgan trigonometrik funksiya berilgan bo‘lsa, uni qanday hisoblaymiz?

Ikki burchak yig‘indisining sinusi formulasini maktab kursidan bilamiz:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

Ushbu formulada $b \to bi$ deb, almashtirish bajarsak, u holda

$$\sin(a+bi) = \sin a \cos bi + \cos a \sin bi = \sin a \cdot chb + i\cos a \cdot shb,$$

ya’ni

$$\sin(a+bi) = \sin a \cdot chb + i\cos a \cdot shb \qquad (1.22)$$

tenglikka kelamiz. Xuddi shuningdek, ikki burchak yig‘indisining kosinusi

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

formulasini ham kompleks ko‘rinishda yozish mumkin:

$$\cos(a+bi) = \cos a \cos bi - \sin a \sin bi = \cos a \cdot chb - i\sin a \cdot shb.$$

Kompleks argumentli kosinus funksiya formulasini topamiz:

$$\cos(a+bi) = \cos a \cdot chb - i\sin a \cdot shb \qquad (1.23)$$

1.1.14-misol. $\cos(3+4i)$ ni hisoblang.

Yechilishi: ► Ushbu misolni yechish uchun (1.23) formuladan foydalanamiz:

$$\cos(3+4i) = \cos 3 \cdot ch4 - i\sin 3 \cdot sh4 \approx -27.03 + 3.85i$$ ◄

Mavzu yuzasidan savollar:

1. Kompleks son deb nimaga aytiladi?
2. Qanday kompleks sonlarga teng kompleks sonlar deyiladi?
3. Qarama-qarshi, qo‘shma kompleks son deganda nimani tushunasiz?
4. Kompleks sonning algebraik va trigonometrik shakllari orasida qanday bog‘liqlik mavjud?
5. Algebraik shakldagi kompleks sonlarni qo‘shish, ayirish, ko‘paytirish va bo‘lish qoidalari qanday?
6. Trigonometrik shakldagi kompleks sonlarni ko‘paytirish va bo‘lish formulalarini keltirib chiqaring.
7. Muavr formulasini yozing.
8. Eyler formulasini yozing.
9. Haqiqiy sonni kompleks darajaga oshirish qanday amalga oshiriladi?
10. Kompleks argumentli trigonometrik funksiyalarni qanday hisoblaymiz?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Berilgan ifodalarni hisoblang:

a) $z_1 = 2+3i$, $z_2 = 3-4i$ va $z_3 = 5-2i$ bo‘lsa, $(2z_1 + z_2)z_3$ ni hisoblang.

b) $z_1 = 2+3i$, $z_2 = 3-4i$ va $z_3 = 1-2i$ bo‘lsa, $\dfrac{(z_1+z_3)z_2}{z_3}$ ni hisoblang.

c) $z_1 = 2-i, z_2 = 3+4i, z_3 = 1-3i$ bo‘lsa, $z = \dfrac{z_1(z_3-3z_2)}{z_1^3 + z_2 z_3}$ ni hisoblang

d) $z_1 = 2-3i, z_2 = 2i-5, z_3 = 1-i$ bo‘lsa, $z = \dfrac{(1+iz_1)(z_3-z_2)}{z_1^2 + z_2(2-i+z_3)}$ ni hisoblang.

2. Algebraik shakldagi kompleks sonlarni trigonometrik shaklda ifodalang:

a) $z = -1+i$, b) $z = -4+2i$, c) $z = 1-i\sqrt{2}$,
d) $z = 3i$; i) $z = 2+i$, f) $z = 3-3i$,
j) $z = -4$; k) $z = 1-i\sqrt{3}$, l) $z = 2\sqrt{3}-2i$.

3. Algebraik shakldagi kompleks sonlarni darajaga oshiring:

a) $\left(1-\sqrt{3}i\right)^5$; b) $(1-i)^{45}$; c) $(2+2i)^{15}$;
d) $(3-3i)^{23}$; i) $\left(1+\dfrac{1-\sqrt{2}i}{2+i}\right)^6$.

4. Algebraik shakldagi kompleks sonlarni barcha ildizlarini toping:

a) $\sqrt[12]{2+2i}-\sqrt[12]{i}$; b) $\sqrt[4]{i}$; c) $\sqrt[6]{1}$;

d) $\sqrt[3]{2-2i}$; i) $\sqrt[3]{-64}$; f) $\sqrt[5]{-1}$;

j) $\sqrt[4]{-8-8i\sqrt{3}}$; k) $\sqrt[3]{-2+2i}$.

5. Hisoblang: a) 5^{3-i}; b) 4^{6+3i}; c) 9^{1+2i}.

d) $\cos(1+2i)$; i) $\sin(3+4i)$; f) $\cos(3-i)$.

TESTLAR

1. $z_1 = 3-4i$ va $z_2 = 2-i$ bo‘lsa, $\dfrac{z_1}{z_2}$ ni hisoblang.

A) $3-2i$, **B)** $-2-i$, **C)** $2-i$, **D)** $0{,}4-i$.

2. $z = 2\sqrt{3}-2i$ kompleks sonning trigonometrik shaklini toping.

A) $4(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6})$; **B)** $4(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6})$;

C) $4(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$; **D)** $4(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$.

3. $z = 2\sqrt{3}+2i$ kompleks sonning ko‘rsatkichli shaklini toping.

A) $z = 4e^{i\frac{\pi}{3}}$; **B)** $z = 4e^{i\frac{\pi}{6}}$; **C)** $z = 4e^{i\frac{5\pi}{6}}$; **D)** $z = 4e^{-i\frac{\pi}{3}}$.

4. $\left(\sqrt{3}-i\right)^4$ ni hisoblang.

A) $8+8i\sqrt{3}$, **B)** $8-8i\sqrt{3}$, **C)** $-8+8i\sqrt{3}$, **D)** $-8-8i\sqrt{3}$.

5. $z^3+8=0$ tenglamaning yechimi noto‘g‘ri berilgan javobni aniqlang.

A) $-1+i\sqrt{3}$, **B)** $1+i\sqrt{3}$, **C)** -2, **D)** $1-i\sqrt{3}$.

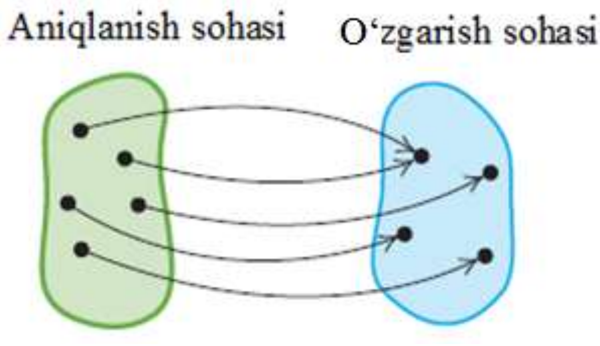
II BOB. DIFFERENSIAL HISOB

2.1-§. Bir o‘zgaruvchili funksiya va uning berilish usullari

Bir o‘zgaruvchili funksiya tushunchasi matematikada juda muhim tushunchalardan biri hisoblanadi. Bir o‘zgaruvchili funksiyani oddiylik uchun bundan buyon funksiya deb ataymiz. Funksiya ikkita to‘plam orasidagi munosabatni aniqlaydigan maxsus ko‘rinishdir. Misol keltiramiz:

1. Telefon klaviaturasidagi har bir belgiga son yozib qo‘yiladi.
2. Do‘kondagi har bir qo‘l telefonining modeliga uning bahosi yoziladi.
3. Har bir haqiqiy songa mos uning kubini yozish mumkin.

Ushbu misollardagi birinchi to‘plamga **aniqlanish sohasi**, ikkinchi to‘plamga **o‘zgarish sohasi** deyiladi.

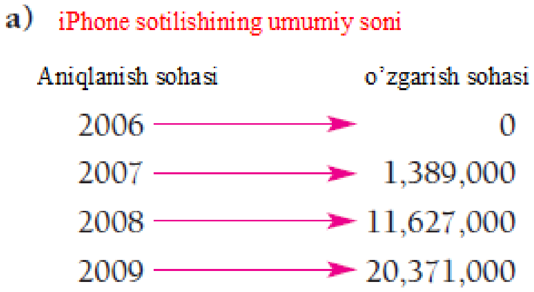


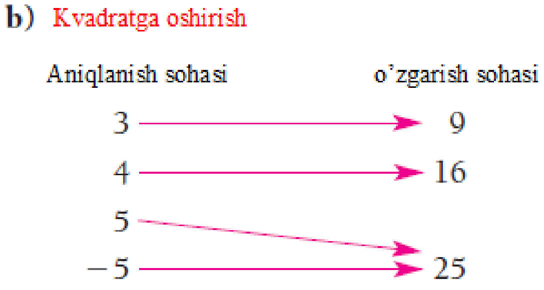
2.1-rasm.To‘plamlarning akslanishi

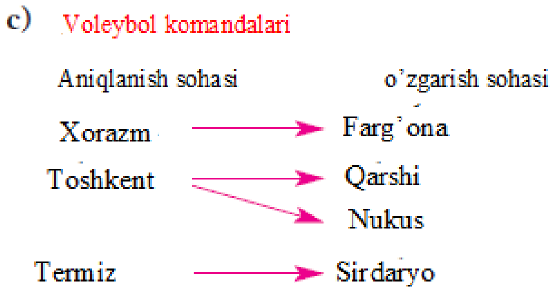
Aniqlanish sohasidan olingan har bir elementga biror qoida asosida o‘zgarish sohasidan faqat bitta element mos qo‘yilgan bo‘lsa, bu moslikka **funksiya** deyiladi.

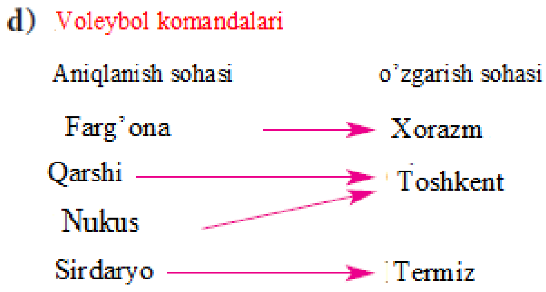
Funksiya 3 xil usulda: analitik, jadval va grafik (diagramma) usullarida beriladi.

2.1.1-misol. Berilgan diagrammalarning qaysi biri funksiya bo‘ladi?









Yechilishi: ►

a) - moslik **funksiya bo‘ladi**, chunki aniqlanish sohasining har bir elementiga o‘zgarish sohasining faqat bitta elementi mos qo‘yilgan.

b) -moslik ham **funksiya.**

c) Voleybol bo‘yicha musobaqa o‘tkazilayotgan bo‘lsin. Ushbu moslik **funksiya emas**, chunki aniqlanish sohasidagi bitta Toshkent komandasiga o‘zgarish sohasidan 2 ta Qarshi va Nukus komandalari mos qo‘yilgan. Bitta komanda bir vaqtning o‘zida ikkita komanda bilan musobaqalasha olmaydi.

d) Funksiya ta’rifiga ko‘ra, aniqlanish sohasining har bir elementiga o‘zgarish sohasining faqat bitta element mos qo‘yilgan bo‘lishi kerak. Ushbu misol **funksiya bo‘ladi**. Bitta komanda navbat bilan ikkita komanda bilan musobaqalashishi mumkin.

Funksiyani sonlar juftligi sifatida qarash mumkin, bunda juftlikning 1-koordinatasi aniqlanish sohasining elementi, 2-koordinatasi esa o‘zgarish sohasiga tegishli bo‘ladi. 2.1.1-misolning b) shartidagi funksiyani f deb belgilasak, u quyidagi ko‘rinishda bo‘ladi:

$$f = \{(-5;25),(3;9),(4;16),(5;25)\}.$$

Aniqlanish sohasi: $D(f) = \{3; 4; 5; -5\}$,

o‘zgarish sohasi: $E(f) = \{9; 16; 25\}$ ◄

2.1.2-misol. Quyidagilarning qaysilari funksiya bo‘la oladi?

	Aniqlanish sohasi	Moslik	O‘zgarish sohasi
a)	Oila	Oila a‘zosining vazni	Musbat sonlar to‘plami
b)	Butun sonlar to‘plami $\{\ldots-3,-2,-1,0,1,2,3,\ldots\}$	Har bir sonning kvadrati	Nomanfiy sonlar to‘plami $\{0,1,4,9,16,25,\ldots\}$

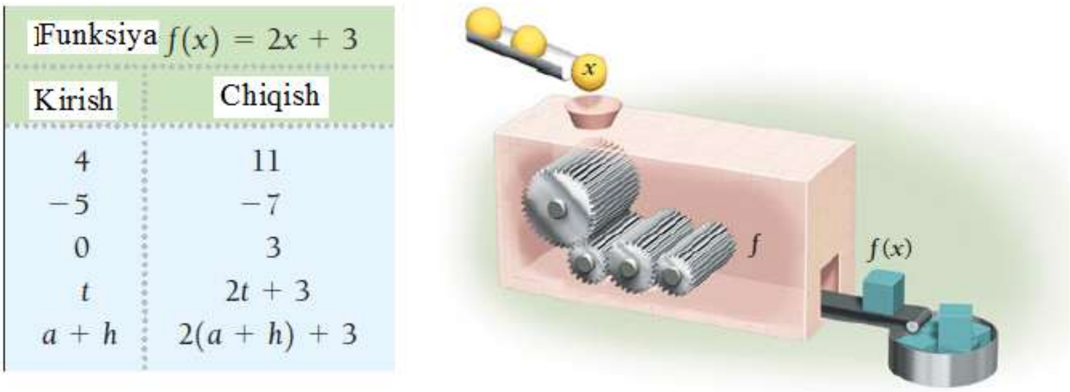
Yechilishi: ►

a) **- funksiya bo‘ladi,** chunki har bir kishiga faqat bitta vazn to‘g‘ri keladi.

b) **- funksiya bo‘ladi,** chunki bitta butun sonning faqat bitta kvadrati mavjud.◄

Funksiya analitik usulda $y=f(x)$ ko‘rinishda beriladi.

Funksiyani mashina deb faraz qilaylik, kirishda x elementni beramiz, chiqishda $f(4)=11$ funksiya qiymati hosil bo‘ladi. Mashina ichida $f(x)=2x+3$ ga mos $2\cdot4+3$ amal hisoblanadi.



2.2-rasm. Funksiyaning kirish va chiqish qiymatlari

2.1.1. Sonli ketma-ketliklar

Natural sonlar to‘plamida aniqlangan funksiya, ya’ni $x=f(n)$, $n\in N$ funksiya **sonli ketma-ketlik** deyiladi.

Agar n ga $1, 2, 3,\ldots, n,\ldots$ qiymatlar bersak, funksiyaning xususiy qiymatlarini olamiz, ular **ketma-ketlikning hadlari** deyiladi:

$$x_1=f(1),\ x_2=f(2),\ \ldots,\ x_n=f(n),\ldots$$

Sonli ketma-ketlik $\{x_n\}$ yoki $\{f(n)\}$ orqali belgilanadi. Ketma-ketlikning $n-$hadi uning **umumiy hadi** deyiladi. Ketma-ketlikning umumiy hadi ma’lum bo‘lsa, ketma-ketlik berilgan hisoblanadi.

2.1.3-misol. $x = \dfrac{n}{n+1}$ funksiyani ketma-ketlik rasmida yozing.

Yechilishi: ► $x = \dfrac{n}{n+1}$ da $n = 1,\ 2,\ 3,\ldots$ natural sonlarni qo‘yib, to‘g‘ri kasrlar ketma-ketligini hosil qilamiz:

$$\{x_n\} = \left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}.$$ ◄

2.1.3-misolda $n \in N$ ketma-ketlik **cheksiz ketma-ketlikdir**, ya’ni uning oxirgi hadi mavjud emas.

Barcha hadlari bir xil qiymat qabul qiladigan $\{x_n\}$ ketma-ketlik **o‘zgarmas ketma-ketlik** deyiladi.

Shunday M son mavjud bo‘lsaki, barcha $n \in N$ uchun $x_n < M$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik **yuqoridan chegaralangan ketma-ketlik** deyiladi.

Shunday $M > 0$ son mavjud bo‘lsaki, istalgan $n \in N$ uchun $x_n > M$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik **quyidan chegaralangan ketma-ketlik** deyiladi.

Ham quyidan, ham yuqoridan chegaralangan $\{x_n\}$ ketma-ketlik **chegaralangan ketma-ketlik** deyiladi. Bu holda shunday $M > 0$ son mavjud bo‘ladiki, istalgan $n \in N$ uchun $|x_n| < M$ tengsizlik bajariladi.

Agar istalgan $n \in N$ uchun $x_n < x_{n+1}$ $(x_n > x_{n+1})$ tengsizlik bajarilsa, $\{x_n\}$ **monoton o‘suvchi (kamayuvchi) ketma-ketlik** deyiladi.

Agar istalgan $n \in N$ uchun $x_n \geq x_{n+1}$ $(x_n \leq x_{n+1})$ tengsizlik bajarilsa, $\{x_n\}$ **o‘smaydigan (kamaymaydigan) ketma-ketlik** deyiladi.

2.1.4-misol. Ketma-ketliklar turlarini aniqlang:

1) $\{x_n\} = \{n\} = \{1,\ 2,\ 3\ ,\ldots, n, \ldots\}$ - o‘suvchi, quyidan chegaralangan;
2) $\{x_n\} = \{1 - 2n\} = \{-1,\ -3,\ -5\ ,\ldots\}$ -kamayuvchi, yuqoridan chegaralangan;
3) $\{x_n\} = \left\{\dfrac{1}{n}\right\} = \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots, \dfrac{1}{n}, \ldots\right\}$ kamayuvchi, yuqoridan chegaralangan ketma-ketlik.

2.1.2. Ketma-ketlikning limiti

Limitni qanday tushuntirish mumkin?

Faraz qiling, ustoz bugungi darsda oʻquvchilarga shunday oʻyin e’lon qildi, agar oʻquvchi 1-boʻlib misolni toʻgʻri bajarsa, bir parta oldinga oʻtib oʻtiradi. Oʻquvchi ketma-ket misollarni hammadan oldin bajaraverdi, u holda oxirgi 5-partada oʻtirgan oʻquvchi boʻlsa, u dastlab 4-partaga, keyin 3-, 2- va 1-partaga oʻtiradi, lekin hech qachon 1-partadan oldinga oʻtmaydi. Aytmoqchimizki, limit – bu oʻquvchi bilan 1-parta orasidagi masofa nolga teng boʻlgandagi holat deyish mumkin (2.3-rasm).

2.3-rasm. Oʻquvchi va 1-parta masalasi

Hisob fani tadqiq qiladigan muhim masala - argument qiymatlari oʻzgarganda, funksiya qiymati qanday oʻzgarishini **baholashdir**. Bunday tadqiqotlarda limit asosiy oʻrin tutadi.

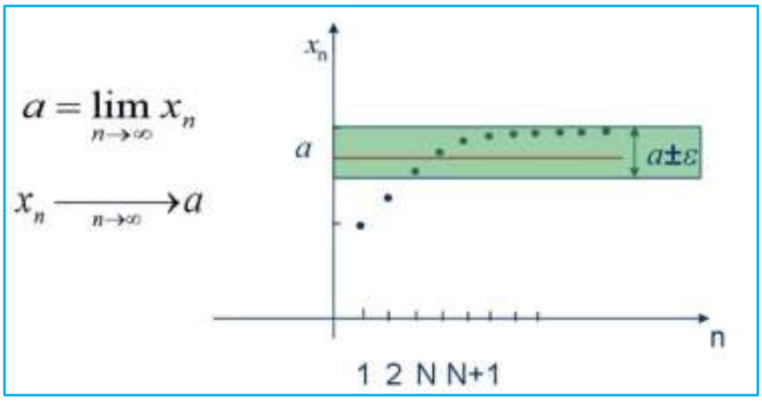
a oʻzgarmas son va $\{x_n\}$ ketma-ketlik berilgan boʻlsin.

Agar istalgan $\varepsilon > 0$ son uchun shunday $N = N(\varepsilon) > 0$ son mavjud boʻlsaki, barcha $n \geq N$ lar uchun $|x_n - a| < \varepsilon$ tengsizlik bajarilsa, a oʻzgarmas son $\{x_n\}$ **ketma-ketlikning limiti** deyiladi va quyidagicha yoziladi:

$$\lim_{n\to\infty} x_n = a \qquad (2.1)$$

Agar $\{x_n\}$ ketma-ketlik chekli limitga ega boʻlsa, u **yaqinlashuvchi ketma-ketlik**, aks holda esa **uzoqlashuvchi ketma-ketlik** deyiladi.

$|x_n - a| < \varepsilon$ tengsizlik $a - \varepsilon < x_n < a + \varepsilon$ tengsizliklarga teng kuchli ekanini bilamiz. Buni hisobga olsak, limit tushunchasini geometrik nuqtai nazardan bunday tushuntirish mumkin (2.4-rasm):

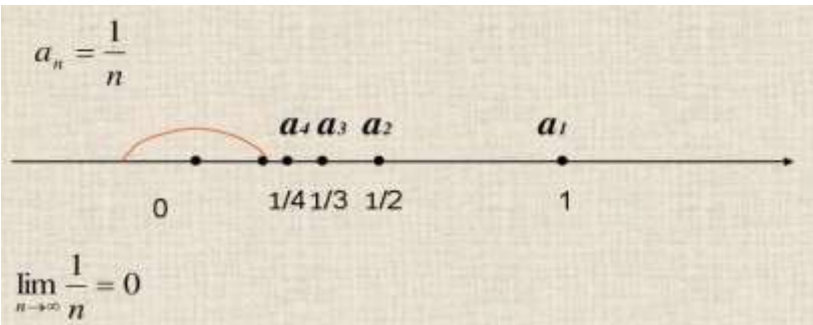


2.4-rasm. Limit tushunchasining geometrik ma’nosi

2.1.5-misol. 0 soni $\{x_n\} = \left\{\frac{1}{n}\right\}$ ketma-ketlikning limiti ekanligini, ya’ni $\lim_{n\to\infty}\frac{1}{n} = 0$ ni ta’rifga ko‘ra isbotlang.

Yechilishi: ► **I usul.** Ixtiyoriy $\varepsilon > 0$ sonni olaylik. $\left|\frac{1}{n} - 0\right| < \varepsilon$ yoki $\left|\frac{1}{n}\right| < \varepsilon$ tengsizlikni tuzamiz. Biroq $n > 0$, shuning uchun $\frac{1}{n} < \varepsilon$ yoki $n < \frac{1}{\varepsilon}$. Bundan ko‘rinadiki, $N = N(\varepsilon)$ sifatida $\frac{1}{\varepsilon}$ dan katta istalgan son, ya’ni $N(\varepsilon) > \frac{1}{\varepsilon}$ olinsa, u holda barcha $n > N(\varepsilon)$ uchun $\left|\frac{1}{n}\right| < \varepsilon$ yoki $\left|\frac{1}{n} - 0\right| < \varepsilon$ tengsizlik bajariladi. Bu esa $\lim_{n\to\infty}\frac{1}{n} = 0$ ekanini bildiradi. Masalan, $\varepsilon = 0{,}01$ bo‘lganda $N(\varepsilon) = 100$ va $n > 100$ uchun $\left|\frac{1}{n}\right| \le 0{,}01$ mos keladi.

II usul. Chizma yordamida ketma-ketlik limiti 0 ga yaqinlashishini ko‘rsatish mumkin (2.5-rasm):



2.5-rasm. $\lim_{n\to\infty}\frac{1}{n} = 0$ limitning geometrik ma’nosi

Yoki jadval yordamida ko‘rsatish mumkin:

n	$\lim\limits_{n\to\infty}\frac{1}{n}$
1	1
2	0.5
3	0.333…
4	0.25
10	0.1
100	0.01
1000	0.001
10000	0.0001
100000	0.00001

◀

2.1.6-misol. $x_n=\frac{1-2n^2}{2+4n^2}$ ketma-ketlik limiti $a=-\frac{1}{2}$ ekanini ta’rifga ko‘ra isbotlang.

Yechilishi: ▶ $\forall\varepsilon>0$ son uchun unga mos $N=N(\varepsilon)>0$ son mavjudligini ko‘rsatamiz: barcha $n\ge N$ larda $|x_n-a|<\varepsilon$ shart bajarilishi kerak.

$$|x_n-a|=\left|\frac{1-2n^2}{2+4n^2}+\frac{1}{2}\right|=\left|\frac{1-2n^2}{2(1+2n^2)}+\frac{1}{2}\right|=\left|\frac{1-2n^2+1+2n^2}{2+4n^2}\right|=\left|\frac{2}{2+4n^2}\right|=\frac{1}{1+2n^2},$$

berilgan tengsizlik quyidagi ko‘rinishga keladi: $\frac{1}{1+2n^2}<\varepsilon.$

Tengsizlikdan n ni topib olamiz: $n>\sqrt{\frac{1}{2\varepsilon}-\frac{1}{2}}.$

Demak, limit ta’rifidagi $N=N(\varepsilon)>0$ sifatida $N(\varepsilon)=\left[\sqrt{\frac{1}{2\varepsilon}-\frac{1}{2}}\right]+1$ olinsa, $\left|x_n-\left(-\frac{1}{2}\right)\right|<\varepsilon$ bo‘ladi. Bu esa $\lim\limits_{n\to\infty}\frac{1-2n^2}{2+4n^2}=-\frac{1}{2}$ bo‘lishini bildiradi.◀

Yaqinlashuvchi ketma-ketlikning xossalari:

1⁰. $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo‘lsa, u chegaralangan bo‘ladi.

2⁰. Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty}x_n=a$ bo‘lib, $a>p$ $(a<q)$ bo‘lsa, u holda $\exists n_0\in N$ topiladiki, $\forall n>n_0$ bo‘lganda $x_n>p$ $(x_n<q)$ bo‘ladi.

3⁰. Agar $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lib,

1) $\lim\limits_{n\to\infty}x_n=a$, $\lim\limits_{n\to\infty}y_n=b$;

2) $\forall n \in N$ uchun $x_n \le y_n$ $(x_n \ge y_n)$ bo‘lsa, u holda $a \le b$ $(a \ge b)$ bo‘ladi.

4⁰. Agar $\{x_n\}$ va $\{z_n\}$ ketma-ketliklar yaqinlashuvchi bo‘lib,

1) $\lim\limits_{n\to\infty} x_n = a,$ $\lim\limits_{n\to\infty} z_n = a$

2) $\forall n \in N$ uchun $x_n \le y_n \le z_n$ bo‘lsa, u holda $\{y_n\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty} y_n = a$ bo‘ladi.

2.1.7-misol. Ushbu $\lim\limits_{n\to\infty} \dfrac{n^3-81}{3n^3+4n^2+2}$ limitni hisoblang.

Yechilishi: ► Agar $n \to \infty$ bo‘lsa hamda limit ostidagi ifoda kasr ratsional ko‘rinishda bo‘lsa, bunday limitni hisoblash uchun kasrning surat va maxrajini n ning eng katta darajasiga bo‘lib chiqish kerak:

$$\lim_{n\to\infty} \frac{n^3-81}{3n^3+4n^2+2} = \lim_{n\to\infty} \frac{\frac{n^3}{n^3}-\frac{81}{n^3}}{\frac{3n^3}{n^3}+\frac{4n^2}{n^3}+\frac{2}{n^3}} = \frac{1-0}{3+0+0} = \frac{1}{3}. \quad ◄$$

Yaqinlashuvchi ketma-ketliklar ustida amallar:

Faraz qilaylik, $\{x_n\}$ hamda $\{y_n\}$ ketma-ketliklar berilgan bo‘lsin: Quyidagi

$$x_1+y_1,\ \ x_2+y_2,\ \ x_3+y_3,\ \dots,\ \ x_n+y_n,\dots$$

$$x_1-y_1,\ \ x_2-y_2,\ \ x_3-y_3,\ \dots,\ \ x_n-y_n,\dots$$

$$x_1\cdot y_1,\ \ x_2\cdot y_2,\ \ x_3\cdot y_3,\dots,x_n\cdot y_n,\dots$$

$$\frac{x_1}{y_1},\ \frac{x_2}{y_2},\ \frac{x_3}{y_3},\ \dots,\ \frac{x_n}{y_n},\dots \qquad (y_n \ne 0,\ \ n=1,2,3,\ \dots)$$

ketma-ketliklar mos ravishda $\{x_n\}$ va $\{y_n\}$ ketma-ketliklarning **yig‘indisi, ayirmasi, ko‘paytmasi hamda nisbati** deyiladi va ular

$$\{x_n+y_n\},\ \ \{x_n-y_n\},\ \ \{x_n\cdot y_n\},\ \ \left\{\frac{x_n}{y_n}\right\}$$

kabi belgilanadi.

5⁰. Aytaylik $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar berilgan bo‘lib,

$$\lim_{n\to\infty} x_n = a,\qquad \lim_{n\to\infty} y_n = b,\qquad (a\in R,\quad b\in R)$$

bo‘lsin. U holda $n\to\infty$ da

$$(c\cdot x_n)\to c\cdot a;\qquad x_n+y_n\to a+b;\qquad x_n\cdot y_n\to ab;\qquad \frac{x_n}{y_n}\to\frac{a}{b}\ \ (b=0)$$

o‘rinli bo‘ladi, ya’ni

a) $\forall c\in R$ da $\lim\limits_{n\to\infty}(c\cdot x_n) = c\cdot\lim\limits_{n\to\infty} x_n;$

b) $\lim\limits_{n\to\infty}(x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$;

c) $\lim\limits_{n\to\infty}(x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$;

d) $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}, \quad (b \neq 0)$.

2.1.8-misol. $\lim\limits_{n\to\infty}\dfrac{3+6+9+...+3n}{n^2+4}$ limitni hisoblang.

Yechilishi: ► Limit ostidagi yig‘indi arifmetik progressiyadan iborat, shuning uchun arifmetik progressiyaning yig‘indisini topish $a_1 + a_2 + ... + a_n = \dfrac{a_1 + a_n}{2} \cdot n$ formulasidan foydalanamiz:

$$\lim\limits_{n\to\infty}\frac{3+6+9+...+3n}{n^2+4} = \lim\limits_{n\to\infty}\frac{\frac{3+3n}{2}n}{n^2+4} = \frac{1}{2}\lim\limits_{n\to\infty}\frac{3n+3n^2}{n^2+4} = \frac{3}{2}. \quad ◄$$

2.1.9-misol. Ushbu $\lim\limits_{n\to\infty}\dfrac{(2n+1)!+(2n+2)!}{(2n+3)!-(2n+2)!}$ limitni hisoblang.

Yechilishi: ►

$$\lim\limits_{n\to\infty}\frac{(2n+1)!+(2n+2)!}{(2n+3)!-(2n+2)!} = \lim\limits_{n\to\infty}\frac{(2n+1)![1+2n+2]}{(2n+1)![(2n+2)(2n+3)-(2n+2)]} =$$

$$= \lim\limits_{n\to\infty}\frac{2n+3}{4n^2+10n+6-2n-2} = \lim\limits_{n\to\infty}\frac{2n+3}{4n^2+8n+4} = \lim\limits_{n\to\infty}\frac{0}{4} = 0. \quad ◄$$

Agar istalgan $\varepsilon > 0$ son uchun shunday $N = N(\varepsilon) > 0$ natural son topilib, $n \geq N(\varepsilon)$ bo‘ladigan barcha n natural sonlar uchun $|\alpha_n| < \varepsilon$ tengsizlik bajarilsa, u holda $\{\alpha_n\}$ sonli ketma-ketlik **cheksiz kichik sonli ketma-ketlik** deyiladi.

Cheksiz kichik sonli ketma-ketliklar quyidagi xossalarga ega:

1⁰. Cheksiz kichik α_n va β_n sonli ketma-ketliklarning $\alpha_n \pm \beta_n$ algebraik yig‘indisi ham cheksiz kichik sonli ketma-ketlik bo‘ladi.

2⁰. Cheksiz kichik α_n va β_n sonli ketma-ketliklarning $\alpha_n \cdot \beta_n$ ko‘paytmasi ham cheksiz kichik sonli ketma-ketlik bo‘ladi.

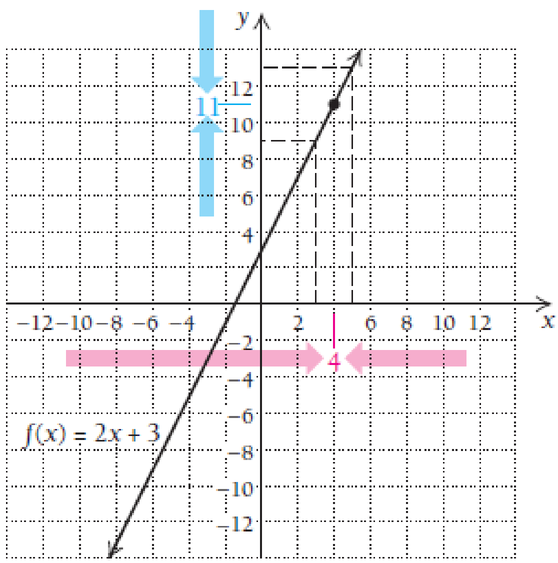
3⁰. Agar α_n cheksiz kichik sonli ketma-ketlik va x_n chegaralangan ketma-ketlik bo‘lsa, u holda $\alpha_n \cdot x_n$ cheksiz kichik sonli ketma-ketlik bo‘ladi.

Agar istalgan $A > 0$ son uchun $\{\gamma_n\}$ sonli ketma-ketlikning shunday bir $N(A)$ tartib raqamini tanlash mumkin bo‘lib, barcha $n > N(A)$ tartib raqamli hadlar uchun $|\gamma_n| > A$ tengsizlik bajarilsa, u holda $\{\gamma_n\}$ sonli ketma-ketlik **cheksiz katta sonli ketma-ketlik** deyiladi.

2.1.3. Funksiyaning nuqtadagi va cheksizlikdagi limiti

Aytaylik, f funksiya berilgan va faraz qilingki, x qiymat biror a soniga asta-sekin yaqinlashib bormoqda. Agar mos ravishda funksiyaning qiymatlari ham qandaydir L soniga yaqinlashib borsa, u holda L soni funksiyaning $x \to a$ nuqtadagi limiti bo‘ladi.

x	$f(x)$
2	7
3.6	10.2
3.9	10.8
3.99	10.98
3.999	10.998
...	...
4.001	11.002
4.01	11.02
4.1	11.2
4.8	12.6
5	13



2.5-rasm. $f(x) = 2x + 3$ funksiyaning nuqtadagi limiti

$f(x) = 2x + 3$ funksiya berilgan bo‘lsin va $x-$qiymat 4 ga asta-sekin yaqinlashib borsin. Jadval va grafikdan ko‘rib turibmizki, agar x qiymat 4 ga chapdan yaqinlashsa, funksiyaning qiymati 11 ga yaqinlashmoqda, agar x qiymat 4 ga o‘ngdan yaqinlashsa, funksiyaning qiymati yana 11 ga yaqinlashmoqda. Shunga ko‘ra, 4 ga ikkala tomondan yaqinlashganda ham funksiyaning qiymati 11 ga teng deyish mumkin (2.5-rasm).

Strelka “$\rightarrow$” belgisi yaqinlashadi degan ma’noni bildiradi: $x \rightarrow 4$ da $f(x) = 2x + 3 \rightarrow 11$. Buni qisqacha $\lim_{x \to 4}(2x+3) = 11$ deb yozamiz. O‘qilishi: x qiymat 4 ga intilganda, $2x+3$ qiymat 11 ga intiladi. ◄

Limitni jadval shaklida ifodalash uning **sonli yondoshuvi**, grafik shakldа ifodalash esa **grafik yondoshuvi** bo‘ladi.

Agar x ning qiymatlari biror a soniga yetarlicha yaqin bo‘lib, lekin teng bo‘lmasa va bunda f funksiyaning barcha qiymatlari L soniga yaqinlashib borsa, u holda L soni x ning a ga intilgandagi **funksiya limiti** deyiladi va quyidagicha belgilanadi:

$$\lim_{x \to a} f(x) = L \quad \text{yoki} \quad f(x) \underset{x \to a}{\rightarrow} L. \tag{2.2}$$

Agar limitni $\lim_{x \to a} f(x) = L$ ko‘rinishida yozsak, x qiymat a soniga ikki tomondan ham yaqinlashishini bildiradi. $\lim_{x \to a-0} f(x)$ yozuv x qiymat a soniga chap tomondan yaqinlashishini, bunda $x < a$ ekanini, $\lim_{x \to a+0} f(x)$ yozuv x qiymat a soniga o‘ng tomondan yaqinlashishini va bunda $x > a$ ekanini bildiradi. Ularni **chap va o‘ng limitlar** deyiladi. Limit mavjud bo‘lishi uchun chap va o‘ng limitlar mavjud va teng bo‘lishi zarur, buni quyidagi teorema isbotlaydi:

2.1-teorema. Agar chap va o‘ng limitlar mavjud hamda L ga teng bo‘lsa, $x \rightarrow a$ da $f(x)$ funksiyaning limiti mavjud va L ga teng bo‘ladi, ya’ni agar $\lim_{x \to a-0} f(x) = \lim_{x \to a+0} f(x) = L$ bo‘lsa, u holda $\lim_{x \to a} f(x) = L$ bo‘ladi.

Teoremaning teskari tasdig‘i ham o‘rinli: agar $\lim_{x \to a} f(x) = L$ mavjud bo‘lsa, u holda $\lim_{x \to a-0} f(x)$ chap va $\lim_{x \to a+0} f(x)$ o‘ng limit mavjud, shuning bilan birga L ga teng.

2.1.10-misol. $f(x) = \dfrac{x^2 - 1}{x - 1}$ funksiya berilgan.

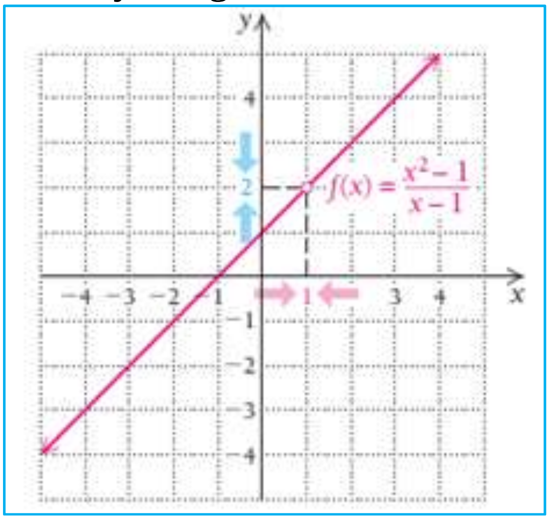
a) Funksiyaning $f(1)$ qiymati nimaga teng?

b) $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ limitini hisoblang.

Yechilishi: ► a) $f(1) = \dfrac{1^2 - 1}{1 - 1} = \dfrac{0}{0}$ kasrning maxrajida 0 hosil bo‘ldi. Shunga ko‘ra, $x = 1$ da funksiyaning qiymati mavjud emas.

b) $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}=\lim\limits_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim\limits_{x\to 1}(x+1)=2$.

Grafikda (1,2) nuqtada teshikcha hosil bo'lgan. Funksiya $x=1$ nuqtada aniqlanmagan, lekin funksiyaning $x\to 1$ da limiti mavjud (2.6-rasm).



2.6-rasm. $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ limit grafigi ◀

Funksiya limitining Koshi ta'rifi. Agar $\forall\varepsilon>0$ son olinganda ham $\exists\delta=\delta(\varepsilon)>0$ topilsaki, $\forall x\in X\bigcap(U_\delta(x_0)\setminus\{x_0\})$ uchun $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, b **soni** $f(x)$ **funksiyaning** x_0 **nuqtadagi limiti** deyiladi va quyidagicha yoziladi:

$$\lim_{x\to x_0} f(x)=b. \quad (2.3)$$

2.1.10-misoldagi $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ limitni Koshi ta'rifiga asosan topamiz: $\forall\varepsilon>0$ soniga ko'ra $\delta=\varepsilon$ deb olsak, u holda $|x-1|<\delta$ $(x\neq 1)$ tengsizlikni qanoatlantiruvchi $\forall x$ da $\left|\frac{x^2-1}{x-1}-2\right|=|x+1-2|=|x-1|<\delta=\varepsilon$ bo'ladi.

Demak, $\lim\limits_{x\to x_0}\frac{x^2-1}{x-1}=2.$

2.1.11-misol. $H(x)=\begin{cases}2x+2, & \text{agar } x<1\\ 2x-4, & \text{agar } x\geq 1\end{cases}$ bo'lakli aniqlangan funksiya uchun quyidagi limitlar mavjud bo'lsa, ularni toping:

a) $\lim\limits_{x\to 1}H(x)$; b) $\lim\limits_{x\to -3}H(x)$

Yechilishi: ▶

a) Chap va o'ng limitlarni hisoblaymiz:

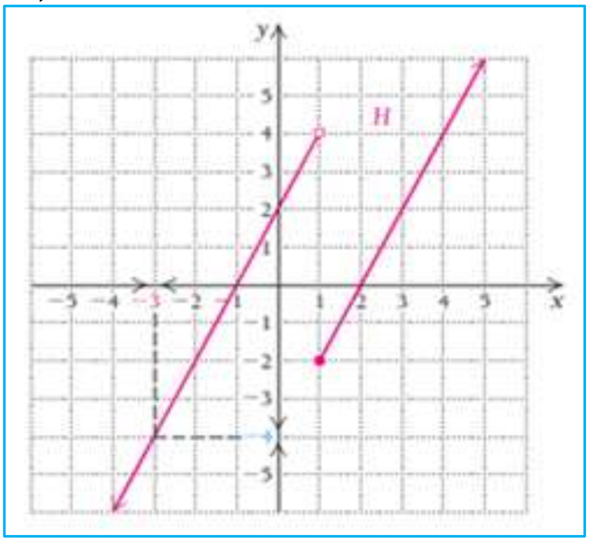
$$\lim_{x\to 1-0}H(x)=4,\quad \lim_{x\to 1+0}H(x)=-2.$$

Bundan kelib chiqadiki, funksiya $x=1$ nuqtada aniqlangan, lekin bu nuqtada $\lim\limits_{x\to 1} H(x)$ limit mavjud emas, chunki chap va o‘ng limitlar turlicha.

b) $x=-3$ da chap va o‘ng limitlarni topamiz:

$$\lim_{x\to -3-0} H(x) = -4 \text{ va } \lim_{x\to -3+0} H(x) = -4.$$

Demak, funksiya $x=-3$ nuqtada aniqlangan va bu nuqtada $\lim\limits_{x\to -3} H(x) = -4$ limit mavjud (2.8-rasm).



2.8-rasm. $H(x) = \begin{cases} 2x+2, & \text{agar } x<1 \\ 2x-4, & \text{agar } x\geq 1 \end{cases}$ funksiya grafigi

Xulosa: Funksiyaning biror nuqtada limiti mavjud bo‘lishi yoki mavjud bo‘lmasligi funksiyaning shu nuqtadagi qiymatiga bog‘liq emas.

Agar $y=f(x)$ funksiya x ning yetarlicha katta qiymatlarida aniqlangan bo‘lib, $\forall\varepsilon>0$ son uchun shunday bir yetarlicha katta $M>0$ son mavjud bo‘lsaki, $|x|>M$ tengsizlikni qanoatlantiradigan barcha x nuqtalar uchun $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, u holda b chekli son $f(x)$ **funksiyaning cheksizlikdagi limiti** deyiladi va quyidagicha belgilanadi:

$$\lim_{x\to\infty} f(x) = b \qquad (2.4)$$

2.2-teorema (Limitga ega funksiyaning chegaralanganligi haqida). Agar $\lim\limits_{x\to a} f(x) = b$ bo‘lib, b chekli son bo‘lsa, u holda $y=f(x)$ funksiya $x=a$ nuqtaning biror atrofida chegaralangandir.

2.3-teorema (Veyershtrass teoremasi).

a) Agar $\{x_n\}$ ketma-ketlik o‘suvchi va yuqoridan chegaralangan bo‘lsa, u yaqinlashuvchi bo‘ladi;

b) Agar $\{x_n\}$ ketma-ketlik kamayuvchi va quyidan chegaralangan bo‘lsa, u yaqinlashuvchi bo‘ladi.

Funksiya limitining xossalari

Ushbu xossalar funksiyaning nuqtadagi limiti uchun ham cheksizlikdagi limiti uchun ham o‘rinli.

Agar $\lim\limits_{x\to a} f(x)=L$ va $\lim\limits_{x\to a} g(x)=M$, c biror o‘zgarmas son bo‘lsa, quyidagilar o‘rinli:

1⁰. O‘zgarmas sonning limiti uning o‘ziga teng: $\lim\limits_{x\to a} c=c$.

2⁰. Darajaning limiti limitning darajasiga teng:

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a} f(x)\right]^n=L^n.$$

Musbat ko‘rsatkichli ildizning limiti limitning shu ko‘rsatkichli ildiziga teng: $\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a} f(x)}=\sqrt[n]{L}$,

agar n juft bo‘lsa, $L\ge 0$ bo‘lishi kerak.

3⁰. Yig‘indi (ayirma)ning limiti limitlar yig‘indisi (ayirmasi)ga teng: $\lim\limits_{x\to a}[f(x)\pm g(x)]=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)=L+M$.

4⁰. Ko‘paytmaning limiti limitlar ko‘paytmasiga teng:

$$\lim_{x\to a}[f(x)\cdot g(x)]=[\lim_{x\to a} f(x)]\cdot[\lim_{x\to a} g(x)]=L\cdot M.$$

5⁰. Bo‘linmaning limiti limitlar bo‘linmasiga teng:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\frac{L}{M},\ \text{bunda}\ M\neq 0.$$

6⁰. O‘zgarmas son ko‘paytmasining limiti limitning o‘zgarmas songa ko‘paytmasiga teng: $\lim\limits_{x\to a} c\cdot f(x)=c\cdot\lim\limits_{x\to a} f(x)=c\cdot L$.

7⁰. (**Oraliq limit haqidagi teorema**). Agar $f(x)\le\varphi(x)\le g(x)$ funksiyalar uchun $\lim\limits_{x\to a} f(x)=L$ va $\lim\limits_{x\to a} g(x)=L$ bo‘lsa, u holda $\lim\limits_{x\to a}\varphi(x)=L$ bo‘ladi.

Funksiya limitini hisoblashga doir misollar keltiramiz:

2.1.12-misol. $\lim\limits_{x\to+\infty} x\left(\sqrt{x^2+4}-\sqrt{x^2-1}\right)$ funksiya limitini hisoblang.

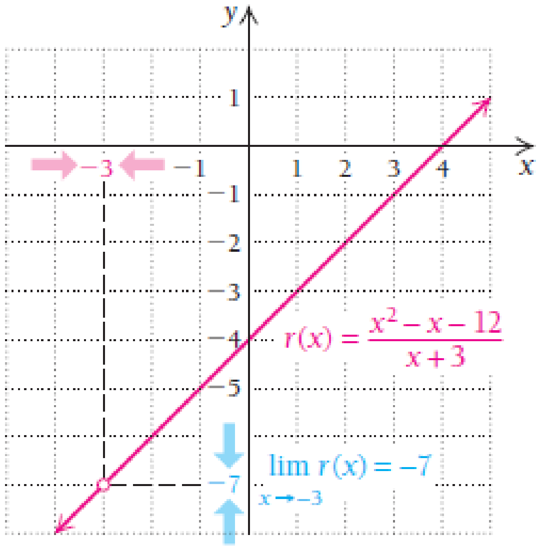
Yechilishi: ►

$$\lim_{x\to+\infty} x\left(\sqrt{x^2+4}-\sqrt{x^2-1}\right)=\lim_{x\to+\infty}\frac{x\left(\sqrt{x^2+4}-\sqrt{x^2-1}\right)\left(\sqrt{x^2+4}+\sqrt{x^2-1}\right)}{\sqrt{x^2+4}+\sqrt{x^2-1}}=$$

$$=\lim_{x\to+\infty}\frac{5x}{\sqrt{x^2+4}+\sqrt{x^2-1}}=\frac{5}{2}.$$ ◄

2.1.13-misol. $r(x)=\dfrac{x^2-x-12}{x+3}$ funksiya berilgan. $\lim\limits_{x\to -3} r(x)$ ni hisoblang.

Yechilishi: ▶ Funksiya $r(-3)$ da aniqlanmagan, chunki x ning o‘rniga -3 qiymatni qo‘ysak, maxraj nolga aylanadi. Bo‘linmaning limiti xossasidan to‘g‘ridan to‘g‘ri foydalanib bo‘lmaydi. Shuning uchun grafikdan foydalanamiz: $x\neq -3$ (2.9-rasm).



2.9-rasm. $\lim\limits_{x\to -3} r(x)$ limit nuqta tasviri

Grafikdan $\lim\limits_{x\to -3}\left(\dfrac{x^2-x-12}{x+3}\right)=-7$ ekanini ko‘rish qiyin emas. Endi algebraik yo‘l bilan, ya’ni xossalardan foydalanib, limitni hisoblaymiz:

$$\lim_{x\to -3}\frac{x^2-x-12}{x+3}=\lim_{x\to -3}\frac{(x+3)(x-4)}{x+3}=\lim_{x\to -3}(x-4)=-7,\quad x\neq -3.◀$$

Grafikdan ko‘rinadiki, (-3, -7) nuqta bo‘yalmagan. $r(-3)$ qiymatda funksiya aniqlanmagan, lekin $x\to 3$ da funksiya limiti mavjud:

$$\lim_{x\to -3}\frac{x^2-x-12}{x+3}=\lim_{x\to -3}\frac{(-3)^2-(-3)-12}{-3+3}=\lim_{x\to -3}\frac{0}{0}$$

bo‘lganligi uchun dastlab kasrni qisqartirib, keyin limitni hisoblash kerak. Bunday “**aniqmaslik**” kasrning surat va mahrajida umumiy bo‘linuvchi borligini bildiradi, bizning misolda bu $x+3$. $\dfrac{0}{0}$ yozuv limit mavjud bo‘lishi mumkinligini bildiradi. Bunday hollarda limitni hisoblash uchun oldin algebraik soddalashtirishlar bajarish yoki jadval va grafiklardan foydalanish maqsadga muvofiq (2.7-§).

2.1.14-misol. $\lim\limits_{x\to 1}\dfrac{x+x^2+x^3+\ldots+x^n-n}{x-1}$ funksiya limitini hisoblang.

Yechilishi: ►

$$\lim_{x\to 1}\frac{x+x^2+x^3+\ldots+x^n-n}{x-1}=\lim_{x\to 1}\frac{(x-1)+(x^2-1)+(x^3-1)+\ldots+(x^n-1)}{x-1}=$$

$$=\lim_{x\to 1}\frac{(x-1)[1+(x+1)+(x^2+x+1)+\ldots+(x^{n-1}+x^{n-2}+x+1)]}{x-1}=$$

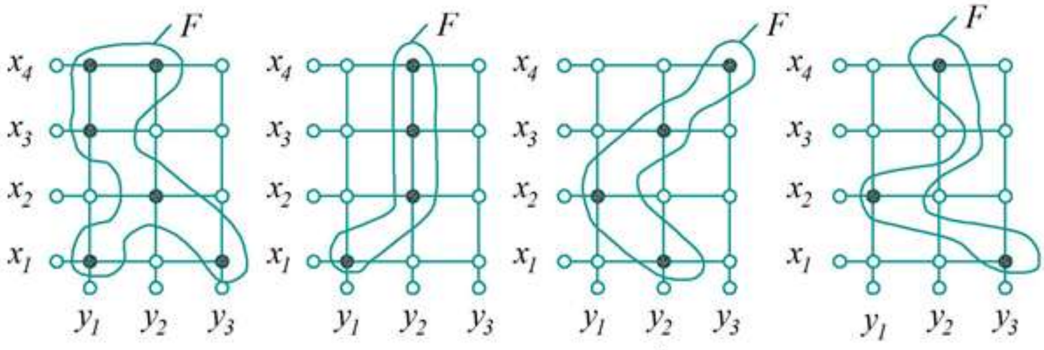
$$=1+2+3+\ldots+n=\frac{n(n+1)}{2}\;. \blacktriangleleft$$

Mavzu yuzasidan savollar:

1. Sonli ketma-ketlikning ta’rifini aytib bering.
2. Qanday ketma-ketliklar yuqoridan (quyidan) chegaralangan deyiladi?
3. Ketma-ketlik limiti ta’rifini ayting.
4. Ketma-ketlik limitining mavjudligi haqidagi teoremani aytib bering.
5. Funksiyaning nuqtadagi limiti deb nimaga aytiladi?
6. Funksiyaning cheksizlikdagi limiti ta’rifini ayting.
7. Funksiya limitining Geyne ta’rifini ayting.
8. Funksiya limiti bilan ketma-ketlik limitini farqini ko‘rsating.
9. Oraliq limit haqidagi teoremani tushuntiring.
10. Funksiya limitining qanday xossalarini bilasiz?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Rasmdagi $A=\{x_1,x_2,x_3,x_4\}$ va $B=\{y_1,y_2,y_3\}$ to‘plamlar mosliklari funksiya bo‘ladimi?



a) b) c) d)

2. Ketma-ketlik limitini ta’rifga ko‘ra isbotlang:

a) $a_n = \dfrac{3n+2}{2n-1}, \quad a = \dfrac{3}{2}.$ b) $a_n = \dfrac{6n-1}{3n+1}, \quad a = 2.$

c) $a_n = \dfrac{3n^2+1}{4n^2+2}, \quad a = \dfrac{3}{4}.$ d) $a_n = \dfrac{9+n^3}{1-2n^3}, \quad a = -\dfrac{1}{2}.$

3. Ketma-ketliklarning limitlarini hisoblang:

a) $\lim\limits_{n\to\infty} \dfrac{2n}{2n^2-1}\cos\dfrac{n+1}{2n-1};$ b) $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+3x-3}-x\right)$

c) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^3+1}-\sqrt{n-1}}{\sqrt[3]{n^3+1}-\sqrt{n-1}}.$ d) $\lim\limits_{n\to\infty}\dfrac{5^n-3}{5^{n+1}+2}$

4. Funksiya limitini hisoblang:

a) $\lim\limits_{x\to2}\dfrac{x^2-5x+6}{x^2-7x+10};$ b) $\lim\limits_{x\to0}\dfrac{2x^3-2x^2}{5x^3-4x^2}.$

5. Chiziqchalar o‘rnini shunday to‘ldiringki, $\lim\limits_{x\to2} f(x)$ limit mavjud bo‘lsin:

a) $f(x)=\begin{cases} -\dfrac{x}{2}+1, & \text{agar } x<2 \\ \dfrac{3}{2}x+\_\_\_, & \text{agar } x>2 \end{cases}$ b) $f(x)=\begin{cases} x^2-9, & \text{agar } x<2 \\ -x^2+\_\_\_\_, & \text{agar } x>2 \end{cases}.$

TESTLAR

1. Hisoblang: $\lim\limits_{x\to\infty}\dfrac{2x^3+x^2-5}{x^3+x-2};$

A) 2 **B)** 1 **C)** 0.5 **D)** 0

2. Hisoblang: $\lim\limits_{n\to\infty}\dfrac{(2n-1)(n-2)(n-3)}{3n^3+2n^2+n}$

A) 2/3 **B)** 3/2 **C)** 2 **D)** 0

3. Hisoblang: $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+3n+1}+n}{2n+3}$

A) 1 **B)** 2 **C)** 1/2 **D)** 0

4. Hisoblang: $\lim\limits_{n\to\infty}\dfrac{1+2+3+...+n}{n^2}$

A) ∞ **B)** 2 **C)** 1/2 **D)** 0

5. Hisoblang: $\lim\limits_{n\to\infty}\dfrac{2^n+3}{2^n-3}$

A) ∞ **B)** 2 **C)** 1 **D)** 0

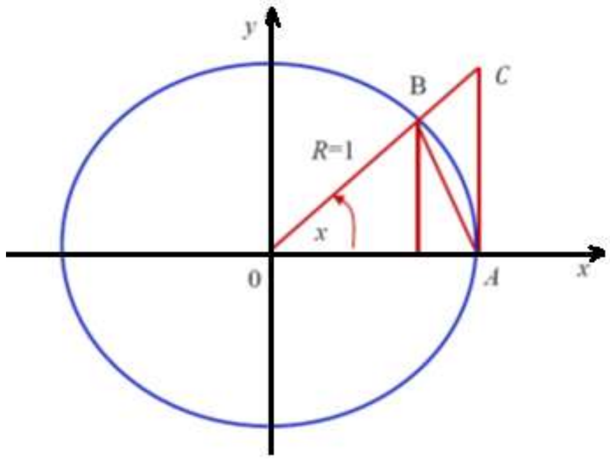
2.2-§. Birinchi va ikkinchi ajoyib limitlar

Quyida **birinchi ajoyib limit** deb ataluvchi muhim limit munosabatni keltirib chiqaramiz:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad (2.5)$$

2.4-teorema. $\frac{\sin x}{x}$ funksiya $x \to 0$ da 1 ga teng limitga ega.

Isboti ► R radiusli aylana olamiz, radianlarda ifodalangan x burchak $0<x<\frac{\pi}{2}$ oraliqda yotadi deb faraz qilaylik (2.10-rasm).



2.10-rasm. Birlik aylana

Rasmdan ko‘rinadiki, $S_{\triangle BOA} < S_{\triangle BOA\, sektor} < S_{\triangle COA}$.

Biroq, $S_{\triangle BOA} = \frac{1}{2}OA \cdot BO \cdot sinx = \frac{R^2}{2} \cdot sinx$

$$S_{\triangle BOA\, sektor} = \frac{1}{2}OA^2 \cdot \overset{\smile}{AB} = \frac{R^2}{2}\cdot x\,,$$

$$S_{\triangle COA} = \frac{1}{2}OA \cdot AC = \frac{1}{2}OA \cdot OA \cdot tgx = \frac{R^2}{2}\cdot tgx$$

Shu sababli tengsizliklar ushbu ko‘rinishni oladi:

$$\frac{R^2}{2}\cdot sinx < \frac{R^2}{2}\cdot x < \frac{R^2}{2}\cdot tgx \quad \text{yoki} \quad sinx < x < tgx.$$

Qo‘sh tengsizlikning barcha hadlarini $sinx > 0$ ga bo‘lamiz, bunda

$$0 < x < \frac{\pi}{2}; \quad 1 < \frac{x}{sinx} < \frac{1}{cosx} \quad \text{yoki} \quad cosx < \frac{sinx}{x} < 1.$$

$\frac{\sin x}{x}$ funksiya bir xil limitga ega bo‘lgan $\lim_{x\to 0}\cos x = 1$ va $\lim_{x\to 0} 1 = 1$ funksiyalar bilan chegaralangan. Oraliq funksiyaning limiti haqidagi teoremaga asosan $\lim_{x\to 0}\frac{\sin x}{x}=1$. ◄

2.2.1-misol. $\lim\limits_{x\to 0}\frac{\sin 3x}{x}$ ni hisoblang.

Yechilishi: ► (2.5) formuladan foydalanamiz:

$$\lim_{x\to 0}\frac{\sin 3x}{x}=\lim_{x\to 0}\frac{3\cdot\sin 3x}{3x}=3\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}=3. \ ◄$$

2.2.2-misol. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$ limitni hisoblang.

Yechilishi: ► $1-\cos x=2\sin^2\frac{x}{2}$ ayniyatni hisobga olib va funksiya limitining xossalaridan foydalanib, hisoblaymiz:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{1}{2}\cdot\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2=$$

$$=\frac{1}{2}\cdot\left[\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]=\frac{1}{2}. \ ◄$$

Birinchi ajoyib limitdan kelib chiqadigan natijalar:

1) $\lim\limits_{x\to 0}\frac{\sin x}{x}=\lim\limits_{x\to 0}\frac{tgx}{x}=\lim\limits_{x\to 0}\frac{shx}{x}=\lim\limits_{x\to 0}\frac{thx}{x}=1;$

2) $\lim\limits_{x\to 0}\frac{\sin\alpha x}{x}=\alpha, (\alpha\in R);$

3) $\lim\limits_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}=\frac{\alpha}{\beta}, \quad (\alpha,\beta\in R);$

4) $\lim\limits_{x\to 0}\frac{\arcsin x}{x}=\lim\limits_{x\to 0}\frac{arctgx}{x}=1;$

5) $\lim\limits_{x\to 0}\frac{x}{\sin x}=1.$

Monoton chegaralangan ketma-ketlikning limiti haqidagi teoremani quyidagi muhim limitga qo‘llaymiz. Bu limitga **ikkinchi ajoyib limit** deyiladi:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e \qquad (2.6)$$

$\{x_n\}=\left\{\left(1+\frac{1}{n}\right)^n\right\}$ sonli ketma-ketlikni qaraymiz, bunda $n\in N$.

2.5-teorema. Umumiy hadi $x_n = \left(1+\frac{1}{n}\right)^n$ bo‘lgan ketma-ketlik $n \to \infty$ da 2 va 3 orasida yotadigan limitga ega.

Isboti: ► Umumiy hadi $x_n = \left(1+\frac{1}{n}\right)^n$ bo‘lgan $\{x_n\}$ ketma-ketlikni qaraymiz. Uni yaqinlashuvchi ekanligini isbotlaymiz. Buning uchun $\{x_n\}$ ketma-ketlikning o‘suvchi va yuqoridan chegaralanganligini ko‘rsatish yetarli (2.3-teorema). Nyuton binomi formulasini qo‘llab, quyidagiga ega bo‘lamiz:

$$x_n = \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots + \frac{n(n-1)(n-2)\ldots[n-(n-1)]}{n!}\cdot\frac{1}{n^n}.$$

Bu ifodani shakl almashtirib, quyidagicha tasavvur qilishimiz mumkin:

$$x_n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right).$$

Hosil bo‘lgan ifodaning qiymati $\approx 2{,}718281828$ ga teng bo‘ladi. Bu sonni buyuk matematik L.Eyler sharafiga e harfi bilan belgilangan. Qatorlar mavzusida biz bu sonni $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$ qatorga yoyish mumkinligini isbotlaymiz.

Shunday qilib, $2 < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots < 3$, ya’ni e soni 2 va 3 sonlari orasida yotuvchi sondir.◄

2.6-teorema. $\left(1+\frac{1}{x}\right)^x$ funksiya $x \to \infty$ da e soniga teng limitga ega:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{2.7}$$

2.2.3-misol. $\lim_{n\to\infty}\left(1-\frac{7}{n}\right)^n$ limitni hisoblang.

Yechilishi: ► (2.6) formuladan foydalanamiz:

$$\lim_{n\to\infty}\left(1-\frac{7}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{-\frac{n}{7}}\right)^{-\frac{n}{7}\cdot(-7)} = e^{-7}.$$ ◄

Ikkinchi ajoyib limitdan kelib chiqadigan natijalar:

1) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim\limits_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e\approx 2{,}71828;$

2) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1;$

3) $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\ln a,\ a>0,\ a\neq 1$

4) $\lim\limits_{x\to 0}\frac{(1+x)^m-1}{x}=m,\ \ m\in R.$

2.2.4-misol. $\lim\limits_{x\to\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^x$ limitni hisoblang.

Yechilishi: ► Berilgan limitni shakl almashtirishlar bajarib, (2.7) ko‘rinishga keltiramiz.

$$\lim_{x\to\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^x=\lim_{x\to\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^x=\lim_{x\to\infty}\left[\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}}\right]^{\frac{x(8x-3)}{x^2-3x+7}}=$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}}\right]^{\frac{8-3/x}{1-3/x+7/x^2}}=e^8$$

◄

Mavzu yuzasidan savollar:

1. Birlik aylanaga ta’rif bering.
2. Uchburchak yuzini topish uchun qanday formulalarni bilasiz?
3. Sector yuzi nimaga teng?
4. Birinchi ajoyib limit deb nimaga aytiladi? Uni isbotlang.
5. 1-ajoyib limitdan qanday natijalar kelib chiqadi?
6. Ikkinchi ajoyib limit deb nimaga aytiladi? Uni isbotlang.
7. 2-ajoyib limitdan kelib chiqadigan natijalarni ayting.
8. $e\approx 2{,}718281828$ soni keltirilganlardan boshqa yana qayerda qo‘llanilgan?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Birinchi ajoyib limitlarni hisoblang:

a) $\lim\limits_{x\to 0}\dfrac{3x\cos 5x}{\sin 7x}$. b) $\lim\limits_{x\to 0}\dfrac{\sin 3x\cdot tg\,4x}{2x^2}$.

2. Ikkinchi ajoyib limitlarni hisoblang:

a) $\lim\limits_{x\to\infty}\left(\dfrac{9x}{9x-7}\right)^{3x}$. b) $\lim\limits_{x\to+\infty}(1-3x)\left[\ln(x+3)-\ln x\right]$.

3. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+9x-6}{x^2+4x+1}\right)^{2x+1}$ limitni hisoblang.

4. $\lim\limits_{x\to 0}\dfrac{\arcsin 2(x-1)}{x^2-1}$ limitni hisoblang.

5. $\lim\limits_{x\to 0}\dfrac{\arcsin(x+2)}{arctg\,3x}$ limitni hisoblang.

TESTLAR

1. Hisoblang: $\lim\limits_{x\to 0}\dfrac{tg\,2x}{\sin 4x}$

A) 4/2 **B)** 2/4 **C)** 1 **D)** 0

2. Hisoblang: $\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{arctg^2\,6x}$

A) 1/36 **B)** 1/4 **C)** 1 **D)** 4

3. $\lim\limits_{x\to 0}\left(1+\dfrac{x}{3}\right)^{\frac{1}{x}}$ hisoblang. **A)** $e^{1/3}$ **B)** e^3 **C)** $e^{1/2}$ **D)** 0

4. Hisoblang: $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-2}\right)^{x}$

A) $e^{1/3}$ **B)** e^5 **C)** $e^{1/5}$ **D)** $e^{1/2}$

5. Hisoblang: $\lim\limits_{x\to+\infty} x[\ln(x+1)-\ln x]$

A) 2 **B)** ∞ **C)** 1 **D)** 0

2.3-§. Cheksiz katta va cheksiz kichik funksiyalar

Limitlar bizga ba'zi funksiyalarga nisbatan cheksizlikning rolini tushunishimizga yordam beradi.

Agar $y = f(x)$ funksiya $x = a$ nuqtaning biror atrofida aniqlangan va istalgan $M > 0$ son uchun shunday $\delta > 0$ son mavjud bo'lsaki, $|x - a| < \delta$ tengsizlikni qanoatlantiradigan barcha $x \neq a$ nuqtalar uchun $|f(x)| > M$ tengsizlik bajarilsa, $x \to a$ da **funksiya cheksizlikka intiladi** deyiladi va quyidagicha yoziladi:

$$\lim_{x\to\infty} f(x) = \infty.$$

Agar $\lim_{x\to a} f(x) = \infty$ $\left(\lim_{x\to\infty} f(x) = \infty\right)$ bo'lsa, u holda $f(x)$ funksiya $x \to a$ da (yoki $x \to \infty$ da) **cheksiz katta funksiya** deyiladi.

2.3.1-misol. $\gamma(x) = \left\{\frac{x^2+1}{5x}\right\}$ va $\eta(x) = \left\{\frac{1-3x^3}{x^2+7}\right\}$ lar cheksiz katta funksiyalar ekanligini ko'rsating.

Yechilishi: ► $\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty}\left\{\frac{n^2+1}{5n}\right\} = \infty$, $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty}\left\{\frac{1-3n^3}{n^2+7}\right\} = \infty$. ◄

Bu ta'rifdan ko'rinadiki, agar $f(x)$ funksiya cheksiz katta funksiya bo'lsa, u holda istalgan $M > 0$ uchun $\exists\delta > 0$ topiladiki, $|x - a| < \delta$ tengsizlikni qanoatlantiradigan barcha x lar uchun $|f(x)| > M$ tengsizlik bajariladi. Bundan **cheksiz katta funksiya chegaralanmagan funksiya** ekani kelib chiqadi.

Agar $\lim_{x\to a} f(x) = 0$ $\left(\text{yoki } \lim_{x\to\infty} f(x) = 0\right)$ bo'lsa, $f(x)$ funksiya $x \to a$ da (yoki $x \to \infty$ da) **cheksiz kichik funksiya** deyiladi.

2.3.2-misol. $\alpha(x) = \left\{\frac{1}{5x}\right\}$ va $\beta(x) = \left\{\frac{3x-1}{x^2+2}\right\}$ lar cheksiz kichik funksiyalar ekanligini ko'rsating.

Yechilishi: ► $\lim_{x\to\infty} \alpha(x) = \lim_{n\to\infty}\left\{\frac{1}{5x}\right\} = 0$, $\lim_{x\to\infty} \beta(x) = \lim_{n\to\infty}\left\{\frac{3x-1}{x^2+2}\right\} = 0$. ◄

2.7-teorema.

1) Agar $f(x)$ funksiya $x \to a$ da ($x \to \infty$ da) cheksiz kichik funksiya bo'lsa, u holda $\frac{1}{f(x)}$ funksiya $x \to a$ da ($x \to \infty$ da) cheksiz

katta funksiyadir.

2) Agar $\varphi(x)$ funksiya $x \to a$ da ($x \to \infty$ da) cheksiz katta funksiya boʻlsa, u holda $\frac{1}{\varphi(x)}$ funksiya $x \to a$ da ($x \to \infty$ da) cheksiz kichik funksiyadir.

Cheksiz kichik funksiyalarning asosiy xossalari:

1^0. Chekli sondagi cheksiz kichik funksiyalarning algebraik yigʻindisi cheksiz kichik funksiyadir.

2^0. Cheksiz kichik funksiyaning chegaralangan funksiyaga koʻpaytmasi cheksiz kichik funksiyadir.

3^0. Cheksiz kichik funksiyalarning koʻpaytmasi cheksiz kichik funksiyadir.

4^0. Cheksiz kichik funksiyaning noldan farqli limitiga ega boʻlgan funksiya cheksiz kichik funksiyadir.

5^0.

1) Agar $y = f(x)$ funksiya $x \to a$ da limitga ega boʻlsa, u holda uni bu limitga teng oʻzgarmas son va cheksiz kichik funksiya yigʻindisi koʻrinishda ifodalash mumkin.

2) Agar $y = f(x)$ funksiyani oʻzgarmas son bilan $x \to a$ da cheksiz kichik funksiyaning yigʻindisi koʻrinishda ifodalash mumkin boʻlsa, u holda oʻzgarmas qoʻshiluvchi bu funksiyaning $x \to a$ dagi limiti boʻladi.

2.3.1. Ekvivalent cheksiz kichik funksiyalar. Cheksiz kichik funksiyalarni taqqoslash

Aytaylik, bir vaqtda bir necha α, β, γ, … cheksiz kichik miqdorlar birgina x argumentning funksiyalaridan iborat boʻlib, x biror a limitga yoki cheksizlikka intilganda ular nolga intilsin.

Agar $\frac{\beta}{\alpha}$ nisbat chekli va noldan farqli limitga ega, ya'ni $\lim\limits_{x\to 0}\frac{\beta}{\alpha} = A \neq 0$ yoki $\lim\limits_{x\to 0}\frac{\alpha}{\beta} = \frac{1}{A} \neq 0$ boʻlsa, u holda β va α cheksiz kichik miqdorlar **bir xil tartibli cheksiz kichik miqdorlar** deyiladi.

2.3.3-misol. $\alpha = x$, $\beta = sin2x$ funksiyalar bir xil tartibli cheksiz kichik miqdorlar boʻladimi?

Yechilishi: ► $\alpha = x$, $\beta = sin2x$ funksiyalar nisbatini $x \to 0$ da limitini hisoblaymiz: $\lim\limits_{x\to 0}\frac{\beta}{\alpha}=\lim\limits_{x\to 0}\frac{\sin 2x}{x}=2.$

Demak, $\alpha = x$ va $\beta = sin2x$ bir xil tartibli cheksiz kichik miqdorlardir.◄

2.3.4-misol. $x \to 0$ da $f(x)=\cos 2x-\cos^3 2x$ va $\varphi(x)=3x^2-5x^3$ funksiyalar bir xil tartibli cheksiz kichik miqdorlar ekanligini isbotlang.

Yechilishi: ► $f(x)$ va $\varphi(x)$ funksiyalar nisbatining $x \to 0$ dagi limitini topamiz:

$$\lim_{x\to 0}\frac{f(x)}{\varphi(x)}=\lim_{x\to 0}\frac{\cos 2x-\cos^3 2x}{3x^2-5x^3}=\lim_{x\to 0}\frac{\cos 2x\left(1-\cos^2 2x\right)}{3x^2-5x^3}=$$

$$=\lim_{x\to 0}\left(2\cdot\frac{\cos 2x\cdot\sin^2 2x}{x^2(3-5x)}\right)=\lim_{x\to 0}\frac{8\cos 2x\cdot\sin 2x\cdot\sin 2x}{2x\cdot 2x(3-5x)}=\frac{8}{3}$$

Limitni qiymati noldan farqli sondan iborat bo‘lgani uchun berilgan funksiyalar bir xil tartibli cheksiz kichik miqdorlardir. ◄

Agar ikkita cheksiz kichik miqdor $\frac{\alpha}{\beta}$ nisbatining limiti nolga intilsa, ya’ni $\lim\limits_{x\to 0}\frac{\alpha}{\beta}=0$ $\left(\lim\limits_{x\to 0}\frac{\beta}{\alpha}=\infty\right)$ bo‘lsa, u holda β cheksiz kichik miqdor α cheksiz miqdorga nisbatan **yuqori tartibli** deyiladi.

2.3.5-misol. $\alpha = x$, $\beta = x^n$, $n>1$ funksiyalar qanday tartibli cheksiz kichik miqdorlar?

Yechilishi: ► $\alpha = x$, $\beta = x^n$, $n>1$. $x \to 0$ da nisbatning limitini hisoblaymiz. β cheksiz kichik miqdor α cheksiz kichik miqdorga nisbatan yuqori tartiblidir, chunki $\lim\limits_{x\to 0}\frac{x^n}{x}=\lim\limits_{x\to 0}x^{n-1}=0$. Bunda α cheksiz kichik miqdor β cheksiz kichik miqdorga nisbatan quyi tartiblidir. ◄

Agar β va α^k bir xil tartibli cheksiz kichik miqdorlar uchun $\lim\limits_{x\to 0}\frac{\beta}{\alpha^k}=A\neq 0$ bo‘lsa, β miqdorga nisbatan α miqdor ***k*** **- tartibli cheksiz kichik miqdor** deyiladi.

2.3.6-misol. $\alpha = x$, $\beta = x^3$ funksiyalar qanday tartibli cheksiz kichik miqdorlar?

Yechilishi: ► $\alpha = x$, $\beta = x^3$ funksiyalar nisbatining $x \to 0$ da limitini hisoblaymiz. β miqdor α miqdorga nisbatan 3-tartibli cheksiz kichik miqdordir, chunki $\lim_{x\to 0}\frac{\beta}{\alpha^3} = \lim_{x\to 0}\frac{x^3}{(x)^3} = 1$. ◄

Agar ikkita cheksiz kichik miqdorning $\frac{\beta}{\alpha}$ nisbati birga intilsa, ya'ni $\lim_{x\to 0}\frac{\alpha}{\beta} = 1$ bo'lsa, u holda β va α miqdorlar **ekvivalent cheksiz kichik miqdorlar** deyiladi va $\alpha \sim \beta$ shaklida yoziladi.

Amaliyotda quyidagi ekvivalent cheksiz kichik miqdorlar ko'p qo'llaniladi:

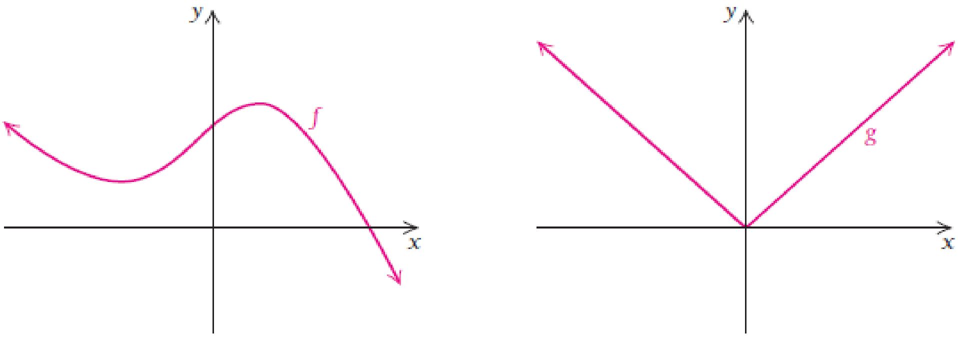
$\sin x \sim x;$	$\arcsin x \sim x;$	$1-\cos x \sim x^2/2;$
$\mathrm{tg} x \sim x;$	$\mathrm{arctg} x \sim x;$	$e^x - 1 \sim x;$
$a^x - 1 \sim x\ln a;$	$\ln(1+x) \sim x;$	$(1+x)^m - 1 \sim mx.$

Mavzu yuzasidan savollar:

1. $y = f(x)$ funksiyaning $x \to a$ va $x \to \infty$ dagi limiti nima?
2. Qanday funksiyaga chegaralangan funksiya deyiladi?
3. Cheksiz katta funksiyaga ta'rif bering.
4. Cheksiz kichik funksiya deb nimaga aytiladi?
5. Bir xil tartibli cheksiz kichik miqdorlar deb nimaga aytiladi?
6. Qanday miqdorlarga ekvivalent cheksiz kichik miqdorlar deyiladi?
7. k - tartibli cheksiz kichik miqdor deb nimaga aytiladi?
8. Amaliyotda qo'llaniladigan ekvivalent cheksiz kichik miqdorlarga misol keltiring.
9. Cheksiz kichik miqdorlarning qanday xossalarini bilasiz?
10. Cheksiz katta miqdor deb nimaga aytiladi?

2.4-§. Funksiya uzluksizligi. Uzilish nuqtalari va ularning turlari

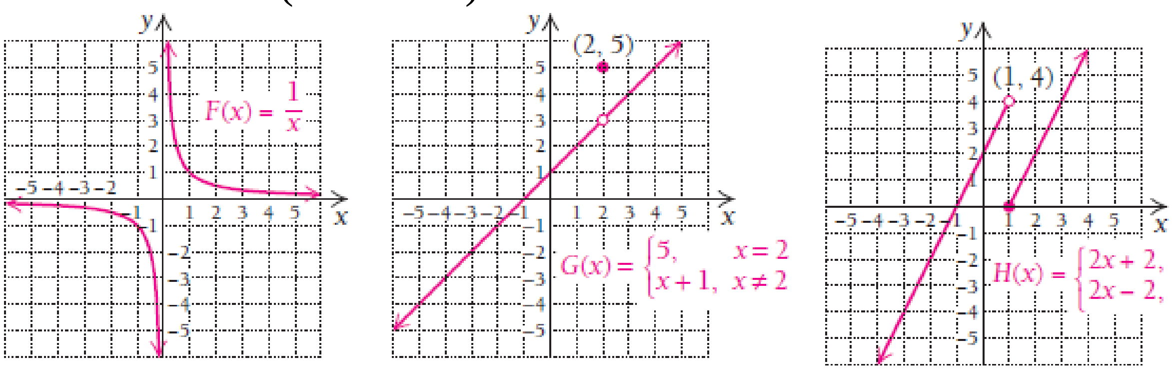
Quyida grafiklari keltirilgan funksiyalar butun haqiqiy sonlar o‘qida, ya’ni $(-\infty, \infty)$ da uzluksiz (2.11-rasm).



2.11-rasm. Uzluksiz funksiyalar

Grafiklarda hech qanday sakrash yoki bo‘yalmay qolgan joylari yo‘q. Shunga ko‘ra, uzluksizlikning sezgilarimizga asoslangan ta’rifini beramiz: Agar grafikning bir uchidan ushlab, uning ustidan qo‘limizni yurgizsak va qo‘limiz chizmadan uzilmasdan ikkinchi uchiga borsa, bu funksiya uzluksiz deymiz. Agar grafikning biror joyida qo‘limizni chizmadan olishga to‘g‘ri kelsa (sakratsak), funksiya uzilishga ega deymiz.

$F(x)$, $G(x)$, $H(x)$ funksiyalar sonlar o‘qida uzilishga ega ekanligini ko‘rish mumkin (2.12-rasm).



2.12-rasm. $F(x)$, $G(x)$, $H(x)$ funksiyalar grafiklari

Bu funksiyalarning uchchalasi ham uzilishga ega, lekin ularning har biri o‘ziga xos.

● $F(x)$ funksiya $x = 0$ nuqtada uziladi. Funksiya 2 bo‘lakdan iborat: $(-\infty, 0)$ va $(0, \infty)$.

● $G(x)$ funksiya $x = 2$ da uziladi va funksiya grafigi 5 ga sakraydi. Shuning uchun uni ham $(-\infty, 2)$ va $(2, \infty)$ qismlarga ajratamiz.

● $H(x)$ funksiya ham uzilishga ega, bu funksiya 1 ga $x \to 1-0$ chapdan intilganda 4 ga, $x \to 1+0$ o‘ngdan intilganda 0 ga teng. Shuning uchun funksiyaning aniqlanish sohasi $(-\infty,1)$ va $(1,\infty)$ qismlardan iborat bo‘ladi.
$F(x)$, $G(x)$ $H(x)$ funksiyalarning barchasida **uzilish nuqtasi** mavjud.
$F(x)$, funksiyaning uzilish nuqtasi $x=0$, $G(x)$ funksiya uchun $x=2$, $H(x)$ funksiya $x=1$ nuqtada uziladi.

Funksiyani nuqtada uzluksizligining 3 xil ta’rifini beramiz, ular o‘zaro teng kuchlidir:

1) Agar $y=f(x)$ funksiya x_0 nuqtada va uning atrofida aniqlangan bo‘lib, $\lim\limits_{x\to x_0} f(x)=f(x_0)$ bo‘lsa, ya’ni funksiyaning x_0 nuqtadagi limiti uning shu nuqtadagi qiymatiga teng bo‘lsa, $y=f(x)$ **funksiya** x_0 **nuqtada uzluksiz** deyiladi.

2) Agar $y=f(x)$ funksiya x_0 nuqtada va uning atrofida aniqlangan bo‘lib, istalgan $\varepsilon>0$ son uchun $\exists\delta>0$ son mavjud bo‘lsaki, $|x-x_0|<\delta$ shartni qanoatlantiradigan istalgan x uchun $|f(x)-f(x_0)|<\varepsilon$ tengsizlik o‘rinli bo‘lsa, $y=f(x)$ **funksiya** x_0 **nuqtada uzluksiz** deyiladi.

3) Agar $y=f(x)$ funksiya x_0 nuqtada va uning atrofida aniqlangan bo‘lib, argumentning cheksiz kichik orttirmasiga funksiyaning cheksiz kichik orttirmasi mos kelsa, ya’ni $\lim\limits_{\Delta x\to 0}\Delta y=0$ bo‘lsa, **funksiya** $\boldsymbol{x_0}$ **nuqtada uzluksiz** deyiladi.

Funksiyaning chap va o‘ng limitlari x_0 nuqtada mavjud va o‘zaro teng bo‘lsa, $y=f(x)$ funksiya x_0 **nuqtada uzluksiz** bo‘ladi.

Agar funksiya x_0 nuqtada uzluksiz bo‘lsa, u holda bu nuqtada limit va funksiya belgilarining o‘rinlarini almashtirish mumkin:

Misol uchun: $\lim\limits_{x\to 1}\ln(x^2+1)=\ln\lim\limits_{x\to 1}(x^2+1)=\ln 2.$

Bir tomonlama uzluksizlik

Agar $y=f(x)$ funksiya $(a,x_0]$ oraliqda aniqlangan va $\lim\limits_{x\to x_0-0} f(x)=f(x_0)$ bo‘lsa, bu funksiya x_0 nuqtada **chapdan uzluksiz** deyiladi.

Agar $y=f(x)$ funksiya $[a,x_0)$ oraliqda aniqlangan va $\lim\limits_{x\to x_0+0} f(x)=f(x_0)$ bo‘lsa, bu funksiya x_0 nuqtada **o‘ngdan uzluksiz** deyiladi.

Nuqtada uzluksiz funksiyalarning xossalari

1⁰. Yig'indining uzluksizligi. Agar $f(x)$ va $\varphi(x)$ funksiyalar x_0 nuqtada uzluksiz bo'lsa, u holda $f(x)\pm\varphi(x)$ funksiya ham x_0 nuqtada uzluksiz funksiyadir, ya'ni

$$\lim_{x\to x_0}\left[f(x)\pm\varphi(x)\right]=\lim_{x\to x_0}f(x)\pm\lim_{x\to x_0}\varphi(x)=f(x_0)\pm\varphi(x_0).$$

2⁰. Ko'paytmaning uzluksizligi. Agar $f(x)$ va $\varphi(x)$ funksiyalar x_0 nuqtada uzluksiz bo'lsa, u holda $f(x)\cdot\varphi(x)$ ko'paytma ham x_0 nuqtada uzluksiz funksiyadir, ya'ni

$$\lim_{x\to x_0}\left[f(x)\cdot\varphi(x)\right]=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}\varphi(x)=f(x_0)\cdot\varphi(x_0).$$

3⁰. Bo'linmaning uzluksizligi. Agar $f(x)$ va $\varphi(x)$ funksiyalar x_0 nuqtada uzluksiz bo'lib, $\varphi(x_0)\neq 0$ bo'lsa, u holda $\frac{f(x)}{\varphi(x)}$ bo'linma ham x_0 nuqtada uzluksiz funksiyadir, ya'ni

$$\lim_{x\to x_0}\left[\frac{f(x)}{\varphi(x)}\right]=\frac{\lim\limits_{x\to x_0}f(x)}{\lim\limits_{x\to x_0}\varphi(x)}=\frac{f(x_0)}{\varphi(x_0)}.$$

Uzilish nuqtalari va ularning turlari

Agar x_0 nuqtada $y=f(x)$ funksiya uchun quyidagi shartladan kamida bittasi bajarilsa, x_0 nuqta $f(x)$ funksiyaning **uzilish nuqtasi**, funksiyaning o'zi esa **uzlukli funksiya** deyiladi:

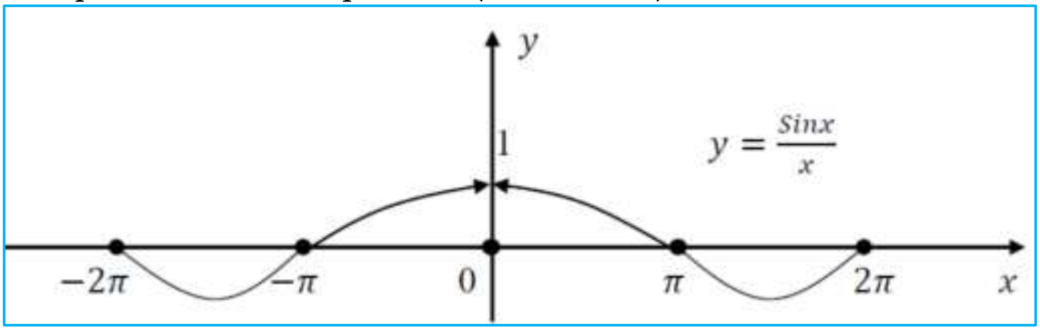
1) funksiya x_0 nuqtada aniqlanmagan;
2) funksiya x_0 nuqtada aniqlangan, lekin $f(x_0-0)$ va $f(x_0+0)$ bir tomonlama limitlardan kamida biri mavjud emas;
3) funksiya x_0 nuqtada aniqlangan, bir tomonlama limitlar mavjud, lekin o'zaro teng emas;
4) funksiya x_0 nuqtada aniqlangan, bir tomonlama limitlar mavjud, o'zaro teng, lekin ular funksiyaning bu nuqtadagi qiymatiga teng emas: $f(x_0-0)=f(x_0+0)\neq f(x_0)$.

Ikki turdagi uzilish farqlanadi: I tur va II tur.
I tur uzilish ham ikkiga ajratiladi: bular bartaraf qilinadigan I tur uzilish hamda sakrashga ega bo'lgan I tur uzilish.

1) $y=f(x)$ funksiya x_0 nuqtada aniqlanmagan, biroq bu nuqtada bir tomonlama limitlar mavjud va o'zaro teng, ya'ni $f(x_0-0)=f(x_0+0)$ bo'lsa, x_0 nuqta **bartaraf qilinadigan I tur uzilish nuqtasi** deyiladi.

2.4.1-misol. $x_0 = 0$ nuqtada $f(x) = \frac{\sin x}{x}$ funksiyani uzluksizlikka tekshiring.

Yechilishi: ► Funksiyaning o‘ng va chap limitlarini hisoblaymiz: $\lim\limits_{x\to+0} \frac{\sin x}{x} = 1$ va $\lim\limits_{x\to-0} \frac{\sin x}{x} = 1$, ya’ni $f(-0) = f(+0)$ bir tomonlama limitlar mavjud va o‘zaro teng, ammo $x_0 = 0$ nuqtada $f(x)$ funksiya qiymati mavjud emas, demak, x_0 yo‘qotiladigan uzilish nuqtasi, $f(0) = f(-0) = f(+0) = 1$ deb uzilish nuqtasini bartaraf qilamiz (2.13-rasm).

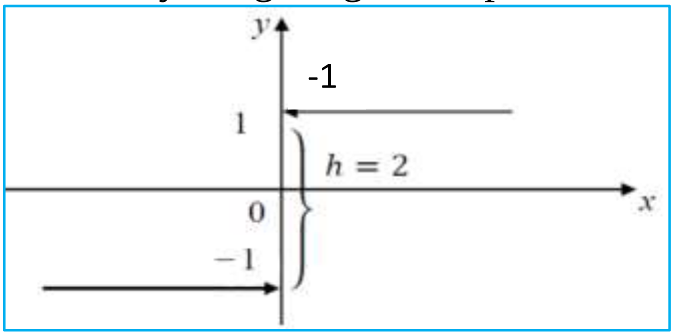


2.13-rasm. $y = \frac{\sin x}{x}$ funksiya grafigi

2) Agar funksiya x_0 nuqtada aniqlangan yoki aniqlanmagan, lekin bir tomonlama limitlar mavjud bo‘lib, ular o‘zaro teng bo‘lmasa, ya’ni $f(x_0 - 0) \neq f(x_0 + 0)$ bo‘lsa, bu nuqta s**akrashga ega bo‘lgan I tur uzilish nuqtasi** deyiladi. $h = f(x_0 - 0) - f(x_0 + 0)$ son funksiyaning x_0 nuqtadagi **sakrashi** deyiladi.

2.4.2-misol. $f(x) = \frac{|x|}{x}$ funksiyani $x = 0$ nuqtada uzluksizlikka tekshiring.

Yechilishi: ► Funksiyaning o‘ng va chap limitlarini hisoblaymiz:



2.14-rasm. $y = \frac{|x|}{x}$ funksiya grafigi

$$f(+0) = \lim_{x\to+0} \frac{|x|}{x} = \lim_{x\to+0} \frac{+x}{x} = 1, \qquad f(-0) = \lim_{x\to-0} \frac{|x|}{x} = \lim_{x\to+0} \frac{-x}{x} = -1,$$

ya’ni $f(+0) \neq f(-0)$ va sakrashi $h = 1 - (-1) = 2$.

Demak, $x = 0$ – birinchi tur uzilish nuqtasi (2.14-rasm). ◄

3) II tur uzilish. Agar x_0 nuqtada bir tomonlama limitlarning kamida biri mavjud bo‘lmasa yoki cheksizlikka intilsa, x_0 nuqta **ikkinchi tur uzilish nuqtasi** deyiladi.

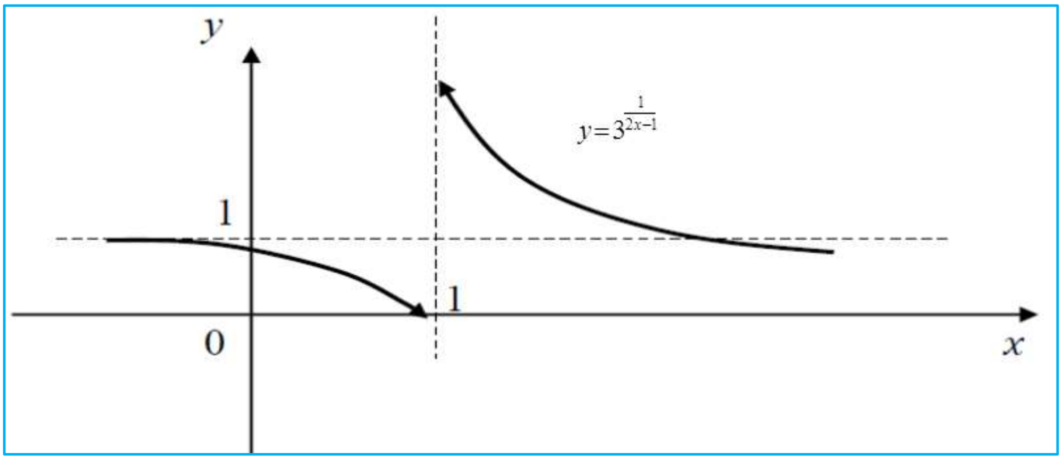
2.4.3-misol. $f(x)=3^{\frac{1}{2x-1}}$ funksiyani $x=\frac{1}{2}$ nuqtada uzluksizlikka tekshiring.

Yechilishi: ► Funksiyaning o‘ng va chap limitlarini hisoblaymiz:

$$f\left(\frac{1}{2}-0\right)=\lim_{x\to\frac{1}{2}-0} 3^{\frac{1}{2x-1}}=\lim_{x\to\frac{1}{2}-0} 3^{\frac{1}{2\left(\frac{1}{2}-0\right)-1}}=3^{-\infty}=0,$$

$$f\left(\frac{1}{2}+0\right)=\lim_{x\to\frac{1}{2}+0} 3^{\frac{1}{2x-1}}=\lim_{x\to\frac{1}{2}+0} 3^{\frac{1}{2\left(\frac{1}{2}+0\right)-1}}=3^{\infty}=\infty$$

Demak, $x=\frac{1}{2}$ - ikkinchi tur uzilish nuqtasi (2.15-rasm).

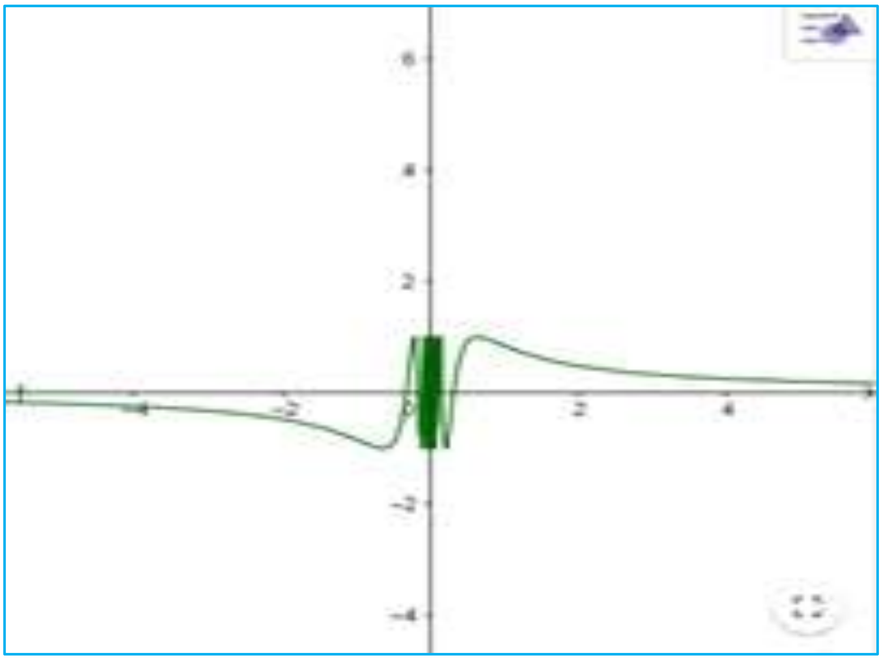


2.15-rasm. $y=3^{\frac{1}{2x-1}}$ funksiya grafigi

2.4.4-misol. $f(x)=\sin\frac{1}{x}$ funksiya $x=0$ nuqtada uzluksizlikka tekshiring.

Yechilishi: ► $f(x)=\sin\frac{1}{x}$ funksiya $x=0$ nuqtada aniqlanmagan.

$f(\pm 0)=\lim\limits_{x\to\pm 0} Sin\frac{1}{x}$ tayin limitga ega emas, demak, $x=0$ nuqta II tur uzilish nuqtasi bo‘ladi (2.16-rasm).



◄

2.16-rasm. $y = \sin\frac{1}{x}$ funksiya grafigi

2.4.5-misol. $f(x) = \begin{cases} x^2, & -\infty < x \le 0, \\ (x-1)^2, & 0 < x \le 2, \\ 5 - x, & 2 < x < +\infty. \end{cases}$ funksiyani uzluksizlikka tekshiring va uzilish oraliqlarini va uzilish nuqtalari turini aniqlang.

Yechilishi: ► $f(x)$ funksiya $(-\infty; 0], (0; 2]$ va $(2; +\infty)$ oraliqda aniqlangan va bu oraliqlarda uzluksiz bo'lgan elementar funksiyalar bilan berilgan. Shunday ekan, funksiya faqat $x_1 = 0$ va $x_2 = 2$ nuqtalarda uzilishi mumkin. $x_1 = 0$ nuqtada funksiya uzilishga ega bo'lsa, uzilish turini aniqlaymiz: $\lim\limits_{x \to 0-0} f(x) = \lim\limits_{x \to 0-0} x^2 = 0$, $\lim\limits_{x \to 0+0} f(x) = \lim\limits_{x \to 0+0} (x-1)^2 = 1$,

$f(0) = x^2\big|_{x=0} = 0$, ya'ni, $f(x)$ funksiya $x_1 = 0$ nuqtada I tur uzilishga ega.

$x_2 = 2$ nuqtada funksiya uzilishga ega bo'lsa, uzilish turini aniqlaymiz:

$$\lim_{x \to 2-0} f(x) = \lim_{x \to 2-0} (x-1)^2 = 1, \qquad \lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} (5 - x) = 3,$$

$f(2) = (x-1)^2\big|_{x=2} = 1$, ya'ni, $f(x)$ funksiya $x_2 = 2$ nuqtada ham I tur uzilishga ega.◄

2.4.6-misol. $f(x) = 8^{1/(x-3)} + 1$ funksiyani $x_1 = 3$ va $x_2 = 4$ nuqtalarda uzluksizlikka tekshiring.

Yechilishi: ► $x_1 = 3$ nuqtada tekshiramiz:

$$\lim_{x \to 3-0} f(x) = \lim_{x \to 3-0} \left(8^{1/(x-3)} + 1\right) = 8^{-\infty} + 1 = 1,$$

$$\lim_{x\to 3+0} f(x) = \lim_{x\to 3+0}\left(8^{1/(x-3)}+1\right) = 8^{\infty}+1 = \infty,$$

$x_1 = 3$ nuqtada $f(x)$ funksiya II tur uzulishga ega.

$x_2 = 4$ nuqtada uzluksizlikka tekshiramiz:

$$\lim_{x\to 4-0} f(x) = \lim_{x\to 4-0}\left(8^{1/(x-3)}+1\right) = 9, \qquad \lim_{x\to 4+0} f(x) = \lim_{x\to 4+0}\left(8^{1/(x-3)}+1\right) = 9,$$

$$f(4) = 8^{1/(4-3)}+1 = 9.$$

Funksiyaning chap va o‘ng limitlari x_0 nuqtada mavjud va o‘zaro teng bo‘lsa, $y = f(x)$ funksiya x_0 **nuqtada uzluksiz** deyiladi.

Demak, funksiya uzluksizligining ta’rifiga ko‘ra, $x_2 = 4$ nuqtada funksiya uzluksiz.◄

Mavzu yuzasidan savollar:

1. $y = f(x)$ funksiyaning x_0 nuqtada uzluksizligi ta’rifi keltiring.
2. $y = f(x)$ funksiyaning x_0 nuqtada chapdan uzluksizligi ta’rifini ayting.
3. $y = f(x)$ funksiyaning x_0 nuqtada o‘ngdan uzluksizligi ta’rifini ayting.
4. Kesmada uzluksiz funksiyaning xossalarini ayting.
5. Funksiyaning uzilish nuqtasi deb nimaga aytiladi?
6. Funksiyaning qanday uzilish turlarini bilasiz?
7. Bartaraf qilinadigan I tur uzilish deb nimaga aytiladi?
8. Sakrashga ega bo‘lgan I tur uzilish deb nimaga aytiladi?
9. Nuqtada uzluksiz funksiyalarning xossalarini ayting.

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Funksiyalarni uzluksizlikka tekshiring, uzilish oraliqlarini va uzilish nuqtalari turini aniqlang:

a) $f(x) = \begin{cases} x+4, & x < -1; \\ x^2+2, & -1 \le x < 1; \\ 2x, & x \ge 1. \end{cases}$ **b)** $f(x) = \begin{cases} -x, & x \le 0; \\ \sin x, & 0 < x \le \pi; \\ x-2, & x > \pi. \end{cases}$

2. Funksiyalarni berilgan nuqtalarda uzluksizlikka tekshiring, agar mavjud bo‘lsa, uzilish oraliqlarini va uzilish nuqtalari turini aniqlang:

a) $f(x) = 9^{\frac{1}{2-x}}$, $x_1 = 0$, $x_2 = 2$. **b)** $f(x) = 10^{1/(7-x)}$, $x_1 = 5$, $x_2 = 7$.

3. $f(x) = \dfrac{x^2-9}{x^2+x-12}$ funksiyaning uzilish nuqtalarini toping.

4. $y = 2^{\frac{1}{x^2-9}}$ funksiyani uzluksizlikka tekshiring, agar mavjud bo‘lsa, uzilish nuqtasini toping.

5. A va B sonlar qanday bo‘lganda funksiya uzluksiz bo‘ladi:

$$f(x) = \begin{cases} -2\sin x; & x \le -\dfrac{\pi}{2} \\ A\sin x + B; & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \cos x; & x \ge \dfrac{\pi}{2} \end{cases}$$

TESTLAR

1. $x = \dfrac{\pi}{2}$ nuqta $y = tgx$ funksiya uchun …. bo‘ladi.

A) Uzluksizlik nuqtasi;
B) Bartaraf qilinadigan I tur uzilish nuqtasi;
C) Sakrashga ega bo‘lgan I tur uzilish nuqtasi;
D) Ikkinchi tur uzilish nuqtasi.

2. $x = 0$ nuqta $y = e^{\frac{1}{x}}$ funksiya uchun …. bo‘ladi.

A) Ikkinchi tur uzilish nuqtasi;
B) Uzluksizlik nuqtasi;
C) Bartaraf qilinadigan I tur uzilish nuqtasi;
D) Sakrashga ega bo‘lgan I tur uzilish nuqtasi.

3. $x = \pm 4$ nuqtalar $y = 2^{\frac{1}{16-x^2}}$ funksiya uchun …. bo‘ladi.

A) Uzluksizlik nuqtasi;
B) Bartaraf qilinadigan I tur uzilish nuqtasi;
C) Sakrashga ega bo‘lgan I tur uzilish nuqtasi;
D) Ikkinchi tur uzilish nuqtasi.

4. $f(x) = \dfrac{1}{\ln|x|}$ funksiyaning nechta uzilish nuqtasi bor?

A) 1 **B)** 2 **C)** 3 **D)** 4

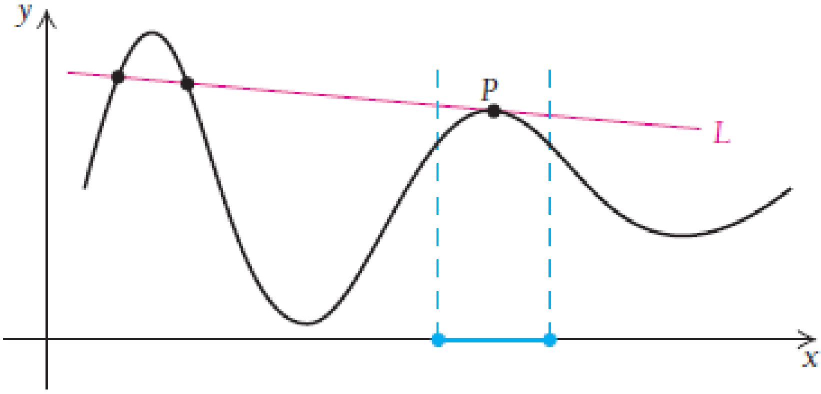
5. $f(x) = \begin{cases} x+1 & agar \ \ x \le 1 \\ 3 - ax^2 & agar \ \ x > 1 \end{cases}$ bo‘lsa, a ning qanday qiymatida funksiya uzluksiz bo‘ladi?

A) $a = 2$ **B)** $a = 1$ **C)** $a = -1$ **D)** $a = 0$

2.5-§. Hosila tushunchasi. Funksiya hosilasini hisoblash. Yuqori tartibli hosila

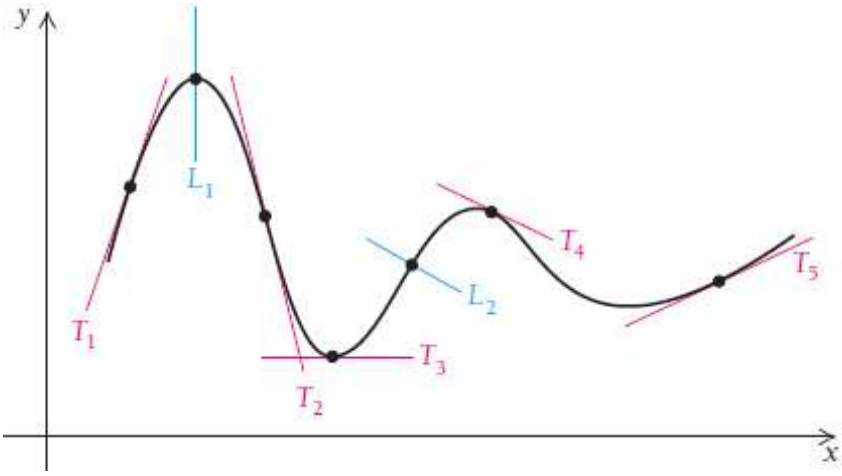
2.5.1. Funksiyaning nuqtadagi hosilasi. Hosilaning geometrik va mexanik ma'nosi

Aylana bilan bitta umumiy nuqtaga ega bo'lgan to'g'ri chiziqqa **urinma** deyiladi. Egri chiziqqa o'tkazilgan urinma ham u bilan bitta umumiy nuqtaga ega bo'ladi. L to'g'ri chiziq egri chiziqqa P nuqtada o'tkazilgan urinma bo'ladi (2.17-rasm).



2.17-rasm. Egri chiziqqa o'tkazilgan urinma

L_1 va L_2 kesuvchilar, T_1, T_2, T_3, T_4, T_5 to'g'ri chiziqlar urinmalardir (2.18-rasm).

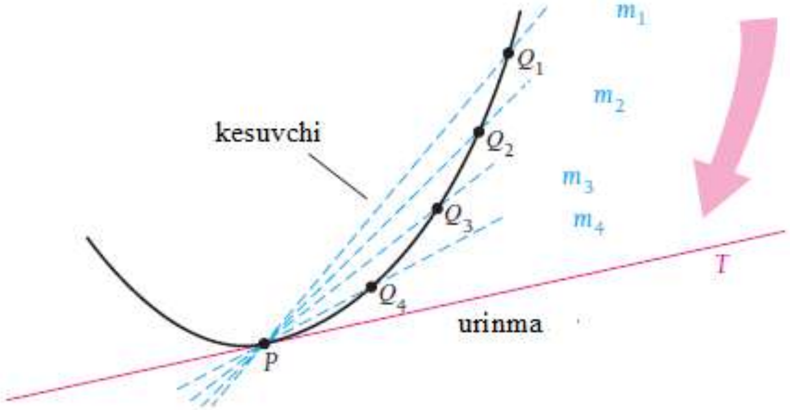


2.18-rasm. Egri chiziqqa o'tkazilgan kesuvchi va urinmalar

Agar egri chiziq silliq bo'lsa (hech qanday uchlari bo'lmasa), u holda egri chiziqning har bir nuqtasidan urinma o'tkazish mumkin.

Endi urinma ixtiyoriy egri chiziq uchun qanday ahamiyatga ega ekanini aniqlaymiz. Buning uchun limit tushunchasidan foydalanamiz.

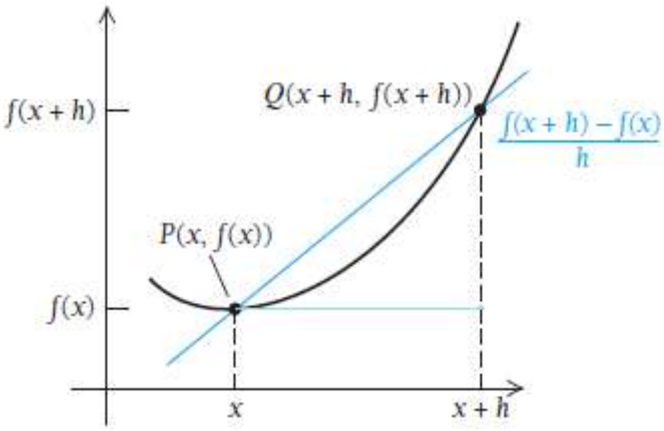
Egri chiziqqa P nuqtada o‘tkazilgan urinmani topish uchun P va Q_1, Q_2, Q_3, Q_4 nuqtalardan o‘tkazilgan kesuvchilarni qaraymiz. Q nuqtalar P ga yaqinlashgani sari kesuvchilar ham T chiziqqa yaqinlashib boradi. Har bir kesuvchi og‘ish koeffitsiyentiga ega. Bular m_1, m_2, m_3, m_4 koeffitsiyentlar bo‘lib, kesuvchilar T chiziqqa yaqinlashganda, ular m ga yaqinlashadi. Biz T chiziqni urinma sifatida aniqlaymiz, bu chiziq P nuqtadan o‘tadi va Q nuqtalar P ga yaqinlashganda ularning og‘ish koeffitsiyentlarining limiti m og‘ish koeffitsiyentiga intiladi (2.19-rasm).



2.19-rasm. Kesuvchilarning og‘ish koeffitsiyentlari

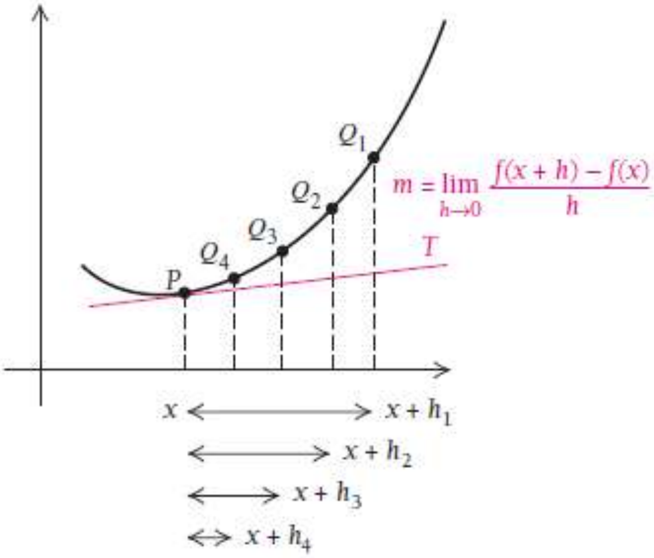
Multiplikatsiya kadrlarini esga soling, daftar varaqlarini o‘tkazganingizda Q nuqtalar P qo‘zg‘almas nuqtaga yaqinlashib borib, barcha kesuvchilar urinma “ustiga” yotadi.

Faraz qilaylik, P nuqtaning koordinatalari $(x, f(x))$. U holda Q nuqtaning koordinatalari $(x+h, f(x+h))$ dan iborat bo‘ladi (2.20-rasm).



2.20-rasm. Kesuvchining og‘ish koeffitsiyenti

PQ kesuvchining og‘ish koeffitsiyenti $\frac{f(x+h)-f(x)}{h}$ ga teng. Bizning holatda Q nuqtalar P ga yaqinlashganda (2.21-rasm)

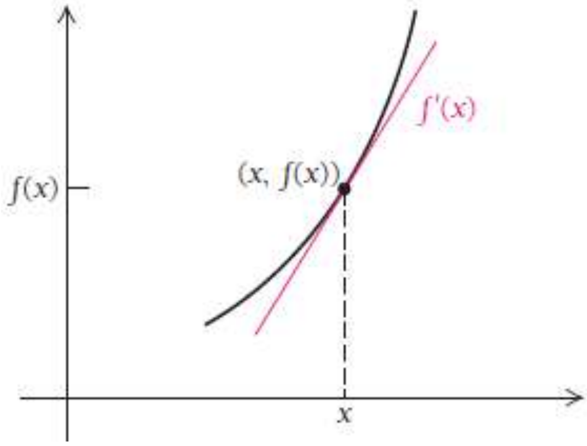


2.21-rasm. Urinmaning og‘ish koeffitsiyenti

$x+h$ qiymatlar ham x ga yaqinlashib boradi. Bunda h ham nolga yaqinlashadi, ya’ni $h \to 0$ deb olish mumkin. Shunda quyidagi tasdiq o‘rinli bo‘ladi:

Urinmaning og‘ish koeffitsiyenti $m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ ga teng. Bu limit $f(x)$ ning x dagi **oniy hosilasiga teng bo‘ladi**.

Demak, hosila egri chiziqqa o‘tkazilgan urinmaning og‘ish koeffitsiyentiga teng bo‘lib, bu hosilaning geometrik ma’nosini beradi (2.22-rasm). Funksiyaning x dagi hosilasini $f'(x)$ deb belgilaymiz. $f'(x)$ ni “**funksiyaning nuqtadagi hosilasi**” yoki “f shtrix x” deb o‘qish mumkin.



2.22-rasm. Hosilaning geometrik ma’nosi

$y = f(x)$ **funksiyaning nuqtadagi hosilasi** deb, quyidagi tenglik bilan aniqlanadigan $f'(x)$ funksiyaga aytiladi:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad (2.8.)$$

Agar hosila ta'rifida $h \to -0$ yoki $h \to +0$ bo'lsa, bir tomonlama hosilalarga ega bo'lamiz, ular $f'_{+0}(x_0)$ va $f'_{-0}(x_0)$ deb belgilanadi hamda quyidagiga teng bo'ladi:

$$f'_{+0}(x) = \lim_{h\to+0}\frac{f(x+h)-f(x)}{h} \quad x \text{ nuqtadagi o'ng hosila,}$$

$$f'_{-0}(x) = \lim_{h\to-0}\frac{f(x+h)-f(x)}{h} \quad x \text{ nuqtadagi chap hosila.}$$

$y=f(x)$ funksiyaning x nuqtada hosilasi mavjud bo'lishi uchun o'ng va chap hosilalar mavjud va teng bo'lishi zarur va yetarlidir, ya'ni

$$f'_+(x) = f'_-(x).$$

Hosilani topish jarayoniga funksiyani **differensiallash** deyiladi.

Funksiya hosilasini ta'rifga ko'ra hisoblash 3 ta qadamdan iborat:

1. Orttirmalar nisbatini yozamiz: $\frac{f(x+h)-f(x)}{h}$;

2. Bu nisbatni soddalashtiramiz;

3. Soddalashgan ifodaning $h\to0$ dagi limitini hisoblaymiz.

2.5.1-misol. $f(x)=x^3$ funksiyaning hosilasini toping.

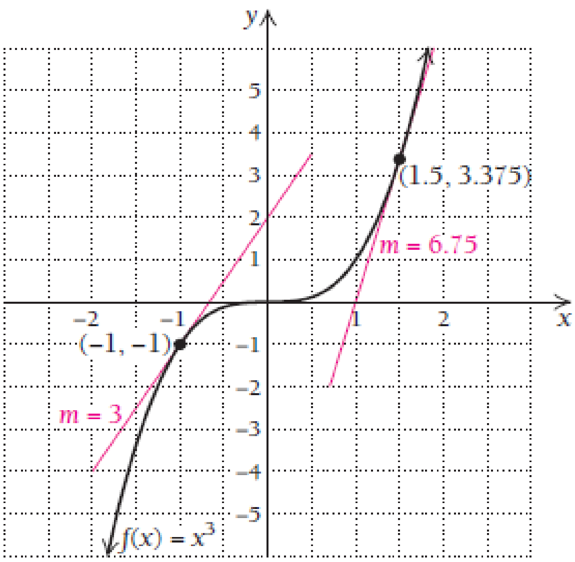
Yechilishi: ▶

1) $$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^3-x^3}{h}, \quad h\neq0;$$

2) $$\frac{f(x+h)-f(x)}{h}=\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\frac{3x^2h+3xh^2+h^3}{h}=3x^2+3xh+h^2;$$

3) $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}(3x^2+3xh+h^2)=3x^2.$$

Demak, $f'(x)=3x^2$.



2.23-rasm. Nuqtalardagi urinmalarning og'ish koeffitsiyentlari

$f'(x)=3x^2$ formuladan $f'(-1)=3\cdot(-1)^2=3$ va $f'(1.5)=3\cdot 1.5^2=6.75$ nuqtalardagi urinmalarning og‘ish koeffitsiyentlarini topishimiz mumkin (2.23-rasm).◀

2.5.2-misol. $f(x)=\dfrac{1}{x}$ funksiya berilgan.

a) $f'(x)$ hosilani toping;

b) $f'(2)$ ni hisoblang;

c) Funksiya grafigiga $x=2$ nuqtada o‘tkazilgan urinma tenglamasini yozing.

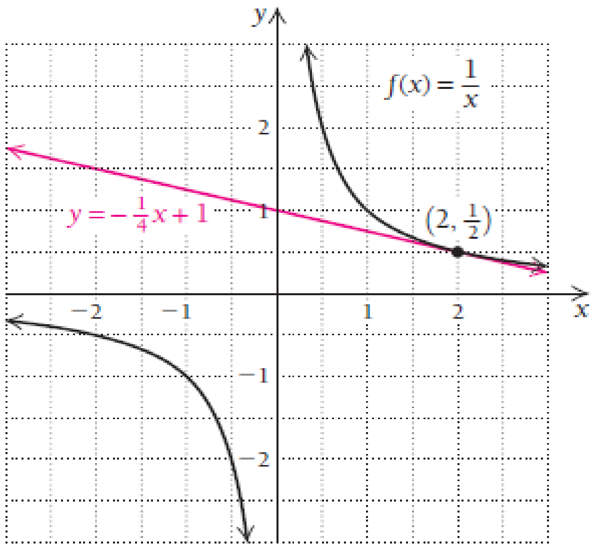
Yechilishi: ▶

a) $f'(x)$ hosilani topamiz:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left(-\frac{1}{x(x+h)}\right)=-\frac{1}{x^2}.$$

b) $f'(2)$ ni hisoblaymiz: $f'(2)=-\dfrac{1}{2^2}=-\dfrac{1}{4}$

c) Funksiya grafigiga $x=2$ nuqtada o‘tkazilgan urinma tenglamasini yozish uchun dastlab, $f(x)=\dfrac{1}{x}$ funksiyaga urinma o‘tkaziladigan nuqtani aniqlaymiz: $f(2)=\dfrac{1}{2}$, bu nuqta $\left(2,\dfrac{1}{2}\right)$ ekan. Og‘ish koeffitsiyenti esa $m=-\dfrac{1}{4}$ ga teng.



2.24-rasm. Funksiya grafigiga $x=2$ nuqtada o‘tkazilgan urinma

Urinma $y-y_1=m(x-x_1)$ tenglamasiga asosan

$$y-\frac{1}{2}=-\frac{1}{4}(x-2),\qquad y-\frac{1}{2}=-\frac{1}{4}x+\frac{1}{2},\qquad y=-\frac{1}{4}x+1.$$

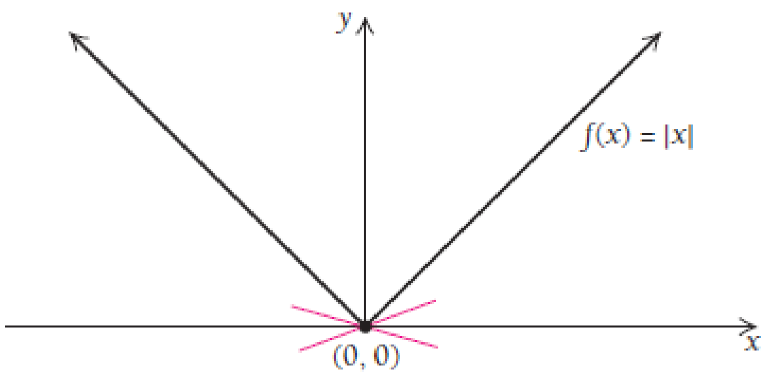
Shunday qilib, $f(x)=\frac{1}{x}$ funksiya grafigiga $x=2$ nuqtada o‘tkazilgan urinma tenglamasi $y=-\frac{1}{4}x+1$ ko‘rinishda bo‘ladi (2.24-rasm).

$f(x)=\frac{1}{x}$ funksiyaning $f(0)$ nuqtadagi qiymati mavjud emas. Shuning uchun bu funksiyaning $f'(0)$ hosilasi ham mavjud emas.◄

Shunday $f(x)$ funksiyalar ham borki, funksiya biror nuqtada aniqlangan, lekin bu nuqtada uning $f'(x)$ hosilasini hisoblab bo‘lmaydi.

2.5.3-misol. $f(x)=|x|$ funksiyaning $x=0$ nuqtada hosilasini toping.

Yechilishi: ► $f(x)=|x|$ funksiya $x=0$ nuqtada uzluksizlikning barcha shartlarini qanoatlantiradi, lekin bu nuqtada funksiyaning hosilasi mavjud emas. Bu nuqtada funksiyaga o‘tkazilgan urinma qanday?



2.25-rasm. $f(x)=|x|$ funksiyaning $x=0$ nuqtadagi urinmalari

Faraz qiling, (0,0) nuqtada funksiyaga urinma o‘tkazmoqchimiz. Funksiya bu nuqtada uchli (silliq emas), bu nuqtada funksiyaga cheksiz ko‘p urinmalar o‘tadi, ularning og‘ish koeffitsiyentlari ham cheksiz bo‘lishi kerak (2.25-rasm). Keling $x=0$ nuqtada funksiya hosilasini topishga harakat qilamiz.

$f(x)=\begin{cases} x, & \text{agar } x>0 \\ -x, & \text{agar } x<0 \end{cases}$ funksiya hosilasi $f'(x)=\begin{cases} 1, & \text{agar } x>0 \\ -1, & \text{agar } x<0. \end{cases}$

O‘ng va chap limitlar bir-biriga teng emas: $\lim\limits_{x\to 0^+} f'(x)\neq \lim\limits_{x\to 0^-} f'(x)$.

Shuning uchun $f(x)=|x|$ funksiyaning $x=0$ nuqtada $f'(0)$ hosilasi mavjud emas.◄

Xulosa. Agar funksiya biror nuqtada aniqlanmagan bo‘lsa, bu funksiyaning shu nuqtada hosilasi mavjud bo‘lmaydi. Yoki agar funksiya biror nuqtada uzilishga ega bo‘lsa, funksiyaning uzilish nuqtasida hosilasi mavjud emas.

Agar funksiya grafigi “uchli” bo‘lsa, funksiyaning shu nuqtadagi hosilasi mavjud bo‘lmaydi.

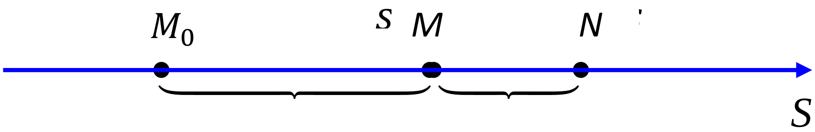
2.5.4-misol. $y = \ln x$ funksiya hosilasini toping.

Yechilishi: ►

$$(\ln x)' = \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) =$$

$$= \lim_{h\to 0} \ln\left(1 + \frac{1}{\frac{x}{h}}\right)^{\frac{1}{h}} = \ln \lim_{h\to 0}\left(1 + \frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}\cdot\frac{1}{x}} = \ln e^{\frac{1}{x}} = \frac{1}{x}$$ ◄

Hosilaning mexanik ma’nosi. Biror M nuqta to‘g‘ri chiziqda harakatlanayotgan bo‘lsin (2.26-rasm).



2.26-rasm. M nuqtaning bosib o‘tgan yo‘li

Biror M_0 boshlang‘ich vaziyatdan M nuqtagacha hisoblanadigan s masofa t vaqtga bog‘liq, ya’ni s masofa t vaqtning funksiyasi bo‘lsin:

$$s = f(t).$$

Vaqtning biror t momentida M nuqta M_0 boshlang‘ich vaziyatdan s masofada, navbatdagi biror $t + \Delta t$ momentda esa bu nuqta N vaziyatda boshlang‘ich vaziyatdan $s + \Delta s$ masofada bo‘lsin. Shunday qilib, Δt vaqt oralig‘ida nuqta Δs masofani o‘tgan, ya’ni s kattalik Δs ga o‘zgargan bo‘ladi. Nuqtaning Δt vaqt ichida o‘rtacha harakat tezligi $v_{o‘rt} = \frac{\Delta s}{\Delta t}$ bo‘lishi ravshan. $\lim_{\Delta t\to 0} v_{o‘rt} = v$ berilgan t momentdagi harakat tezligi, $\lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t} = s'$ esa hosila. Shunday qilib, $v = s'$, ya’ni tezlik yo‘ldan vaqt bo‘yicha olingan hosila ekan.

2.5.2. Funksiyaning differensiallanuvchanligi

Faraz qilaylik, $y = f(x)$ funksiya berilgan bo‘lsin. “y ning x bo‘yicha hosilasi”ning keng tarqalgan belgilanishi nemis matematigi Leybnits tomonidan kiritilgan: $\frac{dy}{dx}$. Bu belgilashga asosan quyidagini yozishimiz mumkin:

Agar $y = f(x)$ funksiya berilgan bo‘lsa, u holda y ning x bo‘yicha hosilasi $\frac{dy}{dx} = f'(x)$ ga teng. Agar hosilani biror nuqtada aniqlamoqchi bo‘lsak, masalan, $\left.\frac{dy}{dx}\right|_{x=2} = f'(2)$ yozuvni ishlatamiz. Hosilani boshqacha $\frac{d}{dx} f(x)$ deb ham yozish mumkin. Bu yozuv $\frac{dy}{dx}$ belgilash bilan bir xil ma’noni beradi. Agar hosila olish belgisi $\frac{d}{dx} f(x)$ funksiyaning oldiga yozib qo‘yilsa, bu hosila olishga berilgan “**buyruq**” sifatida qaraladi, misol uchun: $\frac{d}{dx} x^2 = 2x$, $\frac{d}{dx} x^3 = 3x^2$, $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$.

Agar $y = f(x)$ funksiya $[a,b]$ kesmaning barcha ichki nuqtalarida differensiallanuvchi hamda chekli bir tomonlama $f'_+(a)$ va $f'_-(b)$ hosilalar mavjud bo‘lsa, bu funksiya shu **kesmada differensiallanuvchi** deyiladi.

2.8-teorema. Agar $y = f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo‘lsa, u shu nuqtada uzluksizdir.

2.5.3. Hosila hisoblashning asosiy qoidalari

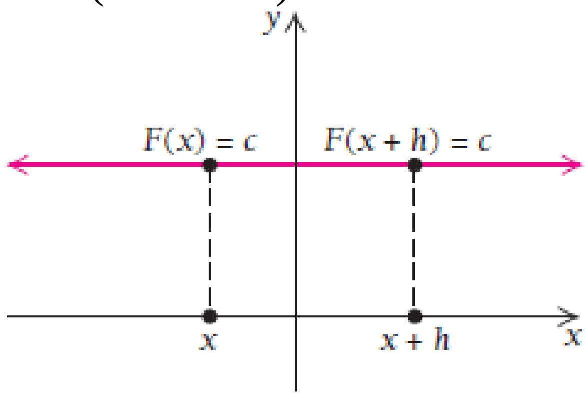
Darajali funksiyaning hosilasini topish:

2.9-teorema. Agar har qanday haqiqiy k son uchun $y = x^k$ funksiya berilgan bo‘lsa, u holda bu funksiyaning hosilasi quyidagiga teng bo‘ladi

$$\frac{d}{dx} x^k = k \cdot x^{k-1} \qquad (2.9)$$

O‘zgarmas funksiyaning hosilasini topish:

O‘zgarmas $F(x) = c$ funksiyaning grafigi og‘ish burchagi 0 ga teng bo‘lgan shakldan iborat (2.27-rasm).



2.27-rasm. O‘zgarmas funksiyaning grafigi

2.10-teorema. O‘zgarmas funksiyaning hosilasi nolga teng: $c' = 0$
.

Isboti: ▶ $\lim\limits_{h \to 0} \dfrac{F(x+h) - F(x)}{h} = \lim\limits_{h \to 0} \dfrac{c - c}{h} = \dfrac{0}{h} = 0$, demak, $F'(c) = 0$. ◀

Funksiya bilan o‘zgarmas sonning ko‘paytmasi hosilasini topish:

2.11-teorema. Funksiya bilan o‘zgarmasning ko‘paytmasidan hosila olishda o‘zgarmas sonni hosila belgisidan tashqariga chiqarish mumkin:

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx} f(x) \quad \text{yoki} \quad [c \cdot f(x)]' = c \cdot f'(x). \qquad (2.10)$$

Isboti: ▶

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = \lim_{h \to 0} \frac{c[f(x+h) - f(x)]}{h} =$$

$$= c \cdot \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = c \cdot f'(x) \quad ◀$$

2.5.5-misol. Tibbiyot masalasi. Saraton kasalligida shar shaklidagi shish hajmini quyidagicha approksimatsiyalash mumkin: $V(r) = \dfrac{4}{3}\pi r^3$, bunda r – shish radiusi, sm.

a) Radiusga nisbatan hajm hosilasini toping.

b) $r = 1.2$ sm bo‘lganda hajm hosilasini toping.

Yechilishi: ▶ a) $V'(r) = \left(\dfrac{4}{3}\pi r^3 \right)' = 4\pi r^2$;

b) $V'(1.2) = S(1.2) = 4\pi (1.2)^2 = 5.76\pi \approx 18 \dfrac{\text{sm}^3}{\text{sm}} = 18 \ \text{sm}^2$.

Ya’ni radiusi $r = 1.2$ sm bo‘lgan shishning radiusi har 1 sm ga kattalashganda, uning hajmi 18 sm^3 ga oshib boradi. ◀

Yig‘indi va ayirmaning hosilasini topish:

2.12-teorema: 1) Yig‘indining hosilasi hosilalar yig‘indisiga teng:

$$[f(x) + g(x)]' = f'(x) + g'(x). \qquad (2.11)$$

2) Ayirmaning hosilasi hosilalar ayirmasiga teng:

$$[f(x) - g(x)]' = f'(x) - g'(x). \qquad (2.12)$$

Isboti: ▶ **1)** Yig‘indining hosilasi:

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h}=$$

$$=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]=f'(x)+g'(x).$$

2) Ayirmaning hosilasini isbotlash uchun yig‘indi hosilasidan foydalanish mumkin:

$$\left[f(x)-g(x)\right]'=\left[f(x)+(-1)g(x)\right]'=f'(x)+(-1)g'(x)=f'(x)-g'(x).\ ◀$$

2.5.6-misol. Hosilalarini hisoblang:

a) $y=\left(3x-4\sqrt[3]{x}+2\right)$;   b) $y=\left(x^2-\frac{1}{x^3}+5\sqrt{x}\right)$.

Yechilishi: ▶ a) $y'=\left(3x-4\sqrt[3]{x}+2\right)'=\left(3x\right)'-\left(4\sqrt[3]{x}\right)'+2'=3-\frac{4}{3\sqrt[3]{x^2}}$;

b) $y'=\left(x^2-\frac{1}{x^3}+5\sqrt{x}\right)'=2x+\frac{3}{x^4}+\frac{5}{2\sqrt{x}}$. ◀

Ko‘paytmaning hosilasini topish

2.13-teorema. $F(x)=f(x)\cdot g(x)$ funksiya hosilasi uchun quyidagi tenglik o‘rinli:

$$F'(x)=f'(x)\cdot g(x)+f(x)\cdot g'(x) \qquad (2.13)$$

Isboti:▶ $F'(x)=\lim\limits_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}=$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}=$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h}=$$

$$=\lim_{h\to 0}f(x+h)\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\frac{f(x+h)-f(x)}{h}=$$

$$=f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=$$

$$=f(x)\cdot g'(x)+g(x)\cdot f'(x)\ ◀$$

2.5.7-misol. $\left(\frac{1}{3}x^3 - x^2 - 3x + 5\right)\cdot\left(4x^2 + x + 1\right)$ ifoda hosilasini toping.

Yechilishi: ▶ $$\left[\left(\frac{1}{3}x^3 - x^2 - 3x + 5\right)\cdot\left(4x^2 + x + 1\right)\right]' =$$

$$= \left(\frac{1}{3}x^3 - x^2 - 3x + 5\right)'\cdot\left(4x^2 + x + 1\right) + \left(\frac{1}{3}x^3 - x^2 - 3x + 5\right)\cdot\left(4x^2 + x + 1\right)' =$$

$$= \left(x^2 - 2x - 3\right)\cdot\left(4x^2 + x + 1\right) + \left(\frac{1}{3}x^3 - x^2 - 3x + 5\right)\cdot\left(8x + 1\right) =$$

$$= \frac{20}{3}x^4 - \frac{44}{3}x^3 - 38x^2 + 32x + 2$$ ◀

Bo‘linmaning hosilasi

2.14-teorema. $R(x) = \frac{f(x)}{g(x)}$ funksiya funksiya hosilasi uchun quyidagi tenglik o‘rinli:

$$R'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2} \qquad (2.14)$$

Isboti: ▶ $$R'(x) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} =$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h) + f(x+h)g(x+h) - f(x+h)g(x+h)}{h\cdot g(x+h)g(x)} =$$

$$= \lim_{h\to 0}\frac{g(x+h)[f(x+h) - f(x)] - f(x+h)[g(x+h) - g(x]}{h\cdot g(x+h)g(x)} =$$

$$= \frac{g(x)}{g(x)g(x)}\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} - \frac{f(x)}{g(x)g(x)}\lim_{h\to 0}\frac{g(x+h) - g(x}{h} = = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$ ◀

2.5.8-misol. $Q(x) = \frac{1 - x^2}{x^3 + 1}$ funksiyaning hosilasini toping.

Yechilishi: ▶

$$Q'(x) = \frac{-2x\cdot(x^3 + 1) - (1 - x^2)\cdot 3x^2}{\left(x^3 + 1\right)^2} = \frac{-2x^4 - 2x - 3x^2 + 3x^4}{(x^3 + 1)^2} = \frac{x^4 - 3x^2 - 2x}{(x^3 + 1)^2}$$ ◀

2.5.9-misol. $f(x)=\dfrac{1-\sin x}{\cos x}$ funksiyaning hosilasini toping.

Yechilishi: ▶ $f'(x)=\left(\dfrac{1-\sin x}{\cos x}\right)'=\dfrac{(1-\sin x)'\cos x-(1-\sin x)(\cos x)'}{\cos^2 x}=$

$$=\frac{-\cos x\cos x-(1-\sin x)(-\sin x)}{\cos^2 x}=\frac{-\cos^2 x+\sin x-\sin^2 x}{\cos^2 x}=\frac{\sin x-1}{\cos^2 x}$$ ◀

2.5.4. Hosilalar jadvali

1) $y=C,\quad y'=0;\quad C$ – const;
2) $y=x,\quad y'=1,\quad x-erkli\ o‘zgaruvchi;$
3) $y=u^{\alpha},\quad y'=\alpha\cdot u^{\alpha-1}\cdot u';$
4) $y=\sqrt{u},\quad y'=\dfrac{1}{2\sqrt{u}}\cdot u'$
5) $y=\dfrac{1}{u},\quad y'=-\dfrac{1}{u^2}\cdot u'$
6) $y=a^u,\quad y'=a^u\cdot lna\cdot u',\quad a-const,a>0,a\neq 1;$
7) $y=e^u,\quad y'=(e^u)'=e^u\cdot u';$
8) $y=u^v,\quad y'=v\cdot u^{v-1}\cdot u'+u^v\cdot lnu\cdot v';$
9) $y=\log_a u,\quad y'=\dfrac{1}{u\cdot lna}\cdot u',a-const,a>0,a\neq 1;$
10) $y=lnu,\quad y'=\dfrac{1}{u}\cdot u';$
11) $y=sinu,\quad y'=cosu\cdot u';$
12) $y=cosu,\quad y'=-sinu\cdot u';$
13) $y=tgu,\quad y'=\dfrac{1}{cos^2u}\cdot u';$
14) $y=ctgu,\quad y'=-\dfrac{1}{sin^2u}\cdot u';$
15) $y=arcsinu,\quad y'=\dfrac{1}{\sqrt{1-u^2}}\cdot u';$
16) $y=arccosu,\quad y'=-\dfrac{1}{\sqrt{1-u^2}}\cdot u';$
17) $y=arctgu,\quad y'=\dfrac{1}{1+u^2}\cdot u';$
18) $y=arcctgu,\quad y'=-\dfrac{1}{1+u^2}\cdot u';$

19) $y = shu,$ $\qquad y' = chu \cdot u';$

20) $y = chu,$ $\qquad y' = shu \cdot u';$

21) $y = thu,$ $\qquad y' = \dfrac{1}{ch^2 u} \cdot u';$

22) $y = cthu,$ $\qquad y' = -\dfrac{1}{sh^2 u} \cdot u';$

23) $y = \dfrac{1}{u^n},$ $\qquad y' = -\dfrac{n}{u^{n+1}} \cdot u'.$

2.5.5. Murakkab funksiya va teskari funksiyaning hosilasi

Murakkab funksiya hosilasi. Faraz qilaylik, $y = f(x)$ funksiya berilgan bo‘lsin. x esa t vaqtning funksiyasi bo‘lsin. y funksiya x ga bog‘liq, x esa t ga bog‘liq, bundan y ning t ga bog‘liq ekanligi kelib chiqadi. Zanjir qoidasidan quyidagi kelib chiqadi:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}. \tag{2.15}$$

Bundan y ning hosilasi x ning hosilasiga bog‘liq ekanini ko‘rish mumkin. Bu istalgan funksiyani har doim vaqtning funksiyasi sifatida qarash zarurligini bildiradi, agar funksiya vaqtga bogliq deb berilmasa ham yoki funksiyaning t bo‘yicha hosilasi 0 ga teng bo‘lsa ham.

Teskari funksiyaning hosilasi. Agar $y = f(x)$ funksiya uchun teskari funksiya mavjud bo`lsa, u holda $x = g(y)$ teskari funksiyaning differensiali quyidagiga teng bo‘ladi (bu yerda $\dfrac{dg}{dy} = g'(y) \neq 0$):

$$f'(x) = \frac{1}{g'(x)}$$

2.5.10-misol. $(arctg\, x)' = \dfrac{1}{1+x^2}$ tenglikni isbotlang.

Yechilishi: ► Tenglikni isbot qilish uchun, $y = arctg\, x$ funksiya hosilasini teskari funksiya hosilasi yordamida hisoblaymiz.

$$y = arctg\, x \quad \Rightarrow \quad \mathrm{tg}(y) = \mathrm{tg}(arctg\, x) \quad \Rightarrow \quad x = \mathrm{tg}\, y$$

$x = \mathrm{tg}\, y$ tenglikning ikkala tomonidan hosila olamiz:

$$(x)' = (\mathrm{tg}\, y)' \quad \Rightarrow \quad 1 = \frac{1}{\cos^2 y} \cdot y'$$

Bu tenglikdan $y' = \cos^2 y$ tenglik kelib chiqadi. Bundan, $y' = \dfrac{1}{1+tg^2 y}$ va $x = \text{tg}\, y$ ni hisobga olsak, $(arctg\, x)' = \dfrac{1}{1+x^2}$ tenglik kelib chiqadi. ◀

Logarifmlab differensiallash.

$y = f(x)$ funksiyani logarifmlab differensiallash uchun quyidagi formuladan foydalanamiz:

$$(\ln f(x))' = \frac{f'(x)}{f(x)} \qquad 2.16)$$

2.5.11-misol. $y = (\sin 2x)^x$ funksiya hosilani toping.

Yechilishi: ►Funksiya hosilasini ikki usul bilan hisoblash mumkin:

1-usul. $y = (\sin 2x)^x$ tenglikni ikkala tomonini logarifmlaymiz va hosila olamiz:

$$\ln y = x \ln \sin 2x.$$

$$(\ln y)' = (x)' \ln \sin 2x + x(\ln \sin 2x)' = \ln \sin 2x + x \cdot \frac{1}{\sin 2x} \cdot 2\cos 2x$$

bu yerdan, $\dfrac{y'}{y} = \ln \sin 2x + 2x \cdot \text{ctg} 2x$.

Demak, $y' = y(\ln \sin 2x + 2x \cdot \text{ctg} 2x) = (\sin 2x)^x (\ln \sin 2x + 2x \cdot \text{ctg} 2x)$.

2-usul: $y = (\sin 2x)^x = e^{\ln (\sin 2x)^x} = e^{x \ln \sin 2x}$ ko‘rinishga keltiramiz.

$$y' = \left(e^{x \ln \sin 2x}\right)' = e^{x \ln \sin 2x}(x \cdot \ln \sin 2x)' = (\sin 2x)^x (\ln \sin 2x + 2x \cdot \text{ctg} 2x). \blacktriangleleft$$

Mavzu yuzasidan savollar

1. Funksiya grafigiga urinma deb nimaga aytiladi?
2. Funksiyaning berilgan x_0 nuqtadagi hosilasi ta’rifini bering.
3. Hosilaning geometrik ma’nosi nimadan iborat?
4. Hosilaning mexanik ma’nosi nimadan iborat?
5. Qanday funksiya nuqtada differensiallanuvchi deyiladi?
6. Qanday funksiya kesmada differensiallanuvchi deyiladi?
7. O‘zgarmas sonning hosilasini keltirib chiqaring.
8. Yig‘indi, ko‘paytma, bo‘linma hosilalari formulalarini keltiring.
9. Funksiyaning differensiali deb nimaga aytiladi?

10. Hosilalar jadvalini yoddan ayting.
11. Murakkab funksiya deb nimaga aytiladi?
12. Murakkab funksiya hosilasi qanday topiladi?
13. Teskari funksiya deb nimaga aytiladi?
14. Teskari funksiya hosilasi formulasini yozing.

MUSTAQIL YECHISH UCHUN MASALALAR

1. Darajali funksiya hosilasini hisoblang:

a) $y=\left(3x-4\sqrt[3]{x}+2\right)^4$;

b) $y=\left(x^2-\dfrac{1}{x^3}+5\sqrt{x}\right)^4$;

c) $y=\left(6x^2-\dfrac{2}{x^4}+5\right)^2$

2. Ko‘rsatkichli va trigonometrik funksiyalar hosilasini hisoblang:

a) $y=2^{3x}\cdot tg\,2x$;

b) $y=e^{tg\,x}\cdot\ln 2x$;

c) $y=x^{x^x}$

3. Funksiya hosilasini hisoblang:

a) $y=4^{\cos x}\cdot arctg\,2x$;

b) $y=\ln\cos 5x$;

c) $y=\dfrac{\sqrt{3-5x^3}}{e^x-ctg\,x}$;

4. Funksiya hosilasini hisoblang:

a) $y=\arcsin\ln 2x$;

b) $y=arctg\ln 8x$;

c) $y=\ln\arcsin 3x$.

5. Funksiya hosilasini hisoblang:

a) $y=x^{1/x}$;

b) $y=x^{\ln x}$.

TESTLAR

1. $y=\sqrt{x}+\frac{2}{\sqrt{x}}+\frac{1}{2}x^{10}$ funksiya hosilasini hisoblang.

A) $y'=\frac{1}{2\sqrt{x}}-\frac{1}{x\sqrt{x}}+5x^9$ **B)** $y'=\frac{1}{2\sqrt{x}}+\frac{1}{x\sqrt{x}}+5x^9$

C) $y'=\frac{1}{2\sqrt{x}}+\frac{1}{x\sqrt{x}}+x^9$ **D)** $y'=\frac{1}{2\sqrt{x}}+\frac{1}{x\sqrt{x}}+10x^9$

2. $y=x+\frac{1}{x^2}-\frac{1}{5x^5}$ funksiya hosilasini hisoblang.

A) $y'=1-\frac{2}{x^3}+\frac{1}{x^6}$ **B)** $y'=1-\frac{1}{x^3}+\frac{1}{x^6}$

C) $y'=1+\frac{2}{x^3}+\frac{1}{x^6}$ **D)** $y'=1-\frac{2}{x^3}-\frac{1}{x^6}$

3. $y=\ln\cos x-\frac{1}{2}\cos^2 x$ funksiya hosilasini hisoblang.

A) $y'=-tgx+\frac{1}{2}\sin 2x$ **B)** $y'=tgx+\frac{1}{2}\sin 2x$

C) $y'=tgx-\frac{1}{2}\sin 2x$ **D)** $y'=-tgx+\sin 2x$

4. $y=\arccos(1-2x)$ funksiya hosilasini hisoblang.

A) $y'=\frac{1}{\sqrt{x-x^2}}$ **B)** $y'=\frac{1}{\sqrt{x^2-x}}$

C) $y'=\frac{1}{\sqrt{2x-x^2}}$ **D)** $y'=\frac{1}{\sqrt{x+x^2}}$

5. $y=\arcsin(e^{2x})$ funksiya hosilasini hisoblang.

A) $y'=\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ **B)** $y'=\frac{e^{2x}}{\sqrt{1-e^{4x}}}$

C) $y'=\frac{2e^{x}}{\sqrt{1-e^{4x}}}$ **D)** $y'=\frac{2e^{2x}}{\sqrt{1-e^{2x}}}$.

2.5.6. Oshkormas funksiya va parametrik funksiyalarni differensiallash

Ko‘pincha biz funksiyani yozganda x erkli o‘zgaruvchi bilan y erksiz o‘zgaruvchini tenglikning turli tomonlarida ajratib yozamiz. Bunday $y=f(x)$ yoki $y=x^3-2x+1$ ko‘rinishdagi funksiyaga **oshkor funksiya** deyiladi. Shunday murakkab tuzilishdagi oshkormas funksiyalar borki, ularni oshkor shaklda yozishning iloji yo‘q. Misol uchun $y^3+y^2x^5-x^3=15$ funksiyani $y=f(x)$ ko‘rinishga keltirish qiyin.

$F(x,y)=0$ ko‘rinishdagi funksiyaga **oshkormas funksiya** deyiladi.

Funksiyani oshkor ko‘rinishga o‘tkazish uchun bu tenglamani y ga nisbatan yechish kerak.

2.5.12-misol. $x^3+2y^3=6$ funksiyani oshkor ko‘rinishda yozing.

Yechilishi: ▶ Buning uchun $x^3+2y^3=6$ tenglamani y ga nisbatan yechamiz: $2y^3=6-x^3 \quad \Rightarrow \quad y=\sqrt[3]{\dfrac{6-x^3}{2}}$.

Demak, $x^3+2y^3=6$ funksiyaning oshkor shakli $y=\sqrt[3]{\dfrac{6-x^3}{2}}$ ekan. ◀

2.5.13-misol. $x^2-6xy+9y^2=5$ funksiyani oshkor ko‘rinishga keltiring.

Yechilishi: ▶ $x^2-6xy+9y^2=5$ tenglamani y ga nisbatan yechamiz: $(x-3y)^2=5$; $\Rightarrow$ $x-3y=\sqrt{5}$; $\Rightarrow$ $y=\dfrac{x-\sqrt{5}}{3}$.

Demak, $x^2-6xy+9y^2=5$ funksiyaning oshkor shakli $y=\dfrac{x-\sqrt{5}}{3}$ ekan. ◀

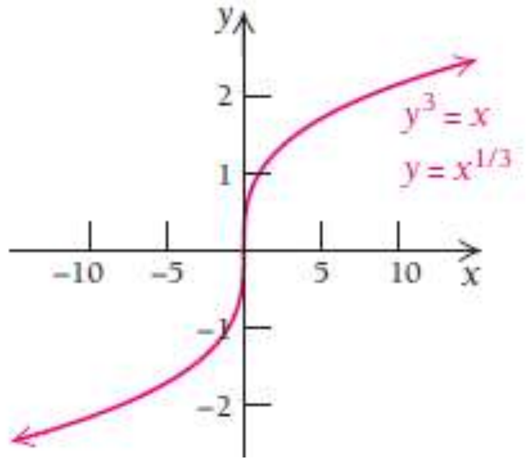
Oshkormas funksiyadan hosila olishning 2 xil usuli mavjud:

1) Funksiyani oshkor ko‘rinishga keltirib, so‘ngra hosila olish;

2) Oshkormas holida to‘g‘ridan – to‘g‘ri hosila olish.

1-usul. $y^3=x$ tenglamani qaraylik. Bu tenglamada y funksiya oshkormas ko‘rinishda (2.28-rasm). Funksiyani oshkor ko‘rinishga keltiramiz:

$y=\sqrt[3]{x}$ yoki $y=x^{\frac{1}{3}}$. Endi undan hosila olamiz: $y'=\left(x^{\frac{1}{3}}\right)'=\dfrac{1}{3}x^{-\frac{2}{3}}$.



2.28-rasm. $y = x^{\frac{1}{3}}$ funksiya grafigi

2-usul. Oshkormas funksiyadan hosila olish uchun tenglikning ikki tomoni ham differensiallanadi. Bunda y ni x ning funksiyasi deb qaraladi: $y^3 = x$.

$$\left(y^3\right)' = x' \quad \Rightarrow \quad 3y^2 \cdot y' = 1 \quad \Rightarrow y' = \frac{1}{3y^2} \text{ yoki } y' = \frac{1}{3}y^{-2} = \frac{1}{3}\left(x^{\frac{1}{3}}\right)^{-2} = \frac{1}{3}x^{-\frac{2}{3}}.$$

Aytaylik, oshkormas $F(x, y) = 0$ funksiya berilgan bo‘lsin. Undan hosila olamiz:

$$F_x'(x, y) + F_y'(x, y) \cdot y_x' = 0;$$

$$F_y'(x, y) \cdot y_x' = -F_x'(x, y);$$

$$y_x' = -\frac{F_x'(x, y)}{F_y'(x, y)} \tag{2.17}$$

(2.17) formula **oshkormas funksiyaning hosilasini topish formulasidir**.

2.5.14-misol. $y^3 + y^2x^5 - x^3 = 15$ funksiya hosilasini toping.

Yechilishi: ► Avvalo $y^3 + y^2x^5 - x^3 = 15$ tenglamani $F(x, y) = 0$ ko‘rinishga keltirib olamiz. $y^3 + y^2x^5 - x^3 - 15 = 0$. So‘ngra hosila olamiz:

$$y_x' = -\frac{F_x'(x, y)}{F_y'(x, y)} = \frac{(y^3 + y^2x^5 - x^3 - 15)_x'}{(y^3 + y^2x^5 - x^3 - 15)_y'} = \frac{5y^2x^4 - 3x^2}{3y^2 + 2yx^5} \blacktriangleleft$$

2.5.15-misol. $e^y + y - x = 0$ funksiyaning $M(1,0)$ nuqtadagi ikkinchi tartibli hosilasini toping.

Yechilishi: ► Berilgan oshkormas funksiyani birinchi tartibli hosilasini topamiz: $\left(e^y + y - x\right)' = 0'$ va $e^y \cdot y' + y' - 1 = 0$

tenglikdan y' ni topamiz: $y'=\frac{1}{e^y+1}$. y' hosilani M nuqtadagi qiymati $y'(M)=\frac{1}{2}$ bo‘ladi. Ikkinchi tartibli hosilasini topamiz:

$$y''=\left(\frac{1}{e^y+1}\right)'=-\frac{\left(e^y+1\right)'}{\left(e^y+1\right)^2}=-\frac{e^y\cdot y'}{\left(e^y+1\right)^2}$$

y'' hosilani M nuqtadagi qiymati $y''(M)=-\frac{e^y\cdot y'}{(e^y+1)^2}=-\frac{e^0\cdot\frac{1}{2}}{(e^0+1)^2}=-\frac{1}{8}$. ◀

Parametrik ko‘rinishda berilgan funksiya va uning hosilasi.

x va y o‘zgaruvchilar orasidagi funksional bog‘lanishni har doim ham $y=f(x)$ oshkor ko‘rinishda yoki $F(x,y)=0$ oshkormas ko‘rinishda yozish qulay bo‘lmaydi. Ba’zan yordamchi o‘zgaruvchi t ni kiritib, x va y o‘zgaruvchilarni t ning funksiyasi sifatida ifodalash qulay bo‘ladi:

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}.$$

Bu tenglama funksiyaning parametrik berilishi bo‘lib, t ning ixtiyoriy qiymatlariga x va y ning aniq qiymati mos keladi. x va y ning qiymatlari juftiga tekislikda $M(x,y)$ nuqtani aniqlaydi. t parametr aniqlanish sohasidan hamma qiymatlarni qabul qilganda $M(x,y)$ nuqta xOy tekislikda biror chiziqni chizadi. Yuqoridagi tenglamani shu **chiziqning parametrik tenglamasi** deyiladi. y ning x ga oshkor bog‘liqligini topish uchun sistema tenglamalaridan t parametrni topish kerak. Buning uchun bu sistemaning birinchi tenglamasidan t ni x ning funksiyasi sifatida ifodalaymiz: $t=u(x)$, uni ikkinchi tenglamaga qo‘yib, $y=\psi(u(x))$ ga yoki $y=f(x)$ ga ega bo‘lamiz.

2.5.16-misol. To‘g‘ri chiziqning tekislikdagi parametrik tenglamasi

$$\begin{cases} x=x_0+mt \\ y=y_0+nt \end{cases}$$

berilgan bo‘lsa, uning kanonik tenglamasini tuzing (bunda $m,n-$ yo‘naltiruvchi vektor koordinatalari).

Yechilishi: ► Kanonik tenglamani yozish uchun sistemadagi tenglamalardan t o‘zgaruvchini topib olamiz va o‘zaro tenglashtiramiz:

$$\frac{x-x_0}{m}=t\,,\qquad \frac{y-y_0}{n}=t\,.$$

Bundan to‘g‘ri chiziqning kanonik tenglamasi kelib chiqadi:

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}.$$ ◀

2.5.17-misol. Aylananing parametrik tenglamasi $\begin{cases} x = Rcost \\ y = Rsint \end{cases}$ berilgan. Aylananing umumiy tenglamasini yozing.

Yechilishi: ► Tenglamaning har birini kvadratga ko‘taramiz va qo‘shamiz, natijada

$$\begin{cases} x^2 = R^2cos^2t \\ y^2 = R^2sin^2t \end{cases} \Rightarrow \quad x^2 + y^2 = R^2(cos^2t + sin^2t)$$

aylana tenglamasi kelib chiqadi: $x^2 + y^2 = R^2$. ◄

Parametrik berilgan $\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$ funksiya hosilasini topish uchun formula keltirib chiqaramiz; bunda $x = \varphi(t)$ funksiya teskari funksiyaga ega bo‘lsin. Bu yerda y ni x ning murakkab funksiyasi deb hisoblash mumkin, bunda t oraliq argument. Shu sababli murakkab funksiyani differensiallash qoidasiga ko‘ra: $y'_x = y'_t \cdot t'_x$. Ammo bunda x o‘zgaruvchining t funksiyasi emas, balki t o‘zgaruvchining x funksiyasi berilgan, shu sababli teskari funksiyani differensiallash qoidasiga ko‘ra, $t'_x = \frac{1}{x'_t}$ ifodani yuqoridagi tenglikka qo‘yib, parametrik berilgan funksiya uchun hosila formulasini topamiz:

$$y'_x = \frac{y'_t}{x'_t} \tag{2.18}$$

Parametrik berilgan funksiyaning 2-tartibli hosilasi quyidagicha topiladi:

$$y''_x = \frac{(y'_x)'_t}{x'_t} = \frac{\left(\frac{y'_t}{x'_t}\right)'_t}{x'_t} = \frac{\frac{y''_{tt} \cdot x'_t - y'_t \cdot x''_{tt}}{(x'_t)^2}}{x'_t} = \frac{y''_{tt} \cdot x'_t - y'_t \cdot x''}{(x'_t)^3},$$

Demak,

$$y''_x = \frac{y''_{tt} \cdot x'_t - y'_t \cdot x''}{(x'_t)^3}. \tag{2.19}$$

2.5.18-misol. $\begin{cases} x = a(t - sint) \\ y = a(1 - cost) \end{cases}$ funksiya hosilasini toping.

Yechilishi: ► (2.17) formulaga ko‘ra,

$$y'_x = \frac{y'_t}{x'_t} = \frac{\big(a(1 - cost)\big)'}{\big(a(t - sint)\big)'} = \frac{a \cdot sint}{a(1 - cost)} = ctg\frac{t}{2}.$$

◄

2.5.19-misol. $\begin{cases} y = t^3 + t^2 + 1 \\ x = \dfrac{1}{t} \end{cases}$ tenglama bilan berilgan funksiyaning ikkinchi tartibli hosilasini toping.

Yechilishi: ► Dastlab (2.18), so‘ngra (2.19) formulalardan foydalanamiz.

1) $\dfrac{dy}{dx} = \dfrac{\left(t^3 + t^2 + 1\right)'_t}{\left(\dfrac{1}{t}\right)'_t} = \dfrac{3t^2 + 2t}{-\dfrac{1}{t^2}} = -3t^4 - 2t^3$;

2) $y''_x = \dfrac{y''_{tt} \cdot x'_t - y'_t \cdot x''}{(x'_t)^3} = \dfrac{(-3t^4 - 2t^3)' \cdot \left(-\dfrac{1}{t^2}\right) - (-3t^4 - 2t^3) \cdot \left(-\dfrac{1}{t^2}\right)'}{\left(-\dfrac{1}{t^2}\right)^3} =$

$$= \frac{(-12t^3 - 6t^2) \cdot \left(-\dfrac{1}{t^2}\right) - (-3t^4 - 2t^3) \cdot \dfrac{2}{t^3}}{\left(-\dfrac{1}{t^2}\right)^3} = \frac{12t + 6 + 6t + 4}{\left(-\dfrac{1}{t^2}\right)^3} = -18t^7 - 10t^6 .$$ ◄

2.5.7. Yuqori tartibli hosilalar

Bizga $y = f(x) = x^5 - 3x^4 + x + 252$ funksiya berilgan bo‘lsin. Undan $f'(x)$ hosila olamiz: $y' = f'(x) = 5x^4 - 12x^3 + 1$.
Funksiyaning hosilasi yana differensiallanuvchi ekan, yana hosila olish mumkin. Keyingi hosilalar uchun belgilashlar kiritamiz. U holda 2-marta olinadigan hosilani $f''(x) = [f'(x)]'$ deb belgilaymiz:

$$y'' = f''(x) = (5x^4 - 12x^3 + 1)' = 20x^3 - 36x^2 .$$

Xuddi shuningdek, 3-, 4- va h.k. tartibli hosilalarini olish mumkin.

$y''' = f'''(x) = (20x^3 - 36x^2)' = 60x^2 - 72x$;

$y^{(IV)} = f^{(IV)}(x) = (60x^2 - 72x)' = 120x - 72$;

$y^{(V)} = f^{(V)}(x) = (120x - 72)' = 120$;

$y^{(VI)} = f^{(VI)}(x) = 120' = 0$;

…

$y^{(n)} = 0$, $n \geq 6$ butun sonlar uchun.

2.5.20-misol. $y=(x^2+10x)^{20}$ funksiyaning y' va y'' hosilalarini toping.

Yechilishi: ► 1-tartibli hosila:

$$y' = 20(x^2+10x)^{19}\cdot(2x+10) =$$
$$= 20(x^2+10x)^{19}\cdot 2(x+5) = 40(x^2+10x)^{19}(x+5);$$

2-tartibli hosila:

$$y'' = [40(x^2+10x)^{19}(x+5)]' = 40\cdot 19(x^2+10x)^{18}(2x+10)(x+5)+(x^2+10x)^{19}\cdot 1 =$$
$$= 40(x^2+10x)^{18}(38x^2+380x+950+x^2+10x) = 40(x^2+10x)^{18}(39x^2+390x+950).$$ ◄

2.5.21-misol. $y=\sin x$ funksiyaning $n-$ tartibli hosilasini toping.

Yechilishi: ► Berilgan funksiyaning $n-$tartibli hosilasini topamiz.

$$y' = \cos x = \sin\left(x+\frac{\pi}{2}\right),$$
$$y'' = \cos\left(x+\frac{\pi}{2}\right) = \sin\left(x+2\cdot\frac{\pi}{2}\right),$$
$$y''' = \cos\left(x+2\cdot\frac{\pi}{2}\right) = \sin\left(x+3\cdot\frac{\pi}{2}\right),\ldots,$$

Umumiy formula chiqarishga harakat qilamiz:

$$y^{(n)} = \cos\left(x+(n-1)\frac{\pi}{2}\right) = \sin\left(x+n\cdot\frac{\pi}{2}\right).$$ ◄

Mavzu yuzasidan savollar

1. Qanday funksiyalarga oshkormas funksiya deyiladi?
2. Oshkormas holda berilgan funksiyalar qanday differensiallanadi?
3. Parametrik ko‘rinishda berilgan funksiyaga ta’rif bering.
4. Parametrik ko‘rinishda berilgan funksiyani differensiallash qanday bajariladi?
5. Parametrik ko‘rinishda berilgan funksiyaning 2-tartibli hosilasi formulasini keltirib chiqaring.
6. Funksiyaning ikkinchi tartibli hosilasi deb nimaga aytiladi?
7. Funksiyaning *n*- tartibli hosilasi deb nimaga aytiladi?

MUSTAQIL YECHISH UCHUN MASALALAR

1. Oshkormas funksiya hosilasini toping:

a) $tg\left(\frac{y}{x}\right) = 5x$**;**

b) $x - y + arctg\, y = 0$**;**

c) $\left(e^x - 1\right)\left(e^y - 1\right) - 1 = 0$**.**

2. Parametrik ko‘rinishda berilgan funksiya hosilasini hisoblang:

a) $\begin{cases} x = a(\cos t + t\sin t) \\ y = a(\sin t - t\cos t) \end{cases}$

b) $\begin{cases} x = \dfrac{1}{t+1} \\ y = \left(\dfrac{t}{t+1}\right)^2 \end{cases}$

c) $\begin{cases} x = t + \ln\cos t \\ y = t - \ln\sin t \end{cases}$

3. Parametrik ko‘rinishda berilgan funksiyaning 2-tartibli hosilasini toping:

a) $\begin{cases} x = \dfrac{2at}{1+t^2} \\ y = \dfrac{a(1-t^2)}{1+t^2} \end{cases}$

b) $\begin{cases} x = a\cos^3 t \\ y = b\sin^3 t \end{cases}$

c) $\begin{cases} x = t^3 + 8 \\ y = t^5 + 2t \end{cases}$

4. Funksiyaning $n-$ tartibli hosilasini toping:

a) $y = sin2x$;

b) $y = cos3x$;

c) $y = x^n$.

5. $d\left(\arcsin\frac{1}{x}\right)$ni hisoblang.

TESTLAR

1. $y=\ln\left(x+\sqrt{1+x^2}\right)$ funksiyaning differensialini toping:

A) $dy=\dfrac{dx}{x\sqrt{1+x^2}}$ **B)** $dy=\dfrac{dx}{\sqrt{1+x^2}}$

C) $dy=\dfrac{xdx}{\sqrt{1+x^2}}$ **D)** $dy=\dfrac{xdx}{1+\sqrt{1+x^2}}$.

2. $y=\arcsin(e^{2x})$ bo‘lsa, $y'=?$

A) $y'=\dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$ **B)** $y'=\dfrac{e^{2x}}{\sqrt{1-e^{4x}}}$

C) $y'=\dfrac{2e^{x}}{\sqrt{1-e^{4x}}}$ **D)** $y'=\dfrac{2e^{2x}}{\sqrt{1-e^{2x}}}$

3. $\begin{cases} x=1-t^2 \\ y=t^2+t \end{cases}$ parametrik funksiya berilgan. $y'_x=?$

A) $y'_x=\dfrac{-2t}{2t+1}$ **B)** $y'_x=\dfrac{2t}{2t+1}$

C) $y'_x=\dfrac{2t+1}{2t}$ **D)** $y'_x=\dfrac{2t+1}{-2t}$.

4. $xy-x^2y^2=0$ oshkormas funksiya berilgan. $y'=?$

A) $y'=\dfrac{y(2xy-1)}{x(1-2xy)}$ **B)** $y'=\dfrac{x(2xy-1)}{y(1-2xy)}$

C) $y'=\dfrac{2xy-1}{1-2xy}$ **D)** $y'=\dfrac{y(2xy+1)}{x(1+2xy)}$

5. $x^2+\ln y=0$ oshkormas funksiya berilgan. $y'=?$

A) $y'=-2xy$ **B)** $y'=2xy$

C) $y'=\dfrac{2x}{y}$ **D)** $y'=\dfrac{y}{2x}$.

2.6-§. Funksiyaning differensiali. Differensial hisobning asosiy teoremalari (Roll, Lagranj va Koshi teoremalari)

2.6.1. Funksiyaning differensiali

$y = f(x)$ funksiya $[a, b]$ kesmada differensiallanuvchi bo‘lsin. Bu har qanday $x \in [a, b]$ uchun $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ chekli hosila mavjud ekanligini bildiradi.

$f(x) \neq 0$ deb faraz qilaylik, u holda yuqoridagi tenglikdan

$$\frac{\Delta y}{\Delta x} = f'(x) + \alpha \tag{2.20}$$

ekani kelib chiqadi, bunda $\Delta x \to 0$ da $\alpha \to 0$. Agar oxirgi tenglikning hamma hadini Δx ga ko‘paytirsak, ushbu

$$\Delta y = f'(x) \cdot \Delta x + \alpha \cdot \Delta x \tag{2.21}$$

yoki $$\Delta y = f'(x) \cdot \Delta x + \beta$$

munosabatga ega bo‘lamiz, bunda $\beta = \alpha \cdot \Delta x$. $\Delta x \to 0$ da (2.21) formuladagi ikkala qo‘shiluvchi nolga intiladi. Ularni Δx bilan taqqoslaymiz:

$$\lim_{\Delta x \to 0} \frac{\beta}{\Delta x} = \lim_{\Delta x \to 0} \frac{\alpha \cdot \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \alpha = 0 .$$

Shunday qilib, birinchi qo‘shiluvchi $f'(x) \cdot \Delta x$ tartibi Δx tartibiga teng bo‘lgan cheksiz kichik miqdordir, u Δx ga nisbatan chiziqli;

ikkinchi qo‘shiluvchi $\beta = \alpha \cdot \Delta x$ darajasi Δx darajasidan yuqori bo‘lgan cheksiz kichik miqdordir, ya’ni u nolga intiladi. Bundan (2.21) formulada birinchi qo‘shiluvchi $f'(x) \cdot \Delta x$ asosiy ekanligi kelib chiqadi. Ana shu qo‘shiluvchiga **funksiyaning differensiali** deyiladi:

$$dy = f'(x) \cdot \Delta x \tag{2.22}$$

Demak, agar $y = f(x)$ funksiya x nuqtada hosilaga ega bo‘lsa, u holda funksiyaning differensiali funksiyaning hosilasi $f'(x)$ ni erkli o‘zgaruvchining Δx orttirmasiga ko‘paytirilganiga teng bo‘ladi, shu bilan birga Δx x ga bog‘liq bo‘lmaydi.

$y = x$ funksiya differensialini topamiz. $y' = 1$ bo‘lgani uchun yoki $dy = dx$, ya’ni erkli o‘zgaruvchining orttirmasi uning differensialiga teng. U holda (2.22) formula bunday yoziladi:

$$dy = f'(x) \cdot dx = y' \cdot dx \tag{2.23}$$

Bu formula hosila bilan differensialni bog‘laydi, shu bilan birga hosila chekli son, differensial esa cheksiz kichik miqdordir.

2.6.1-misol. $y' = \cos(2x+3)$ funksiya differensialini toping.

Yechilishi: ►

$$\frac{dy}{dx} = \left(\cos(2x+3)\right)' = \frac{d(\cos(2x+3))}{dx} = -\sin(2x+3)\cdot(2x+3)' = -2\sin(2x+3)$$

$$dy = -2\sin(2x+3)dx \quad ◄$$

(2.23) tenglikdan $y' = \frac{dy}{dx}$ ga egamiz, ya’ni hosilani funksiya differensialining erkli o‘zgaruvchi differensialiga nisbati deb qarash mumkin. Funksiyaning differensialini topish masalasi hosilani topishga teng kuchli, chunki hosilani erkli o‘zgaruvchi orttirmasiga ko‘paytirib, funksiya differensialiga ega bo‘lamiz. Shunday qilib, hosilalarga tegishli teoremalar va formulalarning ko‘pchiligi differensiallar uchun ham to‘g‘ri bo‘lib qolaveradi.

Agar u va v differensiallanuvchi funksiyalar bo‘lsa, u holda quyidagi formulalar o‘rinli bo‘ladi:

1. $d(u \pm v) = du \pm dv,$

2. $d(Cu) = Cdu, \ C - const.$

3. $d(u \cdot v) = vdu + udv,$

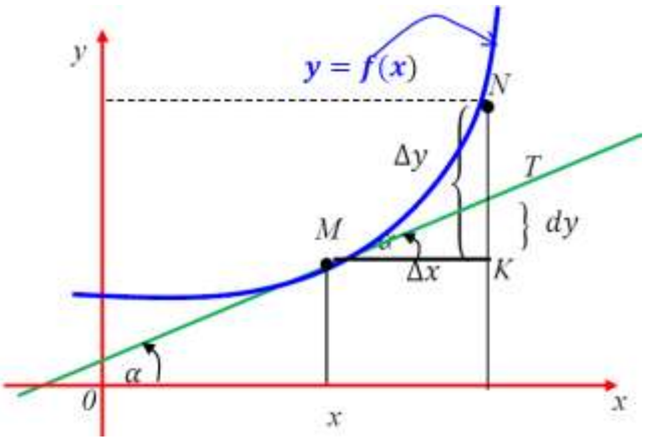
4. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$

2.6.2. Funksiya differensialining geometrik ma’nosi

Aytaylik, $y = f(x)$ funksiya grafigi egri chiziqdan iborat bo‘lsin. Egri chiziqda $M(x, y)$ nuqtani olamiz, shu nuqtada egri chiziqqa urinma o‘tkazamiz, urinma Ox o‘qning musbat yo‘nalishi bilan hosil qiladigan burchakni α bilan belgilaymiz. Erkli o‘zgaruvchi x ga Δx orttirma beramiz, u holda funksiya $\Delta y = f(x+\Delta x) + f(x)$ orttirmani oladi. Chizmada $\Delta y = KN,$ N nuqta esa $N(x+\Delta x, \ f(x+\Delta x))$ yoki ΔMKN dan: $TK = MK \cdot tg\alpha.$ Ammo $tg\alpha = f'(x),$ $MK = \Delta x,$ shu sababli

$$TK = f'(x)\cdot\Delta x.$$

Differensialning ta’rifiga binoan $dy = f'(x)\cdot\Delta x.$ Shunday qilib, $TK = dy.$ Bu differensialning $y = f(x)$ egri chiziqqa x nuqtada o‘tkazilgan urinmaning orttirmasiga teng ekanligini bildiradi. Differensialning geometrik ma’nosi shundan iborat.



2.29-rasm. Differensialning geometrik ma’nosi

Chizmadan $NT = \Delta y - dy$ ekani kelib chiqadi. Ammo $\Delta y \sim dy$ shu sababli $\Delta x \to 0$ da $\frac{NT}{TK} \to 0.$ Chizmada $\Delta y > dy.$ 2.29-rasmdan Δy dy dan kichik bo‘lishi mumkinligini ko‘ramiz. Agar $y = f(x)$ to‘g‘ri chiziq bo‘lsa, u holda $\Delta y = dy$.

2.6.3. Yuqori tartibli differensiallar. Invariantlikning buzilishi

$y = f(x)$ funksiyani qaraymiz, bunda x erkli o‘zgaruvchi. Bu funksiyaning differensiali $dy = f'(x) \cdot dx.$ ham x ning funksiyasi bo‘ladi, bunda $f'(x)$ birinchi ko‘paytuvchi x ga bog‘liq bo‘lishi mumkin, 2-ko‘paytuvchi esa argumentning Δx orttirmasi x ga bog‘liq emas, shu sababli bu funksiyaning differensiali haqida gapirish mumkin.

Funksiya differensialidan olingan differensialga **ikkinchi tartibli differensial** deyiladi $d^2 y$ deb belgilanadi: $d(dy) = d^2 y.$

2-tartibli differensialdan olingan differensialga **uchinchi tartibli differensial** deyiladi va $d^3 y$ deb belgilanadi: $d(d^2 y) = d^3 y.$

$(n-1)$-tartibli differensialdan olingan differensial n-tartibli differensial deyiladi va $d^n y$ deb belgilanadi: $d(d^{n-1} y) = d^n y.$

Endi yuqori tartibli differensiallarni hosilalar orqali ifodalaymiz.

Ikkinchi tartibli differensialning ifodasini topamiz:

$$d^2 y = d(dy) = d(y'dx) = (y'dx)'dx = y''dxdx = y''dx^2.$$

Shunday qilib: $$d^2 y = y''dx^2 \, . \qquad (2.24)$$

Bu yerda $dx^2 = (dx)^2$, chunki differensial darajasini yozishda qavslarni tushirib qoldirish qabul qilingan. Bundan keyin $(dx)^3$ o‘rniga dx^3 deb yozamiz va buni dx ifodaning kubi deb tushinamiz.

Uchinchi tartibli differensialning ifodasini ham shunga o‘xshash topamiz: $d^3y = d(d^2y) = d(y''dx^2) = (y''dx^2)'dx = y'''dx^3 \;\; \Rightarrow$ Shunday qilib,

$$dy^3 = y'''dx^3. \qquad (2.25)$$

Bu jarayonni davom ettirib, $n -$ tartibli differensial ifodasini topamiz:

$$y = d(d^{n-1}y) = d\left(y^{(n-1)}dx^{(n-1)}\right) = \left(y^{(n-1)}dx^{n-1}\right)'dx = y^{(n)}dx^n.$$

Shunday qilib,

$$d^ny = y^{(n)}dx^n. \qquad (2.26)$$

2.6.2-misol. $y = 5x^4 - 12x^3 + 1$ funksiyaning 5-tartibli differensialini toping.

Yechilishi: ►

1- tartibli differensial $\dfrac{dy}{dx} = 5x^4 - 12x^3 + 1$;

2- tartibli differensial $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = 20x^3 - 36x^2$;

3- tartibli differensial $\dfrac{d^3y}{dx^3} = 60x^2 - 72x$;

4- tartibli differensial $\dfrac{d^4y}{dx^4} = 120x - 72$;

5- tartibli differensial $\dfrac{d^5y}{dx^5} = 120$ ko‘rinishda yoziladi. ◄

2.6.3-misol. $y = \dfrac{1}{x}$ funksiya uchun $\dfrac{d^2y}{dx^2}$ ni toping.

Yechilishi: ►

Dastlab, 1-tartibli differensialni $\dfrac{dy}{dx} = \left(x^{-1}\right)' = -1\cdot x^{-2} = -\dfrac{1}{x^2}$ topamiz.

Endi 2-tartibli differensialni aniqlaymiz:

$$\frac{d^2y}{dx^2} = \left(-x^{-2}\right)' = -(-2)x^{-3} = \frac{2}{x^3}. \qquad ◄$$

Yuqori tartibli differensialdan foydalanib, har qanday tartibli hosilani differensiallarning nisbati sifatida tasvirlash mumkin:

$$y' = \frac{dy}{dx},\;\; y'' = \frac{d^2y}{dx^2},\;\; y''' = \frac{d^3y}{dx^3}, \dots ,\;\; y^{(n)} = \frac{d^ny}{dx^n}.$$

Hozirga qadar hamma formulalarda x o‘zgaruvchi erkli bo‘lib keldi. Endi x oraliq argument bo‘lsin, ya’ni $y = f(x)$ va bunda $x = \varphi(t)$. Bu holda ham differensial shakli saqlanishini tekshirib ko‘ramiz. Bilamizki, 1-tartibli differensial, x erkli o‘zgaruvchi yoki oraliq funksiya bo‘lishiga qaramay, o‘z shaklini saqlaydi, ya’ni $dy = y'dx$, bunda

$$dx = \varphi'(t)dt \neq const.$$

2-tartibli differensial uchun ifoda topamiz:

$$d^2y = d(dy) = d(y')dx + y'd(dx) = y''dx^2 + y'd^2x. \quad (2.26)$$

Xuddi shuningdek, 2-tartibli differensialdan boshlab, keyingi differensiallarning hammasi differensial shakli invariantligi xossasiga ega bo‘lmaydi. Invariantlik xossasi faqat 1-tartibli differensial uchun o‘rinli.

2.6.4-misol. $y = cosx$ funksiya uchun dy va d^2y larni toping, x erkli o‘zgaruvchi.

Yechilishi: ► $dy = y'dx = -sinxdx,$

$$d^2y = y''dx^2 = -cosxdx^2. \quad ◄$$

2.6.5-misol. $y = cosx$ va $x = lnt$ murakkab funksiyaning dy va d^2y larni toping.

Yechilishi: ►1) $dy = y'dx = -sinxdx \cdot \frac{dt}{t} = -sinxdx^2,$

chunki $\frac{dt}{t} = dx.$

2) $d^2y = y''dx^2 + y'd^2x = -cosx \cdot \left(\frac{dt}{t}\right)^2 - sinx \cdot \frac{dt^2}{t^2} =$

$$= -cosxdx^2 - sinxd^2x,$$

chunki $\left(\frac{1}{t} \cdot dt\right)^2 = dx^2$, $\left(\frac{dt^2}{t^2}\right) = d^2x.$

Shunday qilib, $d^2y = -\,cosx \cdot dx^2 - sinx \cdot d^2x$ formula o‘rinli. ◄

Mavzu yuzasidan savollar

1. Funksiyaning differensiali deb nimaga aytiladi?
2. Differensialni hisoblashning qanday qoidalarini bilasiz?
3. Differensialning geometrik ma’nosi deganda nimani tushunasiz?
4. n- tartibli differensial ta’rifini ayting.
5. Murakkab funksiyaning differensiali qanday topiladi?
6. Differensialda invariantlikning buzilishi deganda nimani tushunasiz?

MUSTAQIL YECHISH UCHUN MASALALAR

1. Differensial yordamida $\sqrt[3]{84}$ ni taqribiy qiymatini toping.

2. $d\left(\arccos\frac{1}{x}\right)$ ni hisoblang.

3. $y = xtg^3 x$ bo‘lsa, dy -?

4. $d^2\left(\ln\left(1+x^2\right)\right)$ ni hisoblang.

5. $d^4\left(\sin^2 x\right)$ ni hisoblang.

TESTLAR

1. $y = \left(x^2+1\right)arctgx + a$ bo‘lsa, dy -?

A) $\left(2xarctgx+1\right)dx$; **B)** $\left(x^2 arctgx+1\right)dx$;

C) $2\left(xarctgx-1\right)dx$; **D)** $2\left(xarctgx+1\right)dx$.

2. $d^2\left(\frac{\ln x}{x}\right)$ ni hisoblang.

A) $\frac{3\left(\ln x-1\right)}{x^3}dx$; **B)** $\frac{2\ln x-3}{x^3}dx^2$;

C) $\frac{3+2\ln x}{x^4}d^2x$; **D)** $-\frac{3+2\ln x}{x^3}dx^2$.

3. $d^3\left(\sin^2 2x\right)$ ni hisoblang:

A) $-32\sin 4xdx^3$; **B)** $8\sin 4xdx^3$;

C) $-32\cos 4xdx^3$; **D)** $8\cos 4xdx^3$.

4. $d\left(\arcsin\frac{1}{x}\right)$ ni hisoblang:

A) $dy = -\frac{dx}{x\sqrt{x^2-1}}$; **B)** $dy = \frac{dx}{x\sqrt{x^2-1}}$;

C) $dy = \frac{dx}{x\sqrt{1-x^2}}$; **D)** $dy = -\frac{dx}{x\sqrt{1-x^2}}$.

5. $y = e^{-x^3}$ bo‘lsa, dy -?

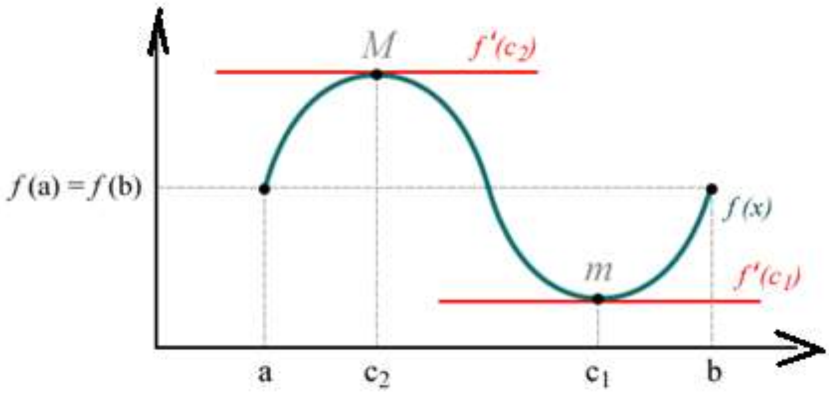
A) $dy = -3x^2e^x dx$; **B)** $dy = e^{-3x^2}dx$;

C) $dy = -3x^2e^{-x^3}$; **D)** $dy = -3x^2e^{-x^3}dx$.

2.6.4. Roll, Lagranj va Koshi teoremalari

2.15-teorema. Roll teoremasi (hosilaning nollari haqida).

Agar $y = f(x)$ funksiya $[a,b]$ kesmada aniqlangan, uzluksiz va differensiallanuvchi bo‘lib, kesmaning oxirlarida teng $f(a) = f(b)$ qiymatlarni qabul qilsa, u holda kesmaning ichida kamida bitta $c \in (a,b)$ nuqta mavjudki, unda hosila nolga teng, ya’ni $f'(c) = 0$.



2.30-rasm. Roll teoremasining geometrik tatbig‘i

Teoremaning shartlaridan aqalli bittasining buzilishi teorema tasdig‘ining buzilishiga olib keladi.

2.15-teoremaning geometrik ma’nosi: shunday $c \in (a,b)$ nuqta mavjudki, bu nuqtada funksiya grafigiga o‘tkazilgan urinma *Ox* o‘qiga parallel bo‘ladi (2.30-rasm).

2.16-teorema. Lagranj teoremasi (chekli orttirmalar haqida).

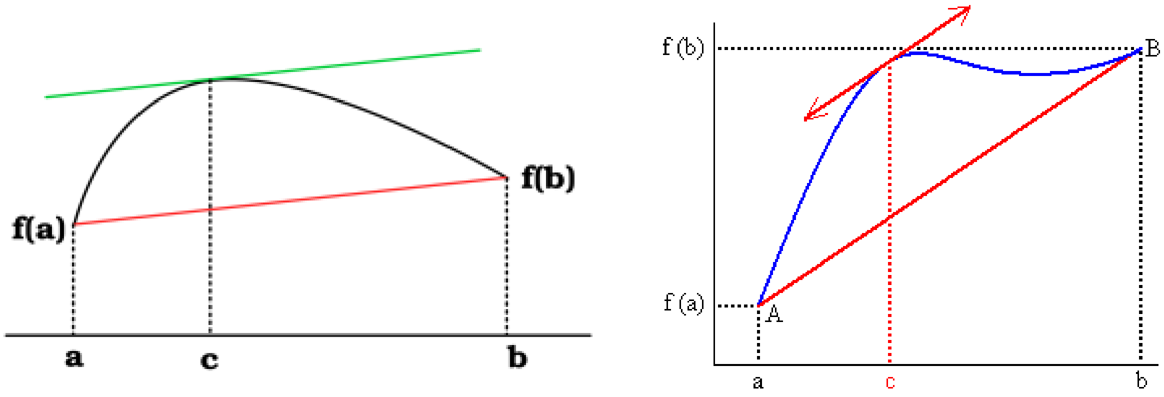
Agar $y = f(x)$ funksiya $[a,b]$ kesmada aniqlangan, uzluksiz va differensiallanuvchi bo‘lsa, u holda $[a,b]$ kesma ichida kamida bitta $c \in (a,b)$ nuqta topiladiki, bu nuqtada $f(b) - f(a) = f'(c)(b-a)$ tenglik bajariladi.

Bu teoremaning geometrik ma’nosini aniqlash uchun Lagranj formulasini

$$\frac{f(b) - f(a)}{b - a} = f'(c) \tag{2.27}$$

ko‘rinishda yozamiz.

2.16-teoremaning geometrik ma’nosi: shunday $c \in (a,b)$ nuqta mavjudki, bu nuqtada funksiya grafigiga o‘tkazilgan urinma $(a, f(a))$ va $(b, f(b))$ nuqtalardan o‘tkazilgan vatarga parallel bo‘ladi (2.31-rasm).



2.31-rasm. Lagrang teoremasining geometrik tatbig‘i

2.6.4-misol. $y = 2x - x^2$ egri chiziqning AB yoyida shunday M nuqtani topingki, bu nuqtada egri chiziqqa o‘tkazilgan urinma AB vatarga parallel bo‘lsin, bunda $A(1;1)$ va $B(3;-3)$.

Yechilishi: ► $y = 2x - x^2$ egri chiziq Lagranj teoremasi shartlarini qanoatlantiradi. Shu sababli (2.27) formula o‘rinli bo‘ladi:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{f(3) - f(1)}{3 - 1} = (2x - x^2)'$$

$$\frac{(2 \cdot 3 - 3^2) - (2 \cdot 1 - 1^2)}{3 - 1} = 2 - 2x \quad \Rightarrow \quad -2 = 2 - 2x \quad \Rightarrow \quad x = 2. \backslash$$

$$y(2) = 2 \cdot 2 - 2^2 = 0.$$

Shunday qilib, M(2;0) nuqtada egri chiziqqa o‘tkazilgan urinma AB vatarga parallel bo‘ladi. ◄

2.17-teorema. Koshi teoremasi (ikki funksiya orttirmasining nisbati haqida).

Agar ikkita $f(x)$ va $\varphi(x)$ funksiya $[a, b]$ kesmada uzluksiz, (a, b) oraliqda differensiallanuvchi, shu bilan birga barcha $x \in (a, b)$ lar uchun $\varphi'(x) \neq 0$ bo‘lsa, u holda $[a, b]$ kesma ichida hech bo‘lmaganda bitta $c \in (a, b)$ nuqta mavjudki, u nuqtada

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(c)}{\varphi'(c)} \qquad (2.28)$$

tenglik bajariladi, bunda $\varphi(b) \neq \varphi(a)$.

2.17-teoremaning geometrik ma’nosi: funksiyaning hosilasi berilgan oraliqning aqalli bitta nuqtasida funksiyaning o‘rtacha qiymatiga teng bo‘ladi.

2.6.5-misol. $f(x)=x^3$ va $\varphi(x)=x^2$ funksiyalar uchun [0;1] kesmada Koshi teoremasini qanoatlantiruvchi c nuqtani toping.

Yechilishi: ► (2.28) formulaga ko‘ra

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(c)}{\varphi'(c)} \quad\Rightarrow\quad \frac{f(1)-f(0)}{\varphi(1)-\varphi(0)}=\frac{3x^2}{2x} \quad\Rightarrow\quad \frac{1^3-0^3}{1^2-0^2}=\frac{3x}{2} \quad\Rightarrow$$

$$1=\frac{3x}{2} \quad\Rightarrow\quad x=\frac{2}{3}.\ \text{ Demak, } c=\frac{2}{3}. \ ◄$$

Koshining o‘rta qiymat haqidagi teoremasi mazmunidan sof nazariy tasdiqqa o‘xshaydi, lekin uni ham amaliy tatbiq qilish mumkin. Teoremaning bitta tatbig‘i aniqmasliklarni ochishda qo‘llaniladigan Lopital qoidasini isbotlashdan iborat.

Mavzu yuzasidan savollar:

1. Roll teoremasini va uning geometrik ma’nosini tushuntirib bering.
2. Lagranj teoremasini va uning geometrik ma’nosini tushuntirib bering.
3. Koshi teoremasini ayting.
4. Koshi teoremasini qayerda tatbiq qilish mumkin?

MUSTAQIL YECHISH UCHUN MASALALAR

1. $y=x^2$ parabolaning $A(1;1)$ va $B(3;9)$ nuqtalari orasidagi yoyiga o‘tkazilgan AB vatariga parallel urinmasining urinish nuqtasi absissasini aniqlang.

2. $y=\ln x$ funksiya $[1;\,2]$ kesmada Lagranj teoremasi shartlarini qanoatlantirsa, $c\in[1;\,2]$ nuqtani aniqlang.

3. $[1;\,3]$ kesmada berilgan $f(x)=x^3$ va $g(x)=x^2$ funksiyalar uchun Koshi formulasidagi c nuqtani aniqlang.

4. $[-1;\,0]$ kesmada berilgan $f(x)=x-x^3$ funksiya Roll teoremasi shartlarini qanoatlantirsa, c nuqtani aniqlang.

5. $\left[0;\,\frac{\pi}{2}\right]$ kesmada berilgan $f(x)=\sin x$ va $g(x)=\cos x$ funksiyalar uchun Koshi formulasidagi c nuqtani aniqlang.

TESTLAR

1. $y = x^2$ parabolaning $A(1;1)$ va $B(3;9)$ nuqtalari orasidagi yoyiga o‘tkazilgan AB vatariga parallel urinmasining urinish nuqtasi ordinatasini aniqlang.

A) $y = 4$; **B)** $y = 2{,}25$; **C)** $y = 6{,}25$; **D)** $y = \frac{16}{9}$.

2. $[0;\ 4]$ kesmada berilgan $f(x) = x^3$ va $g(x) = x^2$ funksiyalar uchun Koshi formulasidagi c nuqtani aniqlang.

A) $3\frac{1}{3}$; **B)** $2\frac{2}{3}$;

C) $2\frac{1}{3}$; **D)** Teorema shartlarini qanoatlantirmaydi.

3. $[1;\ 3]$ kesmada berilgan $f(x) = x^2 - 4x + 3$ funksiya Roll teoremasi shartlarini qanoatlantirsa, c nuqtani aniqlang.

A) $c = 1{,}5$; **B)** $c = 2{,}5$;

C) $c = 2$; **D)** Teorema shartlarini qanoatlantirmaydi.

4. Koshi formulasi qaysi javobda keltirilgan.

A) $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$ **B)** $\frac{f(b) - f(a)}{b - a} = f'(c)$

C) $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(c)}{g(c)}$ **D)** $\frac{f(b)}{b - a} = f'(c)$

5. Lagranj formulasi qaysi javobda keltirilgan.

A) $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$ **B)** $\frac{f(b) - f(a)}{b - a} = f'(c)$

C) $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(c)}{g(c)}$ **D)** $\frac{f(b)}{b - a} = f'(c)$.

2.7-§. Lopital qoidasi va aniqmasliklarni ochish

Agar limitlarni hisoblashda $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0 ko‘rinishdagi natijalar hosil bo‘lsa, ularga **aniqmasliklar** deyiladi.

2.18-teorema. ($\frac{0}{0}$ ko‘rinishdagi aniqmasliklarni ochish haqida).

Agar $f(x)$ va $\varphi(x)$ funksiyalar $x=a$ nuqtaning biror atrofida uzluksiz, a nuqtaning o‘zidan tashqari shu atrofda differensiallanuvchi bo‘lib, $f(a)=0$, $\varphi(a)=0$ va $\varphi'(x)\neq 0$ hamda $\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$ limit (chekli yoki cheksiz) mavjud bo‘lsa, u holda $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}$ limit mavjud va ushbu $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}=\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$ tenglik o‘rinli bo‘ladi.

Agar bir marta hosila olganda yana aniqmaslik hosil bo‘lsa, qayta va qayta Lopital qoidasini qo‘llash mumkin:

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{f''(x)}{\varphi''(x)}=\left(\frac{0}{0}\right)=\ldots=\lim_{x\to a}\frac{f^{(n)}(x)}{\varphi^{(n)}(x)}=A.$$

2.19-teorema. ($\frac{\infty}{\infty}$ ko‘rinishdagi aniqmasliklarni ochish haqida).

Agar $f(x)$ va $\varphi(x)$ funksiyalar $x=a$ nuqtaning biror atrofida uzluksiz, shu oraliqda ($x=a$ nuqtaning o‘zidan tashqari) differensiallanuvchi hamda $\lim\limits_{x\to a}f(x)=\infty$, $\lim\limits_{x\to a}\varphi(x)=\infty$, $\varphi'(x)\neq 0$ bo‘lsa va $\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$ limit (chekli yoki cheksiz) mavjud bo‘lsa, u holda $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}$ limit mavjud va ushbu $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}=\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$ tenglik o‘rinli bo‘ladi.

Agar bir marta hosila olganda yana aniqmaslik hosil bo‘lsa, qayta va qayta Lopital qoidasini qo‘llash mumkin:

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{f''(x)}{\varphi''(x)}=\left(\frac{\infty}{\infty}\right)=\ldots=\lim_{x\to a}\frac{f^{(n)}(x)}{\varphi^{(n)}(x)}=A.$$

2.7.1-misol. $\lim\limits_{x\to 0}\dfrac{\ln x}{ctgx}$ limitni hisoblang.

Yechilishi: ►

$$\lim_{x\to 0}\frac{\ln x}{ctgx}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{(\ln x)'}{(ctgx)'}=\lim_{x\to 0}\frac{-\sin^2 x}{x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(-\sin^2 x)'}{(x)'}=$$

$$=\lim_{x\to 0}(2\sin x\cos x)=0. \quad ◄$$

$0\cdot\infty$ ko‘rinishdagi aniqmasliklarni ochish.

Bunday aniqmasliklarni ochish deganda $\lim\limits_{x\to a}f(x)=0$, $\lim\limits_{x\to a}\varphi(x)=\infty$ bo‘lganda $\lim\limits_{x\to a}f(x)\cdot\varphi(x)$ limitni hisoblash tushuniladi. Agar izlanayotgan ifoda

$$\lim_{x\to a}f(x)\cdot\varphi(x)=\lim_{x\to a}\frac{f(x)}{\dfrac{1}{\varphi(x)}} \quad \text{yoki} \quad \lim_{x\to a}f(x)\cdot\varphi(x)=\frac{\varphi(x)}{\dfrac{1}{f(x)}}$$

ko‘rinishda yozilsa, u holda $0\cdot\infty$ ko‘rinishdagi aniqmaslikdan $x\to a$ da $\frac{0}{0}$ ko‘rinishdagi aniqmaslikka kelamiz. Uni 2.14-teoremaga asosan hisoblaymiz.

2.7.2-misol. $\lim\limits_{x\to 0}x^2\cdot lnx$ ni toping.

Yechilishi: ► Berilgan ifodani rasm almashtiramiz va yuqoridagiga ko‘ra topamiz.

$$\lim_{x\to 0}x^2\cdot lnx=\lim_{x\to 0}\frac{lnx}{\frac{1}{x^2}}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{2}{x^3}}=-\lim_{x\to 0}\frac{x^2}{2}=0.◄$$

$\infty-\infty$ ko‘rinishdagi aniqmasliklarni ochish.

Bunday aniqmasliklarni ochish deganda

$$\lim_{x\to a}f(x)=\infty, \qquad \lim_{x\to a}\varphi(x)=\ \infty$$

bir xil ishorali cheksizlik bo‘lganda $\lim\limits_{x\to a}\big(f(x)-\varphi(x)\big)$ limitni topish tushuniladi. Bunday aniqmasliklarni $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko‘rinishdagi aniqmasliklarga keltiriladi.

2.7.3-misol. $\lim\limits_{x\to\frac{\pi}{2}-0}(secx-tgx)$ ni toping.

Yechilishi: ► $\lim\limits_{x\to\frac{\pi}{2}-0}secx=\infty$. $\mathrm{Lim}_{x\to\frac{\pi}{2}-0}\, tgx=\infty$ bo‘lgani uchun $\infty-\infty$ ko‘rinishdagi aniqmaslikka ega bo‘lamiz. Eng sodda almashtirishlar yordamida $\left(\frac{0}{0}\right)$ ko‘rinishga olib kelamiz:

$$\lim_{x\to\frac{\pi}{2}-0}(secx-tgx)=\lim_{x\to\frac{\pi}{2}-0}\frac{1-sinx}{\cos x}=\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}-0}\frac{-cosx}{-\sin x}=0.$$ ◄

1^{∞}, 0^{0}, ∞^{0} ko‘rinishdagi aniqmasliklarni ochish.

$\lim_{x\to a}(f(x))^{\varphi(x)}$ limitni topish deganda

a) agar $f(x)\to 1$, $\varphi(x)\to\infty$ bo‘lsa, 1^{∞} ko‘rinishdagi aniqmaslikni ochishni;

b) agar $f(x)\to\infty$, $\varphi(x)\to 0$ bo‘lsa, ∞^{0} ko‘rinishdagi aniqmaslikni ochishni;

c) agar $f(x)\to 0$, $\varphi(x)\to 0$ bo‘lsa, 0^{0} ko‘rinishdagi aniqmaslikni ochishni tushinamiz.

Hamma hollarda ham funksiyani oldin logarifmlaymiz, bunda $0\cdot\infty$ ko‘rinishdagi aniqmaslikka ega bo‘lamiz, buni esa o‘z navbatida $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko‘rinishdagi aniqmasliklarga keltiramiz. Shundan keyin logarifmning limiti bo‘yicha berilgan funksiya limitini topamiz. Natijani potensirlaymiz.

2.7.4-misol. $\lim_{x\to 0}(cosx)^{\frac{1}{x^2}}$ ni hisoblang.

Yechilishi: ► $\lim_{x\to 0}cosx=1$, $\lim_{x\to 0}\frac{1}{x^2}=\infty$ bo‘lgani uchun 1^{∞} ko‘rinishga egamiz. Limitni $A=\lim_{x\to 0}(cosx)^{\frac{1}{x^2}}$ deb belgilaymiz.

Bu ifodani e asos bo‘yicha logarifmlaymiz:

$$lnA=ln\lim_{x\to 0}(cosx)^{\frac{1}{x^2}}=\lim_{x\to 0}\frac{\ln(cosx)}{x^2}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(\ln cosx)'}{(x^2)'}=$$

$$=\lim_{x\to 0}\frac{\frac{1}{cosx}(-sinx)}{2x}=-\frac{1}{2}\cdot\lim_{x\to 0}\frac{sinx}{xcosx}=-\frac{1}{2}.$$

Shunday qilib $lnA=-\frac{1}{2}$, buni potensirlab,

$A=e^{-\frac{1}{2}}$ ni hosil qilamiz. ◄

Mavzu yuzasidan savollar

1. $\frac{0}{0}$ ko‘rinishdagi aniqmasliklarni ochish haqidagi teoremani ayting.

2. $\frac{\infty}{\infty}$ ko‘rinishdagi aniqmasliklarni ochish haqidagi teoremani ayting.

3. $0\cdot\infty, \infty-\infty$ aniqmasliklar qanday hisoblanadi?
4. $1^{\infty},\ 0^{0},\ \infty^{0}$ aniqmasliklar qanday hisoblanadi?

MUSTAQIL YECHISH UCHUN MASALALAR

1. $\lim\limits_{x\to\infty}\dfrac{2x^3+x^2-5}{x^3+x-2}$ limitni Lopital qoidasiga ko‘ra hisoblang.

2. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{e^{2x}-e^{x}}$ limitni Lopital qoidasiga ko‘ra hisoblang.

3. $\lim\limits_{x\to 0}\dfrac{e^{x}-1}{arctgx}$ limitni Lopital qoidasiga ko‘ra hisoblang.

4. $\lim\limits_{x\to 0}(\cos 2x)^{3/x^2}$ limitni Lopital qoidasiga ko‘ra hisoblang.

5. $\lim\limits_{x\to 0}\left(1+\dfrac{x}{2}\right)^{1/x}$ limitni Lopital qoidasiga ko‘ra hisoblang.

TESTLAR

1. $\lim\limits_{x\to 3}\dfrac{5x^2-6x+3}{x^2-4x+3}$ limitni Lopital qoidasiga ko‘ra hisoblang.
A) 0 **B)** 1 **C)** 5 **D)** Lopital qoidasini qo‘llab bo`lmaydi.

2. $\lim\limits_{x\to 0}\dfrac{tg^2\frac{x}{2}}{x^2}$ limitni Lopital qoidasiga ko‘ra hisoblang.
A) 0 **B)** 1/4 **C)** 1/5 **D)** 4

3. $\lim\limits_{x\to 0}\dfrac{2x-\arcsin x}{2x+arctgx}$ limitni Lopital qoidasiga ko‘ra hisoblang.
A) 1/3 **B)** 1/4 **C)** 1 **D)** 3

4. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{tgx}\right)$ limitni Lopital qoidasiga ko‘ra hisoblang.
A) 0 **B)** 1 **C)** 1/2 **D)** 4

5. $\lim\limits_{x\to 0}\dfrac{\ln(1+kx)}{x}$ limitni Lopital qoidasiga ko‘ra hisoblang.
A) k **B)** $1+k$ **C)** 1 **D)** $1/k$

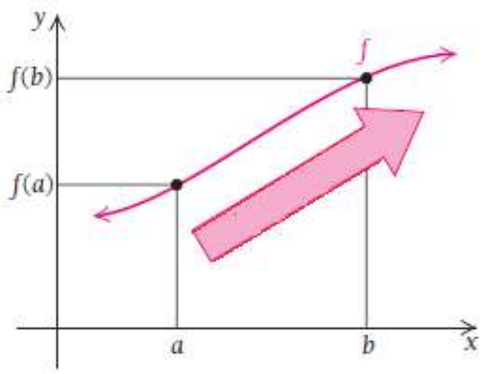
2.8-§. Funksiyani hosila yordamida tekshirish va grafigini yasash

Funksiyaning birinchi va ikkinchi tartibli hosilalari – hisob fanining ish qurollari bo‘lib, ular funksiya grafiklarini chizish va minimal, maksimal qiymatlarini aniqlash uchun axborotlar to‘plashda muhim o‘rin tutadi.

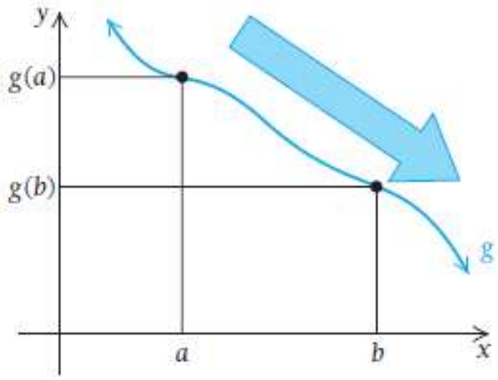
2.8.1. Funksiyaning o‘sish va kamayish shartlari

Agar I oraliqdagi barcha a, b sonlar uchun $a<b$ bo‘lganda $f(a)<f(b)$ tengsizlik bajarilsa, u holda $f(x)$ funksiya I oraliqda **o‘suvchi funksiya** deyiladi (2.32-rasm, a).

Agar I oraliqdagi barcha a, b sonlar uchun $a<b$ bo‘lganda $g(a)>g(b)$ tengsizlik bajarilsa, u holda $g(x)$ funksiya I oraliqda **kamayuvchi funksiya** deyiladi (2.32-rasm, b).



a)



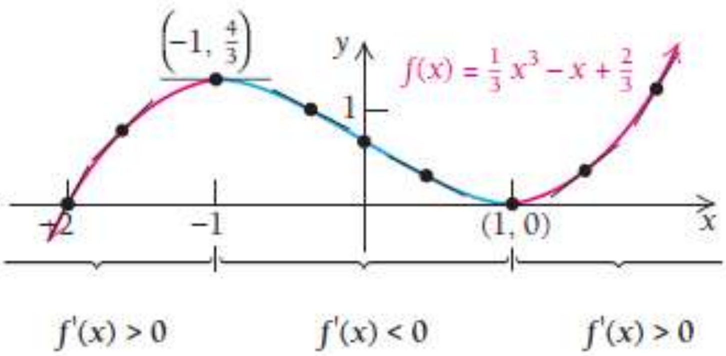
b)

2.32-rasm. O‘suvchi va kamayuvchi funksiyalar

2.20-teorema. Agar (a,b) oraliqning barcha x nuqtalari uchun $f'(x)>0$ bo‘lsa, $f(x)$ funksiya shu oraliqda o‘suvchi bo‘ladi.

Agar (a,b) oraliqning barcha x nuqtalari uchun $f'(x)<0$ bo‘lsa, $f(x)$ funksiya shu oraliqda kamayuvchi bo‘ladi.

Teorema tasdig‘ini chizmadan ko‘rish mumkin (2.33-rasm):



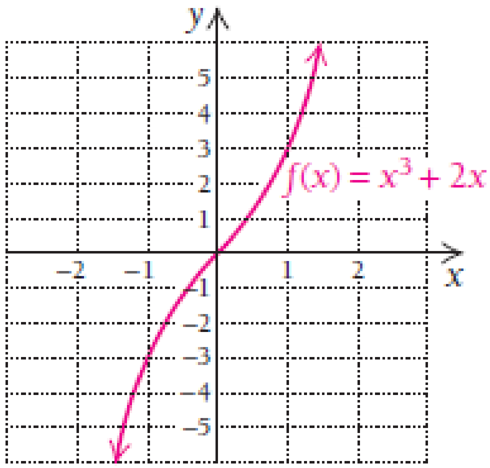
2.33-rasm. O‘sishdan kamayishga va mayishdan o‘sishga o‘tish

$(-\infty;-1)$ va $(1;\infty)$ oraliqda funksiya o‘suvchi, $(-1;1)$ oraliqda kamayuvchi.

Hosila yordamida funksiyaning o‘suvchi yoki kamayuvchi ekanligi ochiq oraliqda tekshiriladi, ya’ni oraliq chetidagi nuqtalar qaralmaydi. E’tibor bering, chizmadagi funksiyaning o‘sish, kamayish oraliqlari yozilganda oraliqqa $x=-1$ va $x=1$ nuqtalar kiritilmaydi. Bu nuqtalarga **kritik nuqtalar** deyiladi.

Ba’zi funksiyalar yoki faqat o‘suvchi yoki faqat kamayuvchi bo‘ladi. Masalan, $f(x)=x^3+2x$ funksiya grafigi butun son o‘qida qat’uy o‘suvchi bo‘lib, uning barcha urinmalari musbat burchak koeffitsiyentiga ega (2.34-rasm).

Ushbu tasdiqni isbotlash uchun hosiladan qanday foydalanamiz?



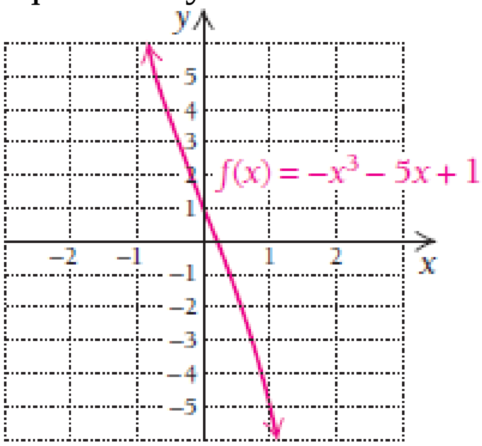
2.34-rasm. $f(x)=x^3+2x$ funksiya grafigi

$f(x)=x^3+2x$ funksiyaning hosilasi x ning barcha qiymatlarida musbat bo‘ladi, chunki $f'(x)=3x^2+2$. Noma’lum x ni aniqlash uchun bu hosilani manfiy qiymatga tenglab, yechib bo‘lmaydi (urinib ko‘ring ☺☺☺). Hosila x ning kvadrati qatnashgani va qolgan sonlar ham musbat bo‘lgani uchun berilgan funksiya faqat o‘sadi deb xulosa qilamiz.

2.8.1-misol. $f(x)=-x^3-5x+1$ funksiya o‘suvchimi yoki kamayuvchimi?

Yechilishi: ► Diagrammaga qarab xulosa qiladigan bo‘lsak, funksiya faqat kamayuvchiga o‘xshaydi. Biz grafikning kichik bir bo‘lagini ko‘rib turibmiz xolos. Biz ko‘ra olmaydigan boshqa bir bo‘lagida balki funksiya o‘suvchidir? Diagrammaning o‘zigina yetarli emas ekan. Hosiladan foydalanib ko‘raylik-chi: $f'(x)=-3x^2-5$.

x^2 had faqat 0 yoki musbat bo‘ladi, $-3x^2$ esa faqat manfiy yoki 0 bo‘ladi. $-3x^2$ dan 5 ni ayirsak, yana manfiy qiymat hosil bo‘ladi. Xulosa qilish mumkinki, x ning barcha qiymatlarida hosila manfiy ekan. Bundan ko‘rinadiki, funksiyaga ixtiyoriy nuqtada o‘tkazilgan urinmaning burchak koeffitsiyenti manfiy va faqat kamayuvchi bo‘ladi.



2.35-rasm. $f(x) = -x^3 - 5x + 1$ funksiya grafigi

Shunga ko‘ra, funksiya grafigi faqat kamayuvchi ekani kelib chiqadi (2.35-rasm).◄

Funksiya grafigini chizish bilan uning o‘suvchi yoki kamayuvchi ekanini aniqlash noto‘g‘ri natijaga olib kelishi mumkin ekan. Misol uchun, $f(x) = x^3 - x^2$ funksiya faqat o‘suvchi bo‘lib ko‘rinadi. Biroq bu funksiyaning shunday bir kichik oralig‘i borki, funksiya bu oraliqda kamayadi.

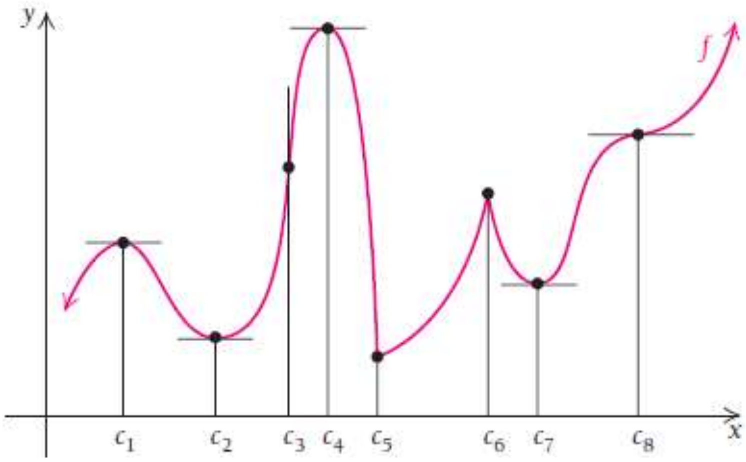
2.8.2. Funksiyaning ekstremum nuqtalari. Ekstremum mavjudligining zaruriy va yetarli shartlari

Bizga biror $f(x)$ funksiyaning grafigi berilgan bo‘lsin (2.36-rasm):

1) $x = c_1,\ c_2,\ c_4,\ c_7,\ c_8$ nuqtalarda $f'(c) = 0$ ga teng. Shuning uchun bu nuqtalarda grafikka o‘tkazilgan urinmalar gorizontal.

2) $x = c_3,\ c_5,\ c_6$ nuqtalarda $f'(c)$ mavjud emas va bu nuqtalarda o‘tkazilgan urinmalar vertikal.

$f(x)$ funksiyaning **kritik nuqtasi** deb, uning aniqlanish sohasiga tegishli qandaydir c nuqtaga aytiladiki, bu nuqtada funksiya hosilasi nolga teng bo‘ladi yoki bu nuqtada funksiya hosilasi mavjud bo‘lmaydi.



2.36-rasm. Funksiyaning kritik nuqtalari

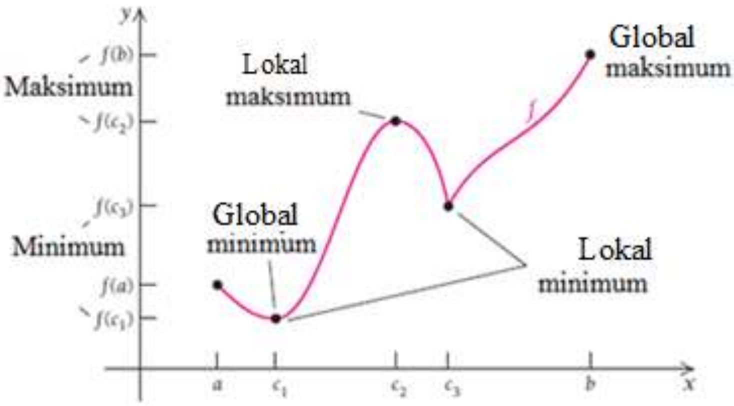
2.36-rasmdan ko‘rinadiki,

1) $c_1,\ c_2,\ c_4,\ c_7,\ c_8$ nuqtalar kritik nuqtalar, chunki bu nuqtalarda $f'(c)=0$.

2) $c_3,\ c_5,\ c_6$ nuqtalar ham kritik nuqtalar, chunki bu nuqtalarda $f'(c)$ mavjud emas.

Shuni ham bilingki, uzluksiz funksiya faqat kritik nuqtalarda o‘sishdan kamayishga yoki kamayishdan o‘sishga o‘tishi mumkin. Funksiya o‘sishdan kamayishga va kamayishdan o‘sishga o‘tadigan oraliqlarini $c_1,\ c_2,\ c_4,\ c_5,\ c_6,\ c_7$ nuqtalar bilan ajratish mumkin. $c_3,\ c_8$ nuqtalar ham kritik nuqtalar, lekin bu nuqtalarda funksiya o‘sishdan kamayishga yoki kamayishdan o‘sishga o‘tmaydi.

2.8.2-misol. Bizga biror $f(x)$ funksiyaning grafigi berilgan bo‘lsin.



2.37-rasm. Funksiyaning maksimum va minimum nuqtalari

Chizmadan ko‘rinadiki (2.37-rasm), funksiyaning bir nechta maksimumi yoki minimumi bo‘lishi mumkin ekan. c_2 nuqta $(c_1,\ c_3)$ oraliqda funksiya eng katta qiymatga erishadigan nuqta va $f(c_2)$ funksiyaning $(c_1,\ c_3)$ **oraliqdagi eng katta qiymati** (lokal maksimum) bo‘ladi. c_3

nuqta esa $[c_2,\ c_3]$ oraliqda funksiya eng kichik qiymatga erishadigan nuqtasi va $f(c_3)$ funksiyaning $[c_2,\ c_3]$ **oraliqdagi eng kichik qiymati** (lokal minimum) bo‘ladi.

I oraliq $f(x)$ funksiyaning aniqlanish sohasi bo‘lsin.

Agar I oraliqda c nuqtani o‘z ichiga olgan I_1 ochiq oraliqning barcha x nuqtalari uchun $f(c) \le f(x)$ tengsizlik o‘rinli bo‘lsa, $f(c)$ qiymatga funksiyaning **oraliqdagi eng kichik qiymati** (lokal minimum) deyiladi.

Agar I oraliqda c nuqtani o‘z ichiga olgan I_2 ochiq oraliqning barcha x nuqtalari uchun $f(c) \ge f(x)$ tengsizlik o‘rinli bo‘lsa, $f(c)$ qiymatga funksiyaning **oraliqdagi eng katta qiymati** (lokal maksimum) deyiladi.

Esda saqlang! Lokal minimum funksiyaning aniqlanish sohasidagi eng kichik qiymati (minimumi) bo‘lmasligi mumkin. Xuddi shuningdek, lokal maksimum ham funksiyaning aniqlanish sohasidagi eng katta qiymati (maksimumi) bo‘lmasligi mumkin.

Maksimum va minimum qiymatlarga **ekstremum qiymatlar** deyiladi. Kritik nuqtalarga **ekstremum nuqtalar** deyiladi.

2.21-teorema. Ekstremum mavjudligining zaruriylik sharti. Agar $f(x)$ funksiya biror ochiq oraliqda $f(c)$ ekstremum qiymatga ega bo‘lsa, u holda c kritik nuqta bo‘ladi va bu nuqtada $f'(c)=0$ bo‘lishi zarur.

● **Kritik nuqta bilan ekstremum nuqtaning farqi nimada?** Kritik nuqtalar ekstremum nuqtalarga shubhali nuqtalar, ya’ni kritik nuqta ekstremum bo‘lishi ham, bo‘lmasligi ham mumkin. Chunki, 2.36-rasmdan ko‘rish mumkinki, hamma kritik nuqtalar ham **maksimum (global maksimum)** yoki **minimum (global minimum)** bo‘la olmaydi.

Hamma kritik nuqtalar ham ekstremum bo‘la olmaydi.

2.8.3-misol. $f(x)=(x-1)^3+2$ funksiyaning maksimum yoki minimum qiymatlarini toping.

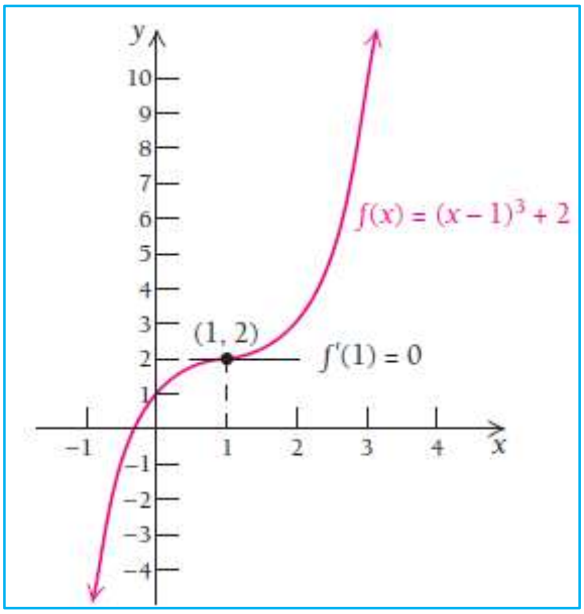
Yechilishi: ►Funksiyadan hosila olamiz: $f'(x)=3(x-1)^2$.

Hosilani nolga tenglaymiz: $f'(x)=3(x-1)^2=0$

va yechimlarni izlaymiz: $x-1=0$

$x=1$ nuqta kritik nuqta bo‘ladi.

Bu nuqta aniqlanish sohasini 2 ta oraliqqa ajratadi: $(-\infty;\ 1)\cup(1;\ \infty)$.

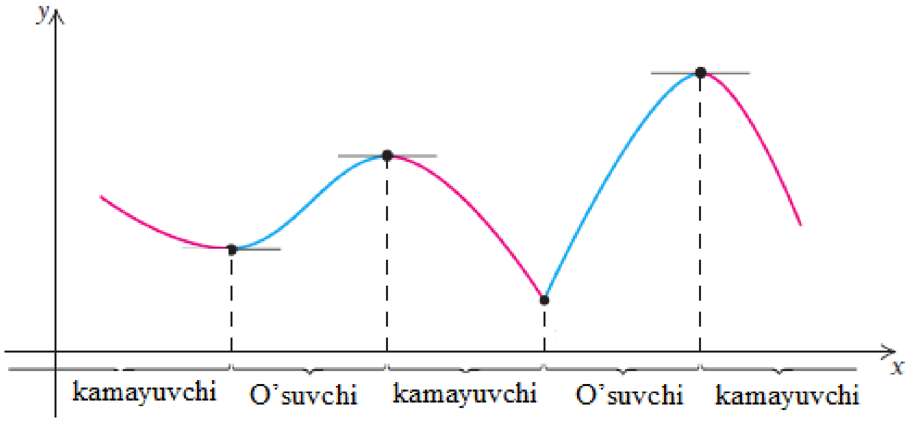


2.38-rasm. $f(x)=(x-1)^3+2$ funksiya grafigi

Lekin funksiya grafigidan ko‘rish mumkinki, $f(1)=2$ qiymat maksimum ham, minimum ham emas (2.38-rasm). ◄

Kritik nuqta qachon ekstremum nuqta bo‘la oladi?

Agar kritik nuqtada funksiya kamayishdan o‘sishga o‘tsa, bu nuqta minimum nuqta bo‘ladi. Agar kritik nuqtada funksiya o‘sishdan kamayishga o‘tsa, bu nuqta maksimum nuqta bo‘ladi.



2.39-rasm. Grafikdan maksimum va minimumni aniqlash

2.22-teorema. Ekstremum mavjudligining yetarlilik sharti. (a,b) oraliqda biror uzluksiz $f(x)$ funksiyaning bitta c kritik nuqtasi bo‘lsa, u holda

1) hosilaning ishorasi (a,c) da $f'(x)<0$, (c,b) da $f'(x)>0$ bo‘lsa, funksiya c nuqtada minimumga erishadi;

2) hosilaning ishorasi (a,c) da $f'(x)>0$, (c,b) da $f'(x)<0$ bo‘lsa, funksiya c nuqtada maksimumga erishadi;

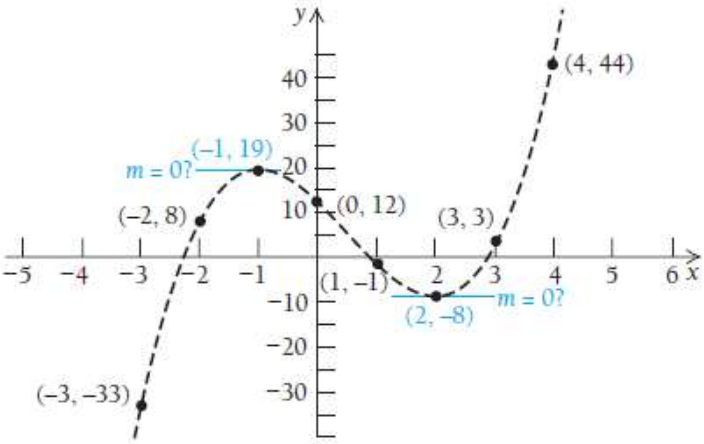
3) hosilaning ishorasi (a,c) va (c,b) oraliqlarda bir xil bo‘lsa, funksiya maksimumga ham, minimumga ham erishmaydi.

2.8.4-misol. $f(x) = 2x^3 - 3x^2 - 12x + 12$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyaning grafigini chizish so‘ralgan bo‘lsin. Grafikni to‘g‘ri chizish uchun juda ko‘p nuqtalarda funksiya qiymatlarini hisoblash kerak bo‘ladi. Hisoblash esa juda murakkab jarayon.

Nima qilsak bo‘ladi? Grafik eskizini chizish uchun uning egilish nuqtalarini topsak yetarli (2.40-rasm).

x	$f(x)$
−3	−33
−2	8
−1	19
0	12
1	−1
2	−8
3	3
4	44



2.40-rasm. $f(x) = 2x^3 - 3x^2 - 12x + 12$ funksiya ekstremumlari

Dastlab funksiya hosilasini topamiz:

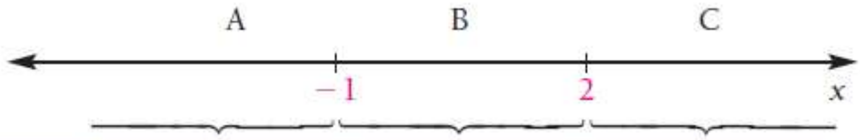
$$f'(x) = \left(2x^3 - 3x^2 - 12x + 12\right)' = 6x^2 - 6x - 12.$$

Topilgan ifodani nolga tenglaymiz: $6x^2 - 6x - 12 = 0$.

Tenglamani yechamiz: $x^2 - x - 2 = 0 \quad \rightarrow \quad x = 2$ va $x = -1$.

Bu nuqtalarni sonlar o‘qiga joylab, uchta oraliq hosil qilamiz.

A oraliq $(-\infty;\ -1)$, B oraliq $(-1;\ 2)$ va C oraliq $(2;\ \infty)$ dan iborat.



	$x = -2$	$x = 0$	$x = 4$
$f'(x)$	$f'(-2) > 0$	$f'(0) < 0$	$f'(4) > 0$
	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$

Endi har bir oraliq ichida bitta nuqta tanlab olib, funksiya ishorasini tekshiramiz.

A oraliqda: $x=-2$, $f'(-2)=6\cdot(-2)^2-6\cdot(-2)-12=24>0$;

B oraliqda: $x=0$, $f'(0)=6\cdot 0^2-6\cdot 0-12=-12<0$;

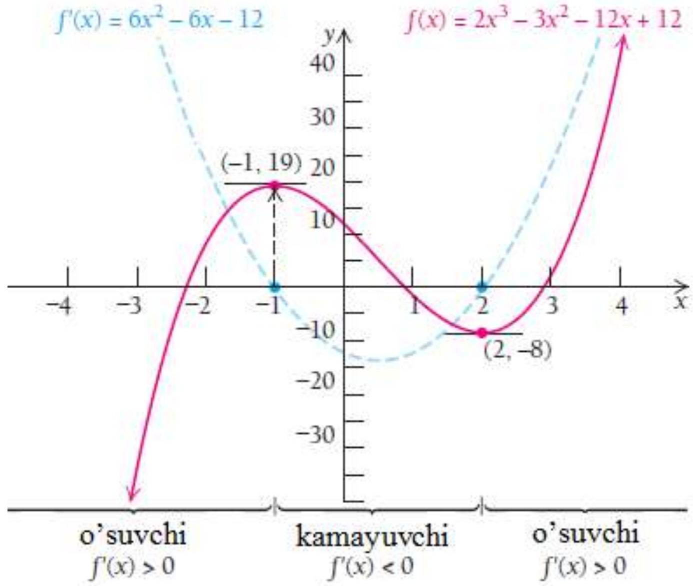
C oraliqda: $x=4$, $f'(4)=6\cdot 4^2-6\cdot 4-12=60>0$.

Jadvalga qarab, funksiya $x=-1$ da maksimumga erishishini bilib olishimiz mumkin: $f_{\max}(-1)=2\cdot(-1)^3-3\cdot(-1)^2-12\cdot(-1)+12=19$;

$x=2$ da funksiya minimumga erishadi:

$$f_{\min}(-1)=2\cdot 2^3-3\cdot 2^2-12\cdot 2+12=-8.$$

Demak, $(-1,\ 19)$ nuqta (lokal) maksimum, $(2,\ -8)$ nuqta (lokal) minimum ekan (2.41-rasm).



2.41-rasm. $f(x)=2x^3-3x^2-12x+12$ funksiya grafigi

$(-\infty;\ -1)\cup(2;\ \infty)$ oraliqda funksiya o‘suvchi;

$(-1;\ 2)$ oraliqda funksiya kamayuvchi. ◄

2.8.5-misol. $f(x)=2x^3-x^4$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyadan hosila olamiz:

$$f'(x)=\left(2x^3-x^4\right)'=6x^2-4x^3$$

Topilgan ifodani nolga tenglaymiz: $6x^2-4x^3=0$.

Tenglamani yechamiz: $2x^2(3-2x)=0 \quad\rightarrow\quad x=0$ va $x=\frac{3}{2}$.

Bu nuqtalarni sonlar o‘qiga joylab, uchta oraliq hosil qilamiz:

A oraliq $(-\infty;\ 0)$, B oraliq $\left(0;\ \frac{3}{2}\right)$ va C oraliq $\left(\frac{3}{2};\ \infty\right)$ dan iborat.

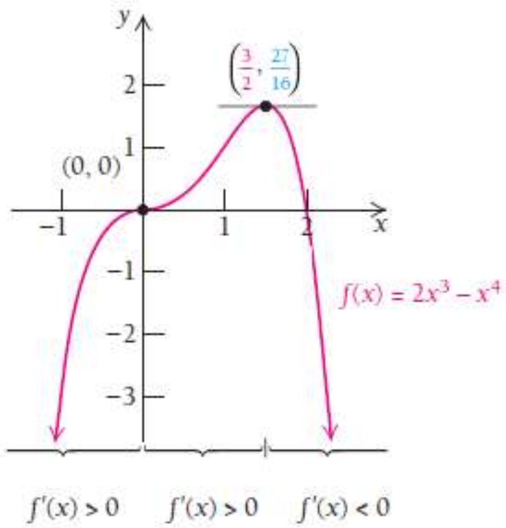
Endi har bir interval ichida funksiya ishorasini tekshiramiz.

A oraliqda: $x=-1$, $\quad f'(-1)=6\cdot(-1)^2-4\cdot(-1)^3=10>0$;

B oraliqda: $x=1$, $\quad f'(1)=6\cdot1^2-4\cdot1^3=2>0$;

C oraliqda: $x=2$, $\quad f'(2)=6\cdot2^2-4\cdot2^3=-8<0$

x	$f(x)$
−1	−3
−0.5	−0.31
0	0
0.5	0.19
1	1
1.25	1.46
2	0



2.42-rasm. $f(x)=2x^3-x^4$ funksiya grafigi

Jadvaldan funksiya $x=0$ da maksimumga ham, minimumga ham ega emasligini ko‘ramiz, chunki funksiya bu nuqtaning chap va o‘ng tomonlarida bir xil ishorali.

$x=\frac{3}{2}$ da funksiya maksimumga erishadi: $f\left(\frac{3}{2}\right)=2\cdot\left(\frac{3}{2}\right)^3-\left(\frac{3}{2}\right)^4=\frac{27}{16}$.

Demak, $\left(\frac{3}{2},\frac{27}{16}\right)$ nuqta maksimum nuqta ekan.

$\left(\frac{3}{2};\ \infty\right)$ oraliqda funksiya kamayuvchi.

$(-\infty;\ 0)$ va $\left(0;\ \frac{3}{2}\right)$ oraliqlarda funksiya o‘suvchi hamda $f(0)=2\cdot0^3-0^4=0$ ga teng, shuning uchun funksiyani $\left(-\infty;\ \frac{3}{2}\right)$ oraliqda o‘suvchi deb aytishimiz mumkin. Chunki bu oraliqda funksiya grafigiga uning ixtiyoriy 2 nuqtasi orqali o‘tkazilgan kesuvchining burchak koeffitsiyenti musbat bo‘ladi (2.42-rasm). ◄

2.8.3. Funksiyalarning kesmadagi eng katta va eng kichik qiymatlari

Ma’lumki, $[a, b]$ kesmada uzluksiz bo‘lgan $y = f(x)$ funksiya shu kesmada o‘zining eng katta va eng kichik qiymatlariga erishadi. Shu qiymatlarni qanday topish mumkin?

Agar $y = f(x)$ funksiya monoton bo‘lsa, u holda funksiyaning eng katta va eng kichik qiymatlari $[a, b]$ kesmaning oxirlari $x = a$ va $x = b$ nuqtalarda bo‘ladi.

Agar $y = f(x)$ funksiya monoton bo‘lmasa, u holda funksiya ekstremumlarga ega bo‘ladi. Bu holda eng katta va eng kichik qiymatlari ekstremumlar bilan bir xil bo‘lishi mumkin, ma’lumki ekstremumlar kritik nuqtalarda bo‘ladi.

Shunday qilib, $y = f(x)$ funksiyaning $[a, b]$ kesmadagi eng katta va eng kichik qiymatlarini topish uchun:

1) funksiyaning kritik nuqtalarini aniqlaymiz;
2) funksiyaning kritik nuqtalardagi va kesmaning oxirlardagi qiymatlarini hisoblaymiz;
3) topilgan qiymatlardan eng katta va eng kichik qiymatlarini tanlaymiz, ana shu qiymatlar funksiyaning $[a, b]$ kesmadagi eng katta va eng kichik qiymatlari bo‘ladi.

2.8.5-misol. $y = x^3 + 3x^2 - 9x + 1$ funksiyaning [-2, 5] kesmadagi eng katta va eng kichik qiymatlarini aniqlang.

Yechilishi: ►

a) Kritik nuqtalarini topamiz: y' hosilani hisoblaymiz:

$y' = 3x^2 + 6x - 9$. $y' = 0$ tenglamani yechamiz:

$$3x^2 + 6x - 9 = 0, \qquad x_1 = 1, \qquad x_2 = -3.$$

Berilgan kesmaga faqat $x_1 = 1$ nuqta kiradi.

b) Funksiyaning $x = 1$, $x = -2$, $x = 5$ nuqtalardagi qiymatlarini hisoblaymiz: $f(1) = -4$, $f(-2) = 23$, $f(5) = 156$.

c) Topilgan qiymatlardan eng katta M ni va eng kichik m ni tanlaymiz:

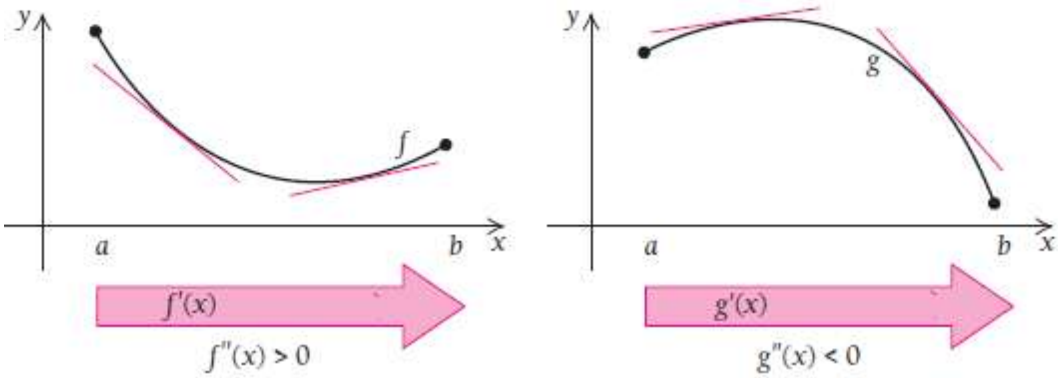
$$M = f(5) = 156, \; m = f(1) = -4.$$

Shunday qilib, funksiyaning eng katta qiymati kesmaning $x = 5$ o‘ng oxirida ekan, eng kichik qiymati esa $x = 1$ nuqtadagi minimum bilan bir xil ekan. ◄

2.8.4. Ekstremumni ikkinchi tartibli hosila yordamida tekshirish.

Funksiyalar grafigini qavariq va botiqlikka tekshirish

Ikkinchi tartibli hosila funksiya grafigining egriligini tahlil qilishda katta ahamiyatga ega. 2.43-rasmga qarang: $f(x)$ funksiyaning grafigi botiq, $g(x)$ ning grafigi esa qavariq ko‘rinishda.



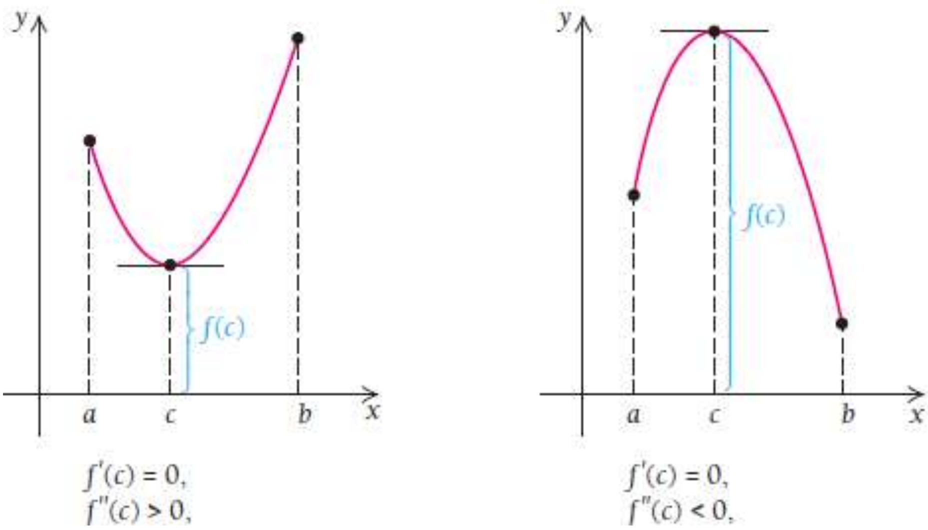
2.43-rasm. Funksiyaning qavariq va botiqligi

2.23-teorema.

1) Agar (a,b) oraliqda $f''(x)>0$ bo‘lsa, $f(x)$ ning grafigi **botiq** bo‘ladi;
2) Agar (a,b) oraliqda $f''(x)<0$ bo‘lsa, $f(x)$ ning grafigi **qavariq** bo‘ladi.

Keling, 2-tartibli hosila yordamida funksiyaning biror (a,b) oraliqda ekstremumi bor yoki yo‘qligini tekshirib ko‘ramiz.

Quyidagi grafiklarda botiq va qavariq funksiyalar berilgan (2.44-rasm). 2-tartibli hosila musbat bo‘lsa, funksiya grafigi botiq va bu yerda – maksimal nuqta mavjud, 2-tartibli hosila manfiy bo‘lsa, funksiya grafigi qavariq – minimal nuqta mavjud bo‘ladi.



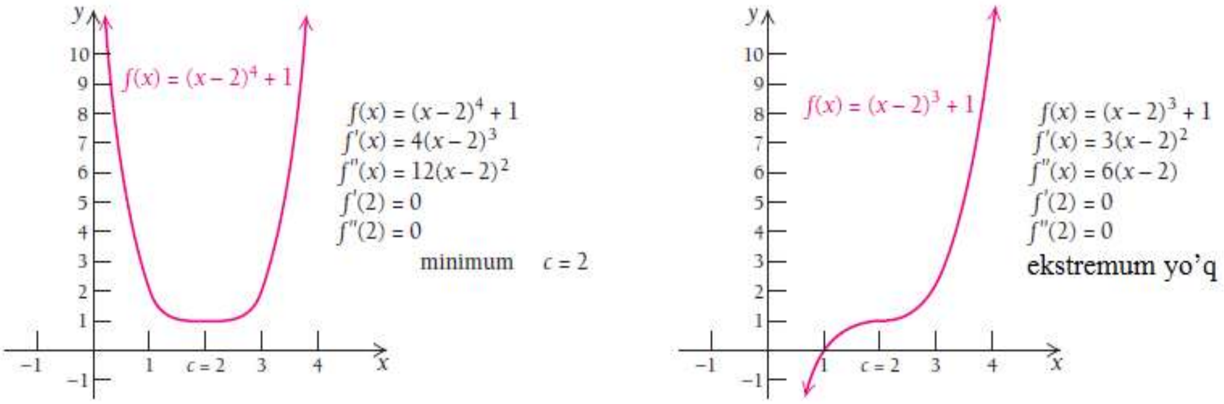
2.44-rasm. Funksiyaning qavariq va botiqligi

2.24-teorema. (Funksiyaning ekstremumlari haqida).

$f(x)$ funksiya (a,b) oraliqning har bir x nuqtasida differensiallanuvchi va oraliqda $f'(x)=0$ shartni qanoatlantiradigan c kritik nuqtaga ega bo‘lsin. U holda

1) Agar $f''(c)>0$ bo‘lsa, $f(c)$ minimum nuqta;
2) Agar $f''(c)<0$ bo‘lsa, $f(c)$ maksimum nuqta bo‘ladi.

2.45-rasmni qaraylik. Ikkala grafikda ham $f'(c)=0$, $f''(c)=0$, lekin birinchi funksiyaning ekstremumi mavjud, ikkinchi funksiyada ekstremum yo‘q. Agar c kritik nuqta bo‘lsa va $f''(c)=0$ bo‘lsa, c nuqtada ekstremum bo‘lishi ham, bo‘lmasligi ham mumkin. Shuningdek, agar c kritik nuqtada $f'(c)$ mavjud bo‘lmasa, u holda $f''(c)$ ham mavjud bo‘lmaydi. Boshqa hollarda $f(c)$ ning ekstremum ekanligini albatta 2-tartibli hosila bilan tekshirib ko‘rish zarur.



2.45-rasm. Ikkinchi tartibli hosila yordamida ekstremumni aniqlash

Demak, 2-tartibli hosiladan ekstremumni aniqlab olish va funksiyaning grafigini to‘liq va to‘g‘ri chizish uchun foydalanilar ekan. Buni quyidagi misolda ham ko‘ramiz.

2.8.6-misol. $f(x)=x^3+3x^2-9x-13$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyadan hosila olamiz:

$$f'(x)=\left(x^3+3x^2-9x-13\right)'=3x^2+6x-9$$

Topilgan hosilani nolga tenglaymiz: $3x^2+6x-9=0$.

Tenglamani yechamiz va kritik nuqtalarni topamiz:

$$x^2+2x-3=0 \quad \Rightarrow \quad (x+3)(x-1)=0$$

$$x=-3 \text{ va } x=1.$$

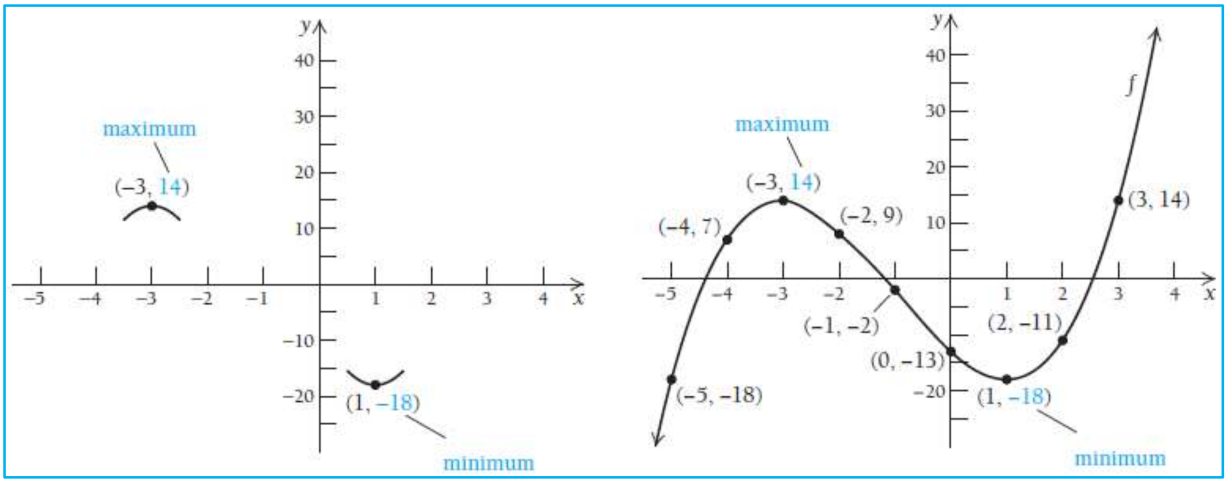
Endi kritik nuqtalarda funksiyaning qiymatlarini topamiz:

$$f(-3)=(-3)^3+3\cdot(-3)^2-9\cdot(-3)-13=14;$$

$$f(1)=1^3+3\cdot1^2-9\cdot1-13=-18.$$

Shunda $(-3;\ 14)$ va $(1;\ -18)$ ekstremum nuqtalar bo‘ladimi?

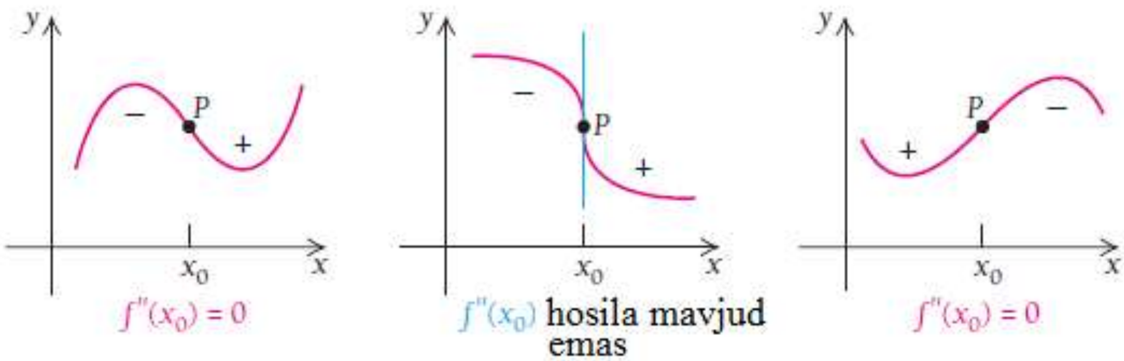
2-tartibli hosilani qaraymiz. $f''(x)=\left(3x^2+6x-9\right)'=6x+6$ hosilaga ktirik qiymatlarni qo‘yib, tekshiramiz: $f''(-3)=6\cdot(-3)+6=-12<0$ (lokal) maksimum $f''(1)=6\cdot1+6=12>0$ (lokal) minimum nuqtalarni topamiz. Shunday qilib, $f(-3)=14$ funksiyaning **maksimum nuqtasi**,
$f(1)=-18$ funksiyaning **minimum nuqtasi** ekanligini aniqlab oldik.
Endi shu ma’lumotlar asosida funksiya grafigini chizamiz (2.46-rasm):



2.46-rasm. $f(x)=x^3+3x^2-9x-13$ funksiyaning ekstremumlari ◄

2.8.5. Funksiya grafigining egilish nuqtalari. Egri chiziqlarning asimptotalari

Egilish nuqtasi botiqlik va qavariqlik orasida joylashib, ularni bir-biridan ajratib turadigan nuqtadir.
Masalan, quyidagi rasmlarda P nuqta egilish nuqtasidir (2.47-rasm):



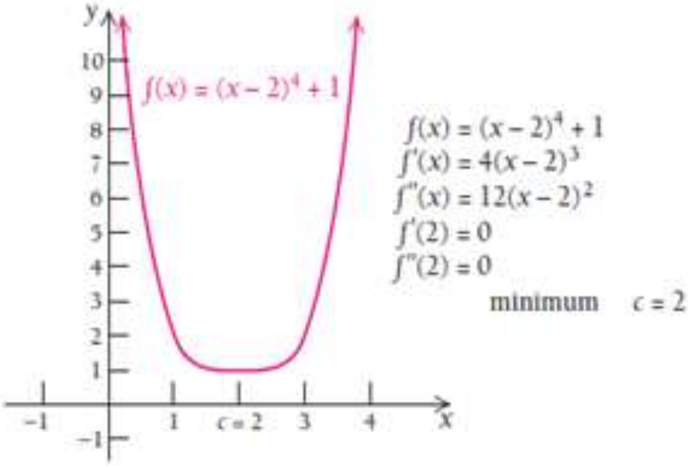
2.47-rasm. Funksiyaning egilish nuqtalari

2-tartibli hosilaning ishorasi P nuqtaning ikki tomonidagi botiqlik, qavariqlik turini aniqlab beradi. Chap va o‘ng tomondagi rasmlarda

$f''(x_0)$ nolga teng, o‘rtadagi rasmda esa 2-tartibli hosila mavjud emas. Har uchchala holda ham P egilish nuqtasi hisoblanadi.

2.25-teorema (Egilish nuqtalari mavjud bo‘lishining yetarlilik sharti). Agar x_0 nuqta $f(x)$ funksiyaning egilish nuqtasi bo‘lsa, u holda bu nuqtada $f''(x_0)=0$ bo‘ladi yoki $f''(x_0)$ mavjud bo‘lmaydi.

Teskari tasdiq har doim ham o‘rinli emas, ya’ni agar $f''(x_0)=0$ bo‘lsa yoki $f''(x_0)$ mavjud bo‘lmasa, u holda x_0 nuqta $f(x)$ funksiyaning egilish nuqtasi bo‘lavermaydi. Misol uchun, $f(x)=(x-2)^4+1$ funksiya grafigidan ko‘rinadiki, $x=2$ nuqta egilish nuqtasi bo‘la olmaydi, lekin bu nuqtada $f''(2)=0$ (2.48-rasm).



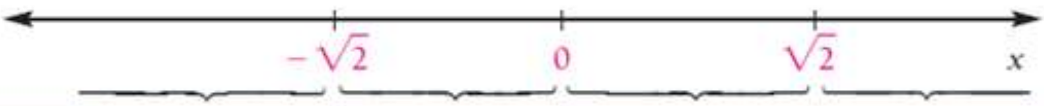
2.48-rasm. $f(x)=(x-2)^4+1$ funksiya grafigi

Demak, egilish nuqtasiga shubhali nuqtani topish uchun biz shunday x_0 nuqtani izlashimiz kerakki, bu nuqtada $f''(x_0)=0$ bo‘lsin yoki $f''(x_0)$ mavjud bo‘lmasin.

2.8.7-misol. $f(x)=3x^5-20x^3$ funksiyaning 2-tartibli hosilasidan foydalanib, egilish nuqtasini toping.

Yechilishi: ► Bizda 2- tartibli hosila $f''(x)=60x^3-120x$ ga teng. Uni nolga tenglab, yechimlarni topamiz: $60x^3-120x=0 \Rightarrow x(x^2-2)=0$.

Bundan, $x=0$ va $x=-\sqrt{2}$ hamda $x=\sqrt{2}$ qiymatlarni hosil qilamiz. So‘ngra shu 3 ta nuqta yordamida ajratilgan oraliqlarda 2-tartibli hosilaning ishoralarini tekshiramiz.



	$x=-2$	$x=-1$	$x=1$	$x=2$
$f''(x)$	$f''(-2)<0$	$f''(-1)>0$	$f''(1)<0$	$f''(2)>0$

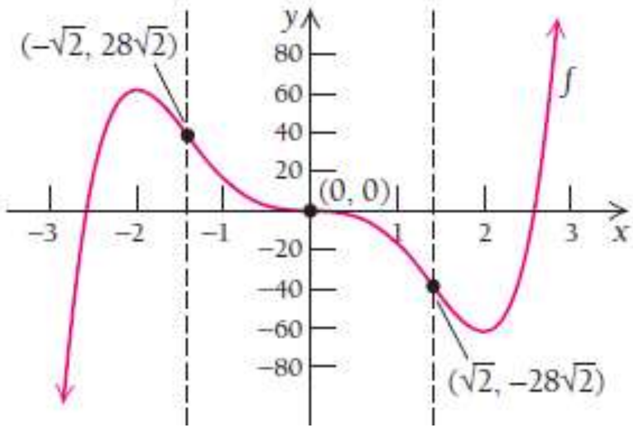
Endi funksiyaning bu nuqtalardagi qiymatlarini hisoblaymiz:

$$f(-\sqrt{2}) = 3\cdot(-\sqrt{2})^5 - 20\cdot(-\sqrt{2})^3 = 28\sqrt{2}$$

$$f(0) = 3\cdot 0^5 - 20\cdot 0^3 = 0;$$

$$f(\sqrt{2}) = 3\cdot(\sqrt{2})^5 - 20\cdot(\sqrt{2})^3 = -28\sqrt{2}.$$

Demak, $(-\sqrt{2};\ 28\sqrt{2})$, $(0;\ 0)$ va $(\sqrt{2};\ -28\sqrt{2})$ nuqtalar funksiyaning egilish nuqtalari ekan (2.49-rasm).



◀

2.49-rasm. $f(x) = 3x^5 - 20x^3$ funksiya grafigi

Funksiyaning asimptotalari

Agar $\lim\limits_{x\to a^-} f(x) = \infty$, yoki $\lim\limits_{x\to a^-} f(x) = -\infty$, (2.29)

$\lim\limits_{x\to a^+} f(x) = \infty$ yoki $\lim\limits_{x\to a^+} f(x) = -\infty$

limit tengliklardan birortasi o‘rinli bo‘lsa, $x = a$ to‘g‘ri chiziqqa **vertikal asimptota** deyiladi.

E’tibor bering:

1) $f(x) = \dfrac{x^2 - 16}{x - 4} = \dfrac{(x-4)(x+4)}{x-4}$ funksiyada $x = 4$ nuqtada mahraj nol bo‘lsa ham, $x = 4$ chiziq vertikal asimptota bo‘la olmaydi, chunki $x - 4$ ifoda kasrning surat va mahraji uchun umumiy ko‘paytuvchisi bo‘ladi va ular qisqarib ketadi.

2) $g(x) = \dfrac{x^2 - 4}{x^2 + x - 12} = \dfrac{(x-2)(x+2)}{(x-3)(x+4)}$ funksiyada esa $x = -4$ va $x = 3$ nuqtalarda funksiya uziladi va bu **uzilish nuqtalaridan o‘tuvchi to‘g‘ri chiziqlar** vertikal asimptotalar bo‘ladi.

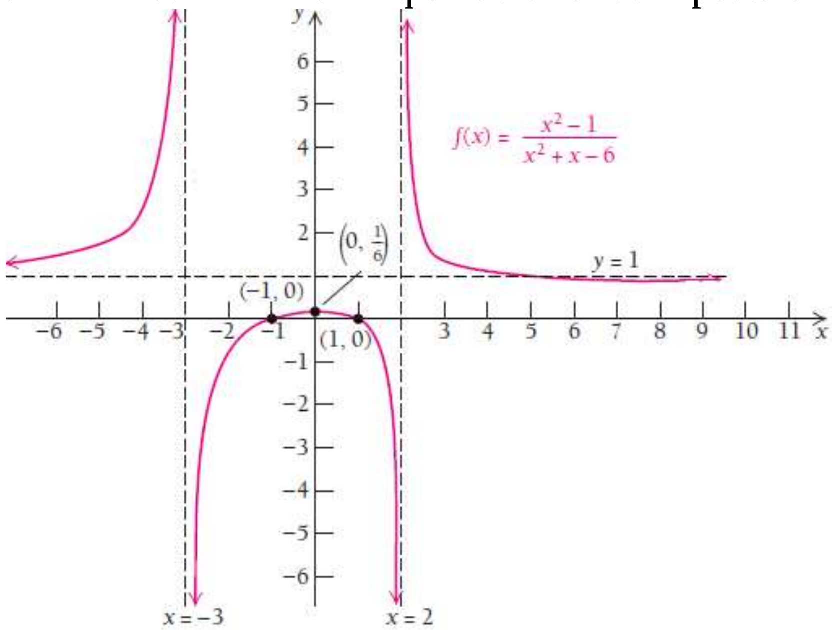
2.8.8-misol. $f(x) = \dfrac{x^2 - 1}{x^2 + x - 6} = \dfrac{(x-1)(x+1)}{(x-2)(x+3)}$ funksiyaning vertikal asimptotalarini toping.

Yechilishi: ► 2.50-rasmdan ko‘rinadiki,

$$\lim_{x\to 2^-} f(x) = -\infty \quad \text{va} \quad \lim_{x\to 2^+} f(x) = \infty;$$

$$\lim_{x \to -3^-} f(x) = \infty \qquad \text{va} \qquad \lim_{x \to -3^+} f(x) = -\infty.$$

Shuning uchun $x=2$ va $x=-3$ chiziqlar vertikal asimptotalar bo‘ladi.



2.50-rasm. Funksiyaning asimptotalari ◄

Vertikal asimptota kasr-ratsional funksiyalarda bo‘ladi, bu funksiyalarning mahrajini nolga aylantiradigan $x=a$ **uzilish nuqtasidan** o‘tuvchi to‘g‘ri chiziq vertikal asimptota hisoblanadi.

2.8.9-misol. $g(x) = \dfrac{2x-1}{x(x-3)(x+5)}$ funksiyaning vertikal asimptotalarini toping.

Yechilishi: ► Vertikal asimptotani topishning eng oson yo‘li, mahrajni nolga aylantiradigan qiymatlarni aniqlaymiz. Bular $x=0$, $x=3$ va $x=-5$ chiziqlardir.

Shuning uchun $x=0$, $x=3$ va $x=-5$ chiziqlardir vertikal asimptotalar bo‘ladi. ◄

2.8.10-misol. $f(x) = \dfrac{x^2-2x}{x^3-x}$ funksiyaning vertikal asimptotalarini toping.

Yechilishi: ► Mahrajni nolga aylantiradigan qiymatlarni aniqlaymiz:

$$f(x) = \frac{x^2-2x}{x^3-x} = \frac{x(x-2)}{x(x-1)(x+1)} = \frac{x-2}{(x-1)(x+1)}, \qquad x \neq 0$$

Shunday qilib, $x=1$ va $x=-1$ vertikal asimptotalarni hosil qilamiz. ◄

Agar $\lim\limits_{x \to -\infty} f(x) = b$ yoki $\lim\limits_{x \to \infty} f(x) = b$ limitlarning bittasi yoki ikkalasi ham o‘rinli bo‘lsa, $y=b$ to‘g‘ri chiziqqa **gorizontal asimptota** deyiladi.

2.8.11-misol. $f(x)=\frac{3x-4}{x}$ funksiyaning gorizontal asimptotasini toping.

Yechilishi: ► Gorizontal asimptotani topish uchun $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{3x-4}{x}$ ni hisoblash kerak. Limit xossalaridan foydalanib, hisoblasak: $\lim\limits_{x\to\infty}\frac{3x-4}{x}=\lim\limits_{x\to\infty}\frac{3x}{x}-\lim\limits_{x\to\infty}\frac{4}{x}=\lim\limits_{x\to\infty}(3-0)=3$.

Shunday qilib, $y=3$ gorizontal asimptota ekan. ◄

2.8.12-misol. $f(x)=\frac{5x^2+3x-4}{2x^2-x-1}$ funksiyaning gorizontal asimptotasini toping (Kasr suratidagi va maxrajidagi ko‘phadlar daraja ko‘rsatkichi teng).

Yechilishi: ► $\lim\limits_{x\to\infty}\frac{5x^2+3x-4}{2x^2-x-1}$ limitni hisoblash uchun kasrning surat va mahrajidagi x ning darajalarini qaraymiz. Argument cheksizlikka intilganda, limitni hisoblash uchun argumentning eng yuqori daraja ko‘rsatkichiga ega bo‘lgan hadga bo‘lamiz.

$$\lim_{x\to\infty}\frac{5x^2+3x-4}{2x^2-x-1}=\lim_{x\to\infty}\frac{\frac{5x^2}{x^2}+\frac{3x}{x^2}-\frac{4}{x^2}}{\frac{2x^2}{x^2}-\frac{x}{x^2}-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{5}{2}=\frac{5}{2}$$

$y=\frac{5}{2}$ to‘g‘ri chiziq gorizontal asimptota bo‘ladi. ◄

2.8.13-misol. $f(x)=\frac{3x+1}{x^3-2x^2+4}$ funksiyaning gorizontal asimptotasini toping (Kasr suratidagi ko‘phadning darajasi maxrajidagi ko‘phad daraja ko‘rsatkichidan katta).

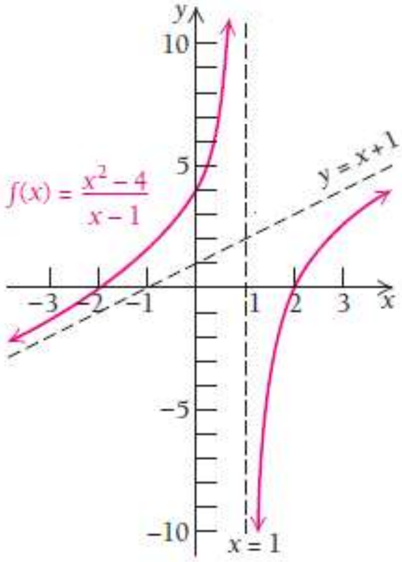
Yechilishi: ► $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{3x+1}{x^3-2x^2+4}=\lim\limits_{x\to\infty}\frac{\frac{3x}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{2x^2}{x^3}+\frac{4}{x^3}}=\lim\limits_{x\to\infty}\frac{0}{1}=0$

$y=0$ to‘g‘ri chiziq gorizontal asimptota bo‘ladi. ◄

Xulosa: Agar ratsional funksiyaning suratidagi ko‘phadning daraja ko‘rsatkichi maxrajdagi ko‘phad daraja ko‘rsatkichidan kichik bo‘lsa, u holda $y=0$ **to‘g‘ri chiziq yoki** Ox **o‘qi gorizontal asimptota** bo‘ladi.

Vertikal ham, gorizontal ham bo‘lmagan chiziqli asimptotaga **og‘ma asimptota** deyiladi.

Ba’zi funksiyalarda vertikal va gorizontal asimptotalardan tashqari og‘ma asismptota ham mavjud. $f(x)=\frac{x^2-4}{x-1}$ funksiyaning asimptotasini qaraylik (2.51-rasm).



2.51-rasm. Funksiyaning asimptotalari

$|x|$ ning qiymatlari cheksizlikka intilgan sari, funksiya grafigi $y=x+1$ to‘g‘ri chiziqqa intiladi. Bu to‘g‘ri chiziq og‘ma asimptota bo‘ladi.

Agar $f(x)=\frac{p(x)}{q(x)}$ ratsional funksiyaning suratidagi ko‘phadning darajasi mahrajdagi ko‘phad darajasidan 1 birlik yuqori bo‘lsa, bu funksiya og‘ma asimptotaga ega bo‘ladi.

Og‘ma asimptotani qanday aniqlaymiz?

I usul.

2.8.14-misol. $f(x)=\frac{x^2-4}{x-1}$ funksiyaning asimptotasini topamiz.

Yechilishi: ► Ratsional ifodaning suratini mahrajiga bo‘lib, butun qismini ajratib olamiz: $f(x)=\frac{x^2-4}{x-1}=(x+1)-\frac{3}{x-1}$.

$|x|$ ning qiymatlari cheksizlikka intilganda funksiyaning $-\frac{3}{x-1}$ qismi nolga intiladi. Funksiyaning butun qismidan tuzilgan $y=x+1$ to‘g‘ri chiziq og‘ma asimptota bo‘ladi. ◄

II usul. Ratsional funksiyaning **og‘ma asimptotasi** $y=kx+b$ tenglama bilan aniqlanadi. k burchak koeffitsiyentini $k=\lim\limits_{x\to\infty}\frac{f(x)}{x}$

limitdan, b ozod hadni esa $b = \lim\limits_{x\to\infty}(f(x) - kx)$ limit yordamida aniqlaymiz.

2.8.15-misol. $y = \dfrac{x^3}{x^2 - 4}$ funksiya grafigining asimptotalarini toping.

Yechilishi: ► Funksiya $x \neq \pm 2$ da aniqlangan.

$\lim\limits_{x\to\pm 2} \dfrac{x^3}{x^2-4} = \pm\infty$ bo‘lgani uchun, $x = -2$ va $x = 2$ to‘g‘ri chiziqlar funksiya grafigining **vertikal asimptotalari** bo‘ladi.

Endi og‘ma asimptotalarni izlaymiz: $k = \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2-4} = 1$,

$$b = \lim_{x\to\infty}(f(x) - kx) = \lim_{x\to\infty}\left(\frac{x^3}{x^2-4} - x\right) = \lim_{x\to\infty} \frac{4x}{x^2-4} = 0.$$

Demak, $y = x$ to‘g‘ri chiziq **og‘ma asimptota** bo‘ladi.◄

2.8.6. Grafik yasashning umumiy sxemasi

Endi funksiyani to‘liq tekshirish algoritmini keltiramiz:

1) **Funksiyaning koordinata o‘qlari bilan kesishish nuqtalarini aniqlash;**
2) **Funksiyaning asimptotalarini topish;**
3) **Funksiyaning aniqlanish sohasi topish;**
4) **Funksiyaning kritik nuqtalarini topish**, ya’ni $f'(x) = 0$ bo‘ladigan yoki $f'(x)$ mavjud bo‘lmaydigan nuqtalarini topish. Ulardan foydalanib, finksiyaning maksimum va minimum qiymatlarini topish; funksiyaning shu nuqtalardagi qiymatlarini topish;
5) **O‘sish va kamayish oraliqlari, ekstremumlarini topish.** x_0 nuqta atrofida $f''(x_0)$ ni ishorasini tekshiramiz. Agar $f''(x_0) < 0$ bo‘lsa, $f(x_0)$ maksimum nuqta, agar $f''(x_0) > 0$ bo‘lsa, $f(x_0)$ minimum nuqta bo‘ladi.
6) **Egilish nuqtalarini topish.** $f''(x_0) = 0$ bo‘ladigan yoki $f''(x_0)$ mavjud bo‘lmaydigan nuqtalarni aniqlaymiz. Bu nuqtalarda funksiya qiymatlarini hisoblaymiz.
7) **Qavariqlik va botiqlik oraliqlarini topish.** 4-shartdagi egilishga shubhali nuqtalardan foydalanamiz. Agar $f''(x_0) > 0$ bo‘lsa, bu oraliqda funksiya botiq, $f''(x_0) < 0$ bo‘lsa, funksiya qavariq bo‘ladi.

8) Grafikni yasash. Yuqoridagi topilganlarga ko‘ra funksiya grafigini chizamiz. Agar zarurat bo‘lsa, qo‘shimcha ma’lumotlarni ham hisoblab topamiz.

2.8.16-misol. $f(x)=\dfrac{x^2+4}{x}$ funksiyani to‘liq tekshiring va grafigini chizing.

Yechilishi: ► 1) Funksiyaning koordinata o‘qlari bilan kesishish nuqtalari: $f(0)=\dfrac{0^2+4}{0}$ funksiya Oy o‘qi bilan kesishmaydi. $f(x)=0$ qiymatlari ham mavjud emas, ya’ni funksiya Ox o‘qi bilan ham kesishmaydi.

2) Funksiyaning asimptotalari: Vertikal asimptota $x=0$ to‘g‘ri chiziq, gorizontal asimptota yo‘q, chunki funksiyada kasrning surati va mahrajidagi ko‘phadlarning daraja ko‘rsatkichlari teng emas. Ular teng bo‘lgandagina kasrning butun qismini ajratar edik. Og‘ma asimptota: $f(x)=\dfrac{x^2+4}{x}=x+\dfrac{4}{x}$, bundan $y=x$ to‘g‘ri chiziqni hosil qilamiz.

3) Funksiyaning aniqlanish sohasi: $x\neq 0$ shu sabali

$$D(f)=(-\infty;\ 0)\cup(0;\ \infty)$$

4) Funksiyaning kritik nuqtalari: Funksiyaning $f'(x)=0$ bo‘ladigan yoki $f'(x)$ mavjud bo‘lmaydigan nuqtalarini topamiz.

$$f'(x)=\left(\frac{x^2+4}{x}\right)'=\frac{x^2-4}{x^2}\,,\qquad \frac{x^2-4}{x^2}=0,$$

Funksiya hosilasi $x=-2$ va $x=2$ nuqtalarda nolga teng, shuning uchun ular kritik nuqtalar bo‘ladi.

5) O‘sish va kamayish oraliqlari, ekstremumlari:

Funksiya $x=-2$ da maksimumga erishadi: $f(-2)=\dfrac{(-2)^2+4}{-2}=-4$, funksiyaning maksimum nuqtasi $(-2;-4)$.

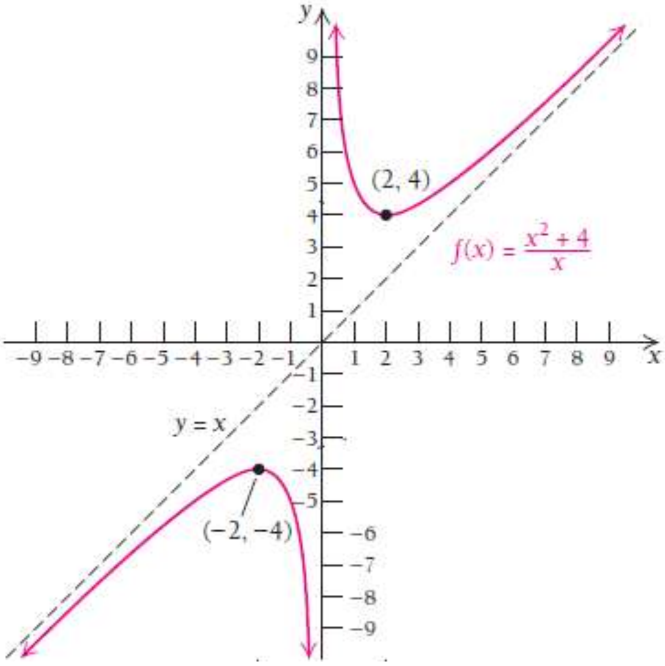
$x=2$ da minimumga erishadi: $f(2)=\dfrac{2^2+4}{2}=4$, minimum nuqtasi $(2;4)$.

6) Egilish nuqtalari: $f''(x_0)=0$ bo‘ladigan yoki $f''(x_0)$ mavjud bo‘lmaydigan nuqtalarni aniqlaymiz va bu nuqtalarda funksiya qiymatlarini hisoblaymiz.

$$f''(x)=\left(\frac{x^2-4}{x^2}\right)'=\frac{2x^3-2x^3+8x}{x^4}=\frac{8}{x^3}=0.$$

Demak, mazkur funksiyaning egilish nuqtasi yo‘q.

7) Qavariqlik va botiqlik oraliqlari va grafigi:



2.52-rasm. $f(x)=\dfrac{x^2+4}{x}$ funksiya grafigi ◄

Biz shu paytgacha funksiya berilgan bo‘lsa, uning asimptotalarini topgan edik. Endi teskari masalani qaraymiz.

Agar asimptotalar berilgan bo‘lsa, kasr-ratsional funksiyaning analitik ifodasini aniqlash mumkinmi?

2.8.17-misol. Vertikal asimptotasi $x=-5$ va $x=2$ to‘g‘ri chiziqlar, gorizontal asimptotasi esa $y=2$ to‘g‘ri chiziq, shu bilan birga $f(1)=3$ qiymat qabul qiluvchi qisqarmaydigan ratsional funksiyani aniqlang va grafigini chizing.

Yechilishi: ► Vertikal asimptotasi $x=-5$ va $x=2$ bo‘lgani uchun kasrning mahrajida $(x+5)(x-2)$ ko‘paytma bo‘ladi.

Gorizontal asimptotasi $y=2$ ekanligidan va mahrajdagi qavslarni ko‘paytirganimizda $(x+5)(x-2)=x^2+3x-10$ kvadrat ko‘phad hosil bo‘lishidan, kasrning butun qismini ajratganda 2 kelib chiqishi uchun suratida $2x^2+C$ ko‘phad bo‘lishi kerak deb faraz qilamiz, bu yerda C qandaydir o‘zgarmas son.

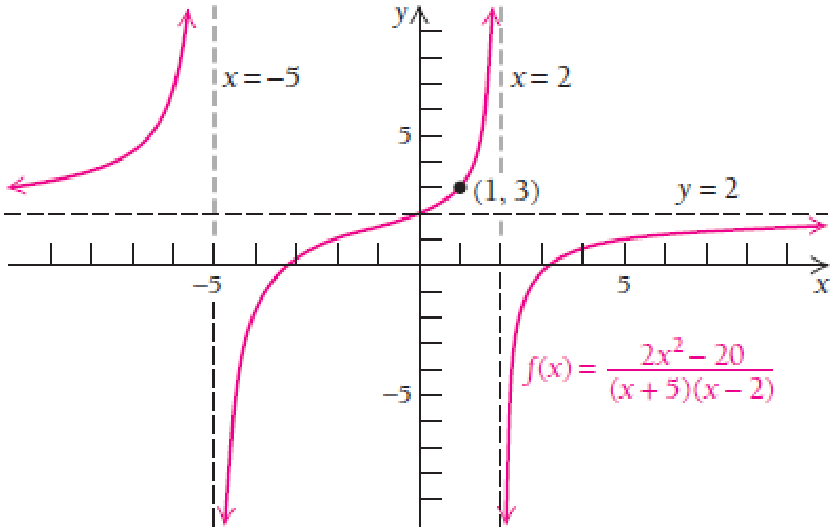
Shunda, bizning ratsional funksiya quyidagi ko‘rinishda bo‘ladi:

$$f(x)=\frac{2x^2+C}{(x+5)(x-2)}$$

Endi C nimaga teng ekanligini aniqlaymiz. Buning uchun $f(1)=3$ tenglikdan foydalanamiz.

$$3=\frac{2\cdot 1^2+C}{(1+5)(1-2)}, \quad \longrightarrow \quad 3=\frac{2+C}{-6}, \quad \longrightarrow \quad -18=2+C \quad \longrightarrow \quad C=-20.$$

U holda $f(x)=\frac{2x^2-20}{(x+5)(x-2)}$ ratsional funksiyani hosil qilamiz.Ushbu funksiyaning grafigi 2.53-rasmda keltirilgan.



2.53-rasm. $f(x)=\frac{2x^2-20}{(x+5)(x-2)}$ funksiya grafigi ◄

Mavzu yuzasidan savollar:

1. Kesmada o‘suvchi va kamayuvchi funksiya ta’rifini ifodalang.
2. Funksiya o‘suvchi bo‘lishining zaruriy va yetarli shartlari qanday?
3. Funksiyaning ekstremum nuqtalarini ta’riflang.
4. Ekstremumning zaruriy shartini ifodalang.
5. Funksiyaning kesmadagi eng katta va eng kichik qiymatlari qanday topiladi?
6. 2-tartibli hosila yordamida funksiya ekstremumining yetarlilik sharti qanday?
7. Funksiya grafigining botiq va qavariq bo‘lish ta’rifini bering.
8. Funksiya grafigining botiqlik va qavariqlilik sharti qanday?
9. Egilish nuqtalari uchun yetarlilik sharti nimadan iborat?
10. Egri chiziq asimptotasining ta’rifini ayting.
11. Qanday asimptotalarni bilasiz?

MUSTAQIL YECHISH UCHUN MASALALAR

1. Funksiyaning asimptotalarini toping:

a) $f(x)=\dfrac{2x-3}{x-5}$ **;** **b)** $f(x)=\dfrac{7x+5}{x^2+7x+6}$

c) $f(x)=\dfrac{6x^4+4x^2-7}{2x^5-x+3}$ **d)** $f(x)=\dfrac{4x^3-3x+2}{x^3+2x-4}$

2. Funksiyaning ekstremumlarini toping:

a) $g(x)=\dfrac{x}{x-1}$ **b)** $g(x)=x^3e^{2x+1}$

3. Funksiyaning qavariqlik va botiqlik oraliqlarini toping:

a) $g(x)=-\dfrac{x^3-2}{x^2+1}$ **b)** $g(x)=\dfrac{e^{3(x-2)}}{x^2-4}.$

4. Funksiyaning kesmadagi eng katta va eng kichik qiymatlarini toping:

a) $y=\dfrac{(x+3)^2}{x-4},\quad [-4;\ 3]$**;** **b)** $y=2\sin x+\cos 2x,\quad [0;\pi/2]$**;**

c) $y=x^3e^{x+1},\quad [-4;\ 0]$**;** **d)** $y=e^{4x-x^2},\quad [1;\ 3]$**;**

e) $y=(x+1)\sqrt[3]{x^2},\quad [-\dfrac{4}{5};\ 3]$**;** **f)** $y=\dfrac{1+\ln x}{x},\quad \left[\dfrac{1}{e};\ e\right]$**.**

5. Funksiyani to‘la tekshiring va grafigini yasang:

a) $g(x)=\dfrac{2}{x^2}$ **b)** $g(x)=x^2\ln(x+1)$

TESTLAR

1. $y=\dfrac{\ln(x+1)}{x^2}+2x$ funksiyaning asimptotalarini toping.

A) $x=0,\ y=4x,\ x=-1$ **B)** $x=0,\ y=2x,\ x=-1$

C) $x=0,\ y=3x,\ x=1$ **D)** $x=0,\ y=3x,\ x=-1$

2. $y=3x^5-5x^4+4$ funksiyaning qavariq va botiq oraliqlarini toping.

A) $(-\infty;1)$ *da qavariq*, $(1;+\infty)$ *da botiq* **B)** $(-\infty;4)$ *da botiq*, $(4;+\infty)$ *da qavariq*

C) $(-\infty;3)$ *da qavariq*, $(3;+\infty)$ *da botiq* **D)** $(-\infty;2)$ *da qavariq*, $(2;+\infty)$ *da botiq*

3. $y = e^{-\frac{x^2}{2}}$ funksiyaning qavariqlik, botiqlik oraliqlari topilsin

A) $(-\infty;1)\, va\, (1;+\infty)\, da$ $qa\,\text{var}\, iq,\ (-1;1)\, da$ $botiq$

B) $(-\infty;1)\, va\, (1;+\infty)\, da\ botiq,$ $(-1;1)\, da\ qa\,\text{var}\, iq$

C) $(-\infty;1)\, da\ qa\,\text{var}\, iq,$ $(1;+\infty)\, da\ botiq$

C) $(-\infty;2)\, da\, qa\,\text{var}\, iq,$ $(2;+\infty)\, da\ botiq$

4. $[-4;\,4]$ kesmada $y = x^3 - 3x^2 - 9x + 35$ funksiyaning eng katta va eng kichik qiymatlari topilsin.

A) 40 eng katta va -41 eng kichik qiymati;
B) 8 eng katta va -41 eng kichik qiymati;
C) 20 eng katta va -41 eng kichik qiymati;
D) 40 eng katta va -21 eng kichik qiymati.

5. $y = x^2 \cdot (1 + \frac{4}{9}x)$ funksiyaning monotonlik oraliqlari topilsin.

A) $(-1{,}5;\,0)$ da kamayuvchi, $(-\infty;-1{,}5) \cup (0;+\infty)$ da o‘suvchi;
B) $(0;+\infty)$da kamayuvchi, $(-\infty;-1{,}5) \cup (-1{,}5;\,0)$ da o‘suvchi;
C) $(-1{,}5;\,0)$ da kamayuvchi, $(-\infty;-1{,}5)$ da o‘suvchi;
D) $(-\infty;-1{,}5) \cup (0;+\infty)$da kamayuvchi, $(-1{,}5;\,0)$da o‘suvchi;

2.9-§. Funksiyalarni Lagranj interpolyatsion formulasi yordamida approksimatsiyalash va egri chiziq yasash

2.9.1. Masalaning qo‘yilishi. Funksiyalarni interpolyatsiyalash

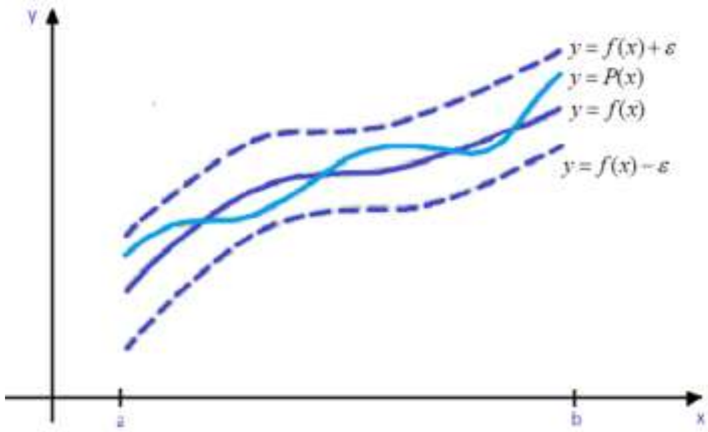
Approksimatsiyalash – **yaqinlashtirish** degan ma’noni bildiradi.

Ko‘pincha amaliy masalalarni yechishda qandaydir $y = f(x)$ funksional bog‘lanishlar qiymatlarini hisoblashga to‘g‘ri keladi. Bunday masalalarda ikkita holat bo‘lishi mumkin:

1. $[a;b]$ oraliqda x va y lar orasidagi oshkor bog‘lanish ma’lum emas, faqat $\{x_i, y_i\}$, $i = \overline{1,n}$ tajriba ma’lumotlari jadvali ma’lum. Bu jadvaldan $[x_i, x_{i/2}] \in [a, b]$ oraliqda $y = f(x)$ bog‘lanishni aniqlash talab qilinadi.

2. $y = f(x)$ bog‘lanish ma’lum va uzluksiz, biroq u amaliy hisoblashlar uchun murakkablik qiladi. Bunday holda $y = f(x)$ funksiyani va uning (hosilasi, maksimum va minimum qiymatlari, funksiya integrali kabi) xarakteristikalarini hisoblash ishlarini soddalashtirish kerak bo‘ladi.

Shuning uchun moddiy resurslarni va vaqtni iqtisod qilish maqsadida qandaydir boshqa $y = P(x)$ funksional bog‘lanish tuziladi. Bu tuzilgan bog‘lanish $y = f(x)$ ga uning asosiy parametrlari bo‘yicha yaqin bo‘lishi, hisoblash oson va qulay bo‘lishi kerak, ya’ni $y = f(x)$ funksiyani aniqlanish sohasida **approksimatsiyalash** kerak (2.54-rasm).



2.54-rasm. Funksiya va uni approksimatsiyalovchi funksiya

$y = P(x)$ funksiyaga **approksimatsiyalovchi funksiya** deyiladi.

Agar yaqinlashishni biror $\{x_i\}$, $i = \overline{1,n}$ diskret to‘plamda bajarsak, u holda approksimatsiyaga **nuqtaviy approksimatsiya** deyiladi.

$[a;b]$ oraliqda yotuvchi, tajriba asosida olingan x_i nuqtalarni o‘sish tartibida raqamlab chiqamiz va ularni **tugunlar** deb ataymiz:

$$a \le x_0 \le x_1 \le x_2 \le ... \le x_n \le b$$

Jadval ma’lumotlarini approksimatsiyalashning quyidagi turlari ma’lum:

1. Jadval ma’lumotlarini algebraik ko‘phadga keltirish

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n;$$

2. Jadval ma’lumotlarini trigonometrik ko‘phadga keltirish

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega t) + b_n \sin(n\omega t) \right);$$

3. Jadval ma’lumotlarini eksponensial ko‘phadga keltirish

$$f(x) = a_0 e^{\alpha_0 x} + a_1 e^{\alpha_1 x} + a_2 e^{\alpha_2 x} + ... + a_n e^{\alpha_n x}.$$

Qaysi turdagi approksimatiyalash formulasidan foydalangan ma’qul?

Algebraik ko‘phad ancha qulay, chunki algebraik ko‘phadni trigonometrik va eksponensial ko‘phadlarga qaraganda xatoligini aniqlash, differensiallash, integrallash oson.

Nuqtaviy approksimatsiyalash turlariga **interpolyatsiyalash va ekstropolyatsiyalash** kiradi.

Interpolyatsiyalashdan maqsad shuki, (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) ma’lumotlar (nuqtalar) ma’lum. Biroq bizga $y = f(x)$ funksiyaning boshqa x nuqtadagi qiymati kerak bo‘lsin. Ma’lumotlar asosida (x, y) ni aniqlash mumkinmi? Misol uchun, raketa tezligining 10, 15, 20-sekundlardagi tezliklarini o‘lchay olganmiz. Lekin 16-sekunddagi tezligi nimaga teng?

Ya’ni berilgan oraliqda yotuvchi nuqtadagi funksiya qiymatini topish masalasiga **interpolyatsiyalash** deyiladi, berilgan oraliqdan tashqaridagi nuqtada funksiya qiymatini topish ekstropolyatsiyalash masalasi deyiladi.

Interpolyatsiyalashning quyidagi turlari mavjud:

1. Chiziqli interpolyatsiya;
2. Kvadratik interpolyatsiya;
3. Kubik interpolyatsiya;
4. Lagranj interpolyatsion formulasi;
5. Nyuton interpolyatsion formulasi;
6. O‘rtacha kvadratik yaqinlashish usuli.

Interpolyatsiya tugunlari sonining oshishi, algebraik ko‘phad darajasini oshishiga olib keladi, bu esa tugunlar oralig‘ida funksiyani juda katta sakrashlariga sabab bo‘ladi. Shu sababli tugunlar sonini ko‘proq olish maqsaddan chetlanib ketishga sabab bo‘ladi.

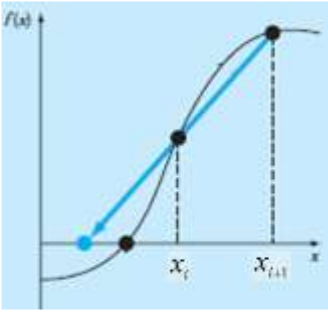
2.9.2. Chiziqli interpolyatsiya

Agar $[a;b]$ oraliqda berilgan jadval yoki diagrammadan foydalanib, algebraik ikkihad tuzsak, unga **chiziqli interpolyatsiya** deyiladi:

$$f(x) = a_0 + a_1 x \tag{2.30}$$

Berilgan oraliqqa tegishli ikkita x_i, x_{i+1} tugunni olamiz (2.55- rasm) va 2 noma’lumli chiziqli tenglamalar sistemasini tuzamiz:

$$\begin{cases} f(x_i) = a_0 + a_1 x_i \\ f(x_{i+1}) = a_0 + a_1 x_{i+1} \end{cases}$$



2.55-rasm. Chiziqli approksimatsiya

Bu formuladan a_0, a_1 ni topib, chiziqli interpolyatsiya formulasiga qo‘yamiz.

2.9.1-misol. Raketaning ko‘tarilish tezligi vaqtning funksiyasi sifatida jadvalda keltirilgan (2.56-rasm). $t = 16$ sekunddagi raketa tezligini hisoblang.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

2.56-rasm. Raketaning ko‘tarilish tezligi

Yechilishi: ► Chiziqli (2.30) interpolyatsiyani tuzib, tezlikni aniqlaymiz: $v(t) = a_0 + a_1 t.$

$a_0,\ a_1$ ni topish uchun $t=16$ sekundga eng yaqin nuqtalarni aniqlaymiz, ular quyidagilar:

$$t_0=15; \quad v(t_0)=362.78$$

$$t_1=20; \quad v(t_1)=517.35$$

Bu nuqtalarni formulaga qo‘yib, tenglamalar sistemasini tuzamiz:

$$\begin{cases} v(15)=a_0+15a_1=362.78 \\ v(20)=a_0+20a_1=517.35 \end{cases} \qquad \begin{matrix} a_0=-100.93 \\ a_1=30.914 \end{matrix}$$

$$v(t)=a_0+a_1t=-100.93+30.914t$$

Ko‘tarilish tezligining chiziqli interpolyatsiyasi:

$$v(t)=30.914t-100.93, \quad 15\le \text{t} \le 20$$

Shunday qilib, raketaning $t=16$ sekunddagi tezligini topamiz:

$$v(16)=30.914\cdot 16-100.93=393.7 \quad \text{m/s}$$ ◀

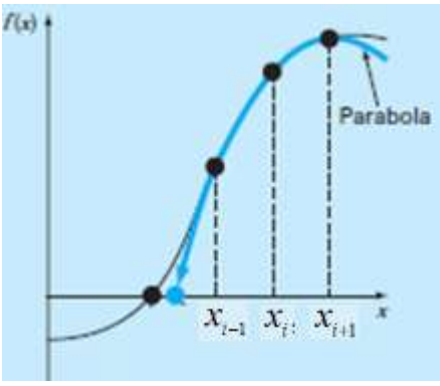
2.9.3. Kvadratik interpolyatsiya

Agar $[a;b]$ oraliqda berilgan jadval yoki diagrammadan foydalanib, 3 ta haddan iborat ko‘phad tuzilsa, unga **kvadratik (parabolik) interpolyatsiya** deyiladi:

$$f(x)=a_0+a_1x+a_2x^2. \qquad (2.31)$$

Bunda oraliqqa tegishli uchta $x_{i-1},\ x_i,\ x_{i+1}$ tugunni olamiz va 3 noma’lumli chiziqli tenglamalar sistemasini tuzamiz:

$$\begin{cases} f(x_{i-1})=a_0+a_1x_{i-1}+a_2x_{i-1}^2 \\ f(x_i)=a_0+a_1x_i+a_2x_i^2 \\ f(x_{i+1})=a_0+a_1x_{i+1}+a_2x_{i+1}^2 \end{cases}$$



2.57-rasm. Kvadratik (parabolik) interpolyatsiya

Bu formuladan $a_0,\ a_1,\ a_2$ koeffitsiyentlarni topib, kvadratik interpolyatsiya formulasiga qo‘yamiz.

2.9.1-misolning 2.56-rasmdagi ma'lumotlaridan $v(16)$ shartni kvadratik interpolyatsiya formulasidan topamiz.

► Kvadratik interpolyatsiya (2.31) ni tuzamiz, bunda a_0, a_1, a_2 larni topish uchun $t = 16$ sekundga eng yaqin bo'lgan 3 ta nuqtani aniqlaymiz, ular quyidagilar:

$$t_0 = 10, \quad v(t_0) = 227.04 \qquad \begin{cases} v(10) = a_0 + 10a_1 + 10^2 a_2 = 227.04 \\ v(15) = a_0 + 15a_1 + 15^2 a_2 = 362.78 \\ v(20) = a_0 + 20a_1 + 20^2 a_2 = 517.35 \end{cases}$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

Hosil qilingan tenglamalar sistemasini yechamiz:

$$\begin{cases} a_0 + 10a_1 + 100a_2 = 227.04 \\ a_0 + 15a_1 + 225a_2 = 362.78 \\ a_0 + 20a_1 + 400a_2 = 517.35 \end{cases} \qquad \begin{aligned} a_0 &= 12.05 \\ a_1 &= 17.733 \\ a_2 &= 0.3766 \end{aligned}$$

Raketa tezligi uchun kvadratik interpolyatsiyani tuzamiz:

$$v(t) = 0.3766t^2 + 17.733t + 12.05, \quad 10 \le t \le 20$$

Shunda raketaning $t = 16$ sekundgagi ko'tarilish tezligi quyidagicha bo'ladi:

$$v(t) = 0.3766 \cdot 16^2 + 17.733 \cdot 16 + 12.05 = 392.19 \text{ m/s}$$ ◄

2.9.4. Lagranj interpolyatsion formulasi[2]

Tugunlar soni ortib borishi bilan hisoblash ishlari ham murakkablashib ketadi. Chunki n noma'lumli tenglamalar sistemasini yechish amaliyoti ko'p vaqt va xotira (EHM) talab qiladi.

Shuning uchun tenglamalar sistemasi tuzishni talab qilmaydigan va faqat arifmetik hisoblashlar bajariladigan ancha sodda, (n+1) ta qo'shiluvchidan iborat Lagranj interpolyatsion formulasidan foydalaniladi.

Jadval ma'lumotlarining berilishiga qarab, teng va tengmas oraliqlar uchun Lagranj interpolyatsion formulalaridan foydalaniladi.

Tengmas oraliqlar uchun Lagranj interpolyatsion formulasi:

$$L_n(x) = \sum_{j=0}^{n} y_j \prod_{\substack{i=0 \\ i \ne j}}^{n} \frac{x - x_i}{x_j - x_i} \qquad (2.31)$$

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2. Autar Kaw, E.Erik Kalu "Numerical Methods with Applications: Abridged", 2-nd edition, 2011. 756 p.

(2.31) munosabatning nazariy xatoligini aniqlash mumkin:

$$R_n(x) = f(x) - L_n(x) = \frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!} \cdot \omega(x), \qquad (2.32)$$

bunda $\xi \in [a, b]$ va $\omega(x) = (x - x_0)(x - x_1)...(x - x_n)$.

Teng oraliqlar uchun Lagranj interpolyatsion formulasi:

$$L_n(x) = \sum_{j=0}^{n} y_j \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{t - i}{j - i} \qquad (2.33)$$

bunda $t = \frac{x - x_0}{h}$, $h = x_{i+1} - x_i = \text{const}$.

(2.33) ning nazariy xatoligini aniqlash mumkin:

$$R_n(x) = h^{n+1} t(t-1)\ldots(t-n) \frac{f^{(n+1)}(\xi)}{(n+1)!}. \qquad (2.34)$$

Lagranj interpolyatsion formulasini qo‘llashda hisoblashlarni soddalashtirish uchun uni 2 qismga ajratib ham foydalanish mumkin:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \qquad (2.35)$$

Bunda $L_i(x)$ ga Lagranj interpolyatsion ko‘phadi deyiladi:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad (2.36)$$

2.9.1-misolning 2.56-rasmdagi ma’lumotlaridan $v(16)$ shartni Lagranj interpolyatsion formulasidan topamiz.

► **a)** 2 ta tugun nuqtani oladigan bo‘lsak, 1-tartibli Lagranj interpolyatsion formulasi hosil bo‘ladi. (2.35) va (2.36) formulalardan foydddalanamiz.

$$v(t) = \sum_{i=0}^{1} L_i(t) v(t_i) = L_0(t) v(t_0) + L_1(t) v(t_1)$$

$$t_0 = 15; \qquad v(t_0) = 362.78$$

$$t_1 = 20; \qquad v(t_1) = 517.35$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1} \qquad L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1) = \frac{t - 20}{15 - 20} \cdot 362.78 + \frac{t - 15}{20 - 15} \cdot 517.35$$

$$v(16)=\frac{16-20}{15-20}\cdot 362.78+\frac{16-15}{20-15}\cdot 517.35=0.8\cdot 362.78+0.2\cdot 517.35=393.69\ \text{m/s}$$

b) 3 ta tugun nuqtani oladigan bo‘lsak, 2-tartibli Lagranj interpolyatsion formulasi hosil bo‘ladi. (2.35) va (2.36) formulalardan foydddalanamiz.

$$v(t)=\sum_{i=0}^{2}L_i(t)v(t_i)=L_0(t)v(t_0)+L_1(t)v(t_1)+L_2(t)v(t_2)$$

$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{2}\frac{t-t_j}{t_0-t_j}=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2},\qquad L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{2}\frac{t-t_j}{t_1-t_j}=\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}$$

$$L_2(t)=\prod_{\substack{j=0\\ j\neq 2}}^{2}\frac{t-t_j}{t_2-t_j}=\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}$$

$$v(t)=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2}v(t_0)+\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}v(t_1)+\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}v(t_2)$$

$$t_0=10,\quad v(t_0)=227.04$$
$$t_1=15,\quad v(t_1)=362.78$$
$$t_2=20,\quad v(t_2)=517.35$$

$$v(16)=\frac{16-15}{10-15}\cdot\frac{16-20}{10-20}\cdot 227.04+\frac{16-10}{15-10}\cdot\frac{16-20}{15-20}\cdot 362.78+$$
$$+\frac{16-10}{20-10}\cdot\frac{16-15}{20-15}\cdot 517.35=392.19\ \text{m/s}$$

c) 3-tartibli Lagranj interpolyatsion formulasi yordamida $v(16)$ ni topamiz. Bunda 4 ta tugunni olamiz. (2.35) va (2.36) formulalardan foydddalanamiz.

$$v(t)=\sum_{i=0}^{3}L_i(t)v(t_i)=L_0(t)v(t_0)+L_1(t)v(t_1)+L_2(t)v(t_2)+L_3(t)v(t_3)$$

$$t_0=10,\quad v(t_0)=227.04$$
$$t_1=15,\quad v(t_1)=362.78$$
$$t_2=20,\quad v(t_2)=517.35$$
$$t_3=22.5,\quad v(t_3)=602.97$$

$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{3}\frac{t-t_j}{t_0-t_j}=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2}\cdot\frac{t-t_3}{t_0-t_3}\qquad L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{3}\frac{t-t_j}{t_1-t_j}=\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}\cdot\frac{t-t_3}{t_1-t_3}$$

$$L_2(t)=\prod_{\substack{j=0\\ j\neq 2}}^{3}\frac{t-t_j}{t_2-t_j}=\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}\cdot\frac{t-t_2}{t_2-t_3}\qquad L_3(t)=\prod_{\substack{j=0\\ j\neq 3}}^{3}\frac{t-t_j}{t_3-t_j}=\frac{t-t_0}{t_3-t_0}\cdot\frac{t-t_1}{t_3-t_1}\cdot\frac{t-t_2}{t_3-t_2}$$

$$v(t)=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2}\cdot\frac{t-t_3}{t_0-t_3}v(t_0)+\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}\cdot\frac{t-t_3}{t_1-t_3}\cdot v(t_1)+$$

$$+\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}\cdot\frac{t-t_3}{t_2-t_3}v(t_2)+\frac{t-t_0}{t_3-t_0}\cdot\frac{t-t_1}{t_3-t_1}\cdot\frac{t-t_2}{t_3-t_2}\cdot v(t_3)$$

$$v(16)=\frac{16-15}{10-15}\cdot\frac{16-20}{10-20}\cdot\frac{16-22.5}{10-22.5}\cdot 227.04+\frac{16-10}{15-10}\cdot\frac{16-20}{15-20}\cdot\frac{16-22.5}{15-22.5}\cdot 362.78+$$

$$+\frac{16-10}{20-10}\cdot\frac{16-15}{20-15}\cdot\frac{16-22.5}{20-22.5}\cdot 517.35+\frac{16-10}{22.5-10}\cdot\frac{16-15}{22.5-15}\cdot\frac{16-20}{22.5-20}\cdot 602.97=$$

$$=392.06\ \ \text{m/s}$$ ◄

Mavzu yuzasidan savollar

1. Approksimatsiyalash deganda nimani tushunasiz?
2. Nuqtaviy approksimatsiyalash va uning turlarini sanab bering.
3. Interpolyatsiya deganda nimani tushunasiz?
4. Ekstropolyatsiya deganda nimani tushunasiz?
5. Teng oraliqlar uchun Lagranj interpolyatsion formulasi qanday tuzilishga ega?
6. Tengmas oraliqlar uchun Lagranj interpolyatsion formulasi qanday tuzilishga ega?
7. Lagranj interpolyatsion formulalarining nazariy xatoliklari qanday aniqlanadi?

MUSTAQIL YECHISH UCHUN MISOLLAR:

1. $x=\frac{1}{6};\frac{1}{4};\frac{1}{3};\frac{1}{2}$ tugun nuqtalari bo‘yicha tengmas oraliqlar uchun Lagranj interpolyatsion formulasidan foydalanib, $f(x)=\cos\pi x$ funksiyaning $x=\frac{5}{12}$ nuqtadagi taqribiy qiymatini toping va butun oraliq bo‘yicha xatolikni baholang.

2. $x=1;\ 8;\ 27;\ 64$ tugun nuqtalari bo‘yicha tengmas oraliqlar uchun Lagranj interpolyatsion formulasiga ko‘ra $f(x)=\sqrt[3]{x}$ funksiya ko‘phadini tuzing va $x=5$ nuqtadagi taqribiy qiymatini toping, butun oraliq bo‘yicha xatolikni baholang.

3. $x=9;\ 10;\ 12;\ 15$ tugun nuqtalari bo‘yicha tengmas oraliqlar uchun Lagranj interpolyatsion formulasidan foydalanib, $f(x)=\ln x$ funksiya ko‘phadini tuzing, bunda $\ln 2=0{,}693;\ \ln 3=1{,}099;\ \ln 5=1{,}609$ qiymatlardan foydalaning.

4. $x=0;\ 1;\ 3$ tugun nuqtalarda $f(x)=x\cdot 2^{-x}$ funksiyaning tengmas oraliqlar uchun Lagranj interpolyatsion ko‘phadini $x=2$ nuqtadagi xatoligini va butun oraliq bo‘yicha xatolikni baholang. Funksiyaning taqribiy qiymatini hisoblang.

5.

x	1	2	4
y	3	8	1

jadval asosida $y=f(x)$ funksiyaning tengmas oraliqlar uchun Lagranj interpolyatsion ko‘phadini tuzing va shu asosida funksiyaning $f(3)$ taqribiy qiymatini toping.

TESTLAR

1. Interpolyatsion ko‘phadning qoldiq hadi yoki xatoligi tugun nuqtalarda nimaga teng bo‘ladi?
   A) $R_n(x)=-1$ **B)** $R_n(x)=0$ **C)** $R_n(x)=1$ **D)** $R_n(x)=2.$

2. Agar $x_0, x_1, x_2, \dots, x_n$- nuqtalar teng oraliqlar bo‘yicha joylashgan bo‘lsa, u holda qanday belgilash kiritib, Lagranj interpolyatsion ko‘phadini hisoblashni soddalashtirish mumkin?
   A) $x+x_0=t\cdot h$ **B)** $h(x-x_0)=t$ **C)** $x-x_0=t\cdot h$ **D)** $x-h\cdot x_0=t$

3. *f(x)* funksiya (0.25;4), (0.5;8) nuqtalardagi qiymati berilgan. Teng oraliqlar uchun Lagranj interpolyatsion ko‘phadi bo‘yicha *f(0.3)* ning taqribiy qiymati qanday hisoblanadi?
   A) $f(0.3)\approx L_n(0.1)$ **B)** $f(0.3)\approx L_n(0.2)$
   C) $f(0.3)\approx L_n(0.3)$ **D)** $f(0.3)\approx L_n(0.4)$

4. Jadvaldagi interpolyatsiya tugunlari soni 8 ta bo‘lsa, u holda qidirilayotgan ko‘phadning darajasi nechaga teng bo‘ladi?
   A) 9 **B)** 8 **C)** 7 **D)** 6

5. $x_0, x_1, x_2, \dots, x_n$ – nuqtalar qanday nomlanadi?
   A) Determinatsiya tugunlari **B)** Interpolyatsiya tugunlari
   C) Approksimatsiya tugunlari **D)** Matematika tugunlari

III BOB. INTEGRAL HISOB

3.1-§. Boshlang‘ich funksiya. Aniqmas integral. Integrallash usullari

3.1.1. Boshlang‘ich funksiya. Aniqmas integral

Faraz qiling, biz differensiallashni teskarisini bajaramiz, ya’ni hosilasi berilgan funksiyaga teng bo‘lgan funksiyani izlaymiz. Bunga **antidifferensiallash (boshlang‘ich funksiyani topish)** deyiladi.

Bizga $f(x)$ funksiya berilgan bo‘lsin. Boshqa $F(x)$ funksiyani shunday aniqlaymiz: $F(x)$ funksiyaning hosilasi $f(x)$ ga teng, ya’ni $F'(x) = f(x)$.

Misol, agar $f(x) = 2x$ bo‘lsa, u holda uning boshlang‘ich funksiyasi $F(x) = x^2$ bo‘ladi, chunki $F'(x) = (x^2)' = 2x.$

Biroq boshqa bir funksiyaning hosilasi ham $2x$ bo‘lishi mumkin, aytaylik, quyidagi funksiyalarning: $y = x^2 + 5$ yo $y = x^2 - 1$ yoki $y = x^2 + 560$.

Sababi x^2 ning hosilasi $2x$, o‘zgarmas sonlarning hosilasi esa nolga teng. Shuning uchun ham $f(x) = 2x$ funksiyaning boshlang‘ich funksiyasini umumiy ko‘rinishda $F(x) = x^2 + C$ deb yoziladi, bunda $C = \text{const}$.

3.1-teorema. $[F(x) + C]' = f(x)$ o‘rinli bo‘lsa, $F(x) + C$ funksiyalar to‘plami $f(x)$ ning boshlang‘ich funksiyasi bo‘ladi. $C -$ o‘zgarmas songa **integrallash o‘zgarmasi** deyiladi.

Natija: Agar $F(x)$ va $G(x)$ funksiyalarning hosilalari bir xil bo‘lsa, u holda ular o‘zgarmas songa farq qiladi: $F(x) = G(x) + C$.

Agar $F(x)$ funksiya $f(x)$ ning boshlang‘ich funksiyasi bo‘lsa, uni

$$\int f(x)dx = F(x) + C \qquad (3.1)$$

ko‘rinishda yozamiz. (3.1) tenglik x o‘zgaruvchiga nisbatan $f(x)$ ning aniqmas integrali $F(x) + C$ ga teng deb o‘qiladi. $\int$ belgi integral belgisi va integral olish uchun buyruq, $f(x)$ integral ostidagi funksiya va dx esa integrallash x o‘zgaruvchi bo‘yicha bajarilishini bildiradi. Chap tomondagi ifodaga **aniqmas integral** deyiladi.

3.2-teorema. Aniqmas integralning ba’zi qoidalari

1⁰. O‘zgarmas sonning integrali: $\int k\,dx = kx + C$;

2⁰. Darajali funksiyaning integrali: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$;

3⁰. Natural logarifm qoidasi: $\int \frac{1}{x}dx = \ln x + C$, $x > 0$;

4⁰. Eksponensial funksiya qoidasi: $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$, $a \neq 0$.

Darajali funksiya x^n ning integralini hisoblaganimizda daraja ko‘rsatkichiga 1 ni qo‘shib, hosil bo‘lgan darajaga bo‘lamiz, faqat $n = -1$ da integral bu qonuniyat asosida hisoblanmaydi:

$$\int \frac{1}{x}dx = \ln x + C, \quad x > 0.$$

Aniqmas integralning xossalari:

1⁰. O‘zgarmas ko‘paytuvchini integral belgisidan tashqariga chiqarish mumkin: $\int k \cdot f(x)\,dx = k \cdot \int f(x)dx$;

2⁰. Funksiyalar algebraik yig‘indisining integrali shu funksiyalar integrallarining algebraik yig‘indisiga teng:

$$\int [f(x) \pm \varphi(x)]\,dx = \int f(x)dx \pm \int \varphi(x)dx.$$

Integrallashdagi C o‘zgarmas son ba’zi amaliy masalalarda ahamiyatli hisoblanadi. Shuning uchun bunday masalalarda funksiyaning aniqmas integralini C ga nisbatan yechib, nuqtani aniqlab olishimiz mumkin. Bu nuqtaga **boshlang‘ich shart** deyiladi.

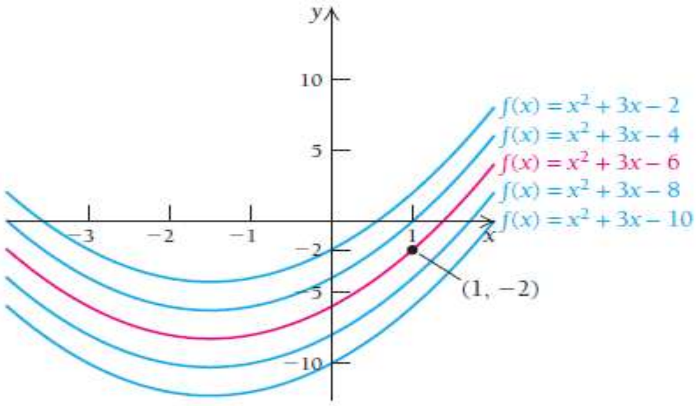
3.1.1-misol. $f'(x) = 2x + 3$ funksiyaning $f(1) = -2$ nuqtadagi boshlang‘ich funksiyasini toping.

Yechilishi: ► Aniqmas integralni hisoblaymiz va quyidagi tenglikni

$$\int (2x+3)dx = x^2 + 3x + C$$

hosil qilamiz. Endi funksiyaning $f(1) = -2$ nuqtadagi qiymatini topiamiz.

$$f(x) = x^2 + 3x + C \quad \Rightarrow \quad -2 = 1^2 + 3 \cdot 1 + C \quad \Rightarrow \quad C = -6.$$



3.1-rasm. $f(x)=x^2+3x+C$ funksiyaning grafiklari

Bundan aytish mumkinki, $f'(x)=2x+3$ funksiyaning $f(1)=-2$ nuqtadagi boshlang‘ich funksiyasi $f(x)=x^2+3x-6$ dan iborat ekan (3.1-rasm). ◄

3.1.2. Bevosita va differensial belgisi ostiga kiritib integrallash

Agar funksiyaning integralini hisoblashda integrallash jadvalidan foydalanib, to‘g‘ridan-to‘g‘ri hisoblash mumkin bo‘lsa, unga **bevosita integrallash** deyiladi.

Integrallash jadvali:

1. $\int du = u + C$

2. $\int u^n du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \dfrac{du}{u^2} = -\dfrac{1}{u} + C$

4. $\int \dfrac{du}{\sqrt{u}} = 2\sqrt{u} + C$

5. $\int \dfrac{du}{u} = \ln|u| + C$

6. $\int a^u du = \dfrac{a^u}{\ln a} + C$

7. $\int e^u du = e^u + C$

8. $\int \sin u du = -\cos u + C$

9. $\int \cos u du = \sin u + C$

10. $\int \dfrac{du}{\cos^2 u} = tgu + C$

11. $\int \dfrac{du}{\sin^2 u} = -ctgu + C$

12. $\int \dfrac{du}{\sin u} = \ln\left|tg\dfrac{u}{2}\right| + C$

13. $\int \dfrac{du}{\cos u} = \ln\left|tg\left(\dfrac{u}{2}+\dfrac{\pi}{2}\right)\right| + C$

14. $\int tgudu = -\ln|\cos u| + C$

15. $\int ctgudu = \ln|\sin u| + C$

16. $\int \dfrac{du}{u^2+a^2} = \dfrac{1}{a} arctg\dfrac{u}{a} + C$

17. $\int \dfrac{du}{u^2-a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right| + C$

18. $\int \dfrac{du}{\sqrt{a^2-u^2}} = \arcsin\dfrac{u}{a} + C$

19. $\int \frac{du}{\sqrt{u^2+a^2}} = \ln\left|u+\sqrt{u^2+a^2}\right| + C$. 20. $\int \frac{dx}{x(a+bx)} = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$

21. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} + \frac{1}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$

22. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left|x+\sqrt{x^2 \pm a^2}\right|\right] + C$

23. $\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx-2a)(a+bx)^{3/2} + C$

24. $\int x^2\sqrt{a+bx}\,dx = \frac{2}{105b^3}(15b^2x^2 - 12abx + 8a^2)(a+bx)^{3/2} + C$

25. $\int \frac{xdx}{\sqrt{a+bx}} = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx} + C$

26. $\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2}{15b^3}(3b^2x^2 - 4abx + 8a^2)\sqrt{a+bx} + C$.

3.1.2-misol. Quyidagi aniqmas integrallarni hisoblang:

a) $\int (2x - \sqrt[5]{x^3} - 4)dx$; b) $\int (x^7 - \frac{2}{\sqrt[3]{x}} + 5^x)dx$; c) $\int x\sqrt{2+3x}\,dx$.

Yechilishi: ► Jadvaldagi mos formulalarga qo‘yib, integrallarni hisoblaymiz:

a) $\int (2x + \sqrt[5]{x^3} - 4)dx = x^2 + \frac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1} - 4x + C = x^2 + \frac{5}{8}\sqrt[5]{x^8} - 4x + C;$

b) $\int (x^7 - \frac{2}{\sqrt[3]{x}} + 5^x)dx = \frac{x^8}{8} - 2\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + \frac{5^x}{\ln 5} + C = \frac{x^8}{8} - 3\sqrt[3]{x^2} + \frac{5^x}{\ln 5} + C.$

c) $\int x\sqrt{2+3x}\,dx = \frac{2}{135}(9x-4)(2+3x)^{3/2} + C$. ◄

Integrallashda differensial belgisi ostiga kiritish usuli. Differensial belgisi ostiga kiritish usulida integral ostidagi ifodani quyidagicha almashtirish mumkin: $\int dx = \int d(x-a) = \int d(x+a)$,

$$\int dx = \frac{1}{k}\int d(kx) = \frac{1}{k}\int d(kx+a) = \frac{1}{k}\int d(kx-a),$$

$$\int xdx = \frac{1}{2}\int d(x^2) = \frac{1}{2}\int d(x^2-a) = \frac{1}{2}\int d(x^2+a),$$

$$\int \cos x dx = \int d(\sin x), \qquad \int \frac{dx}{x} = \int d(\ln x), \ \ldots$$

3.1.3-misol. Integrallarni hisoblang: a) $\int \frac{dx}{x \ln x}$; b) $\int e^{-x^2} x\, dx$

Yechilishi:► a) $\int \frac{dx}{x \ln x} = \int \frac{d(\ln x)}{\ln x} = \ln(\ln x) + C$.

b) $\int e^{-x^2} x\, dx = -\frac{1}{2}\int e^{-x^2} d(-x^2) = -\frac{1}{2} e^{-x^2} + C$. ◄

3.1.3. O‘zgaruvchini almashtirib integrallash va bo‘laklab integrallash usullari

O‘zgaruvchini almashtirib integrallash usuli. Jadvalda berilmagan $\int f(x)dx$ integralni hisoblash kerak bo‘lsin. x ni t erkli o‘zgaruvchining biror differensiallanuvchi funksiyasi sifatida ifodalab, integrallashda yangi t o‘zgaruvchini kiritamiz.

$x = \varphi(t)$, u holda tenglikning har ikki tomonini differensiallasak $dx = \varphi'(t)dt$ hosil bo‘ladi. Topilganlarni integraldagi x va dx larning o‘rniga qo‘yamiz:

$$\int f(x)dx = \int f[\varphi(t)] \cdot \varphi'(t)dt. \qquad (3.2)$$

Shunday qilib, integral yangi t o‘zgaruvchi bo‘yicha hisoblanadi. Tenglikning o‘ng qismida integrallashdan so‘ng yana eski x o‘zgaruvchiga qaytiladi.

3.1.4-misol. Integralni hisoblang: $\int \frac{\cos x}{\sqrt[3]{\sin x}} dx$.

Yechilishi: ► (3.2) formuladan foydalanamiz: *sinx=t* deb belgilaymiz.

$$\int \frac{\cos x}{\sqrt[3]{\sin x}} dx = \int \frac{d(\sin x)}{\sqrt[3]{\sin x}} = \int \frac{dt}{t^{\frac{1}{3}}} = \frac{t^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{3}{2} t^{\frac{2}{3}} + C = \frac{3}{2}\sqrt[3]{\sin^2 x} + C \quad ◄$$

Bo‘laklab integrallash usuli. $u = u(x)$ va $v = v(x)$ funksiyalar berilgan bo‘lsin. Bu ikki funksiya ko‘paytmasini differensiallaymiz:

$$(u \cdot v)' = u \cdot v' + v \cdot u'$$

yoki $d(u \cdot v) = u \cdot dv + v \cdot du$

bundan $u \cdot dv = d(u \cdot v) - v \cdot du$ kelib chiqadi.

Oxirgi tenglikning ikkala tomonini integrallaymiz va quyidagini topamiz:

$$\int u\cdot dv=\int d(u\cdot v)-\int v\cdot du,$$

bunda $\int d(u\cdot v)=u\cdot v$ o‘rinli. Natijada

$$\int udv=uv-\int v\cdot du \qquad (3.3)$$

bo‘laklab integrallash formulasi hosil bo‘ladi.

Odatda, (3.3) formula integral ostidagi funksiya turli sinfdagi funksiyalar ko‘paytmasidan, masalan, darajali va ko‘rsatkichli, darajali va trigonometrik, trigonometrik va ko‘rsatkichli va hakozo funksiyalarning ko‘paytmasidan iborat bo‘lganda qo‘llaniladi. Bunday integrallarning ikki turini ajratib ko‘rsatish mumkin, ularda qaysi funksiyani u deb va nimani dv deb qabul qilishni aniqlab olamiz.

Birinchi tur bo‘laklab integrallashda integral ostidagi ifoda $P_n(x)$ ko‘phad bilan ko‘rsatkichli yoki ko‘phad bilan trigonometrik funksiyaning ko‘paytmasidan iborat bo‘lsa, u holda u orqali $P_n(x)$ ko‘phad belgilanadi, qolgan hamma ifoda dv deb olinadi:

3.1.5-misol. Integralni hisoblang: $\int x\cos 2x\,dx$.

Yechilishi: ▶

$$\int x\cos 2x\,dx=\left|\begin{matrix} x=u & \cos 2xdx=dv \\ dx=du & \dfrac{\sin 2x}{2}=v \end{matrix}\right|=\frac{x\sin 2x}{2}-\int\frac{\sin 2x}{2}dx=$$

$$=\frac{x\sin 2x}{2}-\left(-\frac{\cos 2x}{4}\right)+C=\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+C. \quad ◀$$

3.1.6-misol. Integralni hisoblang: $\int xe^{-x}\,dx$.

Yechilishi:▶ $\int xe^{-x}\,dx=\left|\begin{matrix} x=u & e^{-x}dx=dv \\ dx=du & -e^{-x}=v \end{matrix}\right|=-xe^{-x}-\int -e^{-x}dx=$

$$=-xe^{-x}+\int e^{-x}dx=C-xe^{-x}-e^{-x}=C-e^{-x}(x+1). \quad ◀$$

Ikkinchi tur bo‘laklab integrallashda integral ostidagi ifoda $P_n(x)$ ko‘phad bilan logarifmik funksiya yoki ko‘phad bilan teskari trigonometrik funksiyaning ko‘paytmasidan iborat bo‘lsa, unda dv bilan $P_n(x)dx$ ifoda belgilanadi, qolgan hamma ifoda u deb olinadi:

3.1.7-misol. Integralni hisoblang: $\int x^3\ln x\,dx$.

Yechilishi: ▶ $\int x^3\ln x\,dx=\left|\begin{matrix} \ln x=u & x^3dx=dv \\ \dfrac{1}{x}dx=du & \dfrac{x^4}{4}=v \end{matrix}\right|=\dfrac{x^4\ln x}{4}-\int\dfrac{x^4}{4}\cdot\dfrac{1}{x}dx=$

$$=\frac{x^4\ln x}{4}-\frac{1}{4}\int x^3dx=\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+C=\frac{1}{4}x^4(\ln x-0.25)+C. \blacktriangleleft$$

Agar bo‘laklab integrallashni bir necha marta bajarishga to‘g‘ri kelsa, u holda jadval ko‘rinishida integrallash maqsadga muvofiq bo‘ladi:

3.1.8-misol. Integralni hisoblang: $\int x^3 e^{2x}\,dx$.

Yechilishi: ► Bu integralni hisoblash uchun 3 marta bo‘laklab integrallash formulasini qo‘llaymiz:

1) $$\int x^3 e^{2x}\,dx=\left|\begin{matrix} x^3=u & e^{2x}dx=dv \\ 3x^2dx=du & \frac{1}{2}e^{2x}=v\end{matrix}\right|=\frac{1}{2}x^3e^{2x}-\frac{3}{2}\int x^2e^{2x}dx$$

2) $$\int x^2 e^{2x}\,dx=\left|\begin{matrix} x^2=u & e^{2x}dx=dv \\ 2xdx=du & \frac{1}{2}e^{2x}=v\end{matrix}\right|=\frac{1}{2}x^2e^{2x}-\frac{2}{2}\int xe^{2x}dx$$

3) $$\int x e^{2x}\,dx=\left|\begin{matrix} x=u & e^{2x}dx=dv \\ dx=du & \frac{1}{2}e^{2x}=v\end{matrix}\right|=\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}dx$$

4) $$\int e^{2x}\,dx=\frac{1}{2}e^{2x}+C.$$

5) Topilganlarni joy-joyiga qo‘ysak, quyidagi natijani olamiz:

$$\int x^3 e^{2x}\,dx=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{6}{8}xe^{2x}-\frac{6}{16}e^{2x}+C=e^{2x}\left(\frac{1}{2}x^3-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right)+C.$$

Agar bu integrallarni quyidagi jadval yordamida hisoblaydigan bo‘lsak, juda oson chiqadi:

$f(x)$ va takroriy differensial	Ko’paytma ishorasi	$g(x)$ va takroriy integrallash
x^3	$(+)$	e^{2x}
$3x^2$	$(-)$	$\frac{1}{2}e^{2x}$
$6x$	$(+)$	$\frac{1}{4}e^{2x}$
6	$(-)$	$\frac{1}{8}e^{2x}$
0		$\frac{1}{16}e^{2x}$

$$\int x^3 e^{2x}\,dx=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{6}{8}xe^{2x}-\frac{6}{16}e^{2x}+C=e^{2x}\left(\frac{1}{2}x^3-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right)+C. \blacktriangleleft$$

Mavzu yuzasidan savollar:

1. Berilgan funksiyaning boshlang‘ich funksiyasi deb nimaga aytiladi?
2. Berilgan funksiyaning aniqmas integrali deb nimaga aytiladi?
3. Aniqmas integralning eng sodda xossalarini keltiring.
4. Aniqmas integralda bo‘laklab integrallash formulasini keltirib chiqaring.
5. Aniqmas integralda o‘zgaruvchilarni almashtirish usulini tushuntiring.
6. Bevosita integrallash usulini bayon qiling.
7. Differensial belgisi ostiga kiritish usuli nimadan iborat?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $f(x)=3^x-1$ funksiyaning $N(1;1)$ nuqtadan o‘tuvchi $F(x)$ boshlangich funksiyasini toping.

2. Differensial belgisi ostiga kiritib, integrallang:

a) $\int \frac{2x}{x^2+1}dx$ **;** **b)** $\int \frac{\ln x+3}{x}dx$ **c)** $\int e^{x^3}x^2dx$ **.**

3. Bevosita jadvaldan foydalanib, integralni hisoblang:

a) $\int\left(x^3+5x+\frac{5}{x}\right)dx$ **;** **b)** $\int\left(\frac{5x^2}{\sqrt{x}}-\sqrt[3]{x^2}+2\right)dx$ **;** **c)** $\int\left(x\sqrt{x}-\frac{1}{\sqrt{x^3}}+1\right)dx$ **.**

4. Bo‘laklab integrallash formulasidan foydalanib, hisoblang:

a) $\int e^{-x^2}x\,dx$ **b)** $\int x\cos 2x\,dx$ **c)** $\int\sqrt{x}\ln 3x\,dx$ **.**

5. Yangi o‘zgaruvchi kiritib, integralni hisoblang:

a) $\int \arcsin^2 x\frac{dx}{\sqrt{1-x^2}}$ **b)** $\int\frac{\sin x}{\cos^2 x}dx$ **c)** $\int\frac{x^3}{2+x^4}dx$ **.**

TESTLAR

1. $\int \frac{x^3 dx}{x^8 + 25}$ integralni hisoblang.

A) $\frac{1}{20} arctg \frac{x^4}{5} + C$ **B)** $\frac{1}{5} arctg \frac{x^4}{5} + C$

C) $5ctgx + \frac{1}{\sin^2 x} + C$ **D)** $\mathrm{arc}tg(x+2) + C$

2. $f(x) = \frac{1}{x^2}$ funksiyaning $M_0(1;2)$ nuqtadan o‘tuvchi $F_1(x)$ boshlang‘ich funksiyasini toping.

A) $F_1(x) = 3 - \frac{1}{x^2}$ **B)** $F_1(x) = 6 - \frac{1}{x}$

C) $F_1(x) = 8 - \frac{1}{x}$ **D)** $F_1(x) = 3 - \frac{1}{x}$

3. $\int \frac{dx}{x^2 + 4x + 5}$ integralni hisoblang.

A) $\frac{1}{20} arctg \frac{x^4}{5} + C$ **B)** $\frac{1}{5} arctg \frac{x^4}{5} + C$

C) $5ctgx + \frac{1}{\sin^2 x} + C$ **D)** $\mathrm{arc}tg(x+2) + C$

4. Bo‘laklab integrallash formulasini aniqlang:

A) $\int udv = uv + \int vdu$ **B)** $\int udv = uv - \int vdu$

C) $\int vdu = uv - \int vdu$ **D)** $\int udv = uv + \int udu$

5. $\int \left(e^x + e^{-x}\right)^2 dx$ integralni hisoblang.

A) $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$ **B)** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-x} + C$

C) $e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$ **D)** $\frac{1}{2}e^{2x} + 3x - \frac{1}{2}e^{-2x} + C$

3.2-§. Kasr-ratsional va ba’zi irratsional funksiyalarni integrallash

4.2.1. Kasr-ratsional funksiyalarni sodda kasrlarga ajratish

$$P_n(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_{n-1}x+a_n \qquad (3.4)$$

ko‘rinishidagi funksiyaga $n-$ **darajali ko‘phad** deyiladi, bunda $a_0, a_1, a_2, ..., a_{n-1}, a_n-$ko‘phadning koeffitsiyentlari, n soni daraja ko‘rsatkichi deyiladi.

(3.4) shakldagi ikkita ko‘phadning nisbatiga **kasr-ratsional funksiya** yoki **ratsional kasr** deyiladi:

$$R(x)=\frac{Q_m(x)}{P_n(x)}=\frac{b_0x^m+b_1x^{m-1}+b_2x^{m-2}+...+b_{m-1}x+b_m}{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_{n-1}x+a_n} \qquad (3.5)$$

Agar ratsional kasrning suratidagi ko‘phadning daraja ko‘rsatkichi mahrajidagi ko‘phad daraja ko‘rsatkichidan kichik $m<n$ bo‘lsa, unga **to‘g‘ri ratsioanl kasr**, agar $m\geq n$ bo‘lsa, **noto‘g‘ri ratsional kasr** deyiladi.

$R(x)$ ratsional kasr to‘g‘ri va noto‘g‘ri ratsional kasrga ajratiladi. Agar $R(x)$ to‘g‘ri ratsional kasr bo‘lsa, uni to‘g‘ridan - to‘g‘ri integrallab ketiladi. Agar $R(x)$ noto‘g‘ri ratsional kasr bo‘lsa, u holda ko‘phadni ko‘phadga bo‘lish amalidan foydalanamiz, ya’ni kasrning suratini mahrajiga bo‘lib, uning butun qismini ajratib olamiz:

$$R(x)=\frac{Q_m(x)}{P_n(x)}=q(x)+\frac{r(x)}{P_n(x)}$$

bu yerda $q(x)-$kasrning **butun qismi,** u son yoki ko‘phad bo‘lishi mumkin, $\frac{r(x)}{P_n(x)}-$ esa to‘g‘ri kasrdan iborat bo‘ladi. Natijada, noto‘g‘ri ratsional kasrni integrallash masalasi ko‘phad va to‘g‘ri ratsional kasrni integrallashga keltiriladi.

To‘g‘ri ratsional kasrlarga **oddiy kasr-ratsional funksiyalar** deyiladi: **I.** $\frac{A}{x-a}$,

II. $\frac{A}{(x-a)^k}$, $(k\geq 2$ va butun son)

III. $\frac{Ax+B}{x^2+px+q}$, (maxrajning diskreminanti $D=p^2-4q<0)$.

IV. $\frac{Ax+B}{(x^2+px+q)^k}$, $(k\geq 2$ va butun, $D<0)$.

Bu yerda A, B , a , p, q - haqiqiy sonlar.

Ushbu kasr-ratsional funksiyalarning integrallarini hisoblaymiz:

I. $\int \frac{A}{x-a}dx = A\ln|x-2| + C$.

II. $\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}dx = -\frac{A}{(k-1)(x-a)^{k-1}} + C$.

III. $\int \frac{Ax+B}{x^2+px+q}dx$ integralda $A \neq 0$ bo‘lsa, kasrning suratida mahrajining hosilasini keltirib chiqaramiz va integrallaymiz:

$$\blacktriangleright \int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\int \frac{(2x+p)+\left(\frac{2B}{A}-p\right)}{x^2+px+q}dx == \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx +$$

$$+\left(B-\frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q} = \frac{A}{2}\ln\left(x^2+px+q\right) + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2 + \left(q-\frac{p^2}{4}\right)}$$

Oxirgi integralda $q-\frac{p^2}{4} = \frac{4q-p^2}{4} > 0 \ \ (D<0)$ bo‘lgani uchun, jadvaldagi $\int \frac{du}{u^2+a^2}$ integralga keladi. Demak,

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\ln\left(x^2+px+q\right) + \frac{2B-Ap}{\sqrt{4q-p^2}} arctg \frac{2x+p}{\sqrt{4q-p^2}} + C. \quad (3.6) \blacktriangleleft$$

3.2.1-misol. $\int \frac{3x-2}{2x^2+8x+26}dx$ integralni hisoblang.

Yechilishi: ▶ Kasrning suratida mahrajining hosilasiga teng ifodani hosil qilib olamiz.

$$\frac{1}{2}\int \frac{3x-2}{x^2+4x+13}dx = \frac{3}{4}\int \frac{2x+4-4-\frac{4}{3}}{x^2+4x+13}dx = \frac{3}{4}\int \frac{2x+4}{x^2+4x+13}dx - 4\int \frac{dx}{(x+2)^2+3^2} =$$

$$= \frac{3}{4}\ln\left(x^2+4x+13\right) - \frac{4}{3} arctg \frac{x+2}{3} + C. \blacktriangleleft$$

IV. $\frac{Ax+B}{(x^2+px+q)^k}$, $(k \geq 2$ va butun, $D<0)$ kasrni intagrallaymiz.

$$\blacktriangleright \int \frac{Ax+B}{\left(x^2+px+q\right)^k}dx = \frac{A}{2}\int \frac{2x+p}{\left(x^2+px+q\right)^k}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(\left(x+\frac{p}{2}\right)^2 + \frac{4q-p^2}{4}\right)^k}.$$

Bunda $\frac{A}{2}\int\frac{2x+p}{\left(x^2+px+q\right)^k}dx=-\frac{A}{2}\cdot\frac{1}{(k-1)\left(x^2+px+q\right)^{k-1}}$, oxirgi integralda esa $u=x+\frac{p}{2},\ \ a=\frac{\sqrt{4q-p^2}}{2}$ almashtirish bajaramiz. Shunda quyidagi integralga kelamiz:

$$\int\frac{du}{\left(u^2+a^2\right)^k}=\frac{1}{a^2}\int\frac{\left(u^2+a^2\right)-u^2}{\left(u^2+a^2\right)^k}du=\frac{1}{a^2}\int\frac{du}{\left(u^2+a^2\right)^{k-1}}-\frac{1}{a^2}\int\frac{u^2}{\left(u^2+a^2\right)^k}du.$$

Bunda 1-integral berilgan integralning tartibi bittaga kamaygan holi, ikkinchi integralni bo‘laklab integrallash mumkin.

Natijada, quyidagi rekkurent formulani hosil qilamiz:

$$\int\frac{du}{\left(u^2+a^2\right)^k}=-\frac{u}{2a^2(k-1)\left(u^2+a^2\right)^{k-1}}+\frac{2k-3}{2a^2(k-1)}\int\frac{du}{\left(u^2+a^2\right)^{k-1}}. \qquad (3.7)◀$$

Eslatma. Agar mahrajda ax^2+bx+c ko‘phad bo‘lsa, u holda *a* qavsdan tashqariga chiqariladi: $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

3.2.2-misol. $\int\frac{7x+3}{\left(x^2+2x+5\right)^2}dx$ integralni hisoblang.

Yechilishi: ▶

$$\int\frac{7x+3}{\left(x^2+2x+5\right)^2}dx=\frac{7}{2}\int\frac{2x+2-2+\frac{6}{7}}{\left(x^2+2x+5\right)^2}dx=\frac{7}{2}\int\frac{2x+2}{\left(x^2+2x+5\right)^2}dx-4\int\frac{dx}{\left((x+1)^2+2^2\right)^2}.$$

Birinchi qo‘shiluvchi

$$\frac{7}{2}\int\frac{2x+2}{\left(x^2+2x+5\right)^2}dx=-\frac{7}{2}\cdot\frac{1}{x^2+2x+5}$$ ga teng.

Ikkinchi integral uchun (3.7) rekkurent formulani qo‘llaymiz:

$$\int\frac{dx}{\left((x+1)^2+2^2\right)^2}=-\frac{x+1}{2\cdot2^2(2-1)\left((x+1)^2+2^2\right)^2}+\frac{2\cdot2-3}{2\cdot2^2(2-1)}\int\frac{d(x+1)}{(x+1)^2+2^2}=$$

$$=-\frac{x+1}{8\left((x^2+2x+5\right)^2}+\frac{1}{8}\cdot\frac{1}{2}arctg\,\frac{x+1}{2}.$$

Demak,

$$\int\frac{7x+3}{\left(x^2+2x+5\right)^2}dx=-\frac{7}{2\left(x^2+2x+5\right)}-\frac{x+1}{8(x^2+2x+5)^2}+\frac{1}{16}arctg\,\frac{x+1}{2}+C.◀$$

3.2.2. Integrallashda noma'lum koeffitsiyentlar usuli

Ma'lumki, har qanday haqiqiy koeffitsiyentli ko'phadni quyidagi ko'paytma shaklida ifodalash mumkin:

$$P_n(x) = a_0(x-\alpha_1)^{k_1}\cdot\ldots\cdot(x-\alpha_\beta)^{k_\beta}(x^2+p_1x+q_1)^{t_1}\cdot\ldots\cdot(x^2+p_sx+q_s)^{t_s} \quad (3.8)$$

bu yerda $\alpha_1,\ldots,\alpha_\beta$ lar ko'phadning $k_1,\ldots,k_\beta$ karrali haqiqiy ildizlari,

$$p_i^2-4q_i<0,\ (i=\overline{1,s}) \text{ va } k_1+\ldots+k_\beta+2t_1+\ldots+2t_s=n.$$

3.3-teorema. (To'g'ri kasrni oddiy kasrlar yig'ndisiga ajratish haqida). Mahraji (3.8) ko'rinishda bo'lgan har qanday to'g'ri ratsional kasrni I-IV turdagi oddiy kasrlar yig'indisiga yoyish mumkin. Bu yoyilmada $P_n(x)$ ko'phadning har bir k_r karrali α_r haqiqiy ildiziga

$$\frac{A_1}{x-\alpha_r}+\frac{A_2}{(x-\alpha_r)^2}+\frac{A_3}{(x-\alpha_r)^3}\ldots+\frac{A_{k_r}}{(x-\alpha_r)^{k_r}} \quad (3.9)$$

ko'rinishdagi k_r ta oddiy kasrlar yig'indisi mos keladi. $P_n(x)$ ko'phadning har bir juft qo'shma - kompleks ildiziga

$$\frac{M_1x+N_1}{x^2+p_\gamma x+q_\gamma}+\frac{M_2x+N_2}{(x^2+p_\gamma x+q_\gamma)^2}+\frac{M_3x+N_3}{(x^2+p_\gamma x+q_\gamma)^3}+\ldots+\frac{M_{t_\gamma}x+N_{t_\gamma}}{(x^2+p_\gamma x+q_\gamma)^{t_\gamma}} \quad (3.10)$$

ko'rinishdagi t_γ ta oddiy kasrlar yig'indisi mos keladi.

Demak, integral ostidagi $R(x)$ to'g'ri ratsional kasrni (3.9) va (3.10) formulalar asosida noma'lum koeffitsiyentli oddiy kasrlarga yoyiladi. So'ngra bu kasrlarga umumiy mahraj to'planadi. Yoyilmadagi A, M, N koeffitsiyentlar ayniyatga keltirilib, aniqlanadi.

3.2.3-misol. $\int\frac{x^2-3x+2}{x(x+1)^2}dx$ integralni hisoblang.

Yechilishi:► Mahrajdagi ko'phadning $x=0$ bir karrali va $x=-1$ ikki karrali haqiqiy ildizlari borligi uchun (3.9) ga ko'ra quyidagicha oddiy kasrlarga ajratamiz:

$$\frac{x^2-3x+2}{x(x+1)^2}=\frac{A}{x}+\frac{B}{(x+1)^2}+\frac{C}{x+1}.$$

Unga umumiy mahraj berib, suratdagi ko'phadlarni tenglaymiz:

$$x^2-3x+2\equiv Ax^2+2xA+A+Bx+Cx^2+Cx$$

$$x^2-3x+2\equiv x^2(A+C)+x(2A+B+C)+A.$$

Noma'lum koeffitsiyentlar usulidan foydalanamiz, x ning darajalari oldidagi koeffitsiyentlarni tenglaymiz:

$$x^2: \qquad A+C=1;$$
$$x: \qquad 2A+B+C=-3;$$
$$x^0: \qquad A=2.$$

Bundan , $A=2,\ B=-6,\ C=-1$.

Demak, $\int\frac{x^2-3x+2}{x(x+1)^2}dx=\int\frac{2}{x}dx-\int\frac{6}{(x+1)^2}dx-\int\frac{1}{x+1}dx=$

$$=2\ln|x|+\frac{6}{x+1}-\ln|x+1|+C=\ln\frac{x^2}{|x+1|}+\frac{6}{x+1}+C\,.\blacktriangleleft$$

3.2.4-misol. $\int\frac{(x^2+3)dx}{x(x-1)(x+2)}$ integralni hisoblang.

Yechilishi: ► Integral ostida ifoda to‘g‘ri ratsional kasr bo‘lib, u I turdagi sodda kasrlar yig‘indisiga ajraladi:

$$\frac{x^2+3}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{D}{x+2},$$

bundan $x^2+3=A(x-1)(x+2)+Bx(x+2)+Dx(x-1).$

A, *B*, *D* koeffitsiyentlarni topish uchun o‘rniga qo‘yish usulidan foydalanamiz:

$x=0$ bo‘lganda $3=-2A$, bundan $A=-\frac{3}{2}$;

$x=1$ bo‘lganda $4=3B$, bundan $B=\frac{4}{3}$;

$x=-2$ bo‘lganda, $7=6D$, bundan $D=\frac{7}{6}$.

Shunday qilib , quyidagini hosil qilamiz :

$$\int\frac{(x^2+3)dx}{x(x-1)(x+2)}=-\frac{3}{2}\int\frac{dx}{x}+\frac{4}{3}\int\frac{d(x-1)}{x-1}+\frac{7}{6}\int\frac{d(x+2)}{x+2}=$$

$$=-\frac{3}{2}\ln|x|+\frac{4}{3}\ln|x-1|+\frac{7}{6}\ln|x+2|+C=\ln\sqrt[6]{\frac{(x-1)^8|x+2|^7}{|x|^9}}+C\,.\blacktriangleleft$$

3.2.5-misol. $\int\frac{dx}{x^3+8}$ integralni hisoblang.

Yechilishi: ► Integral ostida to‘g‘ri ratsional kasrning mahrajidagi ko‘phadni ko‘paytuvchilarga ajratamiz va sodda kasrlar yig‘indisi shaklida ifodalaymiz:

$$\frac{1}{x^3+8}=\frac{1}{(x+2)(x^2-2x+4)}=\frac{A}{x+2}+\frac{Mx+N}{x^2-2x+4}.$$

Umumiy mahraj berib, suratlarini tenglaymiz:

$$A(x^2-2x+4)+Bx(x+2)+C(x+2)\equiv 1$$

A, M, N koeffitsiyentlarni topish uchun yuqoridagi usullarni birga qo‘llaymiz:

$$x=-2: \quad 12A=1$$
$$x^2: \quad A+B=0;$$
$$x^0: \quad 4A+2C=1.$$

Bundan , $A=1/12$, $B=-1/12$, $C=1/3$ va $\frac{1}{x^3+8}=\frac{1}{12(x+2)}-\frac{x-4}{12(x^2-2x+4)}$.

Endi integralni hisoblaymiz:

$$\int\frac{dx}{x^3+8}=\frac{1}{12}\int\frac{dx}{x+2}-\frac{1}{12}\int\frac{x-4}{x^2-2x+4}dx=\frac{1}{12}\ln|x+2|-\frac{1}{12\cdot 2}\int\frac{(2x-2)-6}{x^2-2x+4}dx=$$
$$=\frac{1}{12}\ln|x+2|-\frac{1}{24}\int\frac{(2x-2)dx}{x^2-2x+4}+\frac{1}{4}\int\frac{d(x-1)}{(x-1)^2+\left(\sqrt{3}\right)^2}=$$
$$=\frac{1}{12}\ln|x+2|-\frac{1}{24}\ln\left|x^2-2x+4\right|+\frac{1}{4\sqrt{3}}arctg\frac{x-1}{\sqrt{3}}+C=$$
$$=\ln\sqrt[24]{\frac{(x+2)^2}{x^2-2x+4}}+\frac{1}{4\sqrt{3}}arctg\frac{x-1}{\sqrt{3}}+C\ .◄$$

Eslatma. Ratsional kasrni integrallash 4 qadamdan iborat:

1) Ratsional kasrning to‘g‘ri yoki noto‘g‘ri kasr ekanligini tekshiramiz, agar noto‘g‘ri kasr bo‘lsa, uning butun qismini ajratib, ko‘phad va to‘g‘ri ratsional kasr hosil qilamiz;

2) To‘g‘ri ratsional kasrni oddiy kasrlar yig‘indisiga ajratamiz;

3) Ayniyatni tenglab, koeffitsiyentlarini topamiz;

4) Hosil bo‘lgan ifodani integrallaymiz.

3.2.3. Ba’zi irratsional funksiyalarni integrallash

Har qanday irratsional funksiyadan tog‘ridan-to‘g‘ri integral olib bo‘lmaydi. Shunday irratsional funksiyalar borki, ularda shakl almashtirishlar bajarib, kasr-ratsional funksiyalar ko‘rinishiga keltiramiz va integralini hisoblaymiz. Quyida bir nechta shakl almashtirishlarni o‘rganib chiqamiz:

I. Ushbu $\int R\left(x,\left(\frac{ax+b}{cx+d}\right)^{\frac{r_1}{s_1}},\left(\frac{ax+b}{cx+d}\right)^{\frac{r_2}{s_2}},\ldots\left(\frac{ax+b}{cx+d}\right)^{\frac{r_n}{s_n}}\right)dx$, (bu yerda *R*-ratsional funksiya, *a, b, c* , *d* - o‘zgarmas sonlar, r_i , s_i musbat butun sonlar) integralni $\frac{ax+b}{cx+d}=t^m$ almashtirish yordamida ratsional kasr ko‘rinishiga keltirish mumkin. Bu yerda m - $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots \frac{r_n}{s_n}$ kasrlarning umumiy mahraji, ya’ni $m = EKUK(s_1, s_2, \ldots, s_n)$.

$\int R\left(x, x^{\frac{r_1}{s_1}}, x^{\frac{r_2}{s_2}}, \ldots x^{\frac{r_n}{s_n}}\right)dx$ integralni hisoblash uchun esa $x = t^m$ almashtirish bajaramiz va ratsional kasrga keltiramiz.

3.2.6-misol. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+4}dx$ integralni hisoblang.

Yechilishi: ▶ $m = EKUK(2,4) = 4$ bo‘lgani uchun,

$$\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+4}dx = \left| \begin{matrix} x = t^4 \\ dx = 4t^3 dt \end{matrix} \right| = 4\int \frac{t^5}{t^3+4}dt = 4\int \left(t^2 - \frac{4t^2}{t^3+4}\right)dt = \frac{4}{3}t^3 - \frac{16}{3}\ln\left|t^3+4\right| + C =$$

$$= \left|t = \sqrt[4]{x}\right| = \frac{4}{3}\sqrt[4]{x^3} - \frac{16}{3}\ln\left|\sqrt[4]{x^3}+4\right| + C. ◀$$

II. $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ ko‘rinishidagi integralni hisoblaymiz.

Dastlab, kasrning suratida ildiz ostidagi kvadrat uchhadning differensialini hosil qilamiz ($A \neq 0$), kvadrat uchhaddan to‘la kvadrat ajratamiz:

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx = \frac{A}{2a}\int \frac{(2ax+b)dx}{\sqrt{ax^2+bx+c}} + \left(B - \frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}} =$$

$$= \frac{A}{a}\sqrt{ax^2+bx+c} + \left(B - \frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{a\left(x+\frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)}}.$$

Agar $c \neq \frac{b^2}{4a}$, $a > 0$ bo‘lsa, oxirgi integralni $\int \frac{du}{\sqrt{u^2+k}} = \ln\left|u + \sqrt{u^2+k}\right| + C$ integralga keltirib hisoblaymiz.

Agar $c > \frac{b^2}{4a}$, $a < 0$ bo‘lsa, u holda $\int \frac{du}{\sqrt{k^2-u^2}} = \arcsin\frac{u}{k} + C$ integralga keltirib hisoblaymiz.

3.2.7-misol. $\int \frac{5x+3}{\sqrt{x^2-4x+8}}dx$ integralni hisoblang.

Yechilishi: ▶

$$\int \frac{5x+3}{\sqrt{x^2-4x+8}}dx = \frac{5}{2}\int \frac{2x-4}{\sqrt{x^2-4x+8}}dx + \left(3 - \frac{5\cdot 4}{2}\right)\int \frac{dx}{\sqrt{x^2-4x+8}} =$$

$$= 5\sqrt{x^2-4x+8} - 7\int \frac{dx}{\sqrt{(x-2)^2+4}} = 5\sqrt{x^2-4x+8} - 7\ln\left|x-2+\sqrt{(x-2)^2+4}\right| + C ◀$$

3.2.8-misol. $\int \frac{3x-2}{\sqrt{10-8x-2x^2}}dx$ integralni hisoblang.

Yechilishi: ► Qulaylik uchun, avval 2 ni ildizdan chiqarib olamiz.

$$\int \frac{3x-21}{\sqrt{5-8x-2x^2}}dx = \frac{1}{\sqrt{2}}\int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \frac{\sqrt{2}}{2}I_1 .$$

Hosil bo‘lgan integralni hisoblaymiz.

$$I_1 = \int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \int \frac{-\frac{3}{2}(-4-2x)+8}{\sqrt{5-4x-x^2}}dx = -\frac{3}{2}\int \frac{(-4-2x)dx}{\sqrt{5-4x-x^2}} +$$

$$+8\int \frac{dx}{\sqrt{1-(x+2)^2}} = -3\sqrt{5-4x-x^2} + 8\arcsin(x+2) + C_1 .$$

Demak, $\int \frac{3x-2}{\sqrt{10-8x-2x^2}}dx = -\frac{3\sqrt{2}}{2}\sqrt{5-4x-x^2} + 4\sqrt{2}\arcsin(x+2) + C$.◄

III. Agar integral $\int \frac{Ax+B}{(x-\alpha)\sqrt{ax^2+bx+c}}dx$ ko‘rinishda bo‘lsa, uni $x-\alpha = \frac{1}{t}$ almashtirish yordamida hisoblash qulay.

3.2.9-misol. $\int \frac{dx}{(x+1)\sqrt{x^2+2x+10}}$ integralni hisoblang.

Yechilishi: ►

$$\int \frac{dx}{(x+1)\sqrt{x^2+2x+10}} = \left| \begin{matrix} x+1 = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt \end{matrix} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+9}} = -\int \frac{dt}{\sqrt{9t^2+1}} =$$

$$= -\frac{1}{3}\ln\left|3t+\sqrt{9t^2+1}\right| + C = -\frac{1}{3}\ln\left|\frac{3}{x+1} + \sqrt{\frac{9}{(x+1)^2}+1}\right| + C .$$ ◄

3.3-§. Trigonometrik funksiyalarni integrallash

3.3.1. Trigonometrik ifodalarni integrallashda universal almashtirish

Agar faqat trigonometrik funksiyalar qatnashgan ratsional ifoda berilgan bo‘lsa, uni har doim trigonometrik formulalardan foydalanib, $sinx$ va $cosx$ orqali ifodalash mumkin. Bu ifodani $R(sinx,\ cosx)$ deb belgilaymiz. Ushbu $\int R(\sin x, \cos x)dx$ turdagi integralni $tg\frac{x}{2} = z$ o‘rniga qo‘yish bilan z o‘zgaruvchili kasr-rasional funksiyaning integraliga keltirish mumkin.

Agar $tg\frac{x}{2} = z$ belgilash kiritsak, undan x ni topib olamiz: $x = 2arctgz$ Tenglikni har ikki tomonini differensiallaymiz va dx ni aniqlaymiz: $dx = \frac{2dz}{1+z^2}$. $sinx$ va $cosx$ larni ham yarim burchak tangensiga almashtiramiz va berilgan integralga qo‘yamiz:

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2} = \frac{2z}{1+z^2}; \qquad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2} = \frac{1-z^2}{1+z^2};$$

Natijada quyidagini hosil qilamiz:

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right)\cdot\frac{2dz}{1+z^2}. \qquad (3.11)$$

3.3.1-misol. Ushbu $\int\frac{dx}{\sin x + 3\cos x + 1}$ integralni hisoblang.

Yechilishi: ► (3.11) almashtirishdan foydalanamiz:

$$\int\frac{dx}{\sin x + 3\cos x + 4} = \int\frac{\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2} + 3\cdot\frac{1-z^2}{1+z^2} + 4} = \int\frac{2dz}{z^2+2z+7} = \int\frac{dz}{(z+1)^2 + \left(\sqrt{6}\right)^2} =$$

$$= \frac{1}{\sqrt{6}}arctg\frac{z+1}{\sqrt{6}} + C = \frac{1}{\sqrt{6}}arctg\frac{tg\frac{x}{2}}{\sqrt{6}} + C. \ ◄$$

$tg\frac{x}{2} = z$ almashtirish $R(sinx,\ cosx)$ ko‘rinishdagi har qanday funksiyani integrallashga imkon beradi, shuning uchun uni **universal**

trigonometrik almashtirish deyiladi. Biroq amaliyotda bu almashtirish ancha murakkab ratsional funksiyaga olib keladi. Shuning uchun ko‘pincha undan foydalanmasdan, integralning turiga qarab, mos soddaroq almashtirishlardan foydalaniladi.

a) Agar $R(sinx,\ cosx)$ funksiya $sinx$ ga nisbatan toq bo‘lsa, ya’ni $R(-sinx,\ cosx) = -R(sinx,\ cosx)$ bo‘lsa, u holda

$$\cos x = z,\ \sin x dx = -dz \qquad (3.12)$$

o‘rniga qo‘yishdan foydalaniladi.

3.3.2-misol. $I = \int \frac{\sin^3 x}{2+\cos x} dx$ integralni hisoblang.

Yechilishi: ► Integral ostidagi funksiya *sinx* ga nisbatan toq funksiya. Shuning uchun (3.12) almashtirishni bajaramiz:

$$I = \int \frac{\sin^2 x \cdot \sin x dx}{2+\cos x} = \int \frac{(1-\cos^2 x)\sin x dx}{2+\cos x} = -\int \frac{(1-z^2)dz}{2+z} = \int \frac{z^2-1}{2+z} dz =$$

$$= \int \left(z - 2 + \frac{3}{z+2} \right) dz = \frac{z^2}{2} - 2z + 3\ln|z+2| + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln|\cos x + 2| + C \blacktriangleleft$$

b) Agar $R(sinx,\ cosx)$ funksiya $cosx$ ga nisbatan toq bo‘lsa, ya’ni $R(sinx, -cosx) = -R(sinx, cosx)$ bo‘lsa, u holda

$$\sin x = z,\ \cos x dx = dz \qquad (3.13)$$

o‘rniga qo‘yishdan foydalaniladi.

c) Agar $R(sinx,\ cosx)$ funksiya $sinx$ va $cosx$ ga nisbatan juft bo‘lsa, ya’ni $R(-sinx, -cosx) = R(sinx, cosx)$ bo‘lsa, u holda

$$tgx = z,\ x = arctgz,\ dx = \frac{dz}{1+z^2} \qquad (3.14)$$

almashtirishdan foydalaniladi. Bu holda

$$sin^2x = \frac{tg^2x}{1+tg^2x} = \frac{z^2}{1+z^2},\quad cos^2x = \frac{1}{1+tg^2x} = \frac{1}{1+z^2}.$$

3.3.3-misol. $I = \int \frac{dx}{1+\sin^2 x}$ integralni hisoblang.

Yechilishi: ►Integral belgisi ostidagi funksiya juft funksiya, shuning uchun (3.14) almashtirishni bajaramiz.

$$I = \int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{dt}{1+z^2}}{1+\frac{z^2}{1+z^2}} = \int \frac{\frac{dz}{1+z^2}}{\frac{1+z^2+z^2}{1+z^2}} = \int \frac{dz}{1+2z^2} = \frac{1}{2}\int \frac{dz}{\frac{1}{2}+z^2} = \frac{1}{2}\int \frac{dz}{\left(\sqrt{\frac{1}{2}}\right)^2 + z^2} =$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{1}{2}}} arctg \frac{z}{\sqrt{\frac{1}{2}}} + C = \frac{\sqrt{2}}{2} arctg \sqrt{2} Z + C = \frac{\sqrt{2}}{2} arctg \sqrt{2} tgx + C. \blacktriangleleft$$

3.3.2. Ba'zi trigonometrik funksiyalarni integrallashdagi xususiy sodda almashtirishlar

Agar ikkala n va m ko'rsatkichlar juft va nomanfiy bo'lsa, u holda trigonometriyadan ma'lum bo'lgan

$$\sin^2 x = \frac{1-\cos 2x}{2}; \quad \cos^2 x = \frac{1+\cos 2x}{2} \qquad (3.15)$$

darajani pasaytirish formulalaridan foydalanamiz.

3.3.4-misol $I = \int \sin^4 x dx$ integralni hisoblang.

Yechilishi: ► Darajani pasaytirish uchun (3.15) formuladan foydalanamiz:

$$I = \int \sin^4 x dxd = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)dx =$$

$$= \frac{1}{4}\int\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx = \frac{1}{4}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + \frac{1}{8}x + \frac{1}{8}\cdot\frac{\sin 4x}{4} + C =$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \blacktriangleleft$$

Quyidagi ko'rinishdagi integrallarni qarab chiqamiz:

$$\int \cos nx \cdot \cos mx dx,$$

$$\int \sin nx \cdot \cos mx dx,$$

$$\int \sin nx \cdot \sin mx dx.$$

Bunday integrallarni hisoblash uchun trigonometrik funksiyalarning ko'paytmasini yig'indiga almashtiruvchi formulalar qo'llanadi:

$$\sin nx \cos mx = \frac{1}{2}\left[\sin(n+m)x + \sin(n-m)x\right]$$

$$\cos nx \cos mx = \frac{1}{2}\left[\cos(n+m)x + \cos(n-m)x\right]$$

$$\sin nx \sin mx = \frac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right]$$

3.3.5-misol $I = \int \sin 3x \cdot \cos 2x dx$ integralni hisoblang.

Yechilishi: ►Integral ostidagi ko'paytmani yig'indiga almashtirib integrallaymiz.

$$I = \int \sin 3x \cdot \cos 2x dx = \frac{1}{2}\int (\sin 5x + \sin x)dx = -\frac{1}{2}\cdot\frac{\cos 5x}{5} - \frac{1}{2}\cdot\cos x + C =$$

$$= -\frac{1}{10}\cdot\frac{\cos 5x}{1} - \frac{1}{2}\cdot\cos x + C. \blacktriangleleft$$

3.3.3. *sinx* va *cosx* darajalarining ko‘paytmalari ko‘rinishidagi integrallarni hisoblash

Agar $R(\sin x, \cos x)$ funksiya *sinx* va *cosx* darajalarining ko‘paytmasi bo‘lsa, ya’ni $\int \sin^n x \cdot \cos^m x dx$ ko‘rinishdagi integralni hisoblashda, *m* va *n* ga bog‘liq holda turli almashtirishlar bajariladi:

1) agar $n > 0$ va toq bo‘lsa, u holda (3.12) almashtirish;

2) agar $m > 0$ va toq bo‘lsa, u holda (3.13) almashtirish;

3) agar darajalardan biri nolga teng, ikkinchisi manfiy toq son bo‘lsa, u holda (3.11) almashtirish bajariladi.

3.3.6-misol. $I = \int \frac{\sin^3 x}{\cos^4 x} dx$ integralni hisoblang.

Yechilishi: ▶ (3.12) almashtirishni bajaramiz:

$$I = \int \frac{\sin^2 x \sin x}{\cos^4 x} dx = \int \frac{(1-\cos^2 x)\sin x dx}{\cos^4 x} = -\int \frac{(1-z)^2 dz}{z^4} = -\int \frac{dz}{z^4} + \int \frac{z^2}{z^4} dz =$$

$$= \frac{1}{3z^3} - \frac{1}{z} + C = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C. ◀$$

3.3.7-misol. $I = \int \frac{dx}{\sin^3 x}$ integralni hisoblang.

Yechilishi: ▶ (3.11) almashtirish bajaramiz:

$$I = \int \frac{dx}{\sin^3 x} = \int \frac{\frac{2dz}{1+z^2}}{\left(\frac{2z}{1+z^2}\right)^3} = \frac{1}{4}\int \frac{\left(1+z^2\right)^2}{z^3} dz =$$

$$= \frac{1}{4}\int \frac{1+2z^2+z^4}{z^3} dz = \frac{1}{4}\int \left(\frac{1}{z^3} + \frac{2}{z} + z\right) dz = -\frac{1}{8z^2} + \frac{1}{2}\ln|z| + \frac{1}{4}\cdot\frac{z^2}{2} + C =$$

$$= -\frac{1}{8} ctg^2 \frac{x}{2} + \frac{1}{2}\ln\left|tg\frac{x}{2}\right| + \frac{1}{8} tg^2 \frac{x}{2} + C. ◀$$

Agar $m + n = -2k \le 0$ (juft, nomusbat) bo‘lsa, u holda $tgx = z$ yoki $ctgx = z$ almashtirish integralni darajali funksiyalarning integrallari yig‘indisiga olib keladi. Xususan, $n < 0$, $m < 0$ va $m + n = -2k \le 0$ bo‘lsa, kasrning suratini $1 = (\sin^2 x + \cos^2 x)^s$ ifodaga almashtirish mumkin, bu yerda $s = \frac{|m+n|}{2} - 1$.

3.3.8-misol. $I=\int\frac{\sin^2 x}{\cos^6 x}dx$ integralni hisoblang.

Yechilishi: ►Bu yerda $n=2,\ m=-6,\ m+n=-4<0$, (3.16) almashtirishni bajaramiz.

$$\frac{\sin^2 x}{\cos^6 x}=\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^4 x}=tg^2x\left(\frac{1}{\cos^4 x}\right)=tg^2x(1+tg^2x)^2=z^2(1+z^2)^2,$$

Natijada , $$I=\int\frac{\sin^2 x}{\cos^6 x}dx=\int z^2(1+z^2)^2\frac{dz}{1+z^2}=\int(z^2+z^4)dz=$$

$$=\frac{z^3}{3}+\frac{z^5}{5}+C=\frac{tg^3x}{3}+\frac{tg^5x}{5}+C.$$ ◄

3.3.9-misol. $I=\int\frac{dx}{\sin^3 x\cdot\cos x}$ integralni hisoblang.

Yechilishi: ► Bu yerda $n=-3,\ m=-1,\ m+n=-4<0$

$$I=\int\frac{dx}{\sin^3 x\cdot\cos x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cdot\cos x}dx=\int\frac{1}{\sin x\cdot\cos x}dx+\int\frac{\cos x}{\sin^3 x}dx=$$

$$=2\int\frac{dx}{\sin 2x}+\int\frac{d(\sin x)}{\sin^3 x}=\ln|tgx|-\frac{1}{2\sin^2 x}+C.$$ ◄

3.3.4. Trigonometrik almashtirishlardan foydalanib, irratsional ifodalarni integrallash

Agar integral $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ ko‘rinishda bo‘lsa, kvadrat uchhadni to‘la kvadratga ajratib, quyidagi

1) $\int R\left(u,\sqrt{k^2-u^2}\right)du$, $u=k\sin t\,(u=k\cos t)$ almashtirish;

2) $\int R\left(u,\sqrt{k^2+u^2}\right)du$, $u=ktgt\,(u=kctgt)$ almashtirish;

3) $\int R\left(u,\sqrt{u^2-k^2}\right)du$, $u=\frac{k}{\sin t}\left(u=\frac{k}{\cos t}\right)$ almashtirishlar

yordamida hisoblanadigan integrallardan biriga keltirish mumkin.

3.3.10-misol. $\int\sqrt{3+2x-x^2}dx$ integralni hisoblang.

Yechilishi: ► $\int\sqrt{3+2x-x^2}dx=\int\sqrt{4-(x-1)^2}dx=\begin{vmatrix}x-1=2\sin t\\ dx=2\cos tdt\end{vmatrix}=$

$$=\int\sqrt{4-4\sin^2 t}\cdot 2\cos tdt=4\int\cos^2 tdt=2\int(1+\cos 2t)dt=2t+\sin 2t+C=$$

$$=2t+2\sin t\sqrt{1-\sin^2 t}+C=2\arcsin\frac{x-1}{2}+\frac{(x-1)\sqrt{3+2x-x^2}}{2}+C.◀$$

3.3.11-misol. $\int\frac{dx}{\sqrt{\left(x^2+4x+5\right)^3}}$ integralni hisoblang.

Yechilishi: ▶

$$\int\frac{dx}{\sqrt{\left(x^2+4x+5\right)^3}}=\int\frac{dx}{\sqrt{\left((x+2)^2+1\right)^3}}=\left|\begin{matrix}x+2=tgt\\ dx=\dfrac{dt}{\cos^2 t}\end{matrix}\right|=\int\frac{dt}{\cos^2 t\sqrt{\left(tg^2t+1\right)^3}}=$$

$$=\int\frac{dt}{\cos^2 t\sqrt{\left(tg^2t+1\right)^3}}=\int\cos tdt=\sin t+C=\frac{tgt}{\sqrt{tg^2t+1}}+C=\frac{x+2}{\sqrt{x^2+4x+5}}+C.◀$$

Mavzu yuzasidan savollar

1. Kasr -ratsional funksiyaga ta’rif bering.
2. Oddiy ratsional kasrlarni integrallash formulalarini keltiring.
3. To‘g‘ri kasrni oddiy kasrlar yig‘ndisiga ajratish haqidagi teoremani ayting.
4. Ratsional kasrlarni integrallashda noma’lum koeffitsiyentlar usuli algoritmini tushuntiring.
5. Qanday irratsional funksiyalarni integrallash formulalarini bilasiz?
6. Universal almashtirish deb qanday almashtirishga aytiladi?
7. Qanday trigonometrik funksiyalarni bilasiz? Ta’riflarini keltiring.
8. Trigonometrik funksiyalarni integrallashda darajani pasaytirish formulasini yozing.
9. Trigonometrik funksiyalarni integrallashda ko‘paytmadan yig‘indiga o‘tish formulasini yozing.
10. $\int R\left(x,\ \sqrt{ax^2+bx+c}\right)dx$ ko‘rinishidagi irratsional ifodalarni integrallashda qanday trigonometrik almashtirishlar bajariladi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Integrallarni hisoblang:

a) $\int(x^3+2x)dx$ **b)** $\int(8x-2)\sin 5x\,dx$ **;**

2. Integrallarni hisoblang:

a) $\int \frac{x}{\sqrt{1-2x^2}}dx$; **b)** $\int \sin^2 x \sin 3x dx$;

3. Integrallarni hisoblang:

a) $\int \arcsin^2 x \frac{dx}{\sqrt{1-x^2}}$; **b)** $\int \frac{arctg^3 x}{1+x^2}dx$;

4. Integrallarni hisoblang:

a) $\int \frac{\sqrt{arctg\, x}}{1+x^2}dx$; **b)** $\int 2^{\cos x} \sin x\, dx$;

5. Integrallarni hisoblang:

a) $\int \frac{(\sin 2x + \cos 2x)}{\cos 2x}dx$; **b)** $\int \frac{1+tgx}{1-tgx}dx$;

TESTLAR

1. Hisoblang: $\int \frac{1+x^2}{5x}dx$

A) $\frac{1}{5}\ln|x| + \frac{1}{10}x^2 + C$ **B)** $\frac{1}{5}\ln|x| + \frac{1}{10}x + C$

C) $\ln|x| + \frac{1}{10}x^2 + C$ **D)** $\frac{1}{5}\ln|x| - \frac{1}{10}x^2 + C$

2. Hisoblang: $\int \mathrm{tg}^2 x dx$

A) $\mathrm{tg}\, 2x - x + C$ **B)** $\mathrm{tg}\, x - x + C$

C) $2\,\mathrm{tg}\, x - 2x + C$ **D)** $\mathrm{tg}\, x - 2x + C$

3. Hisoblang: $\int \frac{dx}{x \ln x}$

A) $\ln|\ln x| + C$ **B)** $\ln|\ln(\ln x)| + C$

C) $\ln(2\ln x) + C$ **D)** $\ln^2(\ln x) + C$

4. Hisoblang: $\int 2^x 3^{2x} 5^{3x}\, dx$

A) $\frac{2250^x}{\ln 2250} + C$ **B)** $\frac{2250^x}{\ln 250} + C$

C) $\frac{250^x}{\ln 2250} + C$ **D)** $\frac{225^x}{\ln 2250} + C$

5. Hisoblang: $\int \frac{(\sin 3x + \cos 3x)}{\cos 3x}dx$

A) $-\frac{1}{2}\ln|\cos 3x| + x + C$ **B)** $-\frac{1}{3}\ln|\sin 3x| + x + C$

C) $-\frac{1}{3}\ln|\cos 3x| - x + C$ **D)** $-\frac{1}{3}\ln|\cos 3x| + x + C$

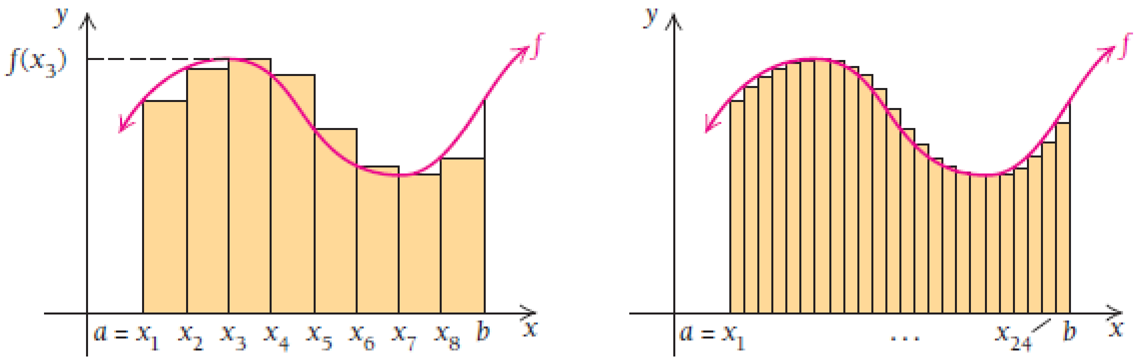
3.4-§. Aniq integral ta’rifi (Riman yig‘indilari). O‘rta qiymat haqidagi teorema. Nyuton – Leybnits formulasi

3.4.1. Aniq integral va uni hisoblash

Aniq integral tushunchasini kiritish uchun quyidagi amallar ketma-ketligini bajaramiz: Bizga $[a,b]$ kesmada uzluksiz bo‘lgan $y = f(x)$ funksiya berilgan bo‘lsin.

1) $[a,b]$ kesmani quyidagi nuqtalar bilan n ta qismga bo‘lamiz, ularni **qismiy intervallar** deb ataymiz (3.2-rasm):

$$a = x_1 < x_2 < . . . < x_{i-1} < x_i < . . . < x_n < b.$$



3.2-rasm. Funksiyani qismiy intervallarga ajratish

2) Qismiy intervallarning uzunliklarini hisoblaymiz:

$$\Delta x_1 = x_1 - a,$$
$$\Delta x_2 = x_2 - x_1,$$
$$. . . ,$$
$$\Delta x_i = x_i - x_{i-1},$$
$$. . . ,$$
$$\Delta x_n = b - x_{n-1}.$$

3) Har bir qismiy interval ichidan bittadan ixtiyoriy nuqta tanlab olamiz: $\xi_1, \xi_2, \ldots, \xi_i, \ldots, \xi_n$.

4) Tanlangan nuqtalarda berilgan funksiyaning qiymatlarini hisoblaymiz: $f(\xi_1), f(\xi_2), \ldots, f(\xi_i), \ldots, f(\xi_n)$.

5) Funksiyaning hisoblangan qiymatlarini mos qismiy interval uzunligiga ko‘paytiramiz:

$$f(\xi_1)\Delta x_1, f(\xi_2)\Delta x_2, \ldots, f(\xi_i)\Delta x_i, \ldots, f(\xi_n)\Delta x_n.$$

6) Hosil bo‘lgan ko‘paytmalarni qo‘shamiz va yig‘indini σ bilan belgilaymiz:

$$\sigma = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_i)\Delta x_i + \ldots + f(\xi_n)\Delta x_n.$$

σ yig'indi $f(x)$ funksiya uchun $[a,b]$kesmada tuzilgan **integral yig'indi (Riman yig'indilari)** deyiladi va quyidagicha belgilanadi: $\sigma=\sum_{i=1}^{n} f(\xi_i)\Delta x_i$.

Endi bo'lishlar soni n ni orttira boramiz $(n\to\infty)$ va bunda eng katta intervalning uzunligi nolga intiladi, ya'ni $max\ \Delta x_i\to 0$ deb faraz qilamiz. Shunda quyidagi tenglik hosil bo'ladi:

$$\int_a^b f(x)dx=\lim_{\max \Delta x_i\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

Agar σ integral yig'indi $[a,b]$ kesmani qismiy $[x_{i-1},x_i]$ kesmalarga ajratish usuliga va ularning har biridan ξ_i nuqtani tanlash usuliga bog'liq bo'lmaydigan chekli songa intilsa, u holda shu son $[a,b]$ kesmada $f(x)$ funksiyadan olingan **aniq integral** deyiladi va bunday belgilanadi:

$$\int_a^b f(x)dx \qquad (3.17)$$

(3.17) ni "$f(x)$ dan x bo'yicha a va b gacha olingan aniq integral" deb o'qiladi. Bu yerda $f(x)$-**integral ostidagi funksiya**, $[a,b]$ kesma-**integrallash oralig'i**, a va b sonlar integrallashning **quyi va yuqori chegaralari** deyiladi.

Aniq integralning ta'rifidan ko'rinadiki, aniq integral hamma vaqt mavjud bo'lavermas ekan. Biz quyida aniq integralning mavjudlik teoremasini keltiramiz.

3.4-teorema. (Aniq integralning mavjudlik sharti). Agar $y=f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, u integrallanuvchidir.

3.4.2. Aniq integralning asosiy xossalari

1^0. Aniq integralning chegaralari almashtirilsa, integralning ishorasi o'zgaradi: $\int_a^b f(x)dx=-\int_b^a f(x)dx$

2^0. Agar aniq integralning chegaralari teng bo'lsa, har qanday funksiya uchun ushbu tenglik o'rinli bo'ladi: $\int_a^a f(x)dx=0.$

3^0. Bir nechta funksiyaning algebraik yig'indisining aniq integrali qo'shiluvchilar integrallarining yig'indisiga teng:

$$\int_a^b[f(x)\pm\varphi(x)]dx=\int_a^b f(x)dx\pm\int_a^b\varphi(x)dx$$

4^0. O‘zgarmas ko‘paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin: $\int_a^b k\ f(x)dx = k\cdot\int_a^b f(x)dx$, bunda $k = const$.

5^0. Agar $[a,b]$kesmada funksiya o‘z ishorasini o‘zgartirmasa, u holda bu funksiya aniq integralning ishorasi funksiya ishorasi bilan bir xil bo‘ladi, ya’ni:

a) agar $[a,b]$ kesmada $f(x)\geq 0$ bo‘lsa, u holda $\int_a^b f(x)dx \geq 0$;

b) agar $[a,b]$ kesmada $f(x)\leq 0$ bo‘lsa, u holda $\int_a^b f(x)dx \leq 0$.

6^0. Agar $[a,b]$kesmada ikki $f(x)$ va $\varphi(x)$ funksiya $f(x)\geq\varphi(x)$ shartni qanoatlantirsa, u holda $\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx$.

7^0. Agar $[a,b]$kesma bir necha qismlarga bo‘linsa, u holda $[a,b]$ kesma bo‘yicha aniq integral har bir qism bo‘yicha olingan aniq integrallar yig‘indisiga teng, agar $a<c<b$ bo‘lsa, u holda

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

8^0. Agar m va M sonlar $f(x)$ funksiyaning $[a,b]$kesmada eng kichik va eng katta qiymati bo‘lsa, u holda quyidagi qo‘shtengsizlik o‘rinli bo‘ladi: $$m\cdot(b-a)\leq\int_a^b f(x)dx \leq M\cdot(b-a).$$

3.4.3. O‘rta qiymat haqidagi teorema

$[a,b]$ kesmada uzluksiz bo‘lgan $y = f(x)$ funksiyaning shu kesmadagi o‘rtacha qiymatini topamiz. Buning uchun

1) $[a,b]$ kesmani $x_0 = a,\ x_1,\ x_2,\dots,\ x_{i-1},\ x_i,\dots,\ x_n = b$ nuqtalar bilan n ta teng qismga bo‘lamiz.
2) Har bir bo‘lakning uzunligi quyidagiga teng bo‘ladi:
$$\frac{b-a}{n} = \Delta x_1 = \Delta x_2 = \dots = \Delta x_i = \dots = \Delta x_n\,.$$
3) Har bir bo‘lakning ichidan bittadan nuqta tanlaymiz:
$$\xi_1\epsilon\Delta x_1\,, \xi_2\epsilon\,\Delta x_2\,,\ .\ .\ .\ ,\xi_i\epsilon\Delta x_i\,,\quad .\ .\ .\quad ,\xi_n\epsilon\Delta x_n.$$
4) Bu nuqtalarda berilgan $y = f(x)$ funksiyaning qiymatlarini hisoblab quyidagi n ta qiymatini hosil qilamiz:
$$f(\xi_1), f(\xi_2)\,,\dots, f(\xi_i)\,,\dots, f(\xi_n).$$
5) Bu qiymatlarning o‘rta arifmetik qiymatini hisoblaymiz va uni $[a,b]$ kesmada $y = f(x)$ funksiyaning o‘rtacha qiymati deb ataymiz:

$$f_{o'rt} = \frac{f(\xi_1) + f(\xi_2) + \ldots + f(\xi_i) + \ldots + f(\xi_n)}{n}$$

6) Bu formulaning har 2 tomonini $(b-a)$ kattalikka ko‘paytiramiz:

$$(b-a) f_{o'rt} = (b-a)\frac{f(\xi_1) + f(\xi_2) + \ldots + f(\xi_i) + \ldots + f(\xi_n)}{n}.$$

Bundan quyidagini hosil qilamiz:

$$f_{o'rt} = \frac{1}{b-a}\left[f(\xi_1)\cdot\frac{b-a}{n} + f(\xi_2)\cdot\frac{b-a}{n} + \ldots + f(\xi_i)\cdot\frac{b-a}{n} + \ldots + f(\xi_n)\cdot\frac{b-a}{n}\right]$$

yoki
$$f_{o'rt} = \frac{1}{b-a}\left[f(\xi_1)\cdot\Delta x_1 + f(\xi_2)\cdot\Delta x_2 + \ldots + f(\xi_i)\cdot\Delta x_i + \ldots + f(\xi_n)\cdot\Delta x_n\right]$$

Buni qisqacha bunday yozish mumkin:

$$f_{o'rt} = \frac{1}{b-a}\sum_{i=1}^{n} f(\xi_i)\cdot\Delta x_i.$$

Demak, $[a,b]$ kesmada $y=f(x)$ funksiya uchun integral yig‘indisini hosil qilamiz. Endi $n\to\infty$ da $\lambda = max\Delta x_i \to 0$ bo‘lgandagi limitga o‘tamiz:

$$\lim_{\lambda\to 0} f_{o'rt} = \lim_{\lambda\to 0}\frac{1}{b-a}\sum_{i=1}^{n} f(\xi_i)\cdot\Delta x_i.$$

va bundan quyidagi tenglikni hosil qilamiz:

$$f_{o'rt} = \frac{1}{b-a}\cdot\int_a^b f(x)dx. \qquad (3.18)$$

Demak, $[a,b]$ kesmada $y=f(x)$ funksiyaning o‘rtacha qiymati shu kesmada bu funksiyaning aniq integralini kesma uzunligiga bo‘linganiga teng.

3.4-teorema. Agar $y=f(x)$ funksiyaning $[a,b]$ kesmada uzluksiz bo‘lsa, bu kesmaning ichida shunday $x=c$ nuqta topiladiki, bu nuqtada funksiyaning qiymati uning shu kesmadagi o‘rtacha qiymatiga teng bo‘ladi, ya’ni $f(c) = f_{o‘rt}$.

3.4.4. Integralning yuqori chegarasi bo‘yicha hosila. Nyuton-Leybnits formulasi

Agar aniq integralda integrallashning quyi chegarasi a ni tayin qilib belgilansa va yuqori chegarasi x esa o‘zgaruvchi bo‘lsa, u holda integralning qiymati ham x o‘zgaruvchining funksiyasi bo‘ladi:

$$F(x) = \int_a^x f(t)dt. \qquad (3.19)$$

3.5-teorema (Barrou teoremasi). Agar $f(t)$ funksiya $t = x$ nuqtada uzluksiz bo‘lsa, u holda $F(x)$ funksiyaning hosilasi integral osti funksiyasining yuqori chegaradagi qiymatiga teng, ya’ni

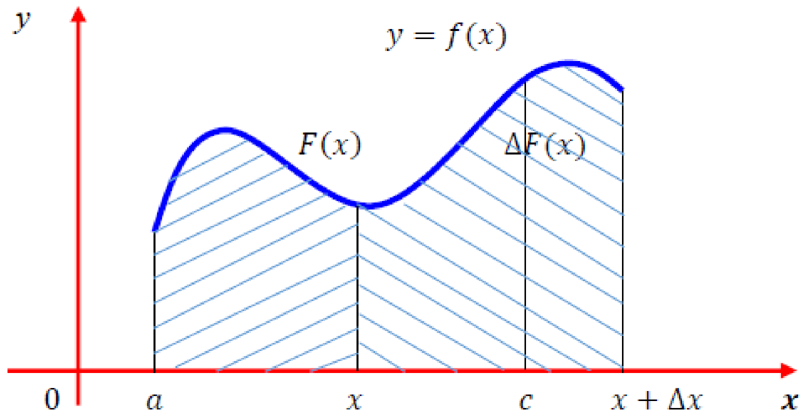
$$\left(\int_a^x f(t)dt\right)' = f(x) \quad \text{yoki} \quad F'(x) = f(x).$$

Isboti: ► $\boldsymbol{x}$ argumentga Δx orttirma beramiz va quydagini hosil qilamiz:

$$F(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt.$$

$F(x)$ funksiyaning orttirmasi quyidagiga teng bo‘ladi (3.3-rasm):

$$\Delta F(x) = F(x + \Delta x) - F(x) = \int_x^{x+\Delta x} f(t)dt \qquad (3.20)$$



3.3-rasm. Funksiyaning orttirmasi

O‘rta qiymat haqidagi teoremani (3.20) integralga qo‘llaymiz,

$$\Delta F = f(c)\Delta x \qquad (3.21)$$

Bunda c nuqta x va $x + \Delta x$ lar orasida yotadi. (3.21) tenglikning ikkala tomonini Δx ga bo‘lamiz: $\frac{\Delta F}{\Delta x} = f(c)$ va $\Delta x \to 0$ bo‘lganda limitga o‘tib, ushbu

$$\lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = \lim_{\Delta x \to 0} f(c)$$

ni hosil qilamiz, biroq

$$\lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = F'(x), \qquad \lim_{\Delta x \to 0} f(c) = f(x).$$

Chunki $\Delta x \to 0$ bo‘lganda $c \to x$ va $f(x)$ funksiya $t \to x$ da uzluksiz.

Shunday qilib, $F'(x) = f(x)$ va $\left(\int_a^x f(t)\,dt\right)' = f(x)$.

Teoremadan $F(x)$ funksiya $f(x)$ ning boshlang‘ich funksiyasi ekanligi kelib chiqadi, chunki $F'(x) = f(x)$. ◄

Nyuton-Leybnits formulasi

Ma’lumki, agar $f(t)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo‘lsa, uning qismi $[a,x]$ kesmada ham integrallanuvchi bo‘lib,

$$G(x)=\int_a^x f(t)dt \tag{3.22}$$

o‘rinli bo‘lar edi. 3.5-teoremaga ko‘ra, $f(x)$ uchun $G(x)$ funksiya boshlang‘ich funksiya bo‘lai, ya’ni $G'(x)=f(x)$.

Faraz qilaylik, $f(x)$ funksiya uchun $F(x)$ funksiya ham boshlang‘ich funksiya bo‘lsin, ya’ni $F'(x)=f(x)$. $f(x)$ uchun $G(x)$ va $F(x)$lar boshlang‘ich funksiyalar bo‘lganliklari sababli ular o‘zaro o‘zgarmas songa farq qilishi kerak:

$$G(x)=F(x)+C \tag{3.23}$$

(3.23) da $x=a$ bo‘lganda $G(a)=F(a)+C$ bo‘ladi, lekin $G(x)=\int_a^a f(t)dt=0$ bo‘lgani uchun $F(a)+C=0$ bo‘ladi, bundan $C=-F(a)$ tenglikka ega bo‘lamiz. Uni (3.23) ga qo‘ysak, quyidagi ko‘rinishni oladi:

$$G(x)=F(x)-F(a) \tag{3.24}$$

(3.24) ga asosan (3.22) quyidagi ko‘rinishni oladi: $\int_a^x f(x)dx=F(x)-F(a)$

bu tenglikda $x=b$ bo‘lsa,

$$\int_a^b f(x)dx=F(b)-F(a) \tag{3.25}$$

bo‘ladi. (3.25) dan ko‘rinadiki, $f(x)$ funksiyaning $[a,b]$ kesmadagi aniq integralini hisoblash uchun uning boshlang‘ich funksiyasi $F(x)$ning aniq integrali yuqori chegarasidan quyi chegarasini ayirish kerak ekan:

$$F(b)-F(a)=F(x)\Big|_a^b \tag{3.26}$$

(3.26) ga asosan (3.25) quyidagi ko‘rinishni oladi:

$$\int_a^b f(x)dx=F(x)\Big|_a^b.$$

Demak, $f(x)$ funksiya $[a;\, b]$ kesmada uzluksiz va $F(x)$ uning boshlang‘ich funksiyasi bo‘lsin. U holda $f(x)$ funksiyaning a dan b gacha **aniq integrali** $\int_a^b f(x)dx=F(b)-F(a)$ ga teng bo‘ladi.

Bu formulaga **Nyuton - Leybnits formulasi** deyiladi.

3.5-§. I va II tur xosmas integrallar. Xosmas integrallarning yaqinlashishi

3.5.1. Chegarasi cheksiz xosmas integrallar

Aniq integrallarni hisoblashda integral ostidagi ifoda integrallash oralig‘ida aniqlangan va uzluksiz bo‘lishi kerak. Agar funksiya integrallash oralig‘ida uzilishga ega bo‘lsa yoki funksiyadan cheksiz oraliqda integral olinadigan bo‘lsa, bunday integrallarga **xosmas integral** deyiladi. Xosmas integrallarni 2 turga ajratish mumkin:

1. Agar integralning quyi yoki yuqori chegarasi yoki ikkala chegarasi ham cheksiz bo‘lsa, unga **I tur xosmas integral** deyiladi.
2. Agar integral ostidagi funksiya integrallash oralig‘ining quyi yoki yuqori chegarasida yoki oraliq ichidagi biror nuqtada uzilishga ega bo‘lsa, unga **II tur xosmas integral** deyiladi.

I tur xosmas integrallarning quyidagi 3 xil turi mavjud:

$[a,\ \infty)$ da uzluksiz bo‘lgan funksiyaning xosmas integrali;

$(-\infty, \mathrm{b}]$ da uzluksiz bo‘lgan funksiyaning xosmas integrali;

$(-\infty, \infty)$ da uzluksiz bo‘lgan funksiyalarning xosmas integrallari.

1) $[a,\ \infty)$ **oraliqda uzluksiz bo‘lgan funksiyaning xosmas integrali** ushbu tenglik bilan aniqlanadi:

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx = \lim_{b\to\infty} F(x)\Big|_a^b = \lim_{b\to\infty}[F(b)-F(a)].$$

Agar ushbu formulada o‘ngdagi limit mavjud bo‘lsa, u holda xosmas integral **yaqinlashuvchi** deyiladi. Bu limit integralning qiymati sifatida qabul qilinadi. Agar limit mavjud bo‘lmasa, xosmas integral **uzoqlashuvchi** deyiladi.

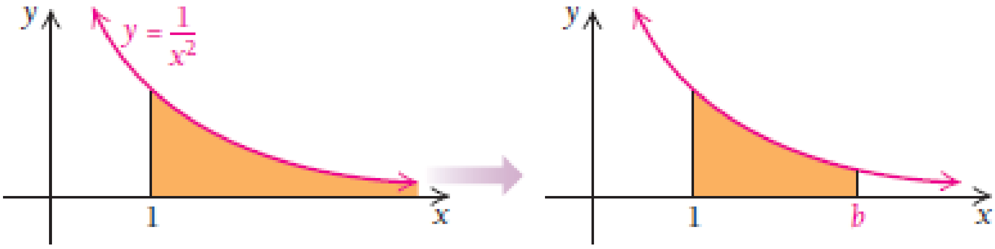
3.5.1-misol. $[1,\ \infty)$ oraliqda $f(x)=\dfrac{1}{x^2}$ funksiya ostidagi soha yuzasini hisoblang.

Yechilishi: ► 3.4-rasmdan ko‘rish mumkinki, bu soha yuzasi cheksiz. Biz haligacha bunday sohaning yuzasini hisoblab ko‘rmagan edik. Keling egri chiziq ostidagi soha yuzasini 1 dan b gacha oraliqda integrallaymiz.

$$\int_1^b \frac{1}{x^2}dx = \left(-\frac{1}{x}\right)\Big|_1^b = \left(-\frac{1}{b}\right)-\left(-\frac{1}{1}\right)=1-\frac{1}{b}.$$

So‘ngra yuzani topish uchun $[1,\ b]$ oraliqda $b\to\infty$ da limit olamiz:

$$S(x)=\lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx=\lim_{b\to\infty}\left(1-\frac{1}{b}\right)=1.$$

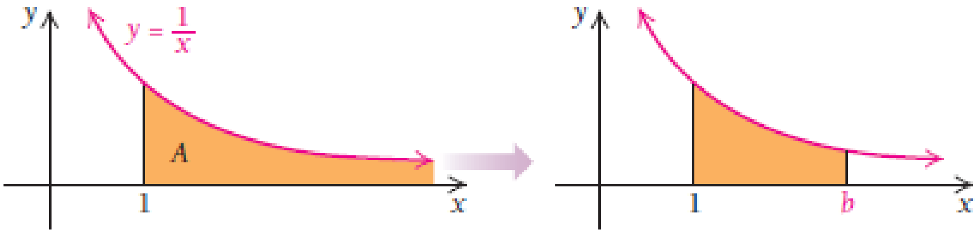


3.4-rasm. $f(x)=\frac{1}{x^2}$ funksiya ostidagi soha

Demak, so‘ralgan yuza chekli bo‘lib, 1 birlikka teng ekan. ◀

3.5.2-misol. $[1,\ \infty)$ oraliqda $f(x)=\frac{1}{x}$ funksiya ostidagi soha yuzasini hisoblang.

Yechilishi: ▶ $A(x)=\lim_{b\to\infty}\int_1^b \frac{1}{x}dx=\lim_{b\to\infty}(\ln x)\Big|_1^b=\lim_{b\to\infty}(\ln b-\ln 1)=\lim_{b\to\infty}\ln b.$



3.5-rasm. $f(x)=\frac{1}{x}$ funksiya ostidagi soha

$y=\ln x$ funksiya grafigidan ma’lumki, u cheksiz o‘suvchi funksiya. Shu sababli $A=\lim_{b\to\infty}\ln b$ yuza cheksiz bo‘ladi (3.5-rasm). ◀

3.5.3-misol. $[0,\ \infty)$ oraliqda $f(x)=e^{-kx}$ funksiyaning xosmas integralini hisoblang.

Yechilishi: ▶ Berilgan funksiya uchun boshlang‘ich funksiya $F(x)=-\frac{e^{-kx}}{k}$ bo‘ladi. Nyuton-Leybnits formulasini qo‘llaymiz:

$$\int_0^\infty e^{-kx}dx=\lim_{b\to\infty}\left(-\frac{e^{-kx}}{k}\Big|_0^\infty\right)=-\frac{1}{k}\lim_{b\to\infty}(e^{-kb}-1).$$

Agar $k>0$ bo‘lsa, $\int_0^\infty e^{-kx}dx=\frac{1}{k}$ integral yaqinlashuvchi.

Agar $k\le 0$ bo‘lsa, $\int_0^\infty e^{-kx}dx=\infty$ integral uzoqlashuvchi. ◀

2) $(-\infty, \mathrm{b}]$ **oraliqda xosmas integral quyidagicha aniqlanadi:**

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx = \lim_{a\to-\infty} F(x)\Big|_a^b = \lim_{a\to-\infty}[F(b)-F(a)].$$

3) $(-\infty, \infty)$ **oraliqda xosmas integral quyidagicha aniqlanadi:**

Agar $y = f(x)$ funksiya butun sonlar o‘qida uzluksiz bo‘lsa, u holda oraliqni c ixtiyoriy olingan c nuqta yordamida ikkita oraliqqa ajratamiz va yuqorida keltirilgan holarlarga olib kelamiz:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx.$$

Agar bu formulada o‘ng tomondagi ikkala integral ham yaqinlashuvchi bo‘lsa, u holda xosmas integral ham yaqinlashuvchi bo‘ladi.

3.5.4-misol. Ushbu integralning yaqinlashuvchiligini tekshiring:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Yechilishi: ► Yuqoridagi formulada c= 0 deb faraz qilib, quyidagini hosil qilamiz: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$

Tenglikning o‘ng qismidagi xosmas integrallar yaqinlashuvchi bo‘ladi, chunki $\int_{-\infty}^{0} \frac{dx}{1+x^2} = arctgx\Big|_{-\infty}^{0} = arctg0 - \lim_{x\to-\infty} arctgx = \frac{\pi}{2};$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = arctgx\Big|_{0}^{\infty} = \lim_{x\to\infty} arctgx - arctg0 = \frac{\pi}{2}$$

Shunga ko‘ra, $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

Demak, integral yaqinlashuvchi va uning qiymati π ga teng. ◄

3.5.2. Chegaralanmagan funksiyaning xosmas integrali

II tur xosmas integrallarning quyidagi turlari mavjud:

1) $(a,b]$ oraliqda uzluksiz va $x = a$ da aniqlanmagan yoki **II** tur uzilishga ega bo‘lgan $f(x)$ funksiyaning xosmas integrali quyidagicha aniqlanadi: $\int_{a}^{b} f(x)dx = \lim_{\varepsilon\to0}\int_{a+\varepsilon}^{b} f(x)dx = \lim_{\varepsilon\to0}[F(b)-F(a+\varepsilon)].$

Agar oxirgi limit mavjud bo‘lsa, u holda xosmas integral **yaqinlashuvchi** deyiladi. Agar ko‘rsatilgan limit mavjud bo‘lmasa, u holda xosmas integral **uzoqlashuvchi** deyiladi.

2) $[a,b)$ oraliqda uzluksiz va $x=b$ da aniqlanmagan yoki **II** tur uzilishga ega bo‘lgan $f(x)$ funksiyaning xosmas integrali ham shunga o‘xshash aniqlanadi:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx = \lim_{\varepsilon\to 0}[F(b-\varepsilon)-F(a)].$$

3) Agarda $f(x)$ funksiya $[a,b]$ kesmaning biror $x=c$ oraliq nuqtasida uzilishga ega yoki aniqlanmagan bo‘lsa, u holda xosmas integral quyidagi integral bilan aniqlanadi:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Agar tenglikning o‘ng tomonidagi integrallardan aqalli bittasi uzoqlashuvchi bo‘lsa, u holda xosmas integral uzoqlashuvchi bo‘ladi.

Agar o‘ng tomondagi ikkala integral yaqinlashuvchi bo‘lsa, u holda xosmas integral ham yaqinlashuvchi bo‘ladi.

4) (a,b) oraliqda uzluksiz, $x=a$ va $x=b$ nuqtalarda aniqlanmagan yoki **II** tur uzilishga ega bo‘lgan $f(x)$ funksiyaning xosmas integrali quyidagicha aniqlanadi:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{b-\varepsilon} f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{c} f(x)dx + \lim_{\varepsilon\to 0}\int_{c}^{b-\varepsilon} f(x)dx =$$

$$= \lim_{\varepsilon\to 0}[F(c)-F(a+\varepsilon)] + \lim_{\varepsilon\to 0}[F(b-\varepsilon)-F(c)].$$

3.5.5-misol. Ushbu $\int_0^4 \frac{dx}{\sqrt{x}}$ integralni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► $x\to 0$ da $f(x)=\frac{1}{\sqrt{x}}\to\infty$. $x=0$ nuqta $[0,4]$ kesmaning chap oxirida yotadi. Shuning uchun quyidagiga ega bo‘lamiz:

$$\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}\int_\varepsilon^4 \frac{dx}{\sqrt{x}} = 2\lim_{\varepsilon\to 0}\sqrt{x}\Big|_\varepsilon^4 = 2\cdot 2 = 4$$

Demak, integral yaqinlashuvchi. ◄

Ishorasi almashinuvchi funksiyalarning xosmas integrallarini hisoblashda nomanfiy funksiya bo‘lgan holga olib kelinadi:

Agar $\int_a^{\infty} |f(x)| dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^{\infty} f(x) dx$ integral ham yaqinlashuvchi bo‘ladi. Bunda oxirgi integral **absolyut yaqinlashuvchi integral** deyiladi.

Agar $\int_a^{\infty} f(x) dx$ integral yaqinlashuvchi, $\int_a^{\infty} |f(x)| dx$ integral esa uzoqlashuvchi bo‘lsa, u holda $\int_a^{\infty} f(x) dx$ integral **shartli yaqinlashuvchi integral** deyiladi.

3.5.6-misol. Ushbu integrallarning yaqinlashuvchiligini tekshiring.

$$\int_0^{\infty} \frac{\cos x dx}{1+x^2} \quad \text{va} \quad \int_0^{\infty} \frac{\sin x dx}{1+x^2}$$

Yechilishi: ► Integral ostidagi funksiyalar ushbu shartlarni qanoatlantiradi:

$$\left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2}, \qquad \left|\frac{\sin x}{1+x^2}\right| \le \frac{1}{1+x^2}.$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = arctgx\Big|_0^{\infty} = \lim_{x\to\infty} arctgx - arctg0 = \frac{\pi}{2}$$

integral yaqinlashuvchi, shuning uchun $\int_0^{\infty} \left|\frac{\cos x}{1+x^2}\right| dx$ va $\int_0^{\infty} \left|\frac{\sin x}{1+x^2}\right| dx$ integrallar ham yaqinlashuvchi bo‘ladi. ◄

Mavzu yuzasidan savollar:

1. Aniq integral ta’rifini keltiring.
2. Riman yig‘indilari qanday hosil qilinadi?
3. O‘rta qiymat haqidagi teoremani ayting.
4. Nyuton-Leybnits formulasini ayting va isbotlang.
5. I tur xosmas integral deb nimaga aytiladi?
6. I tur xosmas integralning qanday turlarini bilasiz?
7. II tur xosmas integral deb nimaga aytiladi?
8. II tur xosmas integralning qanday turlarini bilasiz?
9. Absolyut yaqinlashuvchi integral deganda nimani tushunasiz?
10. Shartli yaqinlashuvchi integral deganda nimani tushunasiz?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. I tur xosmas integrallarni hisoblang:

a) $\int\limits_1^{\infty}\frac{dx}{x^2}$ **b)** $\int\limits_1^{\infty}\frac{dx}{x^p}$ **c)** $\int\limits_2^{\infty}\frac{dx}{\left(x^2-1\right)^2}$

2. I tur xosmas integrallarni hisoblang:

a) $\int\limits_{-\infty}^{\infty}\frac{dx}{x^2+4x+9}$; **b)** $\int\limits_{-\infty}^{\infty}\frac{dx}{1+x^2}$ **c)** $\int\limits_0^{\infty}\frac{dx}{x^3+1}$

3. II tur xosmas integrallarni hisoblang:

a) $\int\limits_0^{1}\frac{dx}{\sqrt{x}}$ **b)** $\int\limits_0^{2}\frac{dx}{x}$ **c)** $\int\limits_0^{1}\frac{dx}{x^p}$ **d)** $\int\limits_0^{1/2}\frac{dx}{x\ln x}$

4. II tur xosmas integrallarni hisoblang:

a) $\int\limits_0^{1/2}\frac{dx}{x\ln^2 x}$ **b)** $\int\limits_0^{1}\frac{e^{\pi-\arcsin x}dx}{\pi\sqrt{1-x^2}}$ **c)** $\int\limits_{1/2}^{2}\frac{\ln(2x-1)}{2x-1}dx$

5. Xosmas integrallarni hisoblang:

a) $\int\limits_0^{\infty}\frac{xdx}{16x^2+1}$ **b)** $\int\limits_0^{1}\frac{dx}{\sqrt{2-4x}}$ **c)** $\int\limits_{1/2}^{1}\frac{dx}{(1-x)\ln^2(1-x)}$

TESTLAR

1. Xosmas integralni hisoblang: $\int\limits_{-\infty}^{1}e^t dt$.

A) e **B)** e^2 **C)** $2e$ **D)** e^3

2. Xosmas integralni hisoblang: $\int\limits_0^{1}\frac{dx}{\sqrt{1-x}}$.

A) -3 **B)** 5 **C)** 2 **D)** 7

3. Xosmas integralni hisoblang: $\int\limits_0^{1}\frac{e^{\frac{1}{x}}dx}{x^3}$.

A) -3 **B)** uzoqlashuvchi **C)** 2 **D)** 8

4. Xosmas integralni hisoblang: $\int\limits_1^{3}\frac{xdx}{\sqrt{4x-x^2-3}}$.

A) 0 **B)** uzoqlashuvchi **C)** 2 **D)** 8

5. Xosmas integralni hisoblang: $\int\limits_{e^2}^{\infty}\frac{dx}{x\left(\ln x-1\right)^2}$.

A) 1 **B)** uzoqlashuvchi **C)** -1 **D)** 0

3.6-§. Aniq integralning tatbiqlari

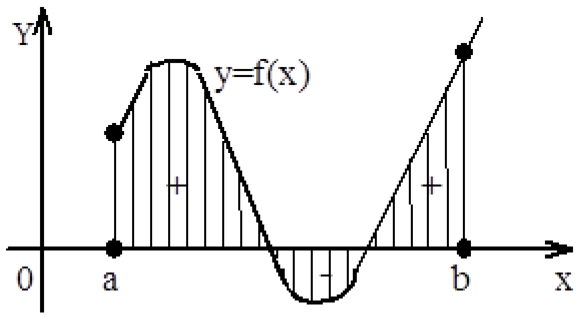
3.6.1. Aniq integral yordamida yassi shakl yuzini hisoblash

Dekart koordinata sistemasida yassi shakl yuzini hisoblash.

Aytaylik, biror $[a,b]$ oraliqda $y=f(x)$ uzluksiz funksiya berilgan bo‘lsin. Yuqoridan $y=f(x)$ egri chiziq, Ox o‘qi va $x=a$ hamda $x=b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzini topish kerak bo‘lsin. Agar $f(x)\geq 0$ bo‘lsa, u holda bu yassi shakl yuzi

$$S=\int_a^b f(x)dx \qquad (3.27)$$

ga teng bo‘ladi. Agar $[a,b]$ kesmada $f(x)\leq 0$ bo‘lsa, u holda aniq integral $\int_a^b f(x)dx \leq 0$ bo‘ladi. Shu sababli egri chiziqli trapetsiyaning yuzini topish uchun uning modulini olamiz: $S=\int_a^b |f(x)|dx$ (3.28)



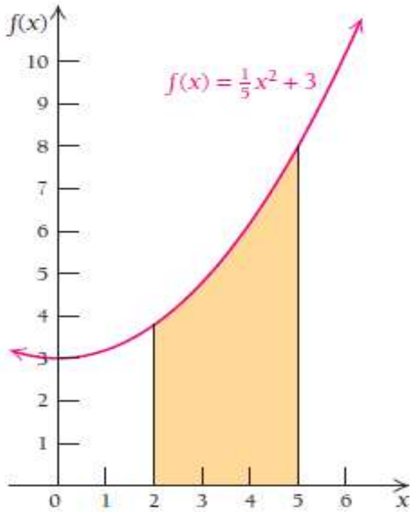
3.6-rasm. $y=f(x)$ funksiya

Agar $y=f(x)$ funksiya $[a,b]$ kesmada ishorasini chekli son marta o‘zgartirsa, u holda integralni butun $[a,b]$ kesmada qismiy kesmachalar bo‘yicha integrallar yig‘indisiga ajratamiz. $f(x)>0$ bo‘lgan kesmalarda integral musbat, $f(x)<0$ bo‘lgan kesmalarda integral manfiy bo‘ladi. Butun kesma bo‘yicha olingan integral Ox o‘qidan yuqorida va pastda yotuvchi yuzalarning tegishli algebraik yig‘indisiga teng bo‘ladi (3.6-rasm).

3.6.1-misol. $f(x)=\frac{1}{5}x^2+3$ funksiya grafigi ostidagi [2; 5] kesmaga mos soha yuzasini hisoblang.

Yechilishi: ► Tasavvur qilish uchun grafigini chizib ko‘ramiz (3.7-rasm). (3.27) formuladan foydalanamiz.

$$S=\int_2^5\left(\frac{1}{5}x^2+3\right)dx=\left(\frac{x^3}{15}+3x\right)\Bigg|_2^5=16\frac{4}{5}.$$

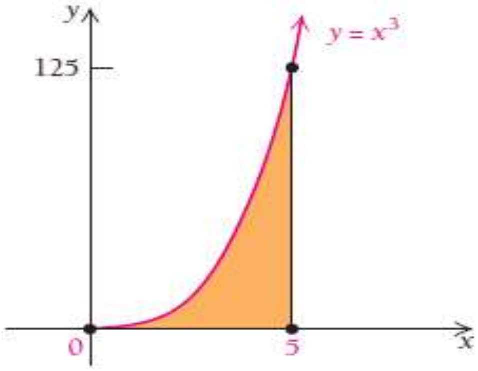


3.7-rasm. $f(x)=\frac{1}{5}x^2+3$ funksiya grafigi ◄

3.6.2-misol. Yangi elektrostansiya kuniga $f(x)=x^3$ kW elektr energiyasi ishlab chiqaradi. Bu stansiya 5 kunda qancha elektr energiya ishlab chiqaradi?

Yechilishi: ► Funksiya grafigini chizamiz (3.8-rasm).

$$\int_0^5 x^3dx=\frac{x^4}{4}\Bigg|_0^5=\frac{5^4}{4}-0=156\frac{1}{4}\text{ kW·kun.}$$

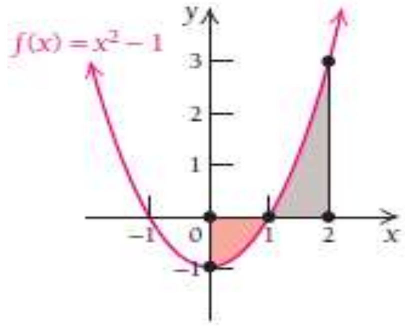


3.8-rasm. $f(x)=x^3$ funksiya grafigi ◄

3.6.3-misol. [0; 2] kesmada $y=x^2-1$ funksiya bilan chegaralangan yuzani toping.

Yechilishi: ► Bu funksiya [0; 1] kesmada manfiy, [1; 2] kesmada musbat qiymatlar qabul qiladi. Agar yuzani hisoblash uchun funksiyani

[0; 2] kesmada integrallasak, unda **noto‘g‘ri yechim hosil bo‘ladi.** Xato qilib qo‘ymaslik uchun kesmani [0; 1] va [1; 2] bo‘laklarga ajratish kerak, so‘ngra yuzalarni qo‘shamiz, 1-bo‘lakda funksiyani absolyut qiymatini olamiz:



3.9-rasm. $y = x^2 - 1$ funksiya grafigi

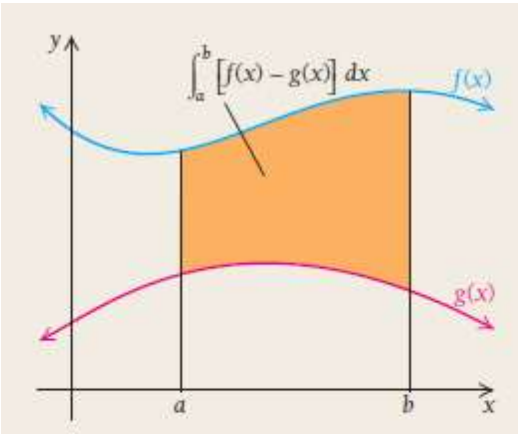
$$S_1 = \int_0^2 (x^2-1)dx = \int_0^1 \left|x^2-1\right|dx = \left|\left(\frac{x^3}{3}-x\right)\right|_0^1 = \left|\left(\frac{1^3}{3}-1\right)-0\right| = \left|\frac{1}{3}-1\right| = \left|-\frac{2}{3}\right| = \frac{2}{3}$$

$$S_2 = \int_1^2 (x^2-1)dx = \left(\frac{x^3}{3}-x\right)\Big|_1^2 = \left(\frac{2^3}{3}-2\right)-\left(\frac{1}{3}-1\right) = \frac{2}{3}+\frac{2}{3} = \frac{4}{3},$$

$$S = S_1 + S_2 = \frac{2}{3}+\frac{4}{3} = 2. \blacktriangleleft$$

3.6-teorema. Agar $f(x)$ va $g(x)$ funksiyalar $[a;b]$ kesmada uzluksiz va $f(x) \geq g(x)$ bo‘lsa, u holda bu funksiyalar hamda $x = a$ va $x = b$ chiziqlar bilan chegaralangan sohaning yuzasi quyidagiga teng bo‘ladi:

$$A = \int_a^b [f(x) - g(x)]dx \qquad (3.29)$$



3.10-rasm. $f(x)$ va $g(x)$ funksiyalar bilan chegaralangan yuza

3.6.4-misol. Ushbu $4y = 8x - x^2$, $4y = x + 6$ chiziqlar bilan chegaralangan shakl yuzini toping.

Yechilishi: ► Bu chiziqlardan biri parabola, ikkinchisi to‘g‘ri chiziq bo‘lib, ularning kesishish nuqtasini topamiz:

$$\begin{cases} 4y = 8x - x^2 \\ 4y = x + 6 \end{cases} \Rightarrow \quad 8x - x^2 = x + 6 \quad \Rightarrow \quad x^2 - 7x + 6 = 0$$

$$x_1 = 1 \text{ va } x_2 = 6, \quad y_1 = \frac{x+6}{4} = \frac{1+6}{4} = \frac{7}{4} \text{ va } y_2 = \frac{x+6}{4} = \frac{6+6}{4} = 3.$$

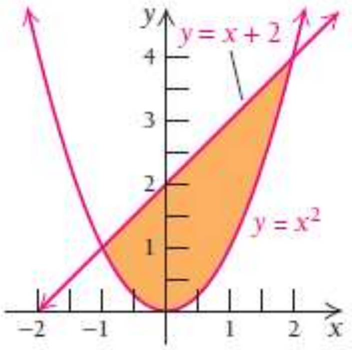
Shunda bu ikki grafik bir-biri bilan A(1;7/4) va B(6;3) nuqtalarda kesishadi. Bunda tepadagi grafik bilan chegaralangan yuzadan pastdagi grafik bilan chegaralangan yuzani ayiramiz:

$$S = \frac{1}{4}\int_1^6 [(8x - x^2) - (x+6)]dx = \frac{1}{4}\int_1^6 (-x^2 + 7x - 6)dx =$$

$$= \frac{1}{4}\left(-\frac{x^3}{3} + \frac{7x^2}{2} - 6x\right)\Bigg|_1^6 = 5\frac{5}{24} \text{ kv.birl.} \quad ◄$$

3.6.5-misol. $y = x^2$ va $y = x+2$ chiziqlar bilan chegaralangan shakl yuzini toping.

Yechilishi: ► 1) Bu chiziqlarning kesishish nuqtalarini topib olamiz (3.11-rasm). Buning uchun tenglamalar sistemasini yechish kerak:



3.11-rasm. $y = x^2$ va $y = x+2$ chiziqlar bilan chegaralangan yuza

$$\begin{cases} y = x^2 \\ y = x+2 \end{cases} \quad \longrightarrow \quad x^2 = x+2 \quad \longrightarrow \quad x^2 - x - 2 = 0 \quad \longrightarrow \quad x = 2, \quad x = -1$$

1) Endi $[-1;\ 2]$ kesmada integralni hisoblaymiz:

$$S = \int_{-1}^{2} (x + 2 - x^2)dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Bigg|_{-1}^{2} = \frac{9}{2}.$$

$y = x^2$ va $y = x+2$ chiziqlar bilan chegaralangan shakl yuzi 4.5 kv. birlikka teng chiqdi. ◄

Parametrik shakldа berilgan egri chiziqlar bilan chegaralangan yuzani topish

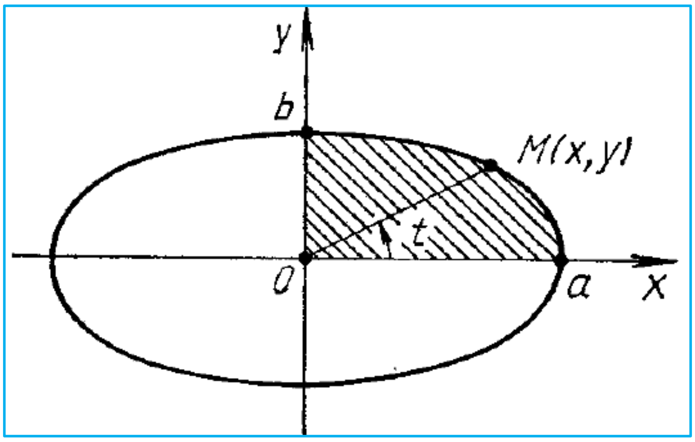
Egri chiziqli trapetsiyaning yuzi tenglamalari parametrik $x = \varphi(t), \quad y = \psi(t)$ shaklda berilgan chiziq bilan chegaralangan bo‘lsin, bunda bu tenglamalar $[a, b]$ kesmadagi biror $y = f(x)$ funksiyani aniqlaydi, bunda $t \in [\alpha, \beta]$ va $\varphi(\alpha) = a, \quad \varphi(\beta) = b$. U holda egri chiziqli trapetsiyaning yuzini $S=\int\limits_a^b ydx$ formula bo‘yicha hisoblashimiz mumkin. Bu integralda o‘zgaruvchini almashtiramiz: $x=\varphi(t), \quad dx=\varphi(t)dt$. Shunda $y=f(x)=f(\varphi(t))$ bo‘ladi.

Demak, parametrik shakldа berilgan egri chiziq bilan chegaralangan shaklning yuzini topish formulasi quyidagicha bo‘ladi:

$$S=\int\limits_{\alpha}^{\beta}\psi(t)\varphi'(t)dt \qquad (3.30)$$

3.6.6-misol. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ellips bilan chegaralangan yuzani hisoblang.

Yechilishi: ► Chizmadan foydalanib, ellips tenglamasini parametrik ko‘rinishga o‘tkazamiz (3.12-rasm): $\begin{cases} x=a\cos t \\ y=b\sin t \end{cases}$



3.12-rasm. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ellips bilan chegaralangan yuza

$$S=4\cdot\int\limits_0^a ydx=4\cdot\int\limits_{\pi/2}^{0}a\sin t\cdot d(b\cos t)=4\cdot\int\limits_{\pi/2}^{0}a\sin t\cdot(-b\sin t)dt=4ab\cdot\int\limits_0^{\pi/2}\sin^2 tdt=$$

$$=4ab\cdot\int\limits_0^{\pi/2}\frac{1-\cos 2t}{2}dt=2ab\left(t-\frac{1}{2}\sin 2t\right)\Bigg|_0^{\frac{\pi}{2}}=\pi ab. \quad ◄$$

Qutb koordinatasida yassi shakl yuzini hisoblash

Bizga $\breve{AB}$ egri chiziq qutb koordinatalar sistemasida

$$\rho=\rho(\theta) \quad , \quad \alpha\le\theta\le\beta \quad (\alpha,\beta\in R)$$

tenglama bilan berilgan bo‘lsin. Tekislikda $\breve{AB}$ egri chiziq hamda OA va OB radius vektorlar bilan chegaralangan shakl yuzasini topish so‘ralgan bo‘lsin.

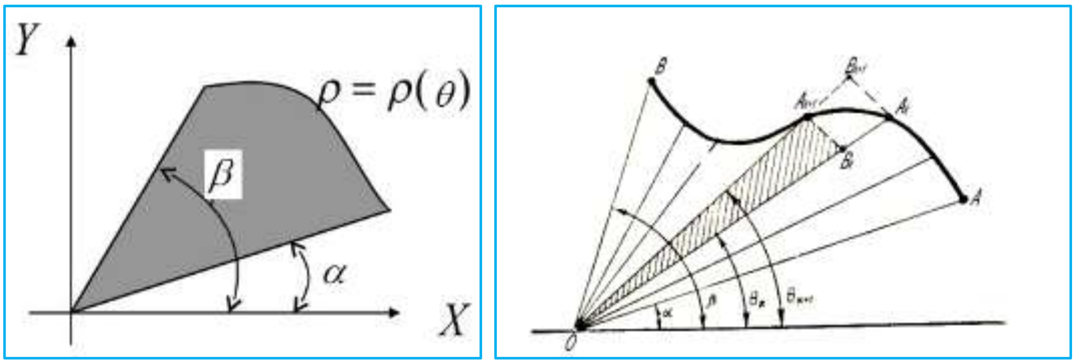
1) $[\alpha,\beta]$ segmentni ixtiyoriy $P=\{\theta_0,\theta_1,...,\theta_n\} \quad (\alpha=\theta_0<\theta_1<...<\theta_n=\beta)$

bo‘laklarga ajratamiz.

2) O nuqtadan har bir qutb burchagi θ_k ga mos OA_k radius vektor o‘tkazamiz. Natijada OAB egri chiziqli sektor

$$OA_k A_{k+1} \ (k = 0,1,2,\ldots,n-1 \ \ ; \ A_0 = A \ , \ A_n = B)$$

egri chiziqli sektorchalarga ajraladi (3.13-rasm).



3.13-rasm. Qutb koordinatasi

3) Endi har bir $[\theta_k, \theta_{k+1}]$ segment uchun radius vektorlari mos ravishda m_k va M_k bo‘lgan doiraviy sektorlarni hosil qilamiz. Ularning yuzi mos ravishda $\frac{1}{2}m_k^2 \cdot \Delta\theta_k \quad , \ \frac{1}{2}M_k^{2\cdot} \cdot \Delta\theta_k \qquad (\Delta\theta_k = \theta_{k+1} - \theta_k)$ bo‘ladi.

4) Radius vektorlari $m_k \quad (k = 0,1,2,\ldots,n-1)$ bo‘lgan barcha doiraviy sektorlar birlashmasidan iborat shaklni Q_1 desak, u holda $Q_1 \subset Q$ bo‘lib, uning yuzi $S(Q_1) = \frac{1}{2}\sum_{k=0}^{n-1} m_k^2 \cdot \Delta\theta_k$ bo‘ladi.

5) Radius vektorlari $M_k \quad (k = 0,1,2,\ldots,n-1)$ bo‘lgan barcha doiraviy sektorlar birlashmasidan iborat rasmni Q_2 desak, u holda $Q \subset Q_2$ bo‘lib, uning yuzi $S(Q_2) = \frac{1}{2}\sum_{k=0}^{n-1} M_k^2 \cdot \Delta\theta_k$ bo‘ladi.

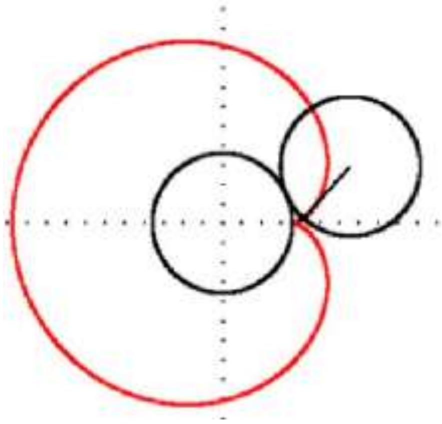
4- va 5- qadamda hosil qilingan yig‘indilar $\frac{1}{2}\rho^2(\theta)$ funksiyaning **Darbu yig‘indilari** deyiladi. $\frac{1}{2}\rho^2(\theta)$ funksiya $[\alpha, \beta]$ oraliqda uzluksiz bo‘lgani uchun u integrallanuvchidir. Demak, Q egri chiziqli sektorning yuzi

$$S(Q) = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\theta) d\theta \tag{3.31}$$

ga teng bo‘ladi.

3.6.7-misol. Ushbu $\rho = \rho(\theta) = a(1 - \cos\theta) \quad (a \in R, \ 0 \le \theta \le 2\pi)$ funksiya grafigi bilan chegaralangan shakl yuzini toping.

Yechilishi: ► $\rho=\rho(\theta)=a(1-\cos\theta)$ $(a\in R,\ 0\le\theta\le 2\pi)$ funksiya grafigini **kardioda** deyiladi (3.14-rasm). Kardioda - bu radiusi r ga teng bo‘lgan aylananing shu radiusli 2-qo‘zg‘almas aylana bo‘ylab harakati (sirpanmasdan dumalashi) natijasida 1-aylana ixtiyoriy nuqtasining chizgan chizig‘idir. Kardioda qutb o‘qiga nisbatan simmetrik bo‘lganligi sababli yuqori yarim tekislikdagi shaklning yuzini topib, so‘ngra uni 2 ga ko‘paytirsak, izlanayotgan yuza kelib chiqadi.



3.14-rasm. Kardioda grafigi

θ o‘zgaruvchi $[0,\pi]$ da o‘zgarganda ρ radius vektor kardiodaning yuqori yarim tekislikdagi qismini chizadi. Bu yuza quyidagiga teng:

$$\mu(Q)=2\cdot\frac{1}{2}\int_0^{\pi}\rho^2(\theta)d\theta=\int_0^{\pi}a^2(1-\cos\theta)^2d\theta=$$

$$=a^2\int_0^{\pi}\left[\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos2\theta\right]d\theta=$$

$$=a^2\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{2}\cdot\frac{1}{2}\sin2\theta\right)\Big|_0^{\pi}=\frac{3}{2}\pi a^2$$ ◄

3.6.8-misol. Grafigi **Dekart yaprog‘i** deb ataluvchi

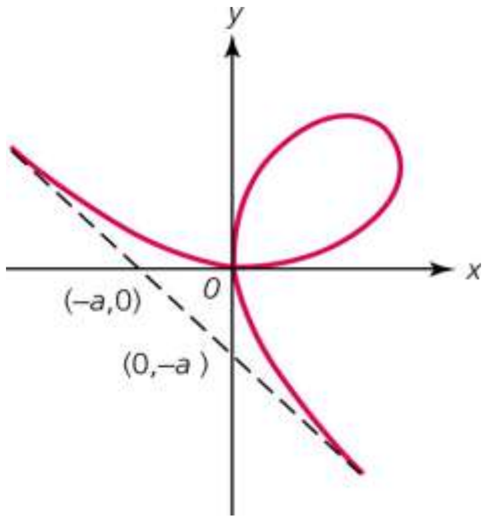
$$\rho=\frac{3a\cos\varphi\sin\varphi}{\sin^3\varphi+\cos^3\varphi},\ \left(0\le\varphi\le\frac{\pi}{2}\right)$$

funksiya bilan chegaralangan shakl yuzini toping (3.15-rasm).

Yechilishi: ► Qaralayotgan shaklning yuzini (3.31) formuladan foydalanib topamiz:

$$S=\frac{1}{2}\int_0^{\pi/2}\left[\frac{3a\cos\varphi\sin\varphi}{\sin^3\varphi+\cos^3\varphi}\right]^2d\varphi=\frac{9a^2}{2}\int_0^{\pi/2}\frac{\cos^2\varphi\sin^2\varphi}{(\sin^3\varphi+\cos^3\varphi)^2}d\varphi=\frac{9a^2}{2}\int_0^{\pi/2}\frac{tg^2\varphi}{(1+tg^3\varphi)^2}\cdot\frac{d\varphi}{\cos^2\varphi}=$$

$$=\frac{9a^2}{2}\int_0^{\pi/2}\frac{tg^2\varphi}{(1+tg^3\varphi)^2}d(tg\varphi)=\frac{9a^2}{2}\cdot\frac{1}{3}\int_0^{\pi/2}\frac{1}{(1+tg^3\varphi)^2}d\left(1+tg^3\varphi\right)=-\frac{9a^2}{2}\cdot\frac{1}{3}\cdot\frac{1}{1+tg^3\varphi}\Big|_0^{\pi/2}=\frac{3a^2}{2}$$



3.15-rasm. Dekart yaprog‘I ◄

3.6.2. Aniq integral yordamida egri chiziq yoyi uzunligini topish

Dekart koordinata sistemasida egri chiziq yoyi uzunligini hisoblash

Tekislikda to‘g‘ri burchakli koordinatalar sistemasida egri chiziq $y = f(x)$ tenglama bilan berilgan bo‘lsin. Bu egri chiziqning $x = a$ va $x = b$ vertikal to‘g‘ri chiziqlar orasidagi $\breve{AB}$ yoyining uzunligini topamiz.

$\breve{AB}$ yoyda absissalari $a = x_0, x_1, x_2, ..., x_i, ..., x_n = b$ bo‘lgan $A, M_1, M_2, ..., M_i, ..., B$ nuqtalarni olamiz va $AM_1, M_1M_2, ..., M_{n-1}B$ vatarlarni o‘tkazamiz, ularning uzunliklarini mos ravishda $\Delta L_1, \Delta L_2, ..., \Delta L_i, ..., \Delta L_n$ bilan belgilaymiz. $\breve{AB}$ yoy ichiga chizilgan siniq chiziqning uzunligi $L_n = \sum_{i=1}^{n} \Delta L_i$ bo‘lgani uchun $\breve{AB}$ yoyning uzunligi $L = \lim_{\max \Delta L_i \to 0} \sum_{i=1}^{n} \Delta L_i$ bo‘ladi. Faraz qilaylik, $y = f(x)$ funksiya va uning $f'(x)$ hosilasi $[a, b]$ kesmada uzluksiz bo‘lsin. U holda $\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + (\frac{\Delta y_i}{\Delta x_i})^2} \cdot \Delta x_i$ yoki Lagranj teoremasiga asosan $\frac{\Delta y_i}{\Delta x_i} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(\xi_i)$. Bunda $x_{i-1} < \xi_i < x_i$ bo‘lgani uchun $\Delta L_i = \sqrt{1 + (f'(\xi_i))^2} \Delta x_i$ bo‘ladi. Ichki chizilgan siniq chiziqning uzunligi esa $L_n = \sum_{i=1}^{n} \sqrt{1 + (f'(\xi_i))^2} \Delta x_i$ bo‘ladi.

Shartga ko‘ra, $f'(x)$ funksiya $[a, b]$ kesmada uzluksiz, demak $\sqrt{1 + (f'(x))^2}$ funksiya ham uzluksizdir. Shuning uchun integral yig‘indining limiti mavjud va u quyidagi aniq integralga teng:

$$L=\lim_{\max \Delta x_i \to 0}\sum_{i=1}^{n}\sqrt{1+(f'(\xi_i))^2}\Delta x_i=\int_a^b\sqrt{1+(f'(x))^2}dx$$

Demak, $\breve{AB}$ yoy uzunligini hisoblash formulasi quyidagicha:

$$L=\int_a^b\sqrt{1+(f'(x))^2}dx=\int_a^b\sqrt{1+(\frac{dy}{dx})^2}dx. \qquad (3.32)$$

3.6.9-misol. Ushbu $f(x)=(x^{\frac{3}{2}})$’ $(0\le x\le 4)$ funksiyaning egri chiziq yoyi uzunligini toping.

Yechilishi: ►Avvalo berilgan funksiyaning hosilasini hisoblaymiz:

$$f'(x)=(x^{\frac{3}{2}})'=\frac{3}{2}x^{\frac{1}{2}}$$

Unda $\qquad 1+f'^2(x)=1+\frac{9}{4}x,\ \sqrt{1+f'^2(x)}=\sqrt{1+\frac{9}{4}x}$

bo‘lib, (3.32) formulaga binoan $L=\int_0^4\sqrt{1+\frac{9}{4}x}\,dx$

bo‘ladi. Keyingi integralda $1+\frac{9}{4}x=t$ almashtirish bajaramiz.

Unda $dx=\frac{9}{4}dt$ va yangi o‘zgaruvchi chegarasi $1\le t\le 10$ bo‘lib, quyidagi integral hosil bo‘ladi:

$$\int_0^4\sqrt{1+\frac{9}{4}x}\,dx=\frac{9}{4}\int_1^{10}t^{\frac{1}{2}}dt=\frac{8}{27}t^{\frac{3}{2}}\Big|_1^{10}=\frac{8}{27}\left(\sqrt{1000}-1\right)$$

$$=\frac{8}{27}\left(10\sqrt{10}-1\right).$$

Demak, yoy uzunligi $L=\frac{8}{27}\left(10\sqrt{10}-1\right)$ ga teng. ◄

Parametrik shakldа berilgan funksiyaning egri chiziq yoyi uzunligini hisoblash

Faraz qilaylik, $\breve{AB}$ yoy (egri chiziq)

$$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases},\ (\alpha\le t\le\beta)$$

tenglamalar sistemasi, ya’ni parametrik rasmda berilgan bo‘lib, $x=\varphi(t), y=\psi(t)$ funksiyalar $[\alpha,\beta]$ da aniqlangan, uzluksiz va $\varphi'(t),\psi'(t)$ uzluksiz hosilalarga ega bo‘lsin. Bunda $\breve{AB}$ yoy uzunlikka ega bo‘lib, uning uzunligi

$$L=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}\ dt \quad \text{yoki} \quad L=\int_\alpha^\beta\sqrt{[x'(t)]^2+[y'(t)]^2}dt \qquad (3.33)$$

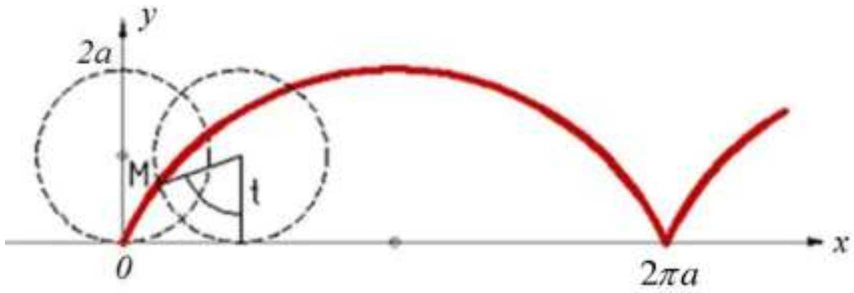
formulalar yordamida topiladi.

3.6.10-misol. $y = a(1-\cos t),\ x = a(t-\sin t)$ **sikloidaning** bitta arkasining uzunligini hisoblang (3.16-rasm).

Yechilishi: ► Parametrik ko‘rinishda berilgan funksiya yoy uzunligini (3.33) formuladan topamiz:

$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt,\ 0 \le t \le 2\pi.$$

$$x' = \big(a(t-\sin t)\big)' = a(1-\cos t), \qquad y' = \big(a(1-\cos t)\big)' = a\sin t.$$



3.16-rasm. Sikloida grafigi

$$l = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\,dt = \int_0^{2\pi} a\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt = a\int_0^{2\pi}\sqrt{2(1-\cos t)}\,dt =$$

$$= 2a\int_0^{2\pi}\sin\frac{t}{2}\,dt = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = 8a. \quad ◄$$

Qutb koordinatasida yoy uzunligini hisoblash

Faraz qilaylik, $\breve{AB}$ egri chiziq qutb koordinata sistemasida

$$\rho = \rho(\theta) \quad (\alpha \le 0 \le \beta) \qquad (3.34)$$

tenglik bilan berilgan bo‘lsin (3.13-rasm). Bunda $\rho = \rho(\theta)$ funksiya $[a;b]$ da uzluksiz va $\rho'(\theta)$ uzluksiz hosilaga ega.

Avvalo (3.34) munosabat bilan berilgan egri chiziq tenglamasini parametrik ko‘rinishda ifodalab olamiz:

$$\begin{cases} \varphi(\theta) = \rho(\theta)\cos\theta \\ \psi(\theta) = \rho(\theta)\sin\theta \end{cases} \quad (\alpha \le 0 \le \beta)$$

So‘ng (3.33) formuladan foydalanib, $\breve{AB}$ egri chiziq yoyining uzunligini topamiz: $l = \int_{\alpha}^{\beta}\sqrt{\varphi'^2(\theta)+\psi'^2(\theta)}\,d\theta = \int_{\alpha}^{\beta}\sqrt{(\rho(\theta)\cos\theta)'^2 + (\rho(\theta)\sin\theta)'^2}\,d\theta$

$$= \int_{\alpha}^{\beta}\sqrt{(\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta)^2 + (\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta)^2}\,d\theta = \int_{\alpha}^{\beta}\sqrt{(\rho'^2(\theta)+\rho^2(\theta)}\,d\theta$$

Demak, (3.34) munosabat bilan berilgan egri chiziq yoyining uzunligi

$$l = \int_{\alpha}^{\beta}\sqrt{(\rho'^2(\theta)+\rho^2(\theta)}\,d\theta \qquad (3.35)$$

bo‘ladi.

3.6.11-misol. Ushbu $\rho = 2a(1 + cos\theta)$, $(0 \leq \theta \leq 2\pi)$ egri chiziq yoyining uzunligini toping.

Yechilishi: ► Bu funksiyaning gragini ham kardioda deyiladi, u yopiq chiziq bo‘lib, qutb o‘qiga nisbatan simmetrik joylashgan. Shuning uchun egri chiziqning uzunligi, uning qutb o‘qining yuqorisida joylashgan qismi uzunligining ikkilanganiga teng bo‘ladi. (3.35) formuladan foydalanib topamiz:

$$l = 2\int_0^{\pi} \sqrt{(\rho'^2(\theta) + \rho^2(\theta)}d\theta = 2\int_0^{\pi} \sqrt{(2a(1+\cos\theta))'^2 + (2a(1+\cos\theta))^2}d\theta =$$

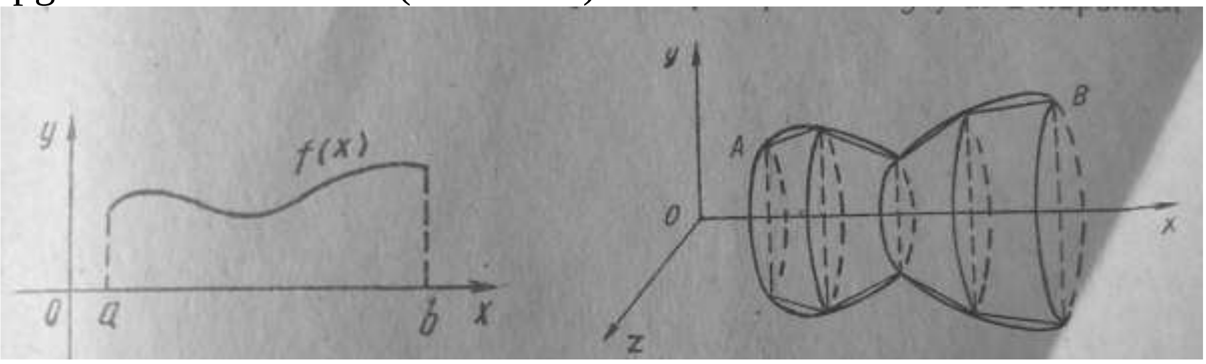
$$= 2\cdot 2a\int_0^{\pi} \sqrt{\sin^2\theta + (1+\cos\theta)^2}d\theta = 4\sqrt{2}a\int_0^{\pi}\sqrt{1+\cos\theta}d\theta = 8a\int_0^{\pi}\cos\frac{\theta}{2}d\theta = 16a$$

Demak, berilgan egri chiziq yoyining uzunligi $l = 16a$ bo‘ladi. ◄

3.6.3. Aylanma sirt yuzini hisoblash

$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan, uzluksiz bo‘lib, $\forall_x \epsilon [a, b]$ uchun $f(x) \geq 0$ bo‘lsin. $f(x)$ funksiya grafigining Ox o‘qi atrofida aylantirishdan aylanma sirt hosil bo‘ladi.

Bu sirt yuzasining aniq integral orqali ifodalanishini ko‘rsatamiz. $[a;b]$ oraliqning ixtiyoriy P=$\{x_0, x_1, \ldots, x_n\}$ (a=$x_0 < x_1 < \cdots < x_n = b$) bo‘linishni olaylik. P bo‘linishning har bir $x_k = (k = 0,1, \ldots, n)$ bo‘luvchi nuqtalari orqali Oy o‘qiga parallel to‘g‘ri chiziqlar o‘tkazib, ularni $\widetilde{AB}$ yoy bilan kesishgan nuqtalarini $A_k(x_k, f(x_k))$ bilan belgilaylik. Bu $A_k(x_k, f(x_k))$ $(k = 0,1, \ldots, n)$, $A_0 = A, A_n = B$ nuqtalarni o‘zaro to‘g‘ri chiziq kesmalari bilan birlashtirib, $\widetilde{AB}$ yoyga L siniq chiziq chizamiz. $\widetilde{AB}$ yoyni va L chiziqni Ox o‘qi atrofida aylantiamiz. Natijada L ning aylanishidan kesik konus sirtlaridan tashkil topgan sirt hosil bo‘ladi (3.17-rasm).



3.17-rasm. $y = f(x)$ funksiyani Ox o‘qi atrofida aylantirishdan hosil bo‘lgan sirt

Bu sirtning yuzi ushbu

$$Q = 2\pi \sum_{k=0}^{n-1} \frac{f(x_k)+f(x_{k+1})}{2} \sqrt{(x_{k+1}-x_k)^2 + [(x_{k+1}) - f(x_k)]^2} \quad (3.36)$$

formula bilan ifodalanadi.

P bo‘linishning diamemtri $\lambda_p \to 0$ da $\breve{AB}$ yoyiga chizilgan L siniq chiziq perimetri $\breve{AB}$ yoy uzunligiga intiladi. Demak, $\lambda_p \to 0$ da L siniq chiziqni Ox o‘qi atrofida aylantirishdan hosil bo‘lgan sirtning yuzasi Q ning limiti biz qarayotgan aylanma sirtning yuzasini aniqlaydi. Bu yuzaning aniq integral orqali ifodasini topamiz.

Buning uchun $f(x)$ funksiya $[a;b]$ da uzluksiz $f'(x)$ hosilaga ega deb olamiz. $f(x)$ funksiya $[a;b]$ oraliqda uzluksiz bo‘lganligi uchun $[(x_{k+1}, x_k]$ oraliqda shunday ξ_k nuqta topiladiki,

$$\frac{f(x_k)+f(x_{k+1})}{2} = f(\xi_k), \xi_n \epsilon [(x_k), x_{k+1}]$$

tenglik o‘rinli bo‘ladi. Ikkinchi tomondan, Lagranj teoremasiga ko‘ra, $[(x_k), x_{k+1}]$ oraliqda shunday τ_k nuqta topiladiki

$$f(x_{k+1}) - f(x_k) = f'(\tau_k)(x_{k+1} - x_k), \qquad \tau_n \epsilon [x_k, x_{k+1}]$$

tenglik ham o‘rinli bo‘ladi. Natijada (3.36) munosabat ushbu

$$Q = 2\pi \sum_{k=0}^{n-1} f(\xi_h) \sqrt{(x_{k+1}-x_k))^2 + f'^2(\tau_{k)}(x_{k+1}-x_k)^2} =$$

$$=2\pi \sum_{k=0}^{n-1} f(\xi_h) \sqrt{1 + f'^2(\tau_k)} \Delta x_k$$

ko‘rinishni oladi. Bu tenglamaning o‘ng tomonidagi

$$\sum_{k=0}^{n-1} f(\xi_k) \sqrt{1 + f'^2(\tau_k)}\, \Delta x \quad (3.37)$$

yig‘indi $\quad f(x)\sqrt{1+f'^2(x)} \quad$ (3.38)

funksiyaning integral yig‘indisini eslatadi. (3.38) funksiya integrallanuvchi bo‘lganligi sababli ξ_k nuqta sifatida τ_k ni olish mumkun. $\lambda_p \to 0$ da (3.37) tenglikdan topamiz:

$$\lim_{\lambda_p \to 0} Q = \lim_{\lambda_p \to 0} 2\pi \sum_{k=0}^{n-1} f(\tau_k) \sqrt{1 + f'^2(\tau_k)} \Delta x_k =$$

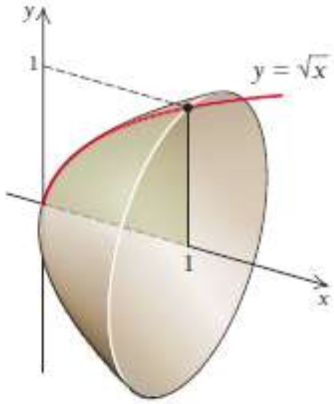
$$= 2\pi \int_a^b f(x) \sqrt{1 + f'^2(x)} dx$$

Shunday qilib, Ox o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma sirtning yuzi uchun ushbu formula o‘rinli:

$$S = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx. \qquad (3.39)$$

3.6.12-misol. $y=\sqrt{x}$ funksiya grafigini $[0;\ 1]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan sirt yuzini toping.

Yechilishi: ► Bu aylanma sirt 3.18-rasmda keltirilgan. (3.39) formuladan foydalanamiz.



3.18-rasm. $y=\sqrt{x}$ funksiya grafigi

$$S = 2\pi\int_0^1 \sqrt{x}\cdot\sqrt{1+\frac{1}{4x}}dx = 2\pi\int_0^1 \cdot\sqrt{x+\frac{1}{4}}dx = \frac{4\pi}{3}\cdot\sqrt{\left(x+\frac{1}{4}\right)^3}\Bigg|_0^1 = \frac{125\pi}{48}. \blacktriangleleft$$

3.6.4. Aniq integral yordamida hajmlarni hisoblash

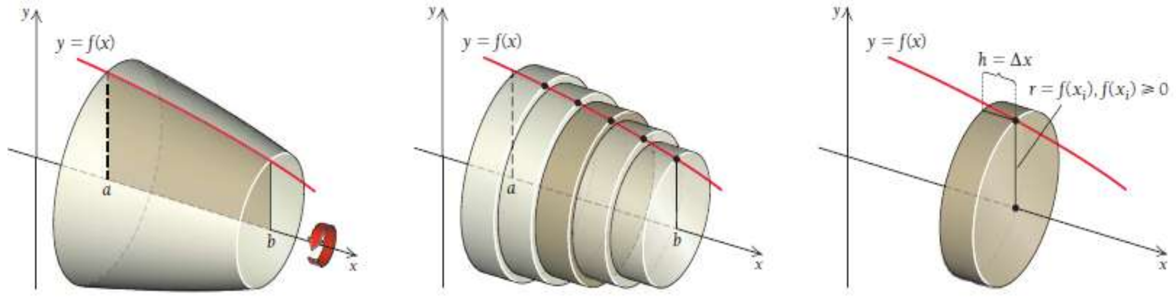
a) Agar jism Ox o‘qining x nuqtasiga o‘tkazilgan perpendikulyar tekisliklar bilan kesishishdan hosil bo‘lgan kesim yuzi *S(x)* berilgan bo‘lsa, jism hajmi $V=\int_a^b S(x)dx$ formula bilan topiladi.

Bu yerda a va b lar x ning o‘zgarish chegaralari, *S(x)* funksiya $[a;b]$ kesmada aniqlangan va uzluksiz deb qaraladi.

b) $y=f(x)$ funksiya grafigini Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini topaylik (3.19-rasm, a).

Agar Ox o‘qidan tepadagi yarim tekislikni Ox o‘qi atrofida aylantirsak, u holda funksiya grafigining har bir nuqtasi aylanma harakat qilib, aylana chizadi va grafikning barcha nuqtasi aylanishidan **aylanma sirt** hosil bo‘ladi.

$y=f(x)$ funksiya grafigi, Ox o‘qi, $x=a,\ x=b$ to‘g‘ri chiziqlar bilan chegaralangan yarim tekislik esa Ox o‘qi atrofida aylanishidan **aylanma jism** hosil qiladi. Bu jismning hajmini topish uchun uni (3.19-rasm, b) qalinligi judayam kichik bo‘lgan chekli sondagi to‘g‘ri silindrlarga yoki disklarga bo‘lib chiqamiz. Bunda $[a;\ b]$ oraliqni har birining uzunligi Δx bo‘lgan qism oraliqlarga ajratamiz. Shunda har bir silindrning balandligi $h=\Delta x$ ga teng bo‘ladi (3.19-rasm, c). Silindr radiusini esa bo‘lakning o‘ng tomonidagi x_i nuqtasiga mos funksiya qiymatiga teng deb olamiz: $r=f(x_i)$. Agar $f(x_i)$ manfiy bo‘lsa, uning modulini $|f(x_i)|$ olamiz.



a) b) c)

3.19-rasm. Aylanma jism

Bizga ma’lumki, to‘g‘ri silindrning hajmi $V=\pi r^2 h$ ga teng. U holda har bir diskning hajmi $V=\pi|f(x_i)|^2\Delta x$ bo‘ladi. Bundan aylanma jismning taxminiy hajmi barcha disklarning hajmlari yig‘indisiga teng bo‘lishi kelib chiqadi: $V\approx\sum_{i=1}^{n}\pi|f(x_i)|^2\Delta x$.

Jismning aniq hajmi esa disklar qalinligi nolga intilgandagi, ya’ni disklar soni cheksizlikka intilgandagi limitga teng bo‘ladi:

$$V\approx\lim_{n\to\infty}\sum_{i=1}^{n}\pi|f(x_i)|^2\Delta x=\int_a^b\pi[f(x)]^2dx$$

Demak, $y=f(x)$, Ox o‘q va to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning Ox o‘qi atrofida aylanishidan hosil bo‘lgan jism hajmi quyidagi formuladan topiladi:

$$V=\int_a^b\pi[f(x)]^2dx \qquad (3.40)$$

3.6.13-misol. $y=\sqrt{x}$ funksiya grafigini $[0;\ 1]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini toping.

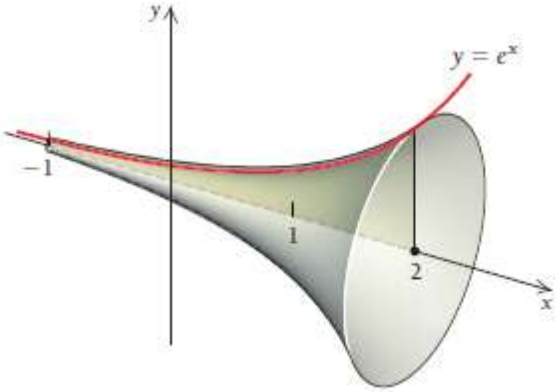
Yechilishi: ► Bu aylanma jism 3.18-rasmda keltirilgan. (3.40) formuladan foydalanamiz.

$$V=\int_0^1\pi\left[\sqrt{x}\right]^2dx=\pi\int_0^1 xdx=\pi\frac{x^2}{2}\bigg|_0^1=\pi\frac{1^2-0^2}{2}=\frac{\pi}{2}.\ ◄$$

3.6.14-misol. $y = e^x$ funksiya grafigini $[-1;\ 2]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini toping (3.20-rasm).

Yechilishi: ► (3.18) formuladan foydalanamiz.

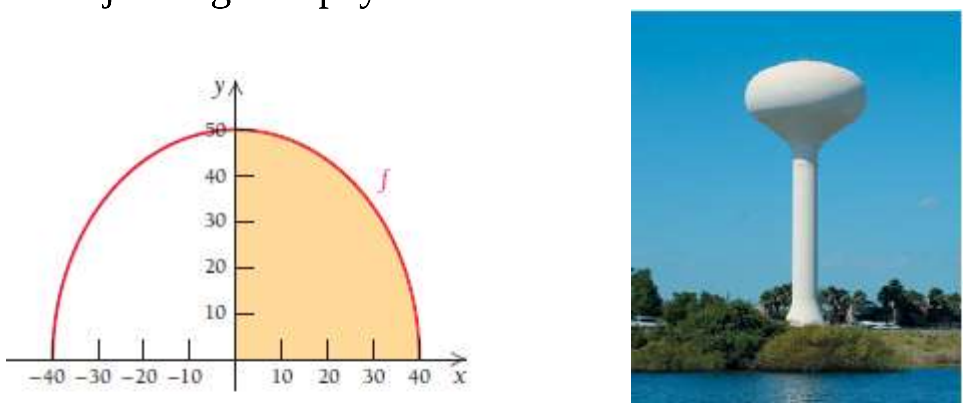
$$V = \int_0^1 \pi\left[e^x\right]^2 dx = \pi \int_0^1 e^{2x} dx = \pi \frac{e^{2x}}{2}\Bigg|_{-1}^{2} = \pi \frac{e^4 - e^{-2}}{2} \approx 85.55$$



3.20-rasm. $y = e^x$ funksiya grafigini Ox o‘qi atrofida aylantirish ◄

3.6.15-misol. Shaharda suv saqlaydigan rezervuar $[-40; 40]$ masofada $f(x) = 50\sqrt{1 - \frac{x^2}{40^2}}$ funksiya grafigini aylantirishdan hosil qilingan aylanma jism shaklida yasalgan. Uning hajmini toping (3.21-rasm).

Yechilishi: ► Rezervuarning bunday shakliga qisilgan sferoid deyiladi. Chunki uning vertikal diametri (80ft=24.384m) bo‘lib, gorizontal diametri(100 ft=30.48m)dan kichik. 2.19-rasmdan ko‘rish mumkinki, u $[-40; 40]$ oraliqda bo‘lishi kerak. Hisoblashni soddalashtirish maqsadida oraliqning yarmi uchun integralni hisoblab, keyin natijani 2 ga ko‘paytiramiz.



3.21-rasm. Rezervuar

$$V_1 = \pi \int_0^{40} \left[50\sqrt{1 - \frac{x^2}{40^2}} \right]^2 dx = 2500\pi \int_0^{40} \left(1 - \frac{x^2}{1600}\right) dx = 2500\pi \left(x - \frac{x^3}{4800} \right)\Bigg|_0^{40} =$$

$$=\frac{200\,000\pi}{3}$$

Shunday qilib, rezervuar hajmi $V=2V_1=\frac{400\,000\pi}{3}$ ga teng. Ma'lumki, 1ft^3 = 0.3048 m^3 hajmda 7.48 gallon=28.3118 l suv bo'ladi, ushbu rezervuar sig'imi $3.13\cdot 10^6$ gallon=11 847050 litr ga teng ekan. ◄

v) $x=\varphi(y)$ egri chiziq, Oy o'qi va $y=c, y=d$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning Oy o'q atrofida aylanishidan hosil bo'lgan jism hajmi

$$V=\pi\int_c^d (\varphi(y))^2 dy=\pi\int_c^d x^2 dy \qquad (3.41)$$

formula bilan hisoblanadi.

g) Agar $y=f(x)$, $(a\le x\le b)$ egri chiziq parametrik usulda, ya'ni $\left.\begin{matrix} x=x(t) \\ y=y(t)\end{matrix}\right\}, \alpha\le t\le\beta$ bo'lsa, egri chiziqli trapetsiyaning Ox o'q atrofida aylanishidan hosil bo'lgan jismning hajmi

$$V=\pi\int_a^b (y(t))^2 x^1(t)dt \qquad (3.42)$$

formula bilan topiladi.

d) Qutb koordinatalar sistemasida $\rho=f(x)$ tenglama bilan berilgan egri chiziq va $\varphi=\alpha, \varphi=\beta$ radius vektorlar bilan chegaralangan rasmning qutb o'qi atrofida aylanishidan hosil bo'lgan jism hajmi

$$V=\pi\int_\alpha^\beta \rho^3\sin^3\varphi d\varphi \qquad (3.43)$$

formula bilan hisoblanadi.

Agar qutb koordinatasida berilgan bo'lib, $0\le\alpha\le\varphi\le\beta\le\pi$ bo'lsa, jism hajmi $V=\frac{2\pi}{3}\int_\alpha^\beta \rho^3\sin^3\varphi d\varphi$ formula bilan hisoblanadi.

3.6.16-misol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsni Ox va Oy o'qlari atrofida aylantirish natijasida hosil qilingan jismlarning hajmlarini hisoblang.

Yechilishi: ►Ellips tenglamasidan

$$y^2=\frac{b^2}{a^2}(a^2-x^2); \qquad x^2=\frac{a^2}{b^2}(b^2-y^2)$$

funksiyalarni topib olamiz. Ellipsni Ox o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmi:

$$V=2V_1=2\pi\int_0^a y^2dx=2\pi\frac{b^2}{a^2}\int_0^a(a^2-x^2)dx=2\pi\frac{b^2}{a^2}(a^2x-\frac{x^3}{3})\Big|_0^a=$$

$$=2\pi\frac{b^2}{a^2}(a^3-\frac{a^3}{3})=\frac{4}{3}\pi ab^2 V=\frac{4}{3}\pi ab^2$$

Ellipsni Oy o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi:

$$V=2V_1=2\pi\int_0^b x^2dy=2\pi\frac{a^2}{b^2}\int_0^b(b^2-y^2)dy=2\pi\frac{a^2}{b^2}(b^2y-\frac{y^3}{3})\Big|_0^b=$$

$$=2\pi\frac{a^2}{b^2}(b^3-\frac{b^3}{3})=\frac{4}{3}\pi a^2b\ \ V=\frac{4}{3}\pi a^2b\ (kub.birl.)$$ ◄

3.6.5. Aniq integralning fizikaviy tatbiqlari

Berilgan kuch ta’sirida bajarilgan ishni hisoblash.

Aytaylik, moddiy nuqta Os to‘g‘ri chiziq bo‘ylab $F(s)$ kuch ta’sirida harakatlanayotgan bo‘lsin. Yo‘lning $[a,b]$ qismida bu kuch ta’sirida bajarilgan ish quyidagi formula bilan hisoblanadi:

$$A=\int_a^b F(s)ds \tag{3.44}$$

3.6.17-misol. Agar prujinani 1 smga cho‘zish uchun 1kN kuch sarflansa, shu prujinani 10 sm ga cho‘zish uchun qancha ish bajarish kerak?

Yechilishi: ► Guk qonuniga ko‘ra, prujinani cho‘zish uchun kerak bo‘ladigan F kuch prujinani cho‘zilishiga to‘g‘ri proportsional, ya’ni $F=kx$, bu yerda $x-$prujinaning cho‘zilishi (metrda), $k-$proportsionallik koeffitsiyenti.

Misol shartidan ma’lumki, prujina $F=1$ kN kuch bilan $x=0{,}01$m cho‘ziladi. Formuladan $1=0{,}01k \quad\rightarrow\quad k=100$ va $F=100x$ ni topamiz. Shunda bajarilgan ish (3.44) formuladan topiladi:

$$A=\int_0^{0,1}100xdx=50x^2\Big|_0^{0,1}=0{,}5\ \text{kJ.}$$ ◄

Tekis shaklning og‘irlik markazini topish

1) Zichligi $\delta=\delta(x)$ bo‘lgan $y=f(x)$ funksiya grafigining moddiy AB yoyining og‘irlik markazining koordinatalari $C(x_c,y_c)$ quyidagi formulalar bilan aniqlanadi:

$$x_C = \frac{\int_a^b x\delta(x)\sqrt{1+y'^2}dx}{\int_a^b \delta(x)\sqrt{1+y'^2}dx}, \qquad y_C = \frac{\int_a^b y\delta(x)\sqrt{1+y'^2}dx}{\int_a^b \delta(x)\sqrt{1+y'^2}dx} \qquad (3.45)$$

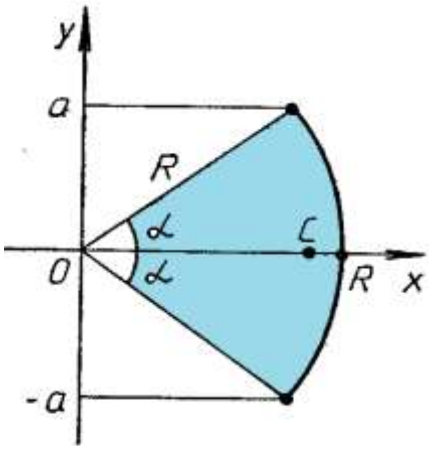
2) Agar rasm pastdan $y = f_1(x)$ va yuqoridan $y = f_2(x)$, shuningdek $[a,b]$ kesmada $f_1(x) \le f_2(x)$ tengsizlik o‘rinli va chiziqli zichlik $\delta = \delta(x)$ bo‘lsa, og‘irlik markazining koordinatalari $C(x_c, y_c)$ quyidagi formulalar bilan aniqlanadi:

$$x_C = \frac{\int_a^b x\delta(x)[f_2(x) - f_1(x)]dx}{\int_a^b \delta(x)[f_2(x) - f_1(x)]dx}, \qquad y_C = \frac{\frac{1}{2}\int_a^b \delta(x)[f_2^2(x) - f_1^2(x)]dx}{\int_a^b \delta(x)[f_2(x) - f_1(x)]dx} \qquad (3.46)$$

3.6.18-misol. Radiusi R, markaziy burchagi 2α bo‘lgan bir jinsli sim yoyning og‘irlik markazini toping, ($\delta = const$).

Yechilishi: ► 3.22-rasmdan ko‘rinadiki, yoy simmetrik va bir jinsli materialdan qilingan, shuning uchun $y_C = 0$ ga teng. (3.45) formulada $\delta = const$ bo‘lganligi uchun x_C quyidagi formuladan topamiz:

$$x_C = \frac{\int_{-a}^{a} x\sqrt{1+y'^2}dx}{\int_{-a}^{a} \sqrt{1+y'^2}dx}$$



3.22-rasm. Radiusi R, markaziy burchagi 2α bo‘lgan bir jinsli sim yoy

Integralni hisoblash uchun qutb koordinatasiga o‘tamiz.

$$\delta = const, \quad x = R\cos t, \quad y = R\sin t.$$

U holda Yakobian: $I = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \cos t & -R\sin t \\ \sin t & R\cos t \end{vmatrix} = R\cos^2 t + R\sin^2 t = R$.

$$x_C = \frac{\int_{-\alpha}^{\alpha} R^2 \cos t dt}{\int_{-\alpha}^{\alpha} R dt} = R\frac{\sin t\big|_{-\alpha}^{\alpha}}{t\big|_{-\alpha}^{\alpha}} = R\frac{\sin\alpha}{\alpha}.$$

Shunday qilib, sim yoyining og‘irlik markazi $x_C = R\frac{\sin\alpha}{\alpha}$, $y_C = 0$. ◀

Mavzu yuzasidan savollar:

1. Yassi shakllar yuzini hisoblash formulasini yozing.
2. Yoy uzunligini hisoblash qanday bajariladi?
3. Aniq integral yordamida hajmlarni hisoblashni bilasizmi?
4. Aylanma jism sirtining yuzasini hisoblash uchun formula yozing.
5. Aniq integralning qanday fizikaviy tatbiqlarini bilasiz?
6. Tekis shaklning og‘irlik markazini topish formulasini yozing.
7. Aniq integral yordamida kuch bajargan ishni topish formulasini yozing.

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Rasmda berilgan aylanma shakl hajmini qanday hisoblash mumkin?

3.23-rasm. Arena

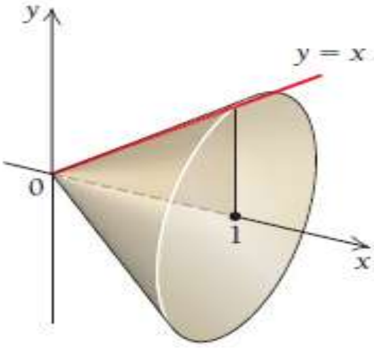
2. Yoy uzunligini toping:

a) $\begin{cases} x = \cos\left(\frac{t}{2}\right) \\ y = t - \sin t \end{cases}$ b) $\begin{cases} x = t^3 + 8 \\ y = t^5 + 2t \end{cases}$ c) $\begin{cases} x = a\cos^2 t \\ y = b\sin^2 t \end{cases}$

3. Marsni o‘rganish uchun kosmik kemada robot olib borildi. Unga $v(t) = -0.42t^2 + 2t$ km/ soat tezlik berildi. Birinchi 3 soatda robot qancha masofani bosib o‘tadi (3.24-rasm)?

3.24-rasm. Mars sayyorasidagi robot

4. $y = x$ funksiya grafigini $[0;\ 1]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini toping (3.25-rasm).



3.25-rasm. Aylanma jism

5. Atom elektr stansiyalarida sovitish mo‘rilari mavjud. Ularning tuzilishi $f(x) = 50\sqrt{1 + \frac{x^2}{22500}}$ giperbolani $-250 \le x \le 250$ masofada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism bo‘lib, 3.26-rasmda keltirilgan. Shu jism hajmini toping.

3.26-rasm. Atom elektr stansiyalaridagi sovitish mo‘rilari

TESTLAR

1. $y^2 = x+1$ va $y = x-1$ chiziqlar bilan chegaralangan shakl yuzasini hisoblang.

A) 9/2 **B)** 2/3 **C)** 9 **D)** 7/2

2. $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$ egri chiziqning $y_1 = 1$ dan $y_2 = e$ gacha yoyi uzunligini toping.

A) $\frac{e^2+1}{3}$ **B)** $\frac{e^3+1}{4}$ **C)** $\frac{e^2-1}{4}$ **D)** $\frac{e^2+1}{4}$

3. Koordinata o‘qlari va $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ parabola bilan chegaralangan shaklni Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini hisoblang.

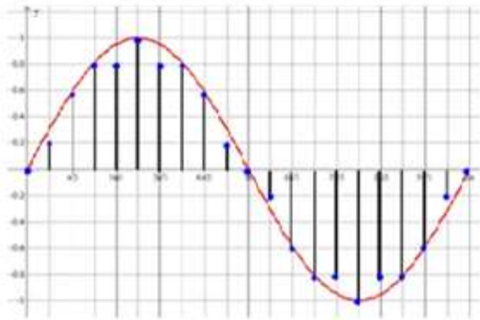
A) $\frac{\pi a}{8}$ **B)** $\frac{\pi a^3}{15}$ **C)** $\frac{2\pi a^3}{15}$ **D)** $\frac{\pi a}{7}$

4. Aniq integral yordamida hajm hisoblash formulasi berilgan variantni aniqlang.

A) $\pi\int_a^b f^2(x)dx$ **B)** $\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$ **C)** $\int_\alpha^\beta y(t)x'(t)dt$ **D)** $\int_\alpha^\beta y(t)x'(t)dt$

5. Quyidagilardan qaysi biri yuzani topish formulasi emas?

A) $\pi\int_a^b f^2(x)dx$ **B)** $\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$ **C)** $\int_\alpha^\beta y(t)x'(t)dt$ **D)** $\int_\alpha^\beta x(t)y'(t)dt$



IV BOB. SONLI VA FUNKSIONAL QATORLAR

4.1. Sonli qatorlar

4.1.1. Sonli qatorlar haqida tushunchalar

Elementlari cheksiz haqiqiy sonlardan iborat bo‘lgan ushbu $a_1, a_2, a_3, ..., a_n, ...$ cheksiz ketma-ketlikni qaraymiz.

Ushbu $a_1 + a_2 + a_3 + ... + a_n + ...$ ifodaga **cheksiz qator** yoki qisqacha **qator** deyiladi va quyidagicha belgilanadi:

$$a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n. \qquad (4.1)$$

$a_1, a_2, a_3, ..., a_n, ...$ qatorning **hadlari** deyiladi. a_n ga qatorning **umumiy hadi** yoki $n-$**hadi** deyiladi. Qatorning umumiy hadi berilgan bo‘lsa, n ning o‘rniga 1 dan boshlab ketma-ket natural sonlarni qo‘yib, uning yordamida qatorning ixtiyoriy hadini hosil qilish mumkin.

Misol uchun, agar $n-$hadi $a_n = \frac{1}{3^n}$ ko‘rinishidagi qator berilgan bo‘lsa, uning umumiy ko‘rinishi quyidagicha bo‘ladi:

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ... + \frac{1}{3^n} + ... = \sum_{n=1}^{\infty} \frac{1}{3^n}.$$

Qatorning hadlari yig‘indilariga **qismiy yig‘indilar** deyiladi:

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &... \\ S_n &= a_1 + a_2 + a_3 + ... + a_n, \ ... \end{aligned} \qquad (4.2)$$

Agar $\sum_{n=1}^{\infty} a_n$ qatorning qismiy yig‘indilari ketma-ketligi $S_1, S_2, S_3, ..., S_n, ...$ chekli $S = \lim_{n\to\infty} S_n$ limitga ega bo‘lsa, u holda $\sum_{n=1}^{\infty} a_n$ qator

yaqinlashuvchi qator deyiladi, chekli limit esa uning **yig‘indisi** deyiladi: $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}a_i=\sum\limits_{i=1}^{\infty}a_i$

Agar $\sum\limits_{n=1}^{\infty}a_n$ qatorning qismiy yig‘indilari ketma-ketligi $S_1,S_2,S_3,...,S_n,...$ cheksiz limitga ega bo‘lsa yoki limiti mavjud bo‘lmasa, u holda $\sum\limits_{n=1}^{\infty}a_n$ qator **uzoqlashuvchi qator** deyiladi.

Eslatma. Sonli qatorlar nazariyasining **asosiy vazifasi** qatorning yaqinlashuvchi yoki uzoqlashuvchi ekanligini aniqlash va qatorning yig‘indisini hisoblashdan iborat.

4.1.1-misol. $\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}$ qatorning qismiy yig‘indilarini toping.

Yechilishi: ► $\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...+\frac{1}{2n-1}+$

$S_1=a_1=1;$

$S_2=a_1+a_2=1+\frac{1}{3}=\frac{4}{3};$

$S_3=a_1+a_2+a_3=1+\frac{1}{3}+\frac{1}{5}=\frac{4}{3}+\frac{1}{5}=\frac{23}{15};$

...

$S_n=a_1+a_2+a_3+...+a_n=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...+\frac{1}{2n-1}.$ ◄

4.1.2-misol. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...+\frac{1}{3^n}+...=\sum\limits_{n=1}^{\infty}\frac{1}{3^n}$ qatorning qismiy yig‘indilarini toping.

Yechilishi: ► $S_1=a_1=\frac{1}{3};$

$S_2=a_1+a_2=\frac{1}{3}+\frac{1}{9}=\frac{4}{9};$

$S_3=a_1+a_2+a_3=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}=\frac{13}{27};$

...

$S_n=a_1+a_2+a_3+...+a_n=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...+\frac{1}{3^n}.$ ◄

4.1.3-misol. $\sum\limits_{n=1}^{\infty}n$ qator yaqinlashuvchimi yoki uzoqlashuvchimi? Agar yaqinlashuvchi bo‘lsa, uning yig‘indisini toping.

Yechilishi: ► Qatorning yaqinlashuvchanligini aniqlash uchun oldin berilgan qatorni n ta hadi yig'indisini yozib olamiz:

$$S_n=\sum_{i=1}^{n} i = 1+2+3+4+5+...+n=\frac{n(n+1)}{2}$$

$$S=\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{n(n+1)}{2}=\infty .$$

Qismiy yig'indilar ketma-ketligi uzoqlashuvchi ekan, demak, qator ham uzoqlashuvchi. ◄

4.1.4-misol. $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ qator yaqinlashuvchimi yoki uzoqlashuvchimi? Agar yaqinlashuvchi bo'lsa, uning yig'indisini toping.

Yechilishi: ► Qatorning qismiy yig'indilar ketma –ketligini yozib olamiz: $S_n=\sum_{i=2}^{n}\frac{1}{i^2-1}$.

i ning o'rniga 2 dan n gacha natural qiymatlarni qo'ysal, u holda quyidagi qator hosil bo'ladi:

$$S_n=\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+...+\frac{1}{n^2-1}=\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+...+\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right).$$

Hosil bo'lgan ifodada kasrlarni ajratib, ayirma rasmida yozish mumkin:

$$S_n=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+...+\frac{1}{n-1}-\frac{1}{n+1}\right)=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right).$$

Endi limitga o'tib, qator yig'indisini hisoblaymiz:

$$S=\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{3}{4}.$$

Demak, qator yaqinlashuvchi va uning yig'indisi 0,75 ga teng ekan. ◄

4.1.5-misol. $\sum_{n=0}^{\infty}(-1)^n$ qator yaqinlashuvchimi yoki uzoqlashuvchimi?

Yechilishi: ► Qatorning qismiy yig'indilar ketma –ketligini yozib olamiz: $S_0=1$;

$S_1=1-1=0$;

$S_2=1-1+1=1$;

$S_3=1-1+1-1=0$ ...

Ko'rinib turibdiki, qatorning limiti mavjud emas:

$$\lim_{n\to\infty} S_n=\begin{cases}0, & \text{agar } n-\text{toq bo'lsa;}\\ 1, & \text{agar } n-\text{juft bo'lsa.}\end{cases}$$

Bunday qatorlar uzoqlashuvchi bo'ladi. ◄

Qatorlarni yozishda indeks qanday harfiy kattalik bilan yozilishi muhim emas, qator shu indeksga mos holda berilsa yetarli. Misol uchun,

$$\sum_{n=1}^{\infty}\frac{3}{n^2+1}=\sum_{i=1}^{\infty}\frac{3}{i^2+1}=\sum_{k=1}^{\infty}\frac{3}{k^2+1}.$$

Geometrik qator.

Sonli qatorga misol qilib amaliyotda ko‘p ishlatiladigan geometrik progressiyani ko‘rsatish mumkin:

$$a+aq+aq^2+...+aq^{n-1}+... \qquad (4.3)$$

Bunda $a-$geometrik progressiya(geometrik qator)ning **birinchi hadi**, $a\cdot q^{n-1}$ - n-hadi, $q-$ **qatorning mahraji** deyiladi.

Birinchi n ta hadining yig‘indisi $S_n=a+aq+aq^2+...+aq^{n-1}=\frac{a(1-q^n)}{1-q}$ ga teng, $(|q|\neq 1)$.

1. Geometrik qatorda $|q|<1$ bo‘lsa $n\to\infty$ da $q^n\to 0$ bo‘lib, uning limiti $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(\frac{a}{1-q}-\frac{aq^n}{1-q}\right)=\frac{a}{1-q}$ bo‘ladi. Demak (4.3) qator yaqinlashuvchi bo‘lib yig‘indisi $S=\frac{a}{1-q}$ bo‘ladi.

2. $|q|>1$ bo‘lsa $n\to\infty$ da $q^n\to\infty$ bo‘lib, (4.3) qator uzoqlashuvchi bo‘ladi.

3. $q=1$ bo‘lsa, (4.3) qator $a+a+a+...+a+...$ ko‘rinishda bo‘lib, uning yig‘indisi $S_n=a+a+a+...+a=na$ bo‘ladi. Limitini hisoblasak, $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}(an)=a\lim\limits_{n\to\infty}n=\infty$ $(a\neq 0)$ ga teng. Demak, qator uzoqlashuvchi.

4. $q=-1, a\neq 0$ bo‘lsa, (4.3) qator $a-a+a-a+...$ ko‘rinishda bo‘lib, n juft son bo‘lganda, qator yig‘indisi nolga teng: $S_n=0$ va n toq son bo‘lganda esa $S_n=a$ ga teng bo‘ladi. Demak, $\lim\limits_{n\to\infty}S_n$ limit mavjud emas va shu sababli qator uzoqlashuvchi deb xulosa qilamiz.

Shunday qilib, geometrik qator, ya’ni (4.3) qator faqat $|q|<1$ bo‘lganda yaqinlashuvchi bo‘lib, $|q|\geq 1$ bo‘lganda uzoqlashuvchi bo‘lar ekan.

4.1.6-misol. $\sum\limits_{n=1}^{\infty}\frac{1}{3^{n-1}}$ qator yaqinlashuvchimi yoki uzoqlashuvchimi? Agar yaqinlashuvchi bo‘lsa, uning yig‘indisini toping.

Yechilishi: ► Ushbu qator geometrik qator bo‘lib, mahraji $q=\frac{1}{3}$, $a=1$. Geometrik progressiya yig‘indisini topish formulasidan foydalanamiz: $S_n = \frac{a(1-q^n)}{1-q}$;

$$S_n = \sum_{i=1}^{n} \frac{1}{3^{i-1}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right);$$

Limitga o‘tamiz: $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}\left(1 - \frac{1}{3^n}\right) = \frac{3}{2}$.

Qator yaqinlashuvchi va uning yig‘indisi 1,5 ga teng. ◄

4.1.7-misol. $\sum_{n=1}^{\infty} 9^{-n+2} \cdot 4^{n+1}$ qator yaqinlashuvchi bo‘lsa, uning yig‘indisini toping.

Yechilishi: ► $\sum_{n=1}^{\infty} 9^{-n+2} \cdot 4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n-1} \cdot 4^2}{9^{n-1} \cdot 9^{-1}} = \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$.

Ko‘rinib turibdiki, bu cheksiz kamayuvchi geometrik qator bo‘lib, uning mahraji $q = \frac{4}{9} < 1$ ga teng, qator yig‘indisi

$$S = \frac{b_1}{1-q} = \frac{144}{1-\frac{4}{9}} = 144 \cdot \frac{9}{5} = \frac{1296}{5}.$$ ◄

4.1.8-misol. $\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$ qator yaqinlashuvchimi, uzoqlashuvchimi?

Yechilishi: ► $\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=1}^{\infty} 5 \cdot \frac{(-64)^n}{5^n} = \sum_{n=1}^{\infty} 5 \cdot \left(-\frac{64}{5}\right)^n$. Bu o‘suvchi geometrik qator bo‘lib, uning maxraji $|q| = \left|-\frac{64}{5}\right| \geq 1$ ga teng, qator yig‘indisi $S = \infty$. ◄

Teleskopik qatorlar (Telescoping Series[4]) – cheksiz qatorlar bo‘lib, ularda qavslarni ochish bilan deyarli barcha qo‘shiluvchilarning yig‘indisi nolga teng bo‘lib qoladi va qatorning yig‘indisi oson topiladi.

Qatorlarga bunday nom berilishiga sabab, teleskopga o‘xshab o‘z uzunligini bir necha marotaba qisqatirishidir.

4.1.9-misol. $\sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2}$ qator yig‘indisini toping.

Yechilishi: ► Qator yig‘indisini topish uchun dastlab uning n ta hadi yig‘indisini aniqlaymiz:

M. L. Bittinger, D. J. Ellenbogen, S. A. Surgent "Calculus and its Applications", USA, Springer, 10-th edition, 2012. -729 p.

$$S_n=\sum_{i=0}^{n}\frac{1}{i^2+3i+2}=\sum_{i=0}^{n}\frac{1}{(i+1)(i+2)}=\sum_{i=0}^{n}\left(\frac{1}{i+1}-\frac{1}{i+2}\right)=$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=1-\frac{1}{n+2}.$$

Limitga o'tamiz: $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\frac{1}{n+2}\right)=1$, demak, berilgan qatorning yig'indisi 1 ga teng: $\sum\limits_{n=0}^{\infty}\frac{1}{n^2+3n+2}=1.$ ◀

4.1.10-misol. $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+4n+3}$ qator yig'indisini toping.

Yechilishi: ▶

$$S_n=\sum_{i=1}^{n}\frac{1}{i^2+4i+3}=\frac{1}{2}\sum_{i=1}^{n}\frac{1}{(i+1)(i+3)}=\frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{i+1}-\frac{1}{i+3}\right)=$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\right]=$$

$$=\frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right].$$

Limitga o'tamiz: $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\frac{1}{2}\left(\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}\right)=\frac{5}{12}.$

Demak, berilgan qatorning yig'indisi $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+4n+3}=\frac{5}{12}$ ga teng. ◀

4.1.11-misol. $\sum\limits_{n=1}^{\infty}\frac{3+2n}{n^2+3n+2}$ qator yig'indisini toping.

Yechilishi: ▶ Qator teleskopik bo'lishi uchun uning hadlarini ishorasi almashinuvchi bo'lishi kerak, bu yerda esa qator hadlarining barchasi musbat. Shu sababli qator uzoqlashuvchi:

$$S_n=\sum_{i=1}^{n}\left(\frac{1}{i+1}+\frac{1}{i+2}\right)=\infty.$$ ◀

Yaqinlashuvchi qatorning xossalari

1⁰. Agar $\sum\limits_{n=1}^{\infty}a_n$ yaqinlashuvchi qator bo'lsa, u holda har qanday $c=const$ son uchun $\sum\limits_{n=1}^{\infty}c\cdot a_n=c\cdot\sum\limits_{n=1}^{\infty}a_n$ tenglik o'rinli.

2⁰. Agar $\sum\limits_{n=1}^{\infty}a_n$ va $\sum\limits_{n=1}^{\infty}b_n$ yaqinlashuvchi qatorlar bo'lsa, u holda $\sum\limits_{n=1}^{\infty}a_n\pm\sum\limits_{n=1}^{\infty}b_n=\sum\limits_{n=1}^{\infty}(a_n\pm b_n)$ tenglik o'rinli.

Qatorlarni ko‘paytirish amali birmuncha murakkab jarayon, ya’ni qator hadlarini mos ravishda ko‘paytirganda hosil bo‘ladigan qatorni analitik shaklda yozish mumkinmi? - degan savolga javob izlaymiz. Ravshanki, $\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n \neq \sum_{n=1}^{\infty} (a_n \cdot b_n)$.

Buni quyidagi misolda ko‘rish mumkin: aytaylik, ikkita chekli qator berilgan bo‘lsin, ularni ko‘paytiramiz: $(2+x)(3-5x+x^2) = 6-7x-3x^2+x^3$.

Endi cheksiz hadli qatorlarni ko‘paytirganda 1-qatorning har bir hadiga 2-qatorning har bir hadini ko‘paytirib, yig‘indisini hisoblash kerak:
$$\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n = (a_1+a_2+a_3+...)(b_1+b_2+b_3+...)$$
$$\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n \text{, bunda } c_n = \sum_{i=1}^{n} a_i b_{n-i} \text{ bo‘ladi.}$$

Bunday qatorning yaqinlashishi haqida ham ko‘p narsa ayta olmaymiz. Berilgan qatorlar yaqinlashuvchi bo‘lsa ham, ko‘paytma qator yaqinlashuvchi ham, uzoqlashuvchi ham bo‘lishi mumkin. Bo‘limning oxirida qatorlarni ko‘paytirish masalasiga yana qaytamiz.

Bu yerda muhokama qilishimiz kerak bo‘lgan yana bir narsa – bu **indekslar siljishi.** Indeksni o‘zgartirishning asosiy g‘oyasi, har qanday sababga ko‘ra (buning qonuniy sabablari bor) boshqa qiymatdagi qatorni boshlashdir.

4.1.12-misol. Quyidagi qator berilgan bo‘lsin: $\sum_{n=2}^{\infty} \frac{n+5}{2^n}$.

► Ushbu qatorda, agar $n=0$ nomerdan boshlasak,
$$\sum_{n=0}^{\infty} \frac{n+5}{2^n} = 5+3+\frac{7}{4}+1+\frac{9}{16}+...+\frac{n+5}{2^n}+...$$
agar $n=2$ deb boshlasak, $\sum_{n=2}^{\infty} \frac{n+5}{2^n} = \frac{7}{4}+1+\frac{9}{16}+...+\frac{n+5}{2^n}+...$

Demak, qatorda indeks $n=0$ dan boshlansa qatorning qiymati $n=2$ dan boshlanganiga qaraganda 8 birlik katta chiqar ekan. Qatorning qiymatlari har xil chiqar ekan. Shuning uchun indekslarni o‘zgartirganda ehtiyot bo‘lish kerak.

Indeksni o‘zgartirish juda oddiy jarayon. Aytaylik, yangi indeks $i=n-2$ ni hosil qilmoqchimiz. Shunda $n=2$ ni o‘rniga $i=0$ deb yozmoqchimiz. U holda e’tibor bering, $n=\infty$ bo‘lganda $i=\infty-2=\infty$ bo‘ladi. Shuning uchun bu yerda faqat pastki chegara o‘zgaradi: $n=i+2$
$$\sum_{n=2}^{\infty} \frac{n+5}{2^n} = \sum_{i=0}^{\infty} \frac{(i+2)+5}{2^{i+2}} = \sum_{i=0}^{\infty} \frac{i+7}{2^{i+2}}.$$

Qatorlarni yozishda indeks qanday harfiy kattalik bilan yozilishining ahamiyati yo‘qligini aytgan edik, shunda

$$\sum_{n=2}^{\infty}\frac{n+5}{2^n}=\sum_{n=0}^{\infty}\frac{n+7}{2^{n+2}}$$

qatorlar bir xil ekanligi kelib chiqadi. Tekshirib ko‘rishingiz mumkin:

$$\sum_{n=2}^{\infty}\frac{n+5}{2^n}=\frac{7}{2^2}+\frac{8}{2^3}+\frac{9}{2^4}+...$$

$$\sum_{n=0}^{\infty}\frac{n+7}{2^{n+2}}=\frac{7}{2^2}+\frac{8}{2^3}+\frac{9}{2^4}+...$$

Shubhasiz,endi ikkala qatorda ham bir xil hadlar qatnashgan. ◀

4.1.13-misol. $\sum_{n=1}^{\infty}\frac{n^2}{1-3^{n+1}}$ qatorning indeksini $n=3$ deb o‘zgartiring.

Yechilishi: ▶ Ketma-ketlikning dastlabki qiymatini 2 ga oshirmoqchimiz, shuning uchun uning $n-$hadini 2 ga kamaytirish kerak:

$$\sum_{n=1}^{\infty}\frac{n^2}{1-3^{n+1}}=\sum_{n=3}^{\infty}\frac{(n-2)^2}{1-3^{(n-2)+1}}=\sum_{n=3}^{\infty}\frac{(n-2)^2}{1-3^{n-1}} \quad ◀$$

Qatorlarni yozishda muqobil usullarni ham qarab chiqaylik:

$\sum_{n=1}^{\infty}a_n=a_1+a_2+a_3+...$ qatorni turlicha ko‘rinishdagi indekslar yordamida yozish mumkin:

$$\sum_{n=1}^{\infty}a_n=a_1+\sum_{n=2}^{\infty}a_n \quad \text{yoki} \quad \sum_{n=1}^{\infty}a_n=a_1+a_2+\sum_{n=3}^{\infty}a_n$$

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+a_3+a_4+\sum_{n=5}^{\infty}a_n \quad \text{yoki} \quad \sum_{n=1}^{\infty}a_n=\sum_{n=1}^{4}a_n+\sum_{n=5}^{\infty}a_n .$$

Demak, $\sum_{n=1}^{\infty}a_n=\sum_{i=1}^{N}a_i+\sum_{j=N+1}^{\infty}a_j$ tenglik o‘rinli.

4.1.14-misol. a) $\sum_{n=0}^{\infty}9^{-n+2}\cdot4^{n+1}$ va b) $\sum_{n=3}^{\infty}9^{-n+2}\cdot4^{n+1}$ qatorlarning yig‘indilarini toping.

Yechilishi: ▶ a) Ushbu qatorda $n=0$ hadini alohida qo‘shiluvchi sifatida yozib olamiz: $\sum_{n=0}^{\infty}9^{-n+2}\cdot4^{n+1}=9^2\cdot4+\sum_{n=1}^{\infty}9^{-n+2}\cdot4^{n+1}=324+\frac{1296}{5}=\frac{2916}{5}$.

b) $\sum_{n=3}^{\infty}9^{-n+2}\cdot4^{n+1}$ qator yig‘indisini topish uchun $n=1$ va $n=2$ hadlarini alohida qo‘shiluvchi sifatida yozib olamiz:

$$\sum_{n=1}^{\infty}9^{-n+2}\cdot4^{n+1}=9\cdot4^2+9^0\cdot4^3+\sum_{n=3}^{\infty}9^{-n+2}\cdot4^{n+1}=208+\sum_{n=3}^{\infty}9^{-n+2}\cdot4^{n+1};$$

Tenglikdan quyidagini topish mumkin:

$$\sum_{n=3}^{\infty} 9^{-n+2} \cdot 4^{n+1} = \sum_{n=1}^{\infty} 9^{-n+2} \cdot 4^{n+1} - 208 = \frac{1296}{5} - 208 = \frac{256}{5}. \blacktriangleleft$$

4.1.2. Qator yaqinlashishining zaruriy sharti. Garmonik qator

4.1-teorema (yaqinlashuvchanlikning zaruriylik sharti). Agar $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo‘lsa, u holda $\lim_{n\to\infty} a_n = 0$ munosabat o‘rinli bo‘ladi.

■ Ushbu teoremani noto‘g‘ri ishlatishdan ehtiyot bo‘ling! Bu teorema bizga yaqinlashish talabini beradi, lekin yaqinlashish kafolatini emas. Boshqacha qilib aytganda, teoremaning teskarisi o‘rinli emas, ya’ni agar $\lim_{n\to\infty} a_n = 0$ bo‘lsa, bu berilgan qator yaqinlashivchi bo‘ladi degani emas.

Natija. Agar $\lim_{n\to\infty} a_n \neq 0$ bo‘lsa, u holda qator uzoqlashuvchi bo‘ladi.

4.1.15-misol. $\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$ qatorni yaqinlashuvchilikka tekshiring.

Yechilishi: ► $\lim_{n\to\infty} \frac{4n^2 - n^3}{10 + 2n^3} = \lim_{n\to\infty} \frac{\frac{4}{n} - 1}{\frac{10}{n^3} + 2} = -\frac{1}{2}$ qatorning umumiy hadining limiti $\lim_{n\to\infty} a_n \neq 0$ ga teng bo‘ldi, demak, berilgan qator uzoqlashuvchi.◄

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$ qatorga **garmonik qator** deyiladi. (4.4)

4.1.16-misol. $\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qator yaqinlashuvchimi?

Yechilishi: ► Garmonik qator uchun $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ zaruriy shart bajariladi. Lekin bu shart yetarli emas. Qatorni quyidagicha yoyib olamiz:

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots = 1 + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots >$$

$$> 1 + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots = 1 + \frac{1}{2} + \frac{1}{2} + \ldots = \infty$$

Hosil bo‘lgan qator chegaralanmagan, bu esa garmonik qator uzoqlashuvchi ekanini bildiradi. ◄

Demak, qator yaqinlashuvchi bo‘lsa, zaruriy shart bajariladi, ammo zaruriy shart bajarilganda qator uzoqlashuvchi ham, yaqinlashuvchi ham bo‘lishi mumkin ekan.

4.1.17-misol. $1+\frac{1}{2\cdot5}+\frac{1}{3\cdot5^2}+\frac{1}{4\cdot5^3}+...$ qator yaqinlashishining zaruriy shartini tekshiring.

Yechilishi: ▶ Qatorning umumiy hadi $u_n=\frac{1}{n\cdot5^{n-1}}$ ni hosil qilamiz va **zaruriy shartni tekshiramiz:** $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{1}{n\cdot5^{n-1}}=0$.

Zaruriy shart bajariladi, lekin qator yaqinlashuvchi deb xulosa qilishga shoshmaymiz. Buning uchun yana bitta shartni – yetarlilik shartini tekshirib ko‘rish kerak bo‘ladi. ◀

4.1.3. Musbat hadli qatorlarning yaqinlashish alomatlari

Qatorlarni yaqinlashuvchanligini aniqlashning bir nechta usullari mavjud:

1) Qatorlarni taqqoslash alomati;
2) Umumlashgan taqqoslash alomati;
3) Dalamber alomati;
4) Koshi alomati;
5) Integral alomati.

Bu alomatlarni dastlab ishorasi o‘zgarmas (hadlari faqat musbat yoki faqat manfiy ishorali bo‘lgan) qatorlarda tushunib olaylik, aniqlik uchun musbat hadli qatorlarni qaraymiz.

Hamma hadlari musbat ishorali bo‘lgan qatorga **musbat hadli qator** deyiladi.

Taqqoslash alomati: $\sum\limits_{n=1}^{\infty}a_n$ va $\sum\limits_{n=1}^{\infty}b_n$ musbat hadli qatorlar bo‘lsin.

4.2-teorema (yaqinlashuvchanlikning yetarlilik sharti).

$\sum\limits_{n=1}^{\infty}a_n\le\sum\limits_{n=1}^{\infty}b_n$ munosabat o‘rinli bo‘lib, $\sum\limits_{n=1}^{\infty}b_n$ qator yaqinlashuvchi bo‘lsa, u holda $\sum\limits_{n=1}^{\infty}a_n$ qator ham yaqinlashuvchi bo‘ladi.

4.1.18-misol. $1+\frac{1}{2\cdot5}+\frac{1}{3\cdot5^2}+\frac{1}{4\cdot5^3}+...$ qatorni yaqinlashishga tekshiring.

Yechilishi: ► Berilgan qator 4.1.17-misoldagi qator bo‘lib, uning umumiy hadi $u_n = \frac{1}{n \cdot 5^{n-1}}$ edi. Yetarlilik shartiga ko‘ra, berilgan qatorni $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots + \frac{1}{5^{n-1}} + \ldots$ qator bilan taqqoslaymiz. Ushbu qatorning mahraji $q = \frac{1}{5}$ ga teng bo‘lgan geometrik progressiya bo‘lib, yaqinlashuvchidir. Hamma n lar uchun $\frac{1}{n \cdot 5^{n-1}} \le \frac{1}{5^{n-1}}$ munosabat o‘rinli. Shu sababli $\sum_{n=1}^{\infty} \frac{1}{n \cdot 5^{n-1}}$ qator ham yaqinlashuvchi. ◄

4.3-teorema (uzoqlashuvchanlikning yetarlilik sharti). Agar $\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$ munosabat o‘rinli bo‘lib, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty} b_n$ qator ham uzoqlashuvchi bo‘ladi.

4.1.19-misol. $1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \ldots + \frac{1}{n \cdot 2^n} + \ldots$ qator yaqinlashishini tekshiring.

Yechilishi: ► Berilgan qatorni $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} + \ldots$ qator bilan taqqoslaymiz. Ma’lumki, keyingi qator mahraji $q = \frac{1}{2}$ ga teng bo‘lgan geometrik progressiya bo‘lib, yaqinlashuvchidir. Hamma n lar uchun

$$\frac{1}{n \cdot 2^n} \le \frac{1}{2^n}$$

tengsizliklar bajariladi, demak taqqoslash alomatiga ko‘ra, berilgan qatorning ham yaqinlashuvchi ekanligi kelib chiqadi. ◄

4.1.20-misol. $\sum_{n=0}^{\infty} \frac{1}{3^n + n}$ qator yaqinlashishini tekshiring.

Yechilishi: ► Berilgan qatorni kattaroq qator bilan almashtiramiz: $\frac{1}{3^n + n} < \frac{1}{3^n}$ va taqqoslash alomatini qo‘llaymiz.

$\sum_{n=0}^{\infty} \frac{1}{3^n}$ qatorning mahraji $q = \frac{1}{3}$ ga teng bo‘lgan geometrik progressiya bo‘lib, $\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$ yaqinlashuvchidir. Shu sababli $\sum_{n=0}^{\infty} \frac{1}{3^n + n}$ qator ham yaqinlashuvchi bo‘ladi. ◄

4.1.21-misol. $\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$ qator yaqinlashishini tekshiring.

Yechilishi: ► Berilgan qatorni kattaroq qator bilan almashtiramiz: $\frac{n^2+2}{n^4+5}<\frac{n^2+2}{n^4}$ va taqqoslash alomatini qo'llaymiz.

$\sum_{n=1}^{\infty}\frac{n^2+2}{n^4}=\sum_{n=1}^{\infty}\frac{n^2}{n^4}+\sum_{n=1}^{\infty}\frac{2}{n^4}=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{2}{n^4}$ qator yaqinlashuvchi.

Shu sababli $\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$ qator ham yaqinlashuvchi bo'ladi. ◄

Umumlashgan taqqoslash alomati

4.4-teorema. $\sum_{n=1}^{\infty}a_n$ va $\sum_{n=1}^{\infty}b_n$ qatorlar uchun $\lim_{n\to\infty}\frac{a_n}{b_n}=k, \quad (0<k<\infty)$ bo'lsa, har ikki qator bir paytda yaqinlashuvchi yoki uzoqlashuvchi bo'ladi.

4.1.22-misol. $\sum_{n=1}^{\infty}\frac{n}{n^2-\cos^2 n}$ qator yaqinlashishini tekshiring.

Yechilishi: ► Berilgan qatorni uzoqlashuvchi $\sum_{n=1}^{\infty}\frac{n}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n}$ qator bilan taqqoslaymiz: $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{n}{n^2-\cos^2 n}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2}{n^2-\cos^2 n}=1<\infty.$

Demak, 4.4-teoremaga ko'ra, berilgan qator ham uzoqlashuvchi. ◄

4.1.23-misol. $\sum_{n=1}^{\infty}\frac{1}{3^n-n}$ qator yaqinlashishini tekshiring.

Yechilishi: ► Berilgan qatorni yaqinlashuvchi $\sum_{n=1}^{\infty}\frac{1}{3^n}$ qator bilan taqqoslaymiz: $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{3^n}}{\frac{1}{3^n-n}}=\lim_{n\to\infty}\frac{3^n-n}{3^n}=1.$

Demak, 4.4-teoremaga ko'ra, berilgan qator ham yaqinlashuvchi. ◄

4.1.24-misol. $\sum_{n=2}^{\infty}\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ qator yaqinlashishini tekshiring.

Yechilishi: ► Berilgan qatorni uzoqlashuvchi $\sum_{n=1}^{\infty}\frac{n^2}{\sqrt[3]{n^7}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$ qator bilan taqqoslaymiz: $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}}{\frac{1}{\sqrt[3]{n}}}=\lim_{n\to\infty}\frac{\sqrt[3]{n}(4n^2+n)}{\sqrt[3]{n^7+n^3}}=4.$

Demak, 4.4-teoremaga ko‘ra, berilgan qator ham uzoqlashuvchi. ◄

Dalamber alomati

4.5-teorema. Agar $\sum_{n=1}^{\infty}u_n=u_1+u_2+u_3+...+u_n+...$ musbat hadli qatorda $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l$, bunda $\begin{cases} l<1 \text{ bo'lsa, qator yaqinlashuvchi;} \\ l>1 \text{ bo'lsa, qator uzoqlashuvchi;} \\ l=1 \text{ bo'lsa, qator yaqinlashishi aniqmas} \end{cases}$

Dalamber alomatidan qatorning umumiy hadida ko‘rsatkichli funksiya yoki faktorial qatnashgan bo‘lsa, foydalanish qulay.

4.1.25-misol. $1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}+...$ qatorni yaqinlashishga tekshiring.

Yechilishi: ► $l=\lim_{n\to\infty}\frac{1/(n+1)!}{1/n!}=\lim_{n\to\infty}\frac{1}{n+1}=0<1.$

Dalamber alomatiga ko‘ra, berilgan qator yaqinlashuvchi. ◄

4.1.26-misol. $1+\frac{2}{3}+\frac{3}{5}+...+\frac{n}{2n-1}+...$ qator yaqinlashishini tekshiring.

Yechilishi: ►

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\frac{n+1}{2(n+1)-1}}{\frac{n}{2n-1}}=\lim_{n\to\infty}\frac{\frac{n+1}{2n+1}}{\frac{n}{2n-1}}=\lim_{n\to\infty}\frac{(n+1)(2n-1)}{(2n+1)n}=\lim_{n\to\infty}\frac{2n^2+n-1}{2n^2+n}=\frac{2}{2}=1$$

Bu holda Dalamber alomati savolga javob bermaydi. Berilgan qator uchun qator yaqinlashishining zaruriy shartini tekshiraylik,

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{n}{2n-1}=\frac{1}{2}\neq 0.$$

Qator yaqinlashishining zaruriy sharti bajarilmaydi, demak, berilgan qator uzoqlashuvchi. ◄

4.1.27-misol. $\sum_{n=0}^{\infty}\frac{n!}{5^n}$ qatorni yaqinlashishga tekshiring.

Yechilishi: ► $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}}=\lim\limits_{n\to\infty}\frac{(n+1)!\cdot 5^n}{5^{n+1}\cdot n!}=\lim\limits_{n\to\infty}\frac{n+1}{5}=\infty$

Dalamber alomatiga ko‘ra, berilgan qator uzoqlashuvchi ekan. ◄

Koshi alomati

4.6-teorema. Agar $\sum_{n=1}^{\infty}u_n=u_1+u_2+u_3+...+u_n+...$ musbat hadli qatorda

$$\lim_{n\to\infty}\sqrt[n]{u_n}=l,\quad \text{bunda}\begin{cases} l<1 \text{ bo'lsa, qator yaqinlashuvchi;}\\ l>1 \text{ bo'lsa, qator uzoqlashuvchi;}\\ l=1 \text{ bo'lsa, qator yaqinlashishi aniqmas}\end{cases}$$

4.1.28-misol. $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n=\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+...+\left(\frac{n}{2n+1}\right)^n+...$ qatorni yaqinlashishga tekshiring.

Yechilishi: ► Koshi alomatidan

$$l=\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n}=\lim_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}<1.$$

Berilgan qator Koshi alomatiga ko‘ra yaqinlashuvchi bo‘ladi. ◄

Qator yaqinlashishining integral alomati

4.7-teorema. Agar $\sum_{n=1}^{\infty}u_n=u_1+u_2+u_3+...+u_n+...$ qatorning hadlari musbat va o‘smaydigan, ya’ni $u_1\geq u_2\geq u_3\geq...\geq u_n\geq...$ bo‘lsa va $f(x)$ funksiya uchun $f(1)=u_1$, $f(2)=u_2$, $f(3)=u_3$, ..., $f(n)=u_n$, ...tengliklar o‘rinli bo‘lsa, u holda

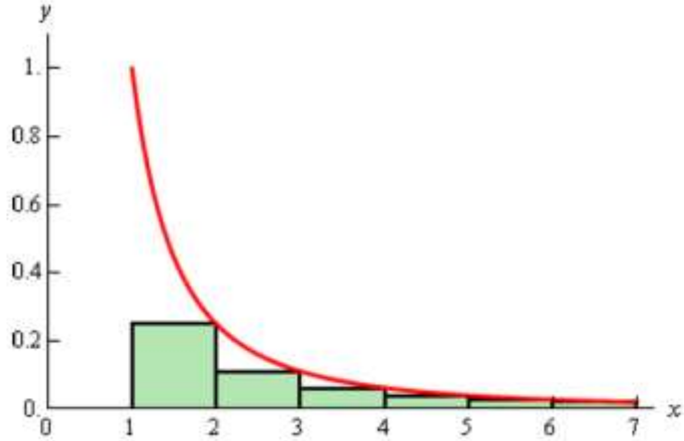
1) agar $\int_1^{\infty}f(x)dx$ xosmas integral yaqinlashsa, $\sum_{n=1}^{\infty}u_n$ qator yaqinlashuvchi,

2) agar $\int_1^{\infty}f(x)dx$ xosmas integral uzoqlashsa, $\sum_{n=1}^{\infty}u_n$ qator uzoqlashuvchi bo‘ladi.

4.1.29-misol. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ qator yaqinlashuvchimi yoki uzoqlashuvchimi?

Yechilishi: ► $\int_1^{\infty}\frac{1}{x^2}dx=-\frac{1}{x}\Big|_1^{\infty}=-\lim_{x\to\infty}\left(\frac{1}{x}-1\right)=1$ dan foydalanamiz.

Berilgan qator yig‘indisi $[1,\infty)$ oraliqda yuqoridan $f(x)=\frac{1}{x^2}$ funksiya bilan chegaralangan shakl yuzasiga teng bo‘ladi.



4.1-rasm. $f(x)=\frac{1}{x^2}$ funksiya bilan chegaralangan shakl

4.1-rasmdan ko‘rish mumkinki, bu yuza asos uzunligi bilan balandlik ko‘paytmasiga teng:

$$S=\frac{1}{2^2}\cdot 1+\frac{1}{3^2}\cdot 1+\frac{1}{4^2}\cdot 1+\ldots$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots<1+\int_1^{\infty}\frac{1}{x^2}dx=1+1=2, \qquad \text{ya’ni } \sum_{n=1}^{\infty}\frac{1}{n^2}<2.$$

Bundan berilgan qatorning yaqinlashuvchi ekanligi kelib chiqadi.◄

4.1.30-misol. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► $f(x)=\frac{1}{x\ln x}$ funksiya ko‘rinishida yozib olamiz va xosmas integralni hisoblaymiz:

$$\int_2^{\infty}\frac{dx}{x\ln x}=\lim_{t\to\infty}\int_2^{t}\frac{dx}{x\ln x}=\lim_{t\to\infty}\left(\ln(\ln x)\right)\Big|_2^t=\lim_{t\to\infty}[\ln(\ln t)-\ln(\ln 2)]=\infty.$$

Integral alomatiga ko‘ra, berilgan qator uzoqlashuvchi bo‘lib chiqdi.◄

4.1.31-misol. $\sum_{n=0}^{\infty}ne^{-n^2}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► Qatorning umumiy hadini $f(x)=xe^{-x^2}$ funksiya ko‘rinishida yozib olamiz va xosmas integralni hisoblaymiz:

$$\int_0^{\infty} xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx = \lim_{t\to\infty}\left(-\frac{1}{2}e^{-x^2})\right|_0^t = \lim_{t\to\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-t^2}\right) = \frac{1}{2}.$$

Integral alomatiga ko‘ra, berilgan qator yaqinlashuvchi ekan.◀

Umumlashgan garmonik qator

4.8-teorema. Agar $k>0$ bo‘lsa, u holda $\sum_{n=k}^{\infty}\frac{1}{n^p}$ qator $p>1$ bo‘lganda yaqinlashuvchi, $p\le 1$ bo‘lganda esa uzoqlashuvchi bo‘ladi.

Haqiqatan ham, $\int_k^{\infty}\frac{1}{x^p}dx$ integral $p>1$ bo‘lganda yaqinlashuvchi, $p\le 1$ bo‘lganda esa uzoqlashuvchi bo‘ladi.

4.1.32-misol. a) $\sum_{n=4}^{\infty}\frac{1}{n^7}$ va

b) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ qatorlarni yaqinlashuvchanlikka tekshiring.

Yechilishi: ▶ a) $p=7>1$, demak, $\sum_{n=4}^{\infty}\frac{1}{n^7}$ qator yaqinlashuvchi;

b) $p=\frac{1}{2}<1$, demak, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ qator uzoqlashuvchi. ◀

Mavzu yuzasidan savollar

1. Sonli qator deb nimaga aytiladi?
2. Qatorning yig‘indisi deb nimaga aytiladi?
3. Sonli qator yaqinlashishining zaruriylik sharti qanday?
4. Garmonik va umumlashgan garmonik qator deb qanday qatorlarga aytiladi?
5. Teleskopik qatorlar qanday bo‘ladi?
6. Geometrik qatorga ta’rif bering.
7. Musbat hadli qatorlarning yaqinlashish alomatlarini sanang.
8. Taqqoslash va umumlashgan taqqoslash alomatlarini tushuntiring.
9. Dalamber va Koshi alomatlarini ayting.
10. Yaqinlashishning integral alomati deganda nimani tushunasiz?

MUSTAQIL YECHISH UCHUN MISOLLAR:

1. $1+\frac{3!}{2\cdot 4}+\frac{5!}{2\cdot 4\cdot 6}+...+\frac{(2n-1)!}{2\cdot 4\cdot 6\cdot 8\cdot ...\cdot 2n}+...$ qatorni yaqinlashishga tekshiring.

2. Dalamber alomatiga asosan qatorni tekshiring: $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+...+\frac{n}{2^n}+...$

3. Koshi alomatiga asosan qatorni tekshiring: $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^{n^2}$.

4. $\frac{1}{4}+\frac{2}{9}+\frac{4}{16}+\frac{8}{25}+...+\frac{2^{n-1}}{(n+1)^2}+...$ qatorni yaqinlashishga tekshiring.

5. Sonli qatorlarni yaqinlashishga tekshiring:

a) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$; **b)** $\sum_{n=1}^{\infty}\frac{2^n}{n\,5^{n+1}}$; **c)** $\sum_{n=1}^{\infty}\frac{1}{(n+2)\,5^n}$;

TESTLAR

1. Qatorning yig‘indisini toping: $\frac{3}{1\cdot 4}+\frac{5}{4\cdot 9}+\frac{7}{9\cdot 16}+...$

A) $S=1$; **B)** $S=2$; **C)** $S=3$; **D)** $S=4$.

2. Qatorning umumiy hadini toping: $\frac{2}{5}+\frac{4}{9}+\frac{6}{13}+..$

A) $a_n=\frac{n}{2n+1}$; **B)** $a_n=\frac{2n}{4n+1}$; **C)** $a_n=\frac{3n+1}{5n+1}$; **D)** $a_n=\frac{2n+1}{5n-1}$.

3. Qatorning birinchi *n* ta hadining S_n yig‘indisiga … deyiladi.

A) qatorning yig‘indisi; **B)** hadlar ko‘paytmasi
C) hadlar ko‘paytmasi; **D)** qatorning n - xususiy yig‘indisi

4. Agar … bo‘lsa, u holda qator yaqinlashuvchi deyiladi.

A) xususiy yig‘indilarining chekli limiti mavjud bo‘lsa;
B) xususiy yig‘indilarning limiti mavjud bo‘lmasa;
C) xususiy yig‘indilarning limiti cheksiz bo‘lsa;
D) xususiy yig‘indilarning chekli limiti mavjud bo‘lmasa.

5. $\sum_{n=1}^{\infty}\frac{4n+3}{5n-7}$ qator uchun

A) yaqinlashishning zaruriy sharti bajarilmaydi;
B) yaqinlashishning zaruriy sharti bajariladi;
C) *n*=0 da zaruriy shart bajariladi;

D) $n=1$ da zaruriy shart bajariladi.

4.1.4. Ishorasi almashinuvchi qatorlar. Leybnits alomati

$$\sum_{n=1}^{\infty}(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}u_n + \ldots \qquad (4.5)$$

ko‘rinishdagi qatorga **ishorasi navbat bilan almashib keladigan** qator deyiladi. Bu yerda $u_1, u_2, u_3, \ldots, u_n, \ldots$ musbat sonlar.

Agar qatorning hadlari orasida musbatlari ham manfiylari ham bo‘lsa, u holda bunday qator **o‘zgaruvchan ishorali qator** deyiladi.

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

Bu yerda $u_1, u_2, u_3, \ldots, u_n, \ldots$ lar har xil ishorali sonlar.

Ishorasi navbatlashuvchi qatorlar o‘zgaruvchan ishorali qatorning xususiy holi hisoblanadi.

4.9-teorema (Leybnits alomati). Agar ishorasi navbat bilan almashinib keluvchi $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ qatorda

1) qatorning u_n umumiy hadi $n \to \infty$ da nolga intilsa, ya’ni

$$\lim_{n\to\infty} u_n = 0;$$

2) qator hadlarining absolyut qiymatlari kamayuvchi bo‘lsa, ya’ni

$$u_1 > u_2 > u_3 > u_4 > \ldots > u_n > \ldots,$$

u holda bu **qator yaqinlashuvchi** bo‘ladi.

4.1.33-misol. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► Qatorning Leybnits alomatini qanoatlantirishini tekshirib ko‘ramiz:

1) qatorning a_n umumiy hadi uchun $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$ o‘rinli.

2) $\frac{1}{n} > \frac{1}{n+1}$ bundan $a_n > a_{n+1}$ ekanligi, ya’ni qator hadlari kamayuvchi;

Bundan qatorning yaqinlashuvchi ekanligi kelib chiqadi. ◄

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ qatorga **ishorasi almashinuvchi garmonik qator** deyiladi.

4.1.34-misol. $\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^2+7}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► Qator yaqinlashuvchanligining zaruriylik shartini tekshirib ko‘ramiz:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2}{n^2+7} = \lim_{n\to\infty}\frac{n^2+7-7}{n^2+7} = \lim_{n\to\infty}\left(1-\frac{7}{n^2+7}\right) = 1 \neq 0 .$$

Leybnits alomatining 1-shartini qanoatlantirmaydi. Bunday holda 2-shartni tekshirib ko‘rishning keragi yo‘q, chunki qatorning a_n umumiy hadi $n\to\infty$ da nolga intilmasa, qator uzoqlashuvchi bo‘ladi.

Keling uzoqlashuvchanlikka tekshirib ko‘ramiz:

$$\lim_{n\to\infty}\frac{(-1)^n n^2}{n^2+7} = \left(\lim_{n\to\infty}(-1)^n\right)\left(\lim_{n\to\infty}\frac{n^2}{n^2+7}\right)$$

limit mavjud bo‘lishi uchun ikkala limit ham mavjud bo‘lishi kerak. Birinchi limit mavjud emas, chunki *n* juft bo‘lganda 1 ga, toq bo‘lganda esa -1 ga intiladi. Shuning uchun umumiy limit mavjud emas. Bundan qatorning uzoqlashuvchi ekanligi kelib chiqadi. ◄

4.1.35-misol. $\sum_{n=1}^{\infty}\frac{(-1)^{n-3}\sqrt{n}}{n+5}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► $(-1)^{n-3}$ ekanligi bizni xavotirga solmasligi kerak. Bu yozuv ham *n* juft bo‘lganda -1 ni, toq bo‘lganda esa 1 ni hosil qiladi.

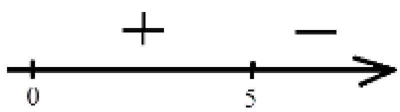
Qator yaqinlashuvchanligining zaruriylik shartini tekshirib ko‘ramiz:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+5} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n}}{\frac{n}{n}+\frac{5}{n}} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}}}{1} = 0 .$$

Qatorning hadlari absolyut qiymat bo‘yicha kamayuvchimi yoki o‘suvchimi ekanligini tekshirib ko‘ramiz. Funksiya ko‘rinishida yozib olamiz va hosila yordamida tekshiramiz: $f(x)=\frac{\sqrt{x}}{x+5}$ bo‘lsa, u holda

$$f'(x)=\left(\frac{\sqrt{x}}{x+5}\right)' = \frac{\frac{1}{2\sqrt{x}}(x+5)-\sqrt{x}}{(x+5)^2} = \frac{5-x}{2\sqrt{x}(x+5)} = 0$$

$x=5,\quad x=0,\quad x=-5$ nuqtalarni topdik. Bu yerda $x\geq 0$ shart ham bor. Topilgan qiymatlarni sonlar o‘qiga qo‘yib, oraliqlarda funksiya hosilasining ishorasini tekshiramiz:



$0\leq x\leq 5$ da funksiya o‘sadi, $x\geq 5$ da kamayadi. Bundan qatorning umumiy hadi ham $0\leq x\leq 5$ oraliqda o‘sadi, $x\geq 5$ oraliqda kamayadi.

Demak, qatorning dastlabki 5 ta hadi o‘sib boradi-da, keyingi hadlari kamaya boshlaydi. Shuning uchun umuman olganda qator 2-Leybnits shartini qanoatlantiradi, shuning uchun yaqinlashuvchi deb xulosa qilamiz. ◄

4.3.3-misoldan ko‘rinadiki, ba’zan qatorning yaqinlashuvchi ekanligini ko‘rsatish uchun bir qancha amallarni bajarish kerak. Keling yana bitta misolni qarab chiqamiz.

4.1.36-misol. $\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► Berilgan qator ishorasi almashinuvchi qator, chunki

$$\cos(n\pi)=(-1)^n.$$

Shunda qatorning ko‘rinishi quyidagicha bo‘ladi: $\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}}$.

Leybnits alomatini tekshiramiz:

1) $\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ zaruriylik sharti bajariladi;

2) $\frac{1}{\sqrt{n}}>\frac{1}{\sqrt{n+1}}$ bundan $a_n>a_{n+1}$ qator hadlari kamayuvchi;

Bundan qatorning yaqinlashuvchi ekanligi kelib chiqadi. Ushbu qatorda *n* butun son va π qatnashganligi uchun yaqinlashuvchi chiqdi, agar n butun son bo‘lmasa va π qatnashmasa qatorni yaqinlashuvchi ekanligiga kafolat berolmas edik.◄

Agar $\sum_{n=1}^{\infty}|u_n|$ qator yaqinlashuvchi bo‘lsa, u holda $\sum_{n=1}^{\infty}u_n$ qatorga **absolyut yaqinlashuvchi qator** deyiladi.

Agar $\sum_{n=1}^{\infty}u_n$ qator yaqinlashuvchi va $\sum_{n=1}^{\infty}|u_n|$ qator uzoqlashuvchi bo‘lsa, u holda berilgan qator **shartli yaqinlashuvchi qator** deyiladi.

4.10-teorema. Agar $\sum_{n=1}^{\infty}u_n$ qator absolyut yaqinlashuvchi bo‘lsa, u holda u albatta yaqinlashuvchi bo‘ladi.

Isboti: ► $|u_n|=\begin{cases}u_n, & \text{agar } u_n\ge 0\\ -u_n, & \text{agar } u_n<0\end{cases}$ modul ta’rifiga ko‘ra, $0\le u_n+|u_n|\le 2|u_n|$ qo‘sh tengsizlikni yozish mumkin.

Agar $\sum_{n=1}^{\infty}|u_n|$ ni yaqinlashuvchi deb faraz qilsak, u holda $\sum_{n=1}^{\infty}2|u_n|$ ham yaqinlashuvchi bo‘ladi. Taqqoslash alomatidan foydalanib, $\sum_{n=1}^{\infty} u_n + |u_n|$ qatorning ham yaqinlashuvchi ekanligi kelib chiqadi.

Shunday qilib, $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} u_n + |u_n| - \sum_{n=1}^{\infty} |u_n|$ tenglik o‘rinli bo‘ladi.

Ikkita yaqinlashuvchi qatorlarning ayirmasi $\sum_{n=1}^{\infty} u_n$ yana yaqinlashuvchi bo‘ladi.◄

Qatorning absolyut yaqinlashuvchiligi yaqinlashishning “kuchliroq” ko‘rinishi hisoblanadi. Chunki, absolyut yaqinlashuvchi bo‘lgan qator albatta yaqinlashuvchi bo‘ladi. Lekin yaqinlashuvchi bo‘lgan qator absolyut yaqinlashuvchi bo‘lmasligi ham mumkin.

4.1.37-misol. $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ qatorni yaqinlashishga tekshiring.

Yechilishi: ► Ushbu qatorning ko‘rinishi ishorasi almashinuvchi qatorga o‘xshamaydi, lekin bu qator ishoralari almashinib keladigan qatordir:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^3} = \frac{\sin 1}{1^3} + \frac{\sin 2}{2^3} + \frac{\sin 3}{3^3} + \ldots$$

Bilamizki, $-1 \le \sin\alpha \le 1 \quad \Rightarrow \quad |\sin\alpha| \le 1$. Bundan $\frac{|\sin n|}{n^3} \le \frac{1}{n^3}$.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ umumlashgan garmonik qator yaqinlashuvchi, taqqoslash alomatiga ko‘ra, $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$ qator ham yaqinlashuvchi bo‘ladi. Shunday qilib, berilgan qator $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ absolyut yaqinlashuvchi, demak yaqinlashuvchi hamdir.◄

Agar ishorasi almashinuvchi qatorda hadlari o‘rinlarini almashtirib yozsak, u holda qatorning yig‘indisi o‘zgarishi mumkin.

4.1.38-misol. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$ qator yig‘indisini toping.

Yechilishi: ► Aytaylik, berilgan qatorning yig‘indisi

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots = S$$

bo‘lsin. Qator hadlarini har bir musbat haddan keyin ikkita manfiy had turadigan qilib almashtiramiz: $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots$

Har bir musbat hadni undan keyin keladigan manfiy had bilan qo‘shamiz:

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\ldots=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\ldots=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)=2S$$

Natijada hadlari berilgan qator hadlarini $\frac{1}{2}$ ga ko‘paytirishdan hosil bo‘lgan qatorga ega bo‘lamiz. ◀

Absolyut va shartli yaqinlashuvchi qatorlarning xossalari:

a) Agar qator **absolyut yaqinlashuvchi** bo‘lsa, u holda bu qator hadlarining o‘rni har qancha almashtirilganda ham u absolyut yaqinlashuvchi bo‘lib qolaveradi, bunda qatorning yig‘indisi uning hadlari tartibiga bog‘liq bo‘lmaydi (bu xossa shartli yaqinlashuvchi qatorlar uchun o‘rinli bo‘lmasligi mumkin);

b) Agar qator **shartli yaqinlashuvchi** bo‘lsa, u holda bu qator hadlarining o‘rinlarini shunday almashtirish mumkinki, natijada uning yig‘indisi o‘zgaradi va almashtirishdan keyin hosil bo‘lgan qator uzoqlashuvchi qator bo‘lib qolishi ham mumkin.

4.1.39-misol. $\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► Dalamber alomatidan foydalanish uchun

$u_n=\frac{(-10)^n}{4^{2n+1}(n+1)}$ va $u_{n+1}=\frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)}=\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$ hadlarini yozib olamiz.

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left|\frac{\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}}{\frac{(-10)^n}{4^{2n+1}(n+1)}}\right|=\lim_{n\to\infty}\left|\frac{-10(n+1)}{16(n+2)}\right|=\frac{5}{8}<1$$

Bundan qator absolyut yaqinlashuvchi ekanligi ko‘rinadi. ◀

4.1.40-misol. $\sum_{n=1}^{\infty}\frac{9^n}{(-2)^{n+1}n}$ qatorni yaqinlashuvchanlikka tekshiring.

Yechilishi: ► Dalamber alomatini qo‘llaymiz:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left|\frac{\frac{9^{n+1}}{(-2)^{n+2}(n+1)}}{\frac{9^n}{(-2)^{n+1}n}}\right|=\lim_{n\to\infty}\left|\frac{9n}{(-2)(n+1)}\right|=\frac{9}{2}>1$$

Qator uzoqlashuvchi.◀

Mavzu yuzasidan savollar:

1. Ishorasi almashinuvchi qator deb qanday qatorga aytiladi?
2. Absolyut yaqinlashuvchi qator deganda nimani tushunasiz?
3. Shartli yaqinlashuvchi qator ta’rifini ayting.
4. Absolyut va shartli yaqinlashuvchi qatorlarning qanday xossalarini bilasiz?
5. Ishorasi almashinuvchi qator hadlarining o‘rni almashtirilsa, qatorning yig‘indisi o‘zgaradimi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n+5}$ qatorni yaqinlashishga tekshiring.

2. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt[3]{n+2}}$ qatorni yaqinlashishga tekshiring. .

3. $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots+\frac{(-1)^{n-1}}{n^2}+\ldots$ qatorni yaqinlashishga tekshiring.

4. $\sum_{n=1}^{\infty}\frac{(-1)^n\cos\frac{\pi n}{3}}{n^2+1}$ qatorni yaqinlashishga tekshiring.

5. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2+3}$ qatorni yaqinlashishga tekshiring.

TESTLAR

1. $\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\ldots+(-1)^{n+1}\frac{1}{(n+1)^2}+\ldots$ qatorni yaqinlashishga tekshiring.

A) qator uzoqlashadi;
B) qator yaqinlashadi;
C) qator shartli yaqinlashadi;
D) qator yaqinlashishini aniqlab bo‘lmaydi.

2. $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots+\frac{(-1)^{n-1}}{n^2}+\ldots$ qatorni yaqinlashishga tekshiring.

A) qator uzoqlashadi;
B) qator absolyut yaqinlashadi;

C) qator shartli yaqinlashadi;
D) qator yaqinlashishini aniqlab bo‘lmaydi.

3. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ qatorni yaqinlashishga tekshiring.
A) qator uzoqlashadi;
B) qator absolyut yaqinlashadi;
C) qator shartli yaqinlashadi;
D) qator yaqinlashishini aniqlab bo‘lmaydi.

4. $\sum_{n=1}^{\infty}\frac{(-1)^{n}\cos\frac{\pi n}{3}}{n^{2}+1}$ qatorni yaqinlashishga tekshiring.
A) qator uzoqlashadi;
B) qator absolyut yaqinlashadi;
C) qator shartli yaqinlashadi;
D) qator yaqinlashishini aniqlab bo‘lmaydi.

5. Agar $\sum_{n=1}^{\infty}|a_n|$ qator yaqinlashuvchi bo‘lsa, u holda o‘zgaruvchan ishorali $\sum_{n=1}^{\infty}a_n$ qator …
A) absolyut yaqinlashuvchi bo‘ladi;
B) shartli yaqinlashuvchi bo‘ladi;
C) yaqinlashuvchi ham, uzoqlashuvchi ham bo‘lishi mumkin;
D) uzoqlashuvchi bo‘ladi.

4.2-§. Funksional qatorlar. Darajali qatorlar, yaqinlashish radiusi va yaqinlashish sohasi

4.2.1. Funksional qatorlar. Tekis yaqinlashish. Veyershtrass alomati

Hadlari x o‘zgaruvchining funksiyalaridan iborat bo‘lgan

$$u_1(x)+u_2(x)+...+u_n(x)+... \tag{4.6}$$

qatorga **funksional qator** deyiladi.

Agar o‘zgaruvchi x ning aniq bir qiymatini olsak, ya’ni $x=x_0$ deb, uni (4.6) ga qo‘ysak $u_1(x_0)+u_2(x_0)+...+u_n(x_0)+...$ sonli qator hosil bo‘ladi.

Demak, o‘zgaruvchi x ga har xil son qiymatlar berish bilan har xil yaqinlashuvchi yoki uzoqlashuvchi bo‘lgan sonli qatorlar hosil qilish mumkin ekan.

Agar (4.6) qator x ning $x_0,\ x_1,\ x_2,...,\ x_n$ aniq son qiymatlarida yaqinlashuvchi bo‘lsa, u holda x ning bu $x_0,\ x_1,\ x_2,...,\ x_n$ son qiymatlar to‘plamiga (4.6) ning **yaqinlashish sohasi** deyiladi.

4.2.1-misol. $1+x^2+x^3+...+x^n+...$ funksional qatorning yig‘indisini va yaqinlashish sohasini toping.

Yechilishi: ► Berilgan qatorning hadlari mahraji $q=x$ bo‘lgan geometrik progressiyani tashkil qiladi. Shunga ko‘ra, uning yaqinlashishi uchun $|x|<1$ bo‘lishi kerak, ya’ni $-1<x<1$. U holda $(-1,1)$ oraliqda qatorning yig‘indisi $\frac{1}{1-x}$ ga teng. Shunday qilib, $(-1,1)$ oraliqda berilgan qator $S(x)=\frac{1}{1-x}$ funksiyani aniqlaydi, bu esa qatorning yig‘indisidir, ya’ni

$$\frac{1}{1-x}=1+x^2+x^3+...+x^n+... \ ◄$$

(4.6) qatorning dastlabki n ta hadi yig‘indisini $S_n(x)$ bilan belgilaymiz:

$$S_n(x)=u_1(x)+u_2(x)+...+u_n(x) \tag{4.7}$$

Agar $\lim\limits_{n\to\infty} S_n(x)=S(x)$ chekli limit mavjud bo‘lsa, u holda (4.6) funksional qatorga **yaqinlashuvchi qator**, $S(x)$ ga esa uning **yig‘indisi** deyiladi. Agar $\lim\limits_{n\to\infty} S_n(x)$ limit mavjud bo‘lmasa yoki cheksiz bo‘lsa, (4.6) funksional qatorga **uzoqlashuvchi qator** deyiladi.

Agar bu qator x ning biror qiymatida yaqinlashsa, u holda

$$S(x) = S_n(x) + r_n(x)$$

bo‘ladi, bu yerda $S(x)$- qatorning yig‘indisi $r_n(x) = u_{n+1}(x) + u_{n+2}(x) + ...$ **qatorning qoldig‘i** deyiladi.

x ning barcha qiymatlari uchun qatorning yaqinlashish sohasida $\lim\limits_{n\to\infty} S_n(x) = S(x)$ munosabat o‘rinli, bundan $\lim\limits_{n\to\infty}[S_n(x) - S(x)] = 0$ deyish mumkin, shu sababli $\lim\limits_{n\to\infty} r_n(x) = 0$, ya’ni yaqinlashuvchi qatorning qoldig‘i $n \to \infty$ da nolga intiladi.

Agar ixtiyoriy ε musbat son uchun ε ga bog‘liq, shunday $N(\varepsilon) > 0$ son topilib, barcha $n \ge N$ da ko‘rsatilgan sohaga tegishli x lar uchun

$$r_n(x) = |S(x) - S_n(x)| < \varepsilon$$

tengsizlik bajarilsa, (4.6) qator ko‘rsatilgan sohada **tekis yaqinlashuvchi qator** deyiladi.

4.11-teorema (Veyershtrass alomati). Agar (4.6) funksional qatorning hadlari biror $[a,b]$ sohada absolyut qiymati bo‘yicha yaqinlashuvchi musbat ishorali biror $c_1 + c_2 + ... + c_n + ...$ qatorning mos hadlaridan katta bo‘lmasa, ya’ni $|u_n(x)| \le c_n$ $(n = 1,2,3,...)$ bo‘lsa, u holda berilgan funksional qator $[a,b]$ sohada tekis yaqinlashadi.

4.2.2-misol. $\frac{\sin^2 x}{1^3} + \frac{\sin^2 2x}{2^3} + ... + \frac{\sin^2 nx}{n^3} + ...$ funksional qator x ning qanday qiymatlarida yaqinlashuvchi?

Yechilishi: ► $\frac{\sin^2 x}{1^3} + \frac{\sin^2 2x}{2^3} + ... + \frac{\sin^2 nx}{n^3} + ...$ funksional qator x ning barcha haqiqiy qiymatlarida tekis yaqinlashadi, chunki barcha x va n larda $\left|\frac{\sin^2 nx}{n^3}\right| \le \frac{1}{n^3}$ tengsizlik o‘rinli. $\frac{1}{1^3} + \frac{1}{2^3} + ... + \frac{1}{n^3} + ...$ qator esa yaqinlashuvchidir. ◄

4.2.3-misol. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n + x}$ qatorni yaqinlashishga tekshiring.

Yechilishi: ► Veyershtrass alomati bu qator uchun bajarilmaydi, chunki berilgan qator shartli yaqinlashuvchi va $x \ge 0$ lar uchun $\sum\limits_{n=1}^{\infty} \frac{1}{n + x}$ qator uzoqlashuvchi. Berilgan qatorni tekis yaqinlashuvchiligini ko‘rsatish uchun Leybnis teoremasidan foydalanamiz. Berilgan qator o‘zgaruvchi ishorali va $x \ge 0$ da absolyut qiymatlari bo‘yicha monoton kamayuvchi va n-hadi $n \to \infty$ da nolga intiladi. Shu sababli, qator $[0, \infty)$ yarim o‘qda yaqinlashuvchi va qator qoldig‘i uchun $|r_n(x)| < \frac{1}{n + 1 + x}$.

$x \geq 0$ da $|r_n(x)| \leq \frac{1}{n+1}$ ga ega bo‘lamiz va $\lim_{n\to\infty}\frac{1}{n+1}=0$ bo‘lgani uchun qator tekis yaqinlashuvchi.◄

Tekis yaqinlashuvchi funksional qatorning xossalari

Tekis yaqinlashuvchi funksional qatorlar uchun funksiyalarning chekli yig‘indisi xossalarini tatbiq qilish mumkin.

1⁰. Agar $u_1(x)+u_2(x)+...+u_n(x)+...$ funksional qatorning har bir hadi $[a,b]$ kesmada uzluksiz bo‘lib, bu funksional qator $[a,b]$ kesmada tekis yaqinlashuvchi bo‘lsa, u holda qatorning yig‘indisi $S(x)$ ham shu kesmada uzluksiz bo‘ladi.

4.2.4-misol. $f(x)=\sum_{n=1}^{\infty}\left(x^2+\frac{1}{n}\right)^n$ funksiyani aniqlanish sohasini toping va uzluksizligini tekshiring.

Yechilishi: ► Berilgan funksional qatorni Koshi alomatiga ko‘ra yaqinlashish sohasini topamiz:

$$\lim_{n\to\infty}\sqrt[n]{\left(x^2+\frac{1}{n}\right)^n}=\lim_{n\to\infty}\left(x^2+\frac{1}{n}\right)=x^2.$$

Shu sababli qator $x^2<1$ da yaqinlashuvchi, $x^2>1$ da uzoqlashuvchi, ya’ni qator (-1,1) oraliqda yaqinlashuvchi. $x=\pm1$ nuqtalarda uzoqlashuvchi, chunki $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\neq0$, qator yaqinlashishining zaruriylik sharti bajarilmaydi.

Funksiyani uzluksizligini tekshiramiz. Buning uchun qatorni $0<a<1$ bo‘lgan ixtiyoriy $[-a,a]$ kesmada tekis yaqinlashuvchi ekanligini ko‘rsatamiz.

$0<a<b<1$ son olamiz va shunday N topiladiki, $n\geq N$ da $a+\frac{1}{\sqrt{n}}\leq b$

U holda $|x|\leq a$ lar uchun $\left(x^2+\frac{1}{n}\right)^n\leq\left(|x|+\frac{1}{\sqrt{n}}\right)^{2n}\leq\left(a+\frac{1}{\sqrt{n}}\right)^{2n}\leq b^{2n}$ tengsizlik bajariladi.

Ravshanki, $b^2+b^4+b^6+...+b^{2m}+...$ qator $[-a,a]$ da yaqinlashuvchi (chunki bu qator mahraji $b^2<1$ bo‘lgan geometrik progressiya), shu sababli berilgan qator tekis yaqinlashuvchi. Demak, $f(x)$ funksiya $[-a,a]$ kesmada uzluksiz. $a\,(0<a<1)$ ning ixtiyoriyligidan $f(x)$ funksiya (-1,1) oraliqda uzluksiz.

2⁰ (Qatorni hadlab integrallash). Agar $u_1(x)+u_2(x)+...+u_n(x)+...$ funksional qatorning har bir hadi $[a,b]$ kesmada uzluksiz bo‘lib, bu funksional qator $[a,b]$ kesmada tekis yaqinlashuvchi bo‘lsa, u holda

$$\int_a^b S(x)dx = \int_a^b u_1(x)dx + \int_a^b u_2(x)dx + ... + \int_a^b u_n(x)dx + ... +$$

tenglik o‘rinli bo‘ladi.

4.2.5-misol. $1-x^2+x^4-...+(-1)^n x^{2n}+...$ funksional qatorning yig‘indisini toping.

Yechilishi: ► Berilgan qator $|x|<1$ da tekis yaqinlashuvchi va

uning yig‘indisi $S(x)=\dfrac{1}{1+x^2}$ ga teng. Berilgan qatorni 0 dan $x<1$ gacha hadlab integrallaymiz va quyidagi qatorga ega bo‘lamiz:

$$\int_0^x dx - \int_0^x x^2dx + \int_0^x x^4dx - ... + (-1)^n \int_0^x x^{2n}dx + ... = x - \frac{x^3}{3} + \frac{x^5}{5} - ... + (-1)^n \frac{x^{2n+1}}{2n+1} + ...$$

Bu qator $|x|<1$ da tekis yaqinlashadi va uning yig‘indisi quyidagiga teng:

$$\int_0^x S(x)dx = \int_0^x \frac{dx}{1+x^2} = arctg\Big|_0^x = arctgx \, .$$

Shunday qilib, $|x|<1$ da tekis yaqinlashuvchi

$$arctgx = x - \frac{x^3}{3} + \frac{x^5}{5} - ... + (-1)^n \frac{x^{2n+1}}{2n+1} + ...$$

qatorga ega bo‘ldik. ◄

3⁰ (Qatorni hadlab differensiallash). Agar $[a,b]$ kesmada hosilalari uzluksiz funksiyalardan tuzilgan $u_1(x)+u_2(x)+...+u_n(x)+...$ funksional qator shu kesmada yaqinlashuvchi va yig‘indisi $S(x)$ bo‘lsa, u holda uning hadlari hosilalaridan tuzilgan $u_1'(x)+u_2'(x)+...+u_n'(x)+...$ qator ham tekis yaqinlashuvchi bo‘lib, yig‘indisi $S'(x)$ bo‘ladi.

4.2.6(a)-misol. ► $arctgx = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - ... + (-1)^n \dfrac{x^{2n+1}}{2n+1} + ...$ qatorni qaraymiz. Tenglikni har ikki tomonini $x \neq 0$ ga hadlab ko‘paytiramiz:

$$x \cdot arctgx = x^2 - \frac{x^4}{3} + \frac{x^6}{5} - ... + (-1)^n \frac{x^{2n+2}}{2n+1} + ...$$

Tenglikning o‘ng tomonida biror qator turibdi. Shu qatorni hadlab differensiallab, quyidagini topamiz:

$$2x - \frac{4x^3}{3} + \frac{6x^5}{5} - ... + (-1)^n \frac{(2n+2)x^{2n+1}}{2n+1} + ...$$

Dalamber alomatiga ko‘ra $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left|\frac{\frac{2n+2}{2n+1}x^{2n+1}}{\frac{2n}{2n-1}x^{2n-1}}\right|=\lim_{n\to\infty}\frac{2(n+1)(2n-1)}{(2n+1)^2}x^2=x^2$

Shunday qilib, qator absolyut yaqinlashuvchi va barcha $|x|<1$ lar uchun tekis yaqinlashuvchi bo‘ladi.

Demak, berilgan qatorning hosilalaridan tuzilgan qator berilgan qator yig‘indisidan olingan hosilaga yaqinlashadi:

$$arctgx+\frac{x}{1+x^2}=2x-\frac{4x^3}{3}+\frac{6x^5}{5}-...+(-1)^n\frac{(2n+2)x^{2n+1}}{2n+1}+...$$

$|x|<1$ da tekis yaqinlashuvchidir. ◄

4.2.2. Darajali qatorlar. Abel teoremasi

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+...+a_n(x-x_0)^n+... \qquad (4.8)$$

ko‘rinishdagi funksional qatorga **darajali qator** deyiladi.

$x_0=0$ bo‘lganda x ning darajalari bo‘yicha yoyilgan darajali qatorga

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+...+a_nx^n+... \qquad (4.9)$$

ega bo‘lamiz.

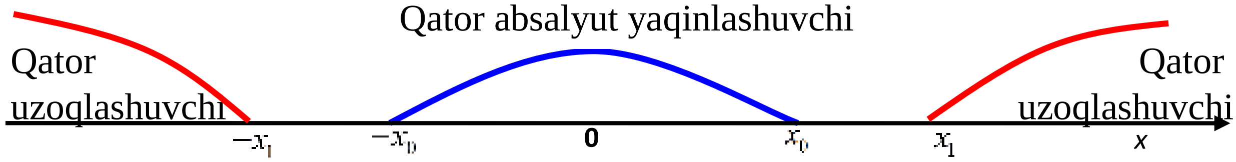
Bu yerda $a_0,\ a_1,\ a_2,...,a_n,...$ o‘zgarmas sonlar bo‘lib, **darajali qatorning koeffitsiyentlari** deyiladi. Darajali qatorlar funksional qatorning xususiy holidan iborat.

Har qanday darajali qator $x=x_0$ nuqtada yaqinlashuvchi bo‘ladi, chunki bu holda (4.9) qator $a_0+a_1\cdot 0+a_2\cdot 0^2+...+a_n\cdot 0^n+...$ ko‘rinishda sonli qatorga aylanadi va $\lim_{n\to\infty}S_n(0)=\lim_{n\to\infty}a_0=a_0$ bo‘ladi.

4.12-teorema (Abel teoremasi). Agar $\sum_{n=1}^{\infty}a_nx^n$ darajali qator x ning $x=x_0$ $(x_0\neq 0)$ qiymatida yaqinlashuvchi bo‘lsa, u holda x ning $|x|<|x_0|$ tengsizlikni qanoatlantiruvchi barcha $(-|x_0|,|x|)$ qiymatlarida (4.9)-darajali qator absolyut yaqinlashuvchi bo‘ladi.

Natija: Agar (4.8) darajali qator $x=x_0$ da uzoqlashuvchi bo‘lsa, u holda x ning $|x|>|x_0|$ tengsizlikni qanoatlantiruvchi hamma qiymatlarida uzoqlashuvchi bo‘ladi.

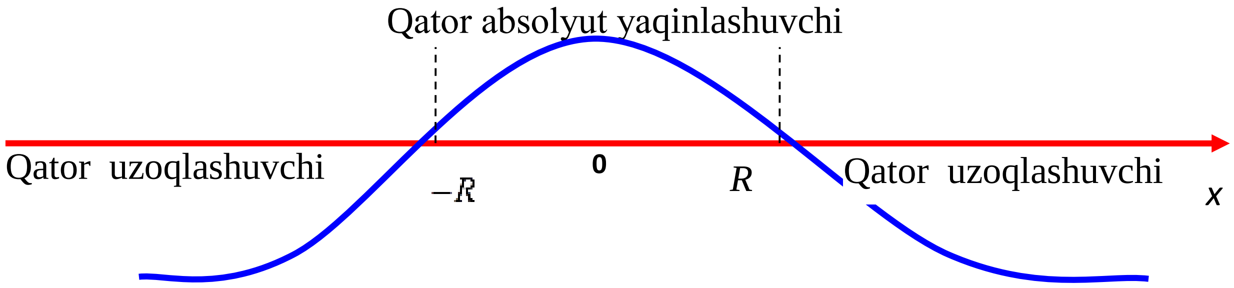
Abel teoremasi darajali qatorning yaqinlashish va uzoqlashish nuqtalarining joylashuvi haqida mulohaza yuritish imkonini beradi. Haqiqatdan, agar $x = x_0$ nuqta yaqinlashish nuqtasi bo‘lsa, u holda $(-|x_0|, |x|)$ oraliqning hammasi absolyut yaqinlashish nuqtalari bilan to‘ldirilgan. Agar $x = x_1$ nuqta uzoqlashish nuqtasi bo‘lsa, u holda $|x_1|$ dan o‘ngdagi cheksiz yarim to‘g‘ri chiziqning va $-|x_1|$ dan chapdagi cheksiz yarim to‘g‘ri chiziqning hammasi uzoqlashish nuqtalaridan iborat bo‘ladi (4.2-rasm).



4.2-rasm. Qarotning yaqinlashish va uzoqlashish oraliqlari

Bundan shunday ***R*** son mavjudligi va $|x| < R$ da absolyut yaqinlashish, $|x| > R$ da esa uzoqlashish nuqtalariga ega bo‘lishi kelib chiqadi. Shunday qilib, darajali qatorning **yaqinlashish sohasi markazi koordinatalar boshida bo‘lgan oraliqdan iborat** (4.3-rasm).

$a_0 + a_1x_0 + a_2x_0^2 + \ldots + a_nx_0^n + \ldots$ darajali qatorning **yaqinlashish sohasi** deb, shunday $(-R, \; R)$ oraliqqa aytiladiki, bu oraliqning ichidagi har qanday x nuqtada qator yaqinlashadi va shu bilan birga absolyut yaqinlashadi, oraliqdan tashqarida yotuvchi x nuqtalarda qator uzoqlashadi. **R** soni darajali **qatorning yaqinlashish radiusi** deyiladi.



4.3-rasm. Qarotning yaqinlashish radiuslari

Oraliqning chetki nuqtalarida, ya’ni $x = R$ va $x = -R$ nuqtalarda berilgan qatorning yaqinlashishi yoki uzoqlashishi masalasi har bir qator uchun alohida hal qilinadi.

Ba’zi qatorlar uchun yaqinlashish intervali nuqtaga aylanib qoladi, u holda $R=0$ bo‘ladi; ba’zilari uchun esa butun Ox o‘qini qamrab oladi, ya’ni $R=\infty$ bo‘ladi.

Darajali qatorning yaqinlashish radiusini aniqlash uchun formula keltirib chiqaramiz, ya'ni (4.9) darajali qator hadlarining absolyut qiymatlaridan tuzilgan

$$\sum_{n=1}^{\infty}\left|a_n x^n\right| = |a_0| + |a_1 x| + \left|a_2 x^2\right| + \ldots + \left|a_n x^n\right| + \ldots \qquad (4.10)$$

musbat hadli qatorni qaraymiz. (4.10) qatorni Dalamber alomatiga ko‘ra yaqinlashishga tekshiramiz:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}\cdot x^{n+1}}{a_n\cdot x^n}\right| = |x|\cdot\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\cdot l$$

mavjud bo‘lsin.

1) agar $|x|\cdot l<1 \to |x|<\frac{1}{l}$ bo‘lsa, qator yaqinlashuvchi,

2) agar $|x|\cdot l>1 \to |x|>\frac{1}{l}$ bo‘lsa, qator uzoqlashuvchi bo‘ladi.

Demak, (4.9) darajali qator $x\in\left(-\frac{1}{l}\,;\frac{1}{l}\right)$ oraliqda absolyut yaqinlashuvchi va $x\in\left(-\infty;-\frac{1}{\ell}\right)U\left(\frac{1}{\ell};+\infty\right)$ oraliqda uzoqlashuvchi bo‘ladi.

Yuqoridagilardan $x\in\left(-\frac{1}{l}\,;\frac{1}{l}\right)$ oraliq (4.9) qatorning yaqinlashish oralig‘i ekanligi kelib chiqadi: $R=\frac{1}{l}=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$.

Bu (4.9) darajali qatorning yaqinlashish radiusini topish formulasidir.

$x\in(-R\ ,R)$ oraliqqa (4.9) darajali qatorning **yaqinlashish oralig‘i** deyiladi.

Eslatma:

1) Agar $R=0$ bo‘lsa, (4.8)-qator $x=0$ dan tashqari barcha nuqtalarda uzoqlashuvchi bo‘lishi mumkin.

2) Agar $R\to\infty$ bo‘lsa, u holda $x\in(-\infty\,;+\infty)$ da (4.8) qator absolyut yaqinlashuvchi bo‘ladi.

3) $\sum\limits_{n=1}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$

umumlashgan darajali qatorning **yaqinlashish oralig‘i**

$$|x-a|<R \Leftrightarrow -R<x-a<R \Leftrightarrow a-R<x<R+a$$

dan iborat bo‘ladi.

Yaqinlashish oralig‘ini aniqlash uchun Koshi alomatidan ham foydalanish mumkin: $R=\frac{1}{l}=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|u_n|}}$.

4.2.7-misol. $\frac{2x}{1}-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-...+(-1)^{n+1}\cdot\frac{(2x)^n}{n}+...$ darajali qatorning yaqinlashish oralig‘ini toping.

Yechilishi: ► Dalamber alomatidan foydalanamiz:

$$a_n=(-1)^{n+1}\cdot\frac{2^n}{n}, \qquad a_{n+1}=(-1)^{n+2}\cdot\frac{2^{n+1}}{n+1}.$$

Bundan $R=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim\limits_{n\to\infty}\frac{2^n\cdot(n+1)}{n\cdot 2^{n+1}}=\frac{1}{2}\lim\limits_{n\to\infty}\frac{n+1}{n}=\frac{1}{2}$.

Demak, $x\in\left(-\frac{1}{2},\ \frac{1}{2}\right)$ yaqinlashish oralig‘i bo‘ladi.

Oraliq chetlarida qator o‘zini qanday tutadi?

$x=\frac{1}{2}$ da $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$ qatorga ega bo‘lamiz, bu qator Leybnits alomatiga ko‘ra yaqinlashuvchi, chunki quyidagi shartlarni qanoatlantiradi:

$a)$ $1>\frac{1}{2}>\frac{1}{3}>\frac{1}{4}>...>\frac{1}{n}>...$

$b)$ $\lim\limits_{n\to\infty}|u_n|=\lim\limits_{n\to\infty}\frac{1}{n}=\frac{1}{\infty}=0$.

$x=-\frac{1}{2}$ da $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-...$ qatorga ega bo‘lamiz, bu garmonik qator bo‘lib, uzoqlashuvchi.

Shunday qilib, qator yaqinlashish oralig‘i quyidagicha: $x\in\left(-\frac{1}{2},\ \frac{1}{2}\right]$ ◄

4.2.8-misol. Ushbu $\sum\limits_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$ funksional qatorning yaqinlashish oralig‘ini toping.

Yechilishi: ► Dalamber alomatidan foydalanib topamiz:

$$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right|=\lim\limits_{n\to\infty}\left|\frac{x^{n+1}}{1+x^{2n+2}}:\frac{x^n}{1+x^{2n}}\right|=\lim\limits_{n\to\infty}\left|\frac{x\cdot(1+x^{2n})}{1+x^{2n+2}}\right|;$$

a) $x\in(-1{,}1)$ da $\lim\limits_{n\to\infty}\left|\frac{x\cdot(1+x^{2n})}{1+x^{2n+2}}\right|=|x|$.

Bu holda berilgan funksional qator $(-1{,}1)$ da yaqinlashuvchi bo‘ladi.

b) $x\in(-\infty,-1)\cup(1,+\infty)$ da $\lim\limits_{n\to\infty}\left|\dfrac{x\cdot\left(1+x^{2n}\right)}{1+x^{2n+2}}\right|=\lim\limits_{n\to\infty}\left|\dfrac{\frac{1}{x^{2n+1}}+\frac{1}{x}}{\frac{1}{x^{2n+2}}+1}\right|=\left|\dfrac{1}{x}\right|$

bo‘lib, funksional qator $x\in(-\infty,-1)\cup(1,+\infty)$ da yaqinlashuvchi bo‘ladi.

v) $x=\pm 1$ oraliq chetlarida berilgan funksional qator mos ravishda $\sum\limits_{n=1}^{\infty}\dfrac{1}{2}$, $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{2}$ sonli qatorlarga aylanadi va ular uzoqlashuvchi bo‘ladi. Shunday qilib, qaralayotgan qatorning yaqinlashish oralig‘i

$R\setminus\{-1,1\}=(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$ bo‘ladi. ◄

4.2.9-misol. $\sum\limits_{n=1}^{\infty}\dfrac{x^{2n}}{(-3)^n}$ darajali qatorning yaqinlashish oralig‘ini va radiusini toping.

Yechilishi: ► Koshi alomatidan foydalanamiz:

$$l=\lim_{n\to\infty}\sqrt[n]{\left|\frac{x^{2n}}{(-3)^n}\right|}=\lim_{n\to\infty}\left|\frac{x^n}{-3}\right|=\frac{x^2}{3}<1$$

$$x^2<3 \quad\Rightarrow\quad \sqrt{x^2}<\sqrt{3} \quad\Rightarrow\quad |x|<\sqrt{3}.$$

Demak, yaqinlashish radiusi $R=\sqrt{3}$, yaqinlashish oralig‘ini topamiz: $-\sqrt{3}<x<\sqrt{3}$. Oraliqning chetki nuqtalarida tekshiramiz:

$x=-\sqrt{3}$ da $\sum\limits_{n=1}^{\infty}\dfrac{(-\sqrt{3})^{2n}}{(-3)^n}=\sum\limits_{n=1}^{\infty}\dfrac{\left((-\sqrt{3})^2\right)^n}{(-3)^n}=\sum\limits_{n=1}^{\infty}\dfrac{3^n}{(-1)^n\cdot 3^n}=\sum\limits_{n=1}^{\infty}(-1)^n$ qator uzoqlashuvchi;

$x=\sqrt{3}$ da ham $\sum\limits_{n=1}^{\infty}\dfrac{(\sqrt{3})^{2n}}{(-3)^n}=\sum\limits_{n=1}^{\infty}(-1)^n$ qator uzoqlashuvchi. Bundan berilgan qatorning yaqinlashish oralig‘i $-\sqrt{3}<x<\sqrt{3}$ ekanini bildiradi◄

Absolyut yaqinlashuvchi darajali qatorning xossalari:

1^0. Agar darajali qator $x\in(-R\,,\;R)$ intervalda yaqinlashsa, u holda bu intervalning hamma nuqtalarida qator yig‘indisi uzluksiz funksiyadir;

2^0. Yaqinlashish intervalining barcha ichki nuqtalarida darajali qatorni hadma-had differensiallash mumkin;

3^0. Darajali qatorni ixtiyoriy $x\in[a\,;b]\subset(-R,\,R)$ oraliqda hadma-had integrallash mumkin.

4.2.10-misol. $f(x)=\dfrac{1}{1+x^3}$ funksiyani darajali qator ko‘rinishida yozing va uning yaqinlashish oralig‘ini toping.

Yechilishi: ► Geometrik qatorni esga olaylik: $\sum\limits_{n=0}^{\infty} aq^n=\dfrac{a}{1-q}$, bu qator $|q|<1$ da yaqinlashuvchi. Aksincha, $|q|\geq 1$ da qator uzoqlashuvchi bo‘ladi. Faraz qilaylik, $a=1$ va $q=x$ bo‘lsin. U holda geometrik qatorni boshqacha yozish mumkin: $\sum\limits_{n=0}^{\infty} x^n=\dfrac{1}{1-x}$, $|x|<1$. Bundan quyidagini hosil qilamiz: $\dfrac{1}{1+x^3}=\dfrac{1}{1-(-x^3)}=\sum\limits_{n=0}^{\infty}(-x^3)^n=\sum\limits_{n=0}^{\infty}(-1)^n x^{3n}$. Yaqinlashish oralig‘i $|x|<1$ bo‘ladi. ◄

4.2.11-misol. $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n n}{4^n}(x+3)^n$ qatorning yaqinlashish oralig‘ini toping.

Yechilishi: ► Yaqinlashish oralig‘ini topish uchun Dalamber alomatidan foydalanamiz:

$$l=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{(n+1)\cdot(x+3)^{n+1}}{4^{n+1}}}{\dfrac{n\cdot(x+3)^n}{4^n}}\right|=\lim_{n\to\infty}\left|\frac{(n+1)\cdot(x+3)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{n(x+3)^n}\right|=\lim_{n\to\infty}\left|\frac{(x+3)}{4}\right|<1;$$

$$|x+3|<4$$

$$-4<x+3<4$$

$$-7<x<1$$

(-7;1) yaqinlashish oralig‘i ekanini aniqladik, endi oraliq chetlarida qator o‘zini qanday tutishini tekshiramiz.

$x=-7$ nuqtada qatorni yaqinlashishga tekshiramiz:

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-7+3)^n=\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-4)^n=\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-1)^n\cdot 4^n=\sum_{n=1}^{\infty}(-1)^{2n}n=\sum_{n=1}^{\infty}n$$

$\lim\limits_{n\to\infty} n=\infty$, demak $x=-7$ nuqtada qator uzoqlashuvchi;

Endi $x=1$ nuqtada qatorni yaqinlashishga tekshiramiz:

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(1+3)^n=\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}4^n=\sum_{n=1}^{\infty}(-1)^n n,$$

$\lim\limits_{n\to\infty}(-1)^n n$ limitdan $x=1$ nuqtada ham qator uzoqlashuvchi ekanligi kelib chiqadi. Shunday qilib, qatorning yaqinlashish sohasi $-7<x<1$, ya’ni $(-7,1)$ oraliqdan iborat. ◄

4.2.12-misol. $\sum_{n=1}^{\infty}\frac{2^n}{n}(4x-8)^n$ qatorning yaqinlashish oralig‘ini toping.

Yechilishi: ► Yaqinlashish oralig‘ini topish uchun Dalamber alomatidan foydalanamiz:

$$l=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(4x-8)^{n+1}}{n+1}}{\frac{2^n(4x-8)^n}{n}}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}(4x-8)^{n+1}}{n+1}\cdot\frac{n}{2^n(4x-8)^n}\right|=\lim_{n\to\infty}\left|2(4x-8)\right|<1$$

$$|4x-8|<\frac{1}{2}\qquad\Rightarrow\qquad -\frac{1}{2}+8<4x<\frac{1}{2}+8$$

$$\frac{15}{2}<4x<\frac{17}{2}\qquad\Rightarrow\qquad \frac{15}{8}<x<\frac{17}{8}$$

$x=\frac{15}{8}$ da qatorni yaqinlashishga tekshiramiz:

$$\sum_{n=1}^{\infty}\frac{2^n}{n}(4x-8)^n=\sum_{n=1}^{\infty}\frac{2^n}{n}\left(4\cdot\frac{15}{8}-8\right)^n=\sum_{n=1}^{\infty}\frac{2^n}{n}\left(-\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

$\lim\limits_{n\to\infty}\frac{(-1)^n}{n}$, demak $x=\frac{15}{8}$ da qator yaqinlashuvchi;

Endi $x=\frac{17}{8}$ da qatorni yaqinlashishga tekshiramiz:

$$\sum_{n=1}^{\infty}\frac{2^n}{n}(4x-8)^n=\sum_{n=1}^{\infty}\frac{2^n}{n}\left(4\cdot\frac{17}{8}-8\right)^n=\sum_{n=1}^{\infty}\frac{2^n}{n}\cdot\left(\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{1}{n}$$

dan $x=\frac{17}{8}$ da ham qator yaqinlashuvchi ekanligi kelib chiqadi. Shunday qilib, qatorning yaqinlashish sohasi $\frac{15}{8}\le x\le\frac{17}{8}$, ya’ni $\left[\frac{15}{8};\frac{17}{8}\right]$ oraliqdan iborat. ◄

4.2.3. Teylor va Makloren qatorlari

$y=f(x)$ funksiya ***a*** nuqtada va uning biror atrofida uzluksiz va a nuqtada istalgan tartibda hosilaga ega bo‘lsin. U holda $y=f(x)$ funksiyani darajali qator ko‘rinishida tasvirlash mumkin va aksincha, ya’ni har doim hosil bo‘lgan darajali qatorni berilgan $y=f(x)$ ko‘rinishida tasvirlash mumkin:

$$f(x)=c_0+c_1\cdot(x-a)+c_2\cdot(x-a)^2+...+c_n\cdot(x-a)^n+... \qquad (4.11)$$

Endi $y=f(x)$ funksiyaning darajali qator koeffitsiyentlari bilan qanday bog‘langanligini ko‘ramiz. (4.11) da $x=a$ deb olsak, $f(a)=c_0$ ekanligini topamiz. Faraz qilaylik, $y=f(x)$ funksiyani yaqinlashish intervali a nuqtaning biror atrofida bo‘lsin, u holda qatorni bu atrofda hadma- had differensiallash mumkin, ya’ni

$$f'(x)=c_1+2c_2(x-a)+...+n\cdot c_n(x-a)^{n-1}+... \quad (4.12)$$

(4.12) da $x=a$ desak $f'(a)=c_1$ bo‘ladi.

$$f''(x)=2\cdot c_2+2\cdot 3\cdot c_3(x-a)+...+n\cdot(n-1)\cdot c_n\cdot(x-a)^{n-2}+... \quad (4.13)$$

(4.13) da $x=a$ desak $f'(a)=2c_2 \Rightarrow c_2=\dfrac{f''(a)}{2!}$

$$f'''(x)=2\cdot 3\cdot c_3+2\cdot 3\cdot 4\cdot c_4(x-a)+........+n(n-1)(n-2)\cdot c_n(x-a)^{n-3}+... \quad (4.14)$$

(4.14) da $x=a$ desak, Teylor koeffitsiyenti quyidagicha topiladi:

$$f'''(a)=2\cdot 3\cdot c_3 \quad \Rightarrow \quad c_3=\frac{f'''(a)}{3!},\ \ldots,\quad c_n=\frac{f^{(n)}(a)}{n!} \quad (4.15)$$

(4.15) Teylor koeffitsiyentlarini (4.11) ga qo‘yamiz.

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+...+\frac{f^{(n)}(a)}{n!}(x-a)^n+... \quad (4.16)$$

(4.16) formula $y=f(x)$ funksiyani a nuqtaning atrofidagi **Teylor qatori** deyiladi.

$y=f(x)$ funksiya $x=a$ nuqtada $(n+1)$ - tartibgacha hosilalarga ega bo‘lsa, u holda quyidagi **Teylor formulasi** o‘rinlidir:

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)^2+\frac{f''(a)}{2!}(x-a)^2+....+\frac{f^{(n)}(a)}{n\,!}(x-a)^n+R_n(x),$$

bu yerda $R_n(x)=\dfrac{f^{(n+1)}[a+\theta(x-a)]}{(n+1)!}(x-a)^{n+1}\ (0<\theta<1)$ bo‘lib, **Lagranj shaklidagi qoldiq hadi** deyiladi.

$a=0$ da Teylor formulasining xususiy holi - **Makloren formulasi** hosil bo‘ladi:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+R_n(x),$$

$$\text{bu yerda}\quad R_n(x)=\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1},\ \ (0<\theta<1).$$

$y=f(x)$ funksiya a nuqta atrofida istalgan marta differensiallanuvchi bo‘lsa va bu nuqtaning biror atrofida $\lim\limits_{n\to\infty}R_n(x)=0$ bo‘lsa, Teylor va Makloren formulalaridan

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+....+\frac{f^{(n)}(a)}{n!}(x-a)^n+....$$

va

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+....$$

qatorlar hosil bo‘ladi. Bularning birinchisi **Teylor qatori,** ikkinchisiga **Makloren qatori** deyiladi. Bu qatorlar x ning $\lim\limits_{n\to\infty}R_n(x)=0$ bo‘ladigan qiymatlarida $y=f(x)$ ga yaqinlashadi.

a nuqtani o‘z ichiga oluvchi biror intervalda istalgan n uchun $\left|f^{(n)}(x)\right|<M$, ($M$ biror musbat son) tengsizlik bajarilsa, $\lim\limits_{n\to\infty}R(x)=0$ bo‘ladi va $y=f(x)$ funksiya Teylor qatoriga yoyiladi.

4.2.13-misol. $f(x)=e^x$ funksiyani darajali qatorga yoying.

Yechilishi:► $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+...$ (4.17)

Makloren qatoriga yoyishda istalgan x uchun quyidagi hosilalarni topamiz:

$$f'(x)=e^x,$$

$$f''(x)=e^x,$$

...

$$f^{(n)}(x)=e^x,...$$

$$x=0 \quad deb \quad f(0)=1,\ \ f'(0)=1,\ \ f''(0)=1,\ ...,\ f^{(n)}(0)=1,...$$

Bularni Makloren qatoriga qo‘yib,

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+... \quad (-\infty<x<+\infty)$$

ni hosil qilamiz. Oxirgi tenglikdan $x=1$ desak,

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}+....$$

bo‘lib, e soni qator yig‘indisi ko‘rinishida ifodalanadi. Bundan foydalanib e sonining taqribiy qiymatini istalgan darajadagi aniqlikkacha hisoblash mumkin.◄

4.2.14 - misol. $f(x)=\sin x$ funksiyani darajali qatorga yoying.

Yechilishi: ► $f(x)=\sin x$ funksiyani (4.17) qatoriga yoyishda istalgan x uchun quyidagi hosilalarni topamiz:

$$f'(x)=\cos x,$$
$$f''(x)=-\sin x,$$
$$f'''(x)=-\cos x,$$
$$f^{IV}(x)=\sin x, \ldots .$$

Bundan $f(0)=0,\ f'(0)=1,\ f''(0)=0,\ f'''(0)=-1,\ f^{IV}(0)=0, \ldots$ bo‘lib, bularni Makloren qatoriga qo‘ysak,

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\ldots$$

hosil bo‘ladi. Bu qator istalgan x uchun yaqinlashuvchi $-\infty<x<+\infty$. Oxirgi qatorni hadlab differensiallasak,

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!}+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots$$

qator hosil bo‘ladi, bu $f(x)=\cos x$ funksiya uchun Makloren qatori bo‘ladi. ◀

Qatorlarning taqribiy hisoblashga tatbiqlari

4.2.15-misol. $\cos x$ ning yoyilmasidan foydalanib $\cos 18^0$ ni $0{,}001$ aniqlikkacha taqribiy hisoblang.

Yechilishi: ▶ $\cos x$ funksiyaning qatorga yoyilmasidan foydalanib,

$$\cos 18^0 = \cos\frac{\pi}{10} = 1-\frac{1}{2!}\left(\frac{\pi}{10}\right)^2+\frac{1}{4!}\left(\frac{\pi}{10}\right)^4-\ldots$$

qatorni hosil qilamiz.

$$\frac{\pi}{10}=0{,}31416;\ \left(\frac{\pi}{10}\right)^2=0{,}09870;\ \left(\frac{\pi}{10}\right)^4=0{,}00974$$

va $\frac{1}{6!}\cdot\left(\frac{\pi}{10}\right)^6<0{,}0001$ bo‘lganligi uchun, taqribiy hisoblashda qatorning birinchi uchta hadi bilan chegaralanamiz, demak

$$\cos 18^0 \approx 1-\frac{0{,}09870}{2}+\frac{0{,}00974}{24} \quad \text{yoki} \quad \cos 18^0\approx 0{,}9511.$$ ◀

4.2.16-misol. $\sqrt[5]{1{,}1}$ *ni* $0{,}0001$ aniqlikkacha taqribiy hisoblang.

Yechilishi: ▶ $\sqrt[5]{1{,}1}=(1+0{,}1)^{\frac{1}{5}}$ deb, binomial qatordan foydalansak:

$$\sqrt[5]{1,1}=(1+0,1)^{\frac{1}{5}}=1+\frac{1}{5}\cdot 0,1+\frac{\frac{1}{5}\cdot(\frac{1}{5}-1)}{2!}0,01+\frac{\frac{1}{5}\cdot(\frac{1}{5}-1)\cdot(\frac{1}{5}-2)}{3!}0,001+$$
$$+...=1+0,02-0,0008+0,000048-....$$

bo‘ladi. To‘rtinchi had $0.000048<0.0001$ bo‘lganligi uchun, hisoblashda birinchi uchta hadini olib, hisoblaymiz:

$$\sqrt[5]{1,1}\approx 1+0,02-0,0008=1,0192\,.$$ ◀

4.2.17-misol. $\sqrt[3]{130}$ *ni* $0,001$ aniqlikkacha taqribiy hisoblang.

Yechilishi: ▶ 5^3 soni 130 ga eng yaqin butun sonning kubi bo‘lganligi uchun $130=5^3+5$ deb olish qulay bo‘lib,

$$\sqrt[3]{130}=\sqrt[3]{5^3+5}=\sqrt[3]{5^3(1+\frac{1}{25})}=5(1+\frac{1}{25})^{\frac{1}{3}}=5(1+\frac{1}{3}0,04+\frac{\frac{1}{3}\cdot(\frac{1}{3}-1)}{2!}0,0016+$$
$$+\frac{(\frac{1}{3}(\frac{1}{3}-1)\cdot(\frac{1}{3}-2))}{3!}0,000064+....)=5+\frac{1}{3}\cdot 0,2\cdot\frac{1}{9}\cdot 0,0016+5\frac{25}{81}\cdot 0,000032-...$$

oxirgi qatorda to‘rtinchi had $0,001$ dan kichik bo‘lganligi uchun, birinchi uchta had bilan chegaralanamiz:

$$\sqrt[3]{130}\approx 5+0,0667-0,0009\approx 5,066.$$ ◀

4.2.18-misol. $\ln 1,04$ *ni* $0,0001$ gacha aniqlikda taqribiy hisoblang.

Yechilishi: ▶ $\ln(1+x)$ funksiyaning darajali qatorga yoyilmasidan foydalanib,

$$\ln(1+0,04)=0,04-\frac{0,04^2}{2}+\frac{0,04^3}{3}-\frac{0,04^4}{4}+....,$$

yoki

$$\ln 1,04=0,04-0,0008+0,000021-0,00000064+...$$

qatorni hosil qilamiz, hamda uchinchi had $0,0001$ dan kichik bo‘lganligi uchun birinchi ikki hadni hisobga olib hisoblaymiz: $\ln 1,04\approx 0,0392.$ ◀

Mavzu yuzasidan savollar:

1. Funksional qator deb nimaga aytiladi?
2. Funksional qatorni yaqinlashish sohasi deb nimaga aytiladi?
3. Funksional qatorning yaqinlashish radiusi deb nimaga aytiladi?
4. Yaqinlashish sohasi bitta nuqtadan iborat qatorga misol keltiring.

5. Funksional qator uzoqlashuvchiligini qanday izohlaysiz?
6. Yaqinlashish alomatlari funksional qatorlarda qanday ishlatiladi?
7. Makloren va Teylor formulalari qanday bo‘ladi?
8. Teylor qatori deb qanday qatorga aytiladi?
9. Qanday funksiyalarni Teylor qatoriga yoyish mumkin?
10. Makloran qatori formulasini yozing.

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $\sum_{n=1}^{\infty}\frac{(2x-3)^n}{2n-1}$ darajali qatorning yaqinlashish sohasini toping.

2. $\sum_{n=1}^{\infty}\frac{n!x^n}{(n+1)^n}$ darajali qatorning yaqinlashish sohasi toping.

3. $y=\ln(1+x)$ funksiyani Makloran qatoriga yoying.

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ qatorning yig‘indisini toping.

5. $\sin 32^0$ ni 0.001 aniqlikda taqribiy hisoblang.

TESTLAR

1. $\sum_{n=1}^{\infty}\frac{(x+1)^n}{(2n-1)!}$ darajali qatorning yaqinlashish sohasini toping.

A) $(0,+\infty)$ **B)** $(-\infty,+\infty)$ **C)** $(-1,1)$ **D)** $[-1,1]$

2. $\sum_{n=1}^{\infty}n!x^{n-1}$ darajali qatorning yaqinlashish sohasini toping.

A) $(0,+\infty)$ **B)** $(-\infty,+\infty)$ **C)** $(-1,1)$ **D)** $[-1,1]$

3. $\sum_{n=1}^{\infty}\frac{(x-3)^{2n}}{(n+1)\ln(n+1)}$ darajali qatorning yaqinlashish sohasini toping.

A) $(2,4]$ **B)** $(2,4)$ **C)** $(-1,1)$ **D)** $[-1,1]$

4. $y=\sin x$ funksiyani Makloran qatoriga yoyilmasini toping.

A) $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n+1)!}$ **B)** $\sum_{n=0}^{\infty}(-1)^{2n-1}\frac{x^{2n-1}}{(2n+1)!}$

C) $\sum_{n=0}^{\infty}(-1)^{2n+1}\frac{x^{2n-1}}{(2n)!}$ **D)** $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$

5. $\sum_{n=1}^{\infty}\frac{x^n}{n}$ qatorning yig‘indisini toping, $(-1<x<1)$.

A) $-\ln(1-x)$ **B)** $\ln(1-x)$ **C)** $-\ln(1+x)$ **D)** $\ln(1+x)$

4.3-§. Furye qatori va uning tatbiqlari

4.3.1. Ortogonal va ortonormal funksiyalar sistemasi

Agar $$\int_a^b \varphi_n(x)\cdot\varphi_m(x)dx = 0 \quad (m \neq n) \qquad (4.18)$$

shart bajarilsa, funksiyalarning $\{\varphi_n(x)\}$: $\varphi_1(x)$, $\varphi_2(x)$, ..., $\varphi_n(x)$ cheksiz sistemasi [a;b] kesmada **ortogonal sistema** deyiladi.

Ushbu $$\|\varphi_n(x)\| = \sqrt{\int_a^b \varphi_n^2(x)dx} \qquad (4.19)$$

ifodaga $\varphi_n(x)$ funksiyaning **normasi** deyiladi.

4.3.1-misol. $(-\pi,\pi)$ oraliqda $f(x)=\sin 5x$ va $\varphi(x)=\cos 2x$ funksiyalarning ortogonalligini tekshiring.

Yechilishi: ► Berilgan funksiyalar ko‘paytmasini $(-\pi,\pi)$ oraliqda integrallaymiz:

$$\int_{-\pi}^{\pi}\cos 2x\cdot\sin 5x dx = \frac{1}{2}\int_{-\pi}^{\pi}(\sin 7x+\sin 3x)dx = \left[-\frac{1}{14}\cos 7x-\frac{1}{6}\cos 3x\right]_{-\pi}^{\pi} = -\frac{1}{14}(\cos 7\pi)-\cos 7\pi)- -\frac{1}{6}(\cos 3\pi-\cos 3\pi)=0-0=0.$$ ◄

4.3.2-misol. $\left(-\frac{\pi}{4},\ \frac{7}{4}\pi\right)$ oraliqda $f(x)=\sin 2x$ va $\varphi(x)=\sin 4x$ funksiyalarning ortogonalligini tekshiring.

Yechilishi: ► $$\int_{-\frac{\pi}{4}}^{\frac{7}{4}\pi}\sin 2\pi\cdot\sin 4x dx = \int_0^{2\pi}\sin 2\pi\cdot\sin 4\pi dx = 0.$$ ◄

Ortogonal funksiyalar sistemasiga misol qilib $[-\pi,\pi]$ da aniqlangan 1; $\cos x$; $\sin x$; $\cos 2x$; $\sin 2x$; ...;$\cos nx$; $\sin nx$; ... trigonometrik funksiyalar sistemasini ko‘rsatish mumkin.

1) $$\int_{-\pi}^{\pi} 1\cdot\cos nx dx = \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi} = 0;$$

2) $$\int_{-\pi}^{\pi} 1\cdot\sin nx dx = -\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi} = 0;$$

3) $$\int_{-\pi}^{\pi}\cos nx\cdot\cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(n+m)x+\cos(n-m)x]dx = 0;$$

4) $\int\limits_{-\pi}^{\pi} \cos nx \cdot \sin mx dx = 0$;

5) $\int\limits_{-\pi}^{\pi} \sin nx \cdot \sin mx dx = 0$.

1) - 5)tengliklardan (4.18) shart o‘rinli ekani kelib chiqadi.

6) $\int\limits_{-\pi}^{\pi} 1^2 dx = x\Big|_{-\pi}^{\pi} = 2\pi \neq 0$

7) $\int\limits_{-\pi}^{\pi} \cos^2 nx dx = \frac{1}{2}\int\limits_{-\pi}^{\pi} (1+\cos 2nx)dx = \pi \neq 0$

8) $\int\limits_{-\pi}^{\pi} \sin^2 nx dx = \frac{1}{2}\int\limits_{-\pi}^{\pi} (1-\cos 2nx)dx = \pi \neq 0$

6)-8) formulalar (4.19) shartdir.

Biz $[-\pi, \pi]$ da trigonometrik funktsiyalar sistemasi ortogonal ekanligini ko‘rsatdik. 6)-8) lardan

$$\| 1 \| = \sqrt{2\pi}, \quad \|\cos nx\| = \sqrt{\pi}, \quad \|\sin nx\| = \sqrt{\pi}$$

ekanligi ko‘rinib turibdi.

Ortogonal funksiyalar sistemasi bilan birga ortonormal funksiyalar sistemasini ham qarash mumkin.

Agar $\int\limits_a^b \varphi_n^2(x)dx = 1$ bo‘lsa, u holda $\{\varphi_n(x)\}$ funksiya sistemasi [a;b]da **normallangan** sistema deyiladi.

$\{\varphi_n(x)\}$ cheksiz funksiyalar sistemasi ortogonal va normallangan, ya’ni $\int\limits_a^b \varphi_n(x)\cdot\varphi_m(x)dx = \begin{cases} 0, \text{ agar } m \neq n \\ 1, \text{ agar } m = n \end{cases}$ bo‘lsa, bu sistema [a;b] kesmada **ortonormallangan sistema** deyiladi.

$[-\pi, \pi]$da ushbu

$$1;\ \cos\frac{\pi x}{l};\ \sin\frac{\pi x}{l};\ \cos\frac{2\pi x}{l};\ \sin\frac{2\pi x}{l};\ \ldots;\ \cos\frac{n\pi x}{l};\ \sin\frac{n\pi x}{l};\ \ldots\ .$$

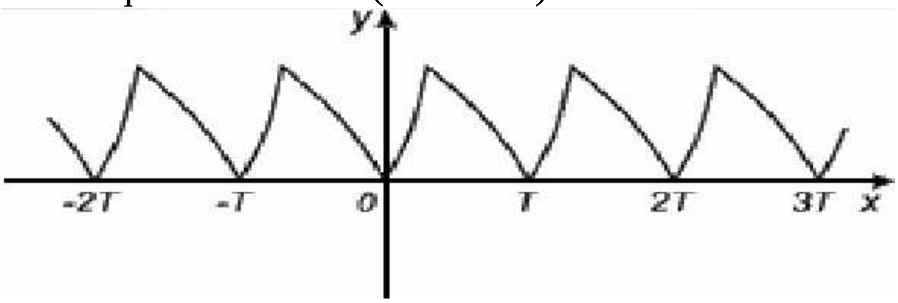
umumlashgan ortogonal trigonometrik funksiyalar sistemasini qaraymiz.

$y = f(x)$ funksiya $R = (-\infty, +\infty)$ to‘plamda berilgan bo‘lsin. Ma’lumki, shunday $T \in R \setminus \{0\}$ son topilsaki, $\forall x \in R$ da $f(x+T) = f(x)$ tenglik bajarilsa, $y = f(x)$ **davriy funksiya**, $T \neq 0$ son esa uning **davri** deyiladi.

Agar $y = f(x)$ davriy funksiya bo‘lib, $T \neq 0$ son uning davri bo‘lsa, kT sonlar $(k = \pm 1, \pm 2, \ldots)$ ham shu funksiyaning davri bo‘ladi.

Agar $f(x)$ va $g(x)$ davriy funksiyalar bo‘lib, $T \neq 0$ ularning davri bo‘lsa, $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$ funksiyalar ham davriy bo‘lib, ularning davri ham T ga teng bo‘ladi.

Aytaylik, $y = f(x)$ davriy funksiya bo‘lib, uning davri T bo‘lsin. Agar bu funksiya grafigining tasviri $[a, a+T]$ oraliqda $(a \in R)$ ma’lum bo‘lsa, uni birin – ketin $x = a + kT$ $(k = \pm 1, \pm 2, \ldots)$ vertikal to‘gri chiziqqa nisbatan simmetrik ko‘chirish natijasida $y = f(x)$ ning $(-\infty, +\infty)$ dagi grafigini hosil qilish mumkin (4.4-rasm):



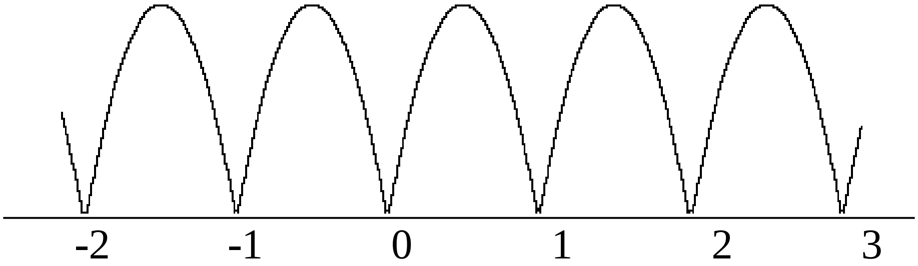
4.4-rasm. Davriy funksiya

Bu jarayonni $[a, a+T]$ da berilgan funksiyani $a+T$ ga davriy davom ettirish deb yuritiladi.

Shuni ta’kidlash lozimki, T davrli $y = f(x)$ funksiya $[a, a+T]$ da uzluksiz bo‘lsa, uni $a+T$ ga davriy davom ettirishdan hosil bo‘lgan funksiya (uni ham $y = f(x)$ deymiz) $a+T$ da uzluksiz yoki bo‘lakli uzluksiz (ya’ni $x = a + kT$ nuqtalarda uzilishga ega bo‘lib, boshqa barcha nuqtalarda uzluksiz) bo‘lishi mumkin.

4.3.3-misol. $f(x) = 2\sqrt{x(1-x)}$ funksiyani $[0,1]$ da berilgan. Uni $a+T$ oraliuqda davriy davom ettirishdan hosil bo‘lgan funksiya grafgini chizing.

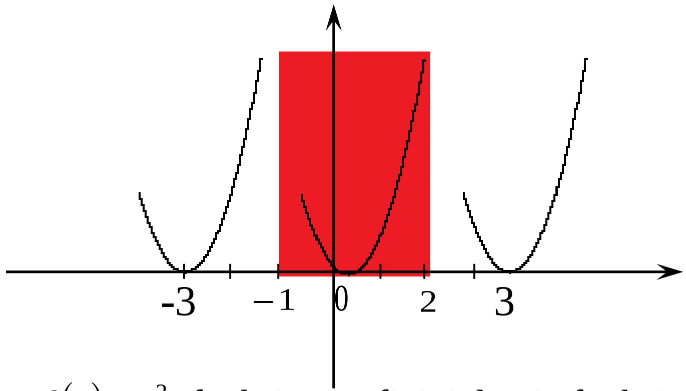
Yechilishi: ► $f(x) = 2\sqrt{x(1-x)}$ funksiya grafigini $[0,1]$ oraliqda chizamiz va uni $a+T$ da takrorlaymiz, ushbu davriy funksiya $a+T$ da uzluksiz bo‘ladi (4.5-rasm):



4.5-rasm. Davriy funksiya ◄

4.3.4-misol. $[-1,2]$ da berilgan $f(x)=x^2$ funksiya grafigini chizing va $a+T$ da davriy davom ettiring.

Yechilishi: ► $f(x)=x^2$ funksiya grafigini $[-1,2]$ kesmada chizamiz. Hosil bo'lgan grafikni $a+T$ da davriy davom ettiramiz, natijada hosil bo'lgan funksiya $a+T$ da bo'lakli uzluksiz bo'ladi (4.6-rasm):



4.6-rasm. $f(x)=x^2$ funksiya grafigini davriy funksiyaga aylantirish ◄

4.1-lemma. Agar $y=f(x)$ davriy funksiya, uning davri T bo'lib, $[a,a+T]$ da integrallanuvchi bo'lsa, u holda

$$\int_a^{a+T} f(x)dx = \int_b^{b+T} f(x)dx, \qquad (b\in R)$$

bo'ladi. Ushbu $f(x)=A\sin(\alpha x+\beta)$ funksiyani qaraylik, bunda A,α,β – haqiqiy sonlar. Bu davriy funksiya bo'lib, uning davri $T=\frac{2\pi}{\alpha}$, $(\alpha\neq 0)$ ga teng. Haqiqatan,

$$f\left(x+\frac{2\pi}{\alpha}\right)=A\sin\left(\alpha\left(x+\frac{2\pi}{\alpha}\right)+\beta\right)=A\sin(\alpha x+\beta+2\pi)=A\sin(\alpha x+\beta)=f(x).$$

Odatda, $f(x)=A\sin(\alpha x+\beta)$ funksiyaga **garmonika** deyiladi.

Garmonikaning grafigi $y=\sin x$ funksiya grafigini Ox va Oy o'qlar bo'yicha qisish (yoki cho'zish) hamda Ox o'qi bo'yicha surish natijasida hosil qilinadi. Garmonikani quyidagicha ham yozish mumkin:

$$f(x)=A\sin(\alpha x+\beta)=A(\cos\alpha x\sin\beta+\sin\alpha x\cos\beta)=$$
$$=A\sin\beta\cdot\cos\alpha x+A\cos\beta\cdot\sin\alpha x=a\cos\alpha x+b\sin\alpha x,$$

bunda $a=A\sin\beta$, $b=A\cos\beta$.

Aksincha, $f(x)=a\cos\alpha x+b\sin\alpha x$ funksiya garmonikani ifodalaydi:

$$f(x)=a\cos\alpha x+b\sin\alpha x=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\alpha x+\frac{b}{\sqrt{a^2+b^2}}\sin\alpha x\right)=$$
$$=A(\sin\beta\cos\alpha x+\cos\beta\sin\alpha x)=A\sin(\alpha x+\beta)$$

bunda, $\sqrt{a^2+b^2}=A$, $\frac{a}{\sqrt{a^2+b^2}}=\sin\beta$, $\frac{b}{\sqrt{a^2+b^2}}=\cos\beta$.

4.3.2. Ortogonal funksiyalar sistemasi bo'yicha funksiyalarni Furye qatoriga yoyish

Har bir hadi $u_n(x)=a_n\cos nx+b_n\sin nx$ $\quad (n=0,1,2,\ldots)$ garmonikadan iborat ushbu

$$\frac{a_0}{2}+(a_1\cos x+b_1\sin x)+\ldots+(a_n\cos nx+b_n\sin nx)+\ldots=$$
$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{4.20}$$

funksional qatorga **trigonometrik qator** deyiladi. Bunda

$$a_0,a_1,b_1,a_2,b_2,\ldots,a_n,b_n,\ldots$$

sonlar trigonometrik **qatorning koeffitsiyentlari** deyiladi.

Aytaylik, $y=f(x)$ funksiya $[-\pi,\pi]$ da berilgan bo'lib, u shu oraliqda integrallanuvchi bo'lsin. Bu oraliqda funksiyani trigonometrik qatorga yoyish mumkin bo'lsin:

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos nx+b_k\sin nx) \tag{4.20*}$$

U holda qator koeffitsiyentlarini aniqlaymiz. a_0 koeffitsiyentni topish uchun tenglikni har ikki tomonini $[-\pi,\pi]$ da integrallaymiz:

$$\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi}^{\pi}\frac{a_0}{2}dx+\int_{-\pi}^{\pi}\sum_{k=1}^{n}(a_k\cos nx+b_k\sin nx)dx;$$

$$\int_{-\pi}^{\pi}f(x)dx=\frac{a_0}{2}\Bigg|_{-\pi}^{\pi}+(a_k\sin nx-b_k\cos nx)\Big|_{-\pi}^{\pi}=\pi a_0;$$

Demak, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$ ga teng. (4.21)

Endi, $k\neq 0$ ning biror qiymatida a_k koeffitsiyentni topish uchun (4.20*) tenglikning ikkala qismini $\cos kx$ ga ko'paytiramiz va hosil bo'lgan ifodani $-\pi$ dan π gacha hadlab integrallaymiz:

$$\int_{-\pi}^{\pi} f(x)\cos kxdx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos kxdx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos kxdx + b_n\int_{-\pi}^{\pi}\sin nx\cos kxdx\right)$$

(1)-(8) formulalarga ko‘ra, o‘ng tomondagi $a_к$ integraldan boshqa hamma integrallarning nolga teng ekanini ko‘ramiz.

Demak, $\int_{-\pi}^{\pi} f(x)\cos kxdx = a_к\int_{-\pi}^{\pi}\cos^2 kxdx = a_к\pi$

bundan $a_к = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx$ ni topamiz. (4.22)

b_k koeffitsiyentni topish uchun (4.20*) tenglikning ikkala qismini $\sin kx$ ga ko‘paytiramiz va hosil bo‘lgan tenglikni $-\pi$ dan π gacha hadlab integrallaymiz:

$$\int_{-\pi}^{\pi} f(x)\sin kxdx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin kxdx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\sin kxdx + b_n\int_{-\pi}^{\pi}\sin nx\sin kxdx\right)$$

(1)-(8) formulalarga ko‘ra, o‘ng tomondagi b_k koeffitsiyentli integraldan boshqa hamma integrallarning nolga teng ekanini ko‘ramiz.

Shunday qilib, $\int_{-\pi}^{\pi} f(x)\sin kxdx = b_к\int_{-\pi}^{\pi}\sin^2 kxdx = b_к\pi$

bundan $b_к = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx$ (4.23)

(4.21), (4.22) va (4.23) formulalar bo‘yicha aniqlangan koeffitsiyentlar 2π davrli $f(x)$ funksiyaning **Furye koeffitsiyentlari** deyiladi. Shunday koeffitsiyentli (4.20*) trigonometrik qator esa $f(x)$ funksiyaning **Furye qatori** deyiladi: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$.

Endi biz dastlabki qo‘yilgan savolga qaytaylik, ya’ni $f(x)$ funksiya qanday shartlarni qanoatlantirganda bu funksiya uchun tuzilgan trigonometrik (Furye) qatori yaqinlashuvchi bo‘lib, yig‘indisi $f(x)$ funksiya bo‘ladi.

$f(x)$ funksiyani $[a,b]$ kesmada **bo‘lakli monoton** deyiladi, agarda bu kesmalarning har birida $f(x)$ funksiya monoton bo‘lsa, ya’ni har birida faqat kamayuvchi, yoki faqat o‘suvchi bo‘lsa.

Agar $f(x)$ funksiya $[a,b]$ kesmada uzilishga ega bo‘lsa, uzilish nuqtalari $a, x_1, x_2, ..., x_{n-1}, b$ da faqat 1-tur uzilishga ega bo‘ladi. Qo‘yilgan savolga, ya'ni $f(x)$ funksiya Furye qatoriga yoyilishining yetarli shartiga quyidagi Dirixle teoremasi javob beradi.

4.3.5-misol. Ushbu $f(x)=e^{\alpha x}$ $(-\pi\le x\le\pi,\alpha\ne 0)$ funksiyaning Furye qatorini toping.

Yechilishi: ▶ (4.21) , (4.22) va (4.23) formulalardan foydalanib, berilgan funksiyaning Furye koeffitsiyentlarini hisoblaymiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}dx=\frac{1}{\alpha\pi}\left(e^{\alpha\pi}-e^{-\alpha\pi}\right)=\frac{2}{\alpha\pi}sh\alpha\pi,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}\cos nxdx=\frac{1}{\pi}\frac{\alpha\cos nx+n\sin nx}{\alpha^2+n^2}e^{\alpha x}\Bigg|_{-\pi}^{\pi}=(-1)^n\frac{1}{\pi}\cdot\frac{2\alpha}{\alpha^2+n^2}sh\alpha\pi,\quad (n=1,2,\dots),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}\sin nxdx=\frac{1}{\pi}\frac{\alpha\sin nx-n\cos nx}{\alpha^2+n^2}e^{\alpha x}\Bigg|_{-\pi}^{\pi}=(-1)^{n-1}\frac{1}{\pi}\cdot\frac{2n}{\alpha^2+n^2}sh\alpha\pi,\quad (n=1,2,\dots).$$

Demak, $f(x)=e^{\alpha x}$ funksiyaning Furye qatori quyidagicha bo'ladi:

$$f(x)=e^{\alpha x}\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=$$

$$=\frac{2sh\alpha\pi}{\pi}\left[\frac{1}{2\alpha}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2+n^2}(\alpha\cos nx-n\sin nx)\right]$$ ◀

4.3.3. 2π davrli funksiya uchun Furye qatori. Dirixle teoremasi

Faraz qilaylik, $y=f(x)$ funksiya $[-\pi,\pi]$ da berilgan juft funksiya bo'lib, u shu oraliqda integrallanuvchi bo'lsin. Bu funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right]=$$

$$=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nxdx\qquad (n=0,1,2,\dots),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right]=$$

$$=\frac{1}{\pi}\left[-\int_{0}^{\pi}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right]=0\qquad (n=1,2,\dots).$$

Demak, juft $f(x)$ funksiyaning Furye koeffitsiyentlari

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \qquad (n = 0,1,2,\ldots)$$

$$b_n = 0 \qquad (n = 1,2,\ldots)$$

bo‘lib, Furye qatori $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ bo‘ladi.

Aytaylik, $y = f(x)$ funksiya $[-\pi, \pi]$ da berilgan toq funksiya bo‘lib, u shu oraliqda integrallanuvchi bo‘lsin. Bu funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx + \int_0^{\pi} f(x)\cos nx dx\right] =$$

$$= \frac{1}{\pi}\left[-\int_0^{\pi} f(x)\cos nx dx - f(x)\cos nx dx\right] = 0 \qquad (n = 0,1,2,\ldots),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx dx + \int_0^{\pi} f(x)\sin nx dx\right] =$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} f(x)\sin nx dx\right] \qquad (n = 1,2,\ldots).$$

Demak, toq $y = f(x)$ funksiyaning Furye koeffitsiyentlari

$$a_n = 0\ , \quad (n = 0,1,2,\ldots),$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx\ , \qquad (n = 1,2,\ldots)$$

bo‘lib, Furye qatori $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$ bo‘ladi.

4.3.6-misol. Ushbu $f(x) = x^2$, $(-\pi \le x \le \pi)$ juft funksiyaning Furye qatorini toping.

Yechilishi: ► Avvalo berilgan funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^2 \quad ,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx = \frac{2}{\pi} x^2 \frac{\sin nx}{n}\Bigg|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x \sin nx dx =$$

$$= \frac{4}{\pi n}\left(\frac{x\cos nx}{n}\Bigg|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \cos nx dx\right) = (-1)^n \cdot \frac{4}{n^2}. \qquad (n = 1,2,\ldots)$$

Demak, $f(x) = x^2$ funksiyaning Furye qatori

$$f(x)=x^2\sim\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$ bo‘ladi. ◀

4.3.7-misol. Ushbu $f(x)=x,\quad (-\pi\le x\le\pi)$ toq funksiyaning Furye qatorini toping.

Yechilishi: ▶ Berilgan funksiyaning Furye koeffitsiyentlarini hisoblaymiz:

$$b_n=\frac{2}{\pi}\int_0^{\pi}x\sin nxdx=\frac{2}{\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nxdx\right)=\frac{2(-1)^{n-1}}{n}.$$

Demak, $f(x)=x$ funksiyaning Furye qatori quyidagicha bo‘ladi:

$$f(x)\sim\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n}\sin nx.$$ ◀

$[-l,l]$ oraliqdagi berilgan funksiyaning Furye qatori

Faraz qilaylik, $f(x)$ funksiya $[-l,l]$ oraliqda $(l>0)$ berilgan bo‘lib, y shu oraliqda integrallanuvchi bo‘lsin.

Ravshanki, ushbu $t=\frac{\pi}{l}x$ almashtirish natijasida $[-l,l]$ oraliq $[-\pi,\pi]$ oraliqqa o‘tadi. Agar $f(x)=f\left(\frac{1}{\pi}t\right)=\varphi(t)$. deyilsa, $\varphi(t)$ funksiya $[-\pi,\pi]$ oraliqda berilgan va shu oraliqda integrallanuvchi funksiya bo‘ladi. Uning Furye qatori $\varphi(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\cos ntdt\quad,\qquad (n=0,1,2,\ldots)$$

bo‘lib,

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\sin ntdt\qquad (n=1,2,\ldots)$$

bo‘ladi. Endi $t=\frac{\pi}{l}x$ bo‘lishini e’tiborga olib topamiz:

$$\varphi\left(\frac{\pi}{l}x\right)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\frac{\pi}{l}x+b_n\sin n\frac{\pi}{l}x\right),$$

$$a_n=\frac{1}{l}\int_{-l}^{l}\varphi\left(\frac{\pi}{l}x\right)\cos n\frac{\pi}{l}xdx\quad,\quad (n=0,1,2\ldots)$$

$$b_n=\frac{1}{l}\int_{-l}^{l}\varphi\left(\frac{\pi}{l}x\right)\sin n\frac{\pi}{l}xdx\quad.\quad (n=1,2\ldots)$$

Natijada berilgan $f(x)$ funksiyaning Furye qatori quyidagicha

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

bo'lib, bunda

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \qquad (n = 0,1,2\ldots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}xdx \qquad (n = 1,2\ldots)$$

ga teng.

4.3.8-misol. Ushbu $f(x) = e^x \quad (-1 \le x \le 1)$ funksiyaning Furye qatorini toping.

Yechilishi: ► Yuqoridagi formulalardan foydalanib, $f(x) = e^x$ funksiyaning Furye koeffitsiyentilarini topamiz:

$$a_0 = \int_{-1}^{1} e^x dx = e - e^{-1} \ ,$$

$$a_n = \int_{-1}^{1} e^x \cos n\pi x dx = \frac{n\pi \sin n\pi x - \cos n\pi x}{1 + n^2\pi^2} e^x \Big|_{-1}^{1} =$$

$$= \frac{1}{1 + n^2\pi^2}\left(e\cos n\pi - e^{-1}\cos n\pi\right) = (-1)^n \frac{e - e^{-1}}{1 + n^2\pi^2} \qquad (n = 1,2,\ldots),$$

$$b_n = \int_{-1}^{1} e^x \cos n\pi x dx = \frac{\sin n\pi x - n\pi \cos n\pi x}{1 + n^2\pi^2} e^x \Big|_{-1}^{1} =$$

$$= \frac{1}{1 + n^2\pi^2}\left(en\pi \cos n\pi + n\pi e^{-1}\cos n\pi\right) = \frac{n\pi(-1)^n}{1 + n^2\pi^2}\left(e^{-1} - e\right) =$$

$$= (-1)^{n+1}\frac{e - e^{-1}}{1 + n^2\pi^2} \qquad (n = 1,2,\ldots).$$

Demak, $f(x) = e^x \quad (-1 \le x \le 1)$ funksiyaning Furye qatori

$$e^x \sim \frac{e - e^{-1}}{2} + \left(e - e^{-1}\right)\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{1 + n^2\pi^2}\cos n\pi + \frac{(-1)^{n+1}}{1 + n^2\pi^2} n\pi \sin n\pi x\right]$$

bo'ladi.◄

Furye qatorining yaqinlashishini isbotlashda muhim bo'lgan lemmalarni keltiramiz.

4.2-lemma. Agar $\varphi(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lsa, u holda

$$\lim_{p\to\infty}\int_a^b \varphi(x)\sin pxdx = 0,$$

$$\lim_{p\to\infty}\int_a^b \varphi(x)\cos pxdx = 0$$

tengliklar o‘rinli bo‘ladi.

Agar $[a,b]$ oraliqni shunday

$$[a_0,a_1],[a_1,a_2],\dots,[a_{n-1},a_n] \quad (a_0=a,\ a_n=b)$$

bo‘laklarga ajratish mumkin bo‘lsaki, har bir (a_k,a_{k+1}), $(k=0,1,2,\dots,n-1)$ oraliqda $f(x)$ funksiya uzluksiz bo‘lib, $x=a_k$ nuqtalarda chekli

o‘ng $\quad f(a_k+0) \quad (k=0,1,2,\dots,n-1)$,

va chap $\quad f(a_k-0) \quad (k=0,1,2,\dots,n)$

limitlarga ega bo‘lsa, $f(x)$ **funksiya** $[a,b]$ **da bo‘lakli-uzluksiz** deyiladi.

Yuqoridagi lemma $\varphi(x)$ funksiya $[a,b]$ oraliqda bo‘lakli uzluksiz funksiya bo‘lgan holda ham o‘rinli bo‘ladi.

4.2-lemmadan quyidagi natija kelib chiqadi.

Natija. Agar $f(x)$ funksiya $[-\pi,\pi]$ oraliqda bo‘lakli uzluksiz bo‘lsa, uning Furye koeffitsiyentlari $n\to\infty$ da nolga intiladi:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = 0,$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0.$$

Furye qatorining yaqinlashuvchiligi. Agar har bir (a_k,a_{k+1}) da $(k=0,1,2,\dots,n-1)$ $f(x)$ funksiya differensiallanuvchi bo‘lib, $x=a_k$ nuqtalarda chekli o‘ng $f'(a_k+0) \quad (k=0,1,2,\dots,n-1)$,

va chap $\quad f'(a_k-0) \quad (k=0,1,2,\dots,n)$

hosilalarga ega bo‘lsa, $f(x)$ funksiya $[a,b]$ da **bo‘lakli-differensiallanuvchi** deyiladi.

4.13-teorema (Dirixle teoremasi). 2π davrli $f(x)$ funksiya $[-\pi,\pi]$ oraliqda bo‘lakli-differensiallanuvchi bo‘lsa, u holda bu funksiyaning Furye qatori $f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$ $[-\pi,\pi]$ da yaqinlashuvchi bo‘lib, uning yig‘indisi $\frac{f(x+0)+f(x-0)}{2}$ ga teng bo‘ladi.

4.3.9 -misol. Ushbu $f(x)=\cos ax \qquad (-\pi\le x\le\pi,\ a\ne n\in Z)$

funksiyani Furye qatoriga yoying va uni yaqinlashishga tekshiring.

Yechilishi: ► Bu funksiyaning Furye koeffitsiyentlarini topamiz. Qaralayotgan funksiya juft bo‘lgani uchun $b_n = 0 \quad (n = 1, 2, 3, \ldots)$ bo‘lib,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos ax \cos nx dx = \int_0^{\pi}[\cos(a-n)x + \cos(a+n)x]dx =$$

$$= \frac{\sin a\pi}{\pi}(-1)^n\left[\frac{1}{a+n} + \frac{1}{a-n}\right]$$

bo‘ladi. Demak,

$$f(x) \sim \frac{\sin a\pi}{\pi}\left[\frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a+n} + \frac{1}{a-n}\right)\cos nx\right].$$

Agar $f(x) = \cos ax \qquad (-\pi \le x \le \pi \,,\, a \ne n \in Z)$ funksiya uchun teoremaning shartlarini bajarishini e’tiborga olsak, unda

$$\cos ax = \frac{\sin a\pi}{\pi}\left[\frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a+n} + \frac{1}{a-n}\right)\cos nx\right],$$

$f(x) = \cos ax \qquad (-\pi \le x \le \pi \,,\, a \ne n \in Z)$ funksiya yoyilmasini topamiz.

Oxirgi tenglikda $x = 0$ deyilsa,

$$\frac{\pi}{\sin a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a+n} + \frac{1}{a-n}\right)$$

bo‘lishi kelib chiqadi. ◄

Mavzu yuzasidan savollar:

1. Funksiyalarning qanday sitemasiga ortogonal sistema deyiladi?
1. Funksiyalarning qanday sitemasiga ortonormal sistema deyiladi?
2. Davri 2π bo‘lgan funksiyalar uchun Furye koeffitsiyentini yozing.
3. Dirixle sharti nima? Dirixle teoremasini ayting
4. $[-\pi,\pi]$ kesmada juft funksiyalar uchun Furye koeffitsiyentlari qanday bo‘ladi?
5. $[-\pi,\pi]$ kesmada toq funksiyalar uchun Furye koeffitsiyentini keltirib chiqaring.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1. $[-\pi,\pi]$da $f(x) = \sin 7x$ va $\varphi(x) = \cos 3x$ funksiyalarning ortogonalligini tekshiring.

2. $[-\pi,\pi]$da $f(x)=\sin x$ va $\varphi(x)=\cos 3x$ funksiyalarning ortonormalligini tekshiring.

3. $f(x)=x$ funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoygandagi b_n koeffitsiyentini toping.

4. $f(x)=x^2$ funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoygandagi b_n koeffitsiyentini toping.

5. $f(x)=x^2$ va $\varphi(x)=\cos 3x$ funksiyalarning ortogonalligini tekshiring.

TESTLAR

1. $f(x)=x$ funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoygandagi b_3 koeffitsiyentini toping.

A) 2/3 **B)** 1/3 **C)** 1 **D)** 3

2. $f(x)=x^2$ funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoygandagi a_n koeffitsiyentini toping.

A) $(-1)^{n+1}\dfrac{2}{n^2}$ **B)** $(-1)^{n+1}\dfrac{n}{2}$ **C)** $(-1)^{n}\dfrac{4}{n^2}$ **D)** $(-1)^{n}\dfrac{2}{n}$

3. $f(x)=\sin x$ funksiyani $\left[0;\dfrac{\pi}{2}\right]$ kesmada kosinuslar bo‘yicha yoyilmasidagi a_0 koeffitsiyentini toping.

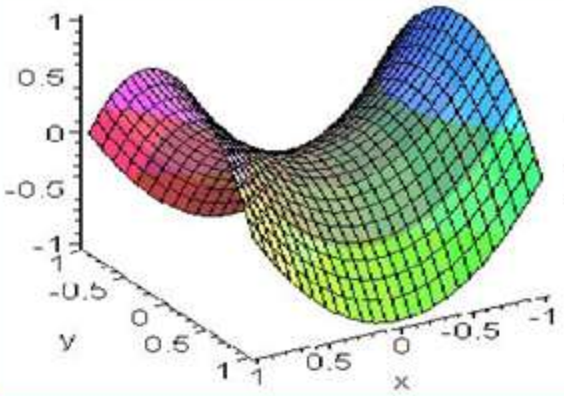
A) $\dfrac{4}{\pi}$ **B)** $\dfrac{\pi}{4}$ **C)** $\dfrac{2}{\pi}$ **D)** $\dfrac{\pi}{2}$

4. $f(x)=x$ funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoygandagi b_n koeffitsiyentni toping.

A) $(-1)^{n+1}\dfrac{2}{n}$ **B)** $(-1)^{n+1}\dfrac{n}{2}$ **C)** $(1)^{n+1}\dfrac{2}{n}$ **D)** $(-1)^{n}\dfrac{2}{n}$

5. $f(x)=x$ funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoygandagi a_2 koeffitsiyentini toping.

A) 3 **B)** 1 **C)** 2 **D)** 0



V BOB.
KO‘P O‘ZGARUVCHILI FUNKSIYALAR

5.1. Ikki argumentli funksiyaning aniqlanish sohasi, grafigi, limiti va uzluksizligi

5.1.1. Ko‘p o‘zgaruvchili funksiyalar haqida umumiy tushunchalar. Ko‘p argumentli funksiyani aniqlanish sohasi

Amaliyotda bir o‘zgaruvchili funksiyalar bilan birga ikki va undan ortiq o‘zgaruvchili funksiyalar bilan ham ish ko‘riladi, bunday miqdorlar bog‘lanishlarida birining sonli qiymati boshqa bir nechtasining sonli qiymati bilan aniqlanadi. Oddiy misol, to‘g‘ri to‘rtburchakning yuzasi uning tomonlari uzunliklariga bog‘liq ravishda o‘zgaradi (ikki o‘zgaruvchi), parallelepipedning hajmi esa uning uchala qirrasining o‘lchovlari o‘zgarishi bilan o‘zgaradi (uch o‘zgaruvchili). Bosib o‘tilgan yo‘lni esa tezlik hamda vaqt funksiyasi sifatida qarash mumkin. Bunday bog‘lanishlarni ko‘p o‘zgaruvchili funksiyalar tushunchasini kiritish bilan tushuntirish mumkin.

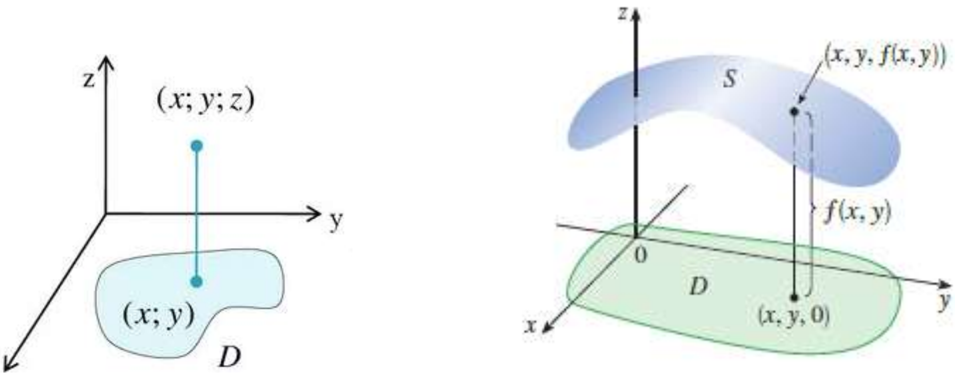
R^2 fazoda biror D to‘plamning bir-biriga bog‘liq bo‘lmagan ixtiyoriy (x, y) haqiqiy sonlari juftligiga biror qoidaga ko‘ra E to‘plamdagi bitta z haqiqiy son mos qo‘yilgan bo‘lsa, D to‘plamda **ikki** x va y **o‘zgaruvchilarning** z **funksiyasi aniqlangan** deyiladi.

Ikki o‘zgaruvchining funksiyasi quyidagicha belgilanadi:

$$z = f(x, y).$$

Bunda x, y erkli o‘zgaruvchilar, z esa funksiya deyiladi.

D to‘plam **funksiyaning aniqlanish** sohasi, E to‘plam **qiymatlar to‘plami** deyiladi.



5.1-rasm. Ikki o‘zgaruvchili funksiyaning aniqlanish va qiymatlar sohasi

Demak, ikki o‘zgaruvchili funksiyada har bir juft haqiqiy songa koordinatalar sistemasida bitta nuqta mos qo‘yilar ekan (5.1-rasm).

Ikki o‘zgaruvchili funksiyalar ham bir o‘zgaruvchili funksiyalar kabi 4 xil usulda berilishi mumkin: ular – analitik usul, geometrik (grafik) usul, jadval usuli yoki so‘z bilan tasvirlash usuli.

Ikki o‘zgaruvchili funksiya analitik usulda berilganda oshkor $z = f(x, y)$ shaklda yoki oshkormas $F(x, y, z) = 0$ shaklda beriladi:

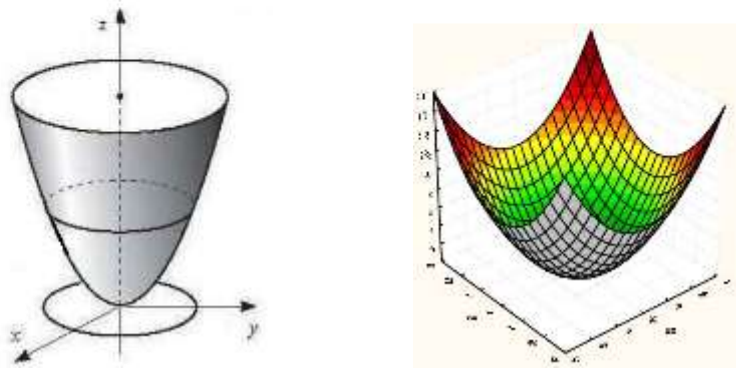
1) $z = \ln(x^2 + y^2 - 2)$ funksiya oshkor shaklda berilgan;

2) $z^2 + 3x^3 + \lg y = 0$ funksiya oshkormas shaklda berilgan.

3) $x^2 + y^2 + z^2 = R^2$ sfera tenglamasi ham ikki o‘zgaruvchili funksiyaning oshkormas shakliga misol bo‘la oladi, uni oshkor shaklga o‘tkazilsa, quyidagicha bo‘ladi: $z = \pm\sqrt{R^2 - x^2 - y^2}$.

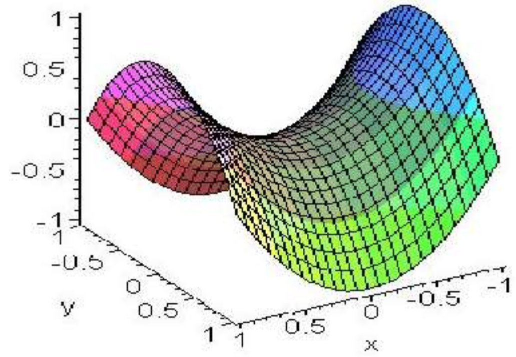
Ikki o‘zgaruvchili funksiya geometrik usulda berilganda, uning tasviri fazoda tenglamasi $z = f(x, y)$ bo‘lgan sirtni ifodalaydi.

1) $z = x^2 + y^2$ funksiya fazoda paraboloidni ifodalaydi (qo‘lda va Maple dasturi yordamida chizilgan, 5.2-rasm):



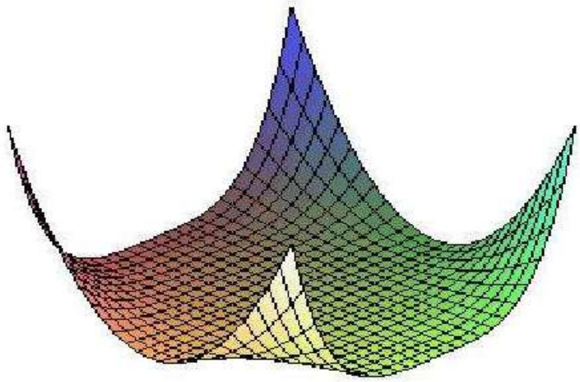
5.2-rasm. $z = x^2 + y^2$ funksiya grafigi

3) $z = x^2 - y^2$ funksiya fazoda egarni tasvirlaydi (5.3-rasm):



5.3-shakl. $z = x^2 - y^2$ funksiya grafigi

4) $z = (x^2 y + 1)^2$ funksiya grafigi (5.4-rasm):



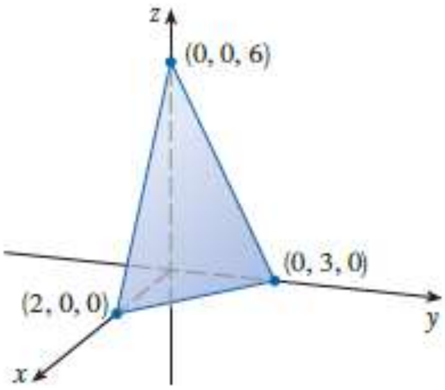
5.4-rasm. $z = (x^2 y + 1)^2$ funksiya grafigi

Ikki o‘zgaruvchili funksiya jadval usulida quyidagicha berilishi mumkin:

X_1X_2 \ X_3X_4	00	01	11	10
00	1	0	0	1
01	1	0	0	1
10	0	1	1	0
11	1	0	0	1

5.1.1-misol. $f(x,y) = 6 - 3x - 2y$ funksiya grafigini chizing.

Yechilishi: ► Berilgan funksiya $z = 6 - 3x - 2y$ yoki $3x + 2y + z = 6$ tekislikni ifodalaydi. Bu tekislikning koordinata o‘qlari bilan kesishish nuqtalarini topish uchun quyidagicha ish tutamiz: O*x* o‘qini kesib o‘tgan nuqtasini topish uchun $y = z = 0$ deb olamiz va $x = 2$ ni aniqlaymiz, ya’ni (2,0,0). Xuddi shuningdek, (0,3,0) va (0,0,6) nuqtalarni topamiz (5.5-rasm).

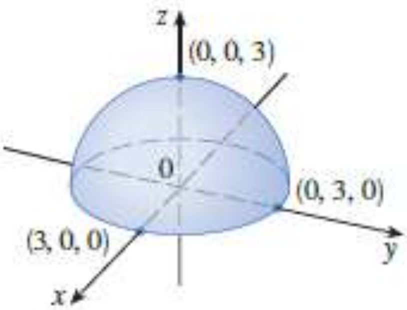


◀

5.5-rasm. $f(x, y) = 6 - 3x - 2y$ funksiya grafigi

5.1.2-misol. $f(x, y) = \sqrt{9 - x^2 - y^2}$ funksiya grafigini chizing.

Yechilishi: ►Berilgan funksiyani $z = \sqrt{9 - x^2 - y^2}$ ko‘rinishda yozib olamiz.



5.6-rasm. $f(x, y) = \sqrt{9 - x^2 - y^2}$ funksiya grafigi

U sfera bo‘lib $x^2 + y^2 + z^2 = 9$, aniqlanish sohasini topish uchun $9 - x^2 - y^2 \geq 0$ yoki $x^2 + y^2 \leq 9$ deb olamiz. Demak, berilgan funksiyaning aniqlanish sohasi markazi koordinatlar boshida, radiusi 3 ga teng bo‘lgan doiradan iborat, qiymatlar sohasi esa $E(z) = [0, 3]$ bo‘ladi (5.6-rasm). ◀

5.1.2. Ikki va ko‘p o‘zgaruvchili funksiya limiti

Ikki o‘zgaruvchili funksiyaning limiti tushunchasini berishdan oldin, berilgan nuqtaning δ - atrofi tushunchasini kiritamiz.

$P_0(x_0, y_0)$ **nuqtaning** δ **atrofi** deb, koordinatalari quyidagi shartni qanoatlantiruvchi $P(x, y)$ nuqtalar to‘plamiga aytiladi:

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta. \tag{5.1}$$

Ushbu belgi P va P_0 nuqtalar orasidagi masofani bildiradi.

Demak, P_0 nuqtaning δ atrofi deganda markazi P_0 nuqtada radiusi δ bo‘lgan doiraning ichida yotuvchi barcha P nuqtalarni tushunamiz.

Fazodagi nuqtaning δ atrofi – markazi $P_0(x_0, y_0, z_0)$ nuqtada radiusi δ bo‘lgan sharning ichki nuqtalari bo‘ladi.

n o‘lchovli $(n > 3)$ fazoda $P_0(x_1, x_2, ..., x_n)$ nuqtaning δ atrofi ham shunga o‘xshash aniqlanadi.

Ikki o‘zgaruvchili $z = f(x, y)$ funksiya P_0 nuqtaning biror atrofida aniqlangan bo‘lsa (P_0 nuqtada aniqlanmagan bo‘lishi ham mumkin) va ixtiyoriy $\varepsilon > 0$ uchun shunday $\delta > 0$ topilsaki, $\rho(P, P_0) = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ tengsizlikni qanoatlantiruvchi barcha $P(x, y)$ nuqtalar uchun $|f(x, y) - A| < \varepsilon$ tengsizlik bajarilsa, A **o‘zgarmas son** $z = f(x, y)$ **funksiyaning** $P \to P_0$ **dagi limiti** deyiladi va

$$\lim_{P \to P_0} f(P) = A \quad \text{yoki} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A \qquad (5.2)$$

kabi belgilanadi.

Uch va undan ortiq o‘zgaruvchi funksiyasining limiti ham yuqoridagiga o‘xshash aniqlanadi.

Limitning ta’rifidan kelib chiqadiki, A son $z = f(x, y)$ funksiyaning limiti bo‘lsa, $|f(x, y) - A|$ ayirma $x \to x_0$, $y \to y_0$ da cheksiz kichik miqdor bo‘ladi. Uch va undan ortiq o‘zgaruvchi funksiyasining limiti ham yuqoridagiga o‘xshash aniqlanadi.

Bir nechta o‘zgaruvchili funksiyaning limiti 0 ga teng bo‘lsa, bunday funksiyaga cheksiz kichik funksiya yoki **cheksiz kichik miqdor** deyiladi.

$y = f(x)$ funksiya uchun limitlar haqidagi barcha asosiy teoremalar ko‘p o‘zgaruvchili funksiya uchun ham o‘rinli.

5.1.3-misol. $\lim\limits_{\substack{x \to 1 \\ y \to -2}} \dfrac{x + 3y^2}{x^2 - 2y}$ limitni hisoblang.

Yechilishi: ► Agar limit ostida kasr-ratsional funksiya bo‘lsa, dastlab x va y larning o‘rniga qiymatlarini qo‘yib ko‘ramiz:

$$\lim_{\substack{x \to 1 \\ y \to -2}} \frac{x + 3y^2}{x^2 - 2y} = \lim_{\substack{x \to 1 \\ y \to -2}} \frac{1 + 3 \cdot (-2)^2}{1^2 - 2 \cdot (-2)} = \frac{13}{5}. \quad ◄$$

5.1.4-misol. $\lim\limits_{\substack{x \to 2 \\ y \to 0}} \dfrac{\sin xy}{y}$ limitni hisoblang.

Yechilishi: ► Ushbu misolda ham limit ostida kasr-ratsional funksiya, x va y larning o‘rniga qiymatlarini qo‘yib ko‘ramiz. P_0 (2;0) nuqtada $\frac{\sin xy}{y}$ funksiya $\frac{0}{0}$ aniqmaslik hosil qiladi. Aniqmaslikni 1-ajoyib limit $\lim\limits_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1$ formulasidan foydalanib topamiz:

$$\lim_{\substack{x\to 2\\ y\to 0}}\frac{\sin xy}{y}=\frac{0}{0}=\lim_{\substack{x\to 2\\ y\to 0}}\left(x\cdot\frac{\sin xy}{xy}\right)=\lim_{\substack{x\to 2\\ y\to 0}}x\cdot\lim_{\substack{x\to 2\\ y\to 0}}\frac{\sin xy}{xy}=2\cdot 1=2\,. \blacktriangleleft$$

5.1.5-misol. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{\sqrt{x^2+y^2+2}-1}$ limitni hisoblang.

Yechilishi: ► Agar limit aniqmaslikdan iborat bo‘lib, limit ostida irratsional funksiya qatnashgan bo‘lsa, uni qo‘shmasiga ko‘paytirib, shakl almashtirish bajaramiz:

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{\sqrt{x^2+y^2+2}-1}=\frac{0}{0}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2(\sqrt{x^2+y^2+1}+1)}{(\sqrt{x^2+y^2+1}-1)(\sqrt{x^2+y^2+1}+1)}=$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2}=\lim_{\substack{x\to 0\\ y\to 0}}\sqrt{x^2+y^2+1}+1=2. \blacktriangleleft$$

5.1.6-misol. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{xy+4}-2}{x+y}$ limitni hisoblang.

Yechilishi: ► Limit aniqmaslikdan iborat, uni hisoblash uchun shakl almashtirish bajaramiz:

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{xy+4}-2}{x+y}=\frac{0}{0}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{(\sqrt{xy+4}-2)(\sqrt{xy+4}+2)}{(x+y)(\sqrt{xy+4}+2)}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{(x+y)(\sqrt{xy+4}+2)}=$$

$$\left|\begin{matrix}y=kx\ \text{to'g'ri chiziq}\\ \text{bo'yicha yaqinlashamiz}\end{matrix}\right|=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x\cdot kx}{(x+kx)(\sqrt{x\cdot kx+4}+2)}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{kx}{(1+k)(\sqrt{kx^2+4}+2)}=0 \blacktriangleleft$$

5.1.7-misol. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{x^3+3y^3}$ limitni hisoblang.

Yechilishi: ► Limitni hisoblash uchun shakl almashtirish bajaramiz:

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{x^3+3y^3}=\left|\begin{matrix}y=kx\ \text{to'g'ri chiziq}\\ \text{bo'yicha yaqinlashamiz}\end{matrix}\right|=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2\cdot kx}{x^3+3(kx)^3}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{k}{1+3k^3}=\left\langle\begin{matrix}k=1\ \text{da}\ 0{,}25\\ k=2\ \text{da}\ 2/25\end{matrix}\right.$$

Ushbu limitning qiymatlari k ning turli qiymatlarida turlicha chiqmoqda, demak bu limit mavjud emas.◄

5.1.3. Ikki va ko‘p o‘zgaruvchili funksiyaning uzluksizligi

$z = f(x,y)$ funksiya $P_0(x_0,y_0)$ nuqtada hamda uning biror atrofida aniqlangan va $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,y_0)$ bo‘lsa, ya’ni funksiyaning $P_0(x_0,y_0)$ nuqtadagi limiti funksiyaning shu nuqtadagi qiymatiga teng bo‘lsa, **funksiya** $P_0(x_0,y_0)$ **nuqtada uzluksiz** deyiladi.

Bu ta’rifga teng kuchli 2-ta’rifni ham keltiramiz.

$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ funksiyaning $P_0(x_0,y_0)$ nuqtadagi to‘liq orttirmasi bo‘lsin.

$z = f(x,y)$ funksiya $P_0(x_0,y_0)$ nuqtada va uning atrofida aniqlangan bo‘lsa, argumentlarning Δx *va* Δy cheksiz kichik orttirmalariga funksiyaning ham Δz cheksiz kichik orttirmasi mos kelsa, ya’ni $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = 0$ bo‘lsa, **funksiya** $P_0(x_0,y_0)$**nuqtada uzluksiz** deyiladi.

Uzluksizlik shartlari bajarilmagan nuqtalar **uzilish nuqtalari** deyiladi. Ikki o‘zgaruvchili funksiya uzilish nuqtalari butun chiziqni hosil qilishi mumkin.

5.1.8-misol. $z = x^2 + y^2$ funksiyaning $P_0\,(2;3)$ nuqtada uzluksizligini ko‘rsating.

Yechilishi: ►Bu nuqtada funksiyaning to‘liq orttirmasini topamiz:

$$\Delta z = (2+\Delta x)^2 + (3+\Delta y)^2 - (2^2 + 3^2) = 2^2 + 2\Delta x + \Delta x^2 +$$

$$+ 3^2 + 6\Delta y + \Delta y^2 - 2^2 - 3^2 = 2\Delta x + \Delta x^2 + 6\Delta y + \Delta y^2$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = \lim\left[2\Delta x + (\Delta x)^2 + 6\Delta y + \Delta y\right] = 2\cdot 0 + 0 + 6\cdot 0 = 0.$$

Shunday qilib, $\Delta x \to 0,\ \Delta y \to 0$ *da* $\Delta z \to 0$. Demak, $P_0\,(2;3)$ nuqtada berilgan funksiya uzluksizdir. Bu holatni istalgan $P_0(x_0,y_0)$ uchun ko‘rsatish mumkin. ◄

$z = f(x,y)$ funksiya biror to‘plamning har bir nuqtasida uzluksiz bo‘lsa, unga shu **to‘plamda uzluksiz** deyiladi.

5.1.9-misol. $z = \dfrac{1}{x^2 - y^2}$ funksiyaning uzilish nuqtalarini toping.

Yechilishi: ► Funksiya koordinatalari $x^2 - y^2 = 0$ tenglamani qanoatlantiruvchi nuqtalarda uzilishga ega. Bu $y = x$ va $y = -x$ to‘g‘ri chiziqlar bo‘lib, bu to‘g‘ri chiziqlarga tegishli har bir nuqtada funksiya uzilishga ega bo‘ladi. ◄

5.1.10-misol. Ushbu $f(x,y)=\begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{agar} \quad x^2+y^2\neq 0 \quad \text{bo'lsa}, \\ 0, & \text{agar} \quad x^2+y^2=0 \quad \text{bo'lsa} \end{cases}$

funksiyani $(0,0)$ nuqtada uzluksizlikka tekshiring.

Yechilishi: ► Berilgan funksiyaning $(0,0)$ nuqtadagi xususiy orttirmalarini yozamiz:

$$\Delta_x f(0,0)=f(0+\Delta x,0)-f(0,0)=0\ ,$$
$$\Delta_y f(0,0)=f(0,0+\Delta y,)-f(0,0)=0\ .$$

Ta'rifga ko'ra, limitni hisoblaymiz:

$$\lim_{\Delta x\to 0}\Delta_x f(0,0)=0, \quad \lim_{\Delta y\to 0}\Delta_y f(0,0)=0\,.$$

Demak, $f(x,y)$ funksiya $(0,0)$ nuqtada har bir o'zgaruvchi bo'yicha uzluksiz.

Endi funksiyaning $(0,0)$ nuqtadagi to'liq orttirmasini tekshiramiz:

$$\Delta f(0,0)=f(0+\Delta x\,,0+\Delta y)-f(0,0)=\frac{2\Delta x\cdot\Delta y}{\Delta x^2+\Delta y^2}\,.$$

Ushbu $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\Delta f(0,0)$ limit mavjud bo'lmaydi. Demak, berilgan funksiya $(0,0)$ nuqtada uzilishga ega. ◄

5.1.11-misol. $f(x,y)=\dfrac{1}{\sin^2\pi x+\sin^2\pi y}$ funksiyaning uzilish nuqtalarini toping.

Yechilishi: ► Ushbu funksiya R^2 to'plamning $\begin{cases}\sin\pi x=0\,,\\ \sin\pi y=0\end{cases}$ sistemasini qanoatlantiruvchi (x,y) nuqtalarida uzilishga ega. Chunki, sistemaning yechimi $\{(x,y)\in R^2;\ x=n\in Z,\ y=m\in Z\}$ to'plamdan iborat. Ko'rish mumkinki, berilgan funksiyaning uzilish nuqtalari cheksiz ko'p bo'lib, ular $\{(nm)\in R^2\ ;\ n\in Z\ ,\ m\in Z\}$ to'plamni tashkil etadi. ◄

Nuqtada uzluksiz funksiyalarning xossalari

Ikki o'zgaruvchili uzluksiz funksiya ham bir o'zgaruvchili uzluksiz funksiya ega bo'lgan asosiy xossalarga ega bo'ladi.

1^0. Yig'indining uzluksizligi. Agar $f(x,y)$ va $g(x,y)$ funksiyalar $P_0(x_0,y_0)$ nuqtada uzluksiz bo'lsa, u holda $f(x,y)\pm g(x,y)$ funksiya ham $P_0(x_0,y_0)$ nuqtada uzluksiz funksiyadir, ya'ni

$$\lim_{\substack{x\to x_0\\ y\to y_0}}\left[f(x,y)\pm g(x,y)\right]=\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)\pm\lim_{\substack{x\to x_0\\ y\to y_0}} g(x,y)=f(x_0,y_0)\pm g(x_0,y_0).$$

2^0.Ko‘paytmaning uzluksizligi. Agar $f(x,y)$ va $g(x,y)$ funksiyalar $P_0(x_0,y_0)$ nuqtada uzluksiz bo‘lsa, u holda $f(x)\cdot\varphi(x)$ ko‘paytma ham $P_0(x_0,y_0)$ nuqtada uzluksiz funksiyadir, ya’ni

$$\lim_{\substack{x\to x_0\\ y\to y_0}}\left[f(x,y)\cdot g(x,y)\right]=\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)\cdot\lim_{\substack{x\to x_0\\ y\to y_0}} g(x,y)=f(x_0,y_0)\cdot g(x_0,y_0).$$

3^0.Bo‘linmaning uzluksizligi. Agar $f(x,y)$ va $g(x,y)$ funksiyalar $P_0(x_0,y_0)$ nuqtada uzluksiz bo‘lib, $g(x_0,y_0)\neq 0$ bo‘lsa, u holda $\dfrac{f(x,y)}{g(x,y)}$ bo‘linma ham $P_0(x_0,y_0)$ nuqtada uzluksiz funksiyadir, ya’ni

$$\lim_{\substack{x\to x_0\\ y\to y_0}}\left[\frac{f(x,y)}{g(x,y)}\right]=\frac{\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)}{\lim\limits_{\substack{x\to x_0\\ y\to y_0}} g(x,y)}=\frac{f(x_0,y_0)}{g(x_0,y_0)}.$$

Mavzu yuzasidan savollar:

1. Qanday funksiyalarga ikki o‘zgaruvchili funksiyalar deyiladi?
2. 2 o‘zgaruvchili funksiyalarning aniqlanish sohalari nima?
3. Ikki o‘zgaruvchili funksiya limiti deb nimaga aytiladi?
4. Nuqtaning δ atrofi tushunchasi nima?
5. Ikki o‘zgaruvchili funksiya limiti qanday xossalarga ega?
6. Ikki o‘zgaruvchili funksiyaning nuqtada uzluksizligini ta’riflang.
7. Ikki o‘zgaruvchili funksiya qanday nuqtalarda uzilishga ega deyiladi?
8. Qanday funksiyalar sohada uzluksiz deyiladi?
9. Ikki o‘zgaruvchili funksiyaning nuqtada uzluksizligi qanday xossalarga ega?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $z=\ln\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)$ funksiyaning aniqlanish sohasini toping.

2. $z=\frac{1}{x-y}$ funksiyaning uzilish nuqtalarini toping.

3. $\lim\limits_{\substack{x\to0\\y\to0}}\frac{\sin(2xy)}{xy}$ limitni hisoblang.

4. $\lim\limits_{\substack{x\to\infty\\y\to\infty}}\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$ limitni hisoblang.

5. $z=\frac{1}{x-1}+\frac{1}{y}$ funksiyaning aniqlanish sohasini toping.

TESTLAR

1. $z=\frac{2x+3y-1}{x-y}+\ln(x-y)$ funksiyaning aniqlanish sohasini toping.

A) $x\neq y$ **B)** $x>y$ **C)** $x\geq0;\ y\geq0$ **D)** $x\geq\frac{1-3y}{2}$

2. $\lim\limits_{\substack{x\to0\\y\to0}}\frac{3-\sqrt{xy+9}}{xy}$ limitni hisoblang.

A) $-\frac{1}{6}$ **B)** 0 **C)** -1 **D)** 6

3. $z=\frac{y}{4x-8}$ funksiya qaysi nuqtada uzilishga ega?

A) (2;1) **B)** (0;0) **C)** (-1;0) **D)** (1;1)

4. $\lim\limits_{\substack{x\to1\\y\to0}}\frac{2\cdot\sin xy}{5y}$ limitni hisoblang.

A) 0,1 **B)** 0,4 **C)** 0,2 **D)** 2,5

5. $z=\sqrt{4-x^2-4y^2}$ funksiyaning aniqlanish sohasini toping.

A) $|x|\leq2;\ |y|\leq1$ **B)** $x^2+4y^2<4$ **C)** $x+2y<2$ **D)** $\frac{x^2}{4}+\frac{y^2}{1}\leq1$

5.2-§. Ikki o‘zgaruvchili funksiya hosilasi

5.2.1. Ikki o‘zgaruvchili funksiyaning xususiy va to‘liq orttirmalari

$z = f(x, y)$ funksiyada x o‘zgaruvchiga biror Δx orttirma berib, y ni o‘zgarishsiz qoldirsak, funksiya $\Delta_x z$ orttirma olib, bu orttirmaga z funksiyaning x **o‘zgaruvchi bo‘yicha xususiy orttirmasi** deyiladi va quyidagicha yoziladi:

$$\Delta_x z = f(x+\Delta x, y) - f(x, y). \qquad (5.3)$$

Xuddi shuningdek, y o‘zgaruvchiga Δy orttirma berib x o‘zgarishsiz qolsa, unga z funksiyaning y **o‘zgaruvchi bo‘yicha xususiy** $\Delta_y z$ **orttirmasi** deyiladi va quyidagicha yoziladi:

$$\Delta_y z = f(x, y+\Delta y) - f(x, y). \qquad (5.4)$$

x va y o‘zgaruvchilar mos ravishda Δx va Δy orttirmalar olsa, unda $z = f(x, y)$ funksiya to‘liq orttirma oladi:

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y). \qquad (5.5)$$

a) $\lim\limits_{\Delta x\to 0}\dfrac{\Delta_x z}{\Delta x}$ chekli limit mavjud bo‘lsa, u holda bu limitga $z = f(x, y)$ funksiyaning x **o‘zgaruvchi bo‘yicha xususiy hosilasi** deyiladi va $\dfrac{\partial z}{\partial x}$ yoki $z'_x = f'_x(x, y)$ ko‘rinishida belgilanadi.

b) $\lim\limits_{\Delta y\to 0}\dfrac{\Delta_y z}{\Delta y}$ chekli limit mavjud bo‘lsa, u holda bu limitga $z = f(x, y)$ funksiyaning y **o‘zgaruvchi bo‘yicha xususiy hosilasi** deyiladi va $\dfrac{\partial z}{\partial y}$ yoki $z'_y = f'_y(x, y)$ ko‘rinishida belgilanadi.

Xususiy hosilalar ta’riflaridan ko‘rinadiki bir argumentli funksiyani differensiallashning hamma qoida va formulalari o‘z kuchida qoladi.

Istalgan chekli sondagi o‘zgaruvchilar funksiyasining xususiy hosilalari ham yuqoridagidek aniqlanadi.

5.2.1-misol. $z = x^2 + 2xy + 3y^2$ xususiy hosilalarni toping.

Yechilishi: ► Oldin y ni o‘zgarmas deb z'_x ni topamiz:

$$z'_x = (x^2 + 2xy + 3y^2)'_x = (x^2)'_x + (2xy)'_x + (3y^2)'_x = 2x + 2y,$$

endi x ni o‘zgarmas deb z'_y ni topamiz:

$$z'_y = (x^2 + 2xy + 3y^2)'_y = (x^2)'_y + (2xy)'_y + (3y^2)'_y = 2x + 6y. \blacktriangleleft$$

5.2.2.-misol. $u = \dfrac{x}{x^2 + y^2 + z^2}$ funksiyaning xususiy hosilalarini toping.

Yechilishi: ► Hosila olish qoidalari va formulalaridan foydalanib quyidagilarni topamiz:

$$u'_x = \left(\frac{x}{x^2+y^2+z^2}\right)'_x = \frac{x'_x(x^2+y^2+z^2) - x(x^2+y^2+z^2)'_x}{(x^2+y^2+z^2)^2} = \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}.$$

$$u'_y = \left(\frac{x}{x^2+y^2+z^2}\right)'_y = \frac{x'_y(x^2+y^2+z^2) - x(x^2+y^2+z^2)'_y}{(x^2+y^2+z^2)^2} = \frac{-2xy}{(x^2+y^2+z^2)^2}.$$

$$u'_z = \left(\frac{x}{x^2+y^2+z^2}\right)'_z = \frac{x'_z(x^2+y^2+z^2) - x(x^2+y^2+z^2)'_z}{(x^2+y^2+z^2)^2} = \frac{-2xz}{(x^2+y^2+z^2)^2}. \blacktriangleleft$$

5.2.3.-misol. $f(x,y) = \ln tg\dfrac{x}{y}$ funksiyaning xususiy hosilalarini toping.

Yechilishi: ► $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(\ln tg\dfrac{x}{y}\right) = \dfrac{1}{tg\dfrac{x}{y}} \cdot \dfrac{1}{\cos^2\dfrac{x}{y}} \cdot \dfrac{1}{y} = \dfrac{2}{y\sin\dfrac{2x}{y}};$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\ln tg\frac{x}{y}\right) = \frac{1}{tg\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right) = \frac{-2}{y^2\sin\frac{2x}{y}}. \blacktriangleleft$$

5.2.4-misol. $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{agar} \quad (x,y) \neq (0,0) \ \text{bo'lsa,} \\ 0, & \text{agar} \quad (x,y) = (0,0) \ \text{bo'lsa} \end{cases}$ funksiyaning xususiy hosilalarini toping.

Yechilishi: ► Aytaylik, $(x,y) \neq (0,0)$ bo‘lsin. U holda xususiy hosilalar quyidagiga teng bo‘ladi:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2xy}{x^2+y^2}\right) = \frac{2y(x^2+y^2) - 2xy \cdot 2x}{(x^2+y^2)^2} = \frac{2y(y^2-x^2)}{(x^2+y^2)^2};$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2xy}{x^2+y^2}\right) = \frac{2x(x^2+y^2) - 2xy \cdot 2y}{(x^2+y^2)^2} = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}$$

Aytaylik, $(x,y) = (0,0)$ bo‘lsin. U holda ta’rifdan foydalanib topsak, quyidagi hosil bo‘ladi, ya’ni $(x,y) = (0,0)$ nuqtada xususiy hosilalar nolga teng:

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\Delta x\cdot 0}{\Delta x^3}=0,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{2\Delta y\cdot 0}{\Delta y^3}=0. \blacktriangleleft$$

5.2.5-misol. $f(x,y)=\sqrt{x^2+y^2}$ funksiyaning xususiy hosilalarini toping.

Yechilishi: ► Aytaylik, $(x,y)\neq(0,0)$ bo‘lsin. U holda xususiy hosilalar quyidagiga teng bo‘ladi:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\sqrt{x^2+y^2}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}},$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\sqrt{x^2+y^2}=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}$$

Aytaylik, $(x,y)=(0,0)$ bo‘lsin. U holda ta’rifdan foydalanib topsak, quyidagi hosil bo‘ladi:

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x},$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{|\Delta y|}{\Delta y}$$

Bu limitlar mavjud emas, shuning uchun berilgan funksiya $(x,y)=(0,0)$ nuqtada xususiy hosilalarga ega bo‘lmaydi.◄

5.2.6-misol. $f(x,y)=\begin{cases}\dfrac{x^3y}{x^6+y^2}, \text{ agar } & (x,y)\neq(0,0) \text{ bo'lsa,}\\ 0 \quad , \text{ agar } & (x,y)=(0,0) \text{ bo'lsa}\end{cases}$

funksiyaning $(0,0)$ nuqtadagi xususiy hosilalarini toping.

Yechilishi: ► Ta’rifga ko‘ra,

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=0$$

bo‘ladi. Biroq berilgan funksiya $(0,0)$ nuqtada uzluksiz bo‘lmaydi, chunki

$$\left(\frac{1}{n},\frac{1}{n^3}\right)\to(0,0) \text{ da } f\left(\frac{1}{n},\frac{1}{n^3}\right)=\frac{1}{2}\to\frac{1}{2}\neq f(0,0). \blacktriangleleft$$

Mavzu yuzasidan savollar:

1. Funksiyaning xususiy orttirmasi deb nimaga aytiladi?
2. Ikki o‘zgaruvchili funksiyaning x o‘zgaruvchi bo‘yicha xususiy orttirmasi qanday topiladi?
3. Ikki o‘zgaruvchili funksiyaning y o‘zgaruvchi bo‘yicha xususiy orttirmasi qanday topiladi?
4. Ikki o‘zgaruvchili funksiyaning to‘la orttirmasi formulasini yozing.
5. Ikki o‘zgaruvchili funksiyaning x o‘zgaruvchisi bo‘yicha xususiy hosilasi deb nimaga aytiladi?
6. Ikki o‘zgaruvchili funksiyaning y o‘zgaruvchisi bo‘yicha xususiy hosilasi deb nimaga aytiladi?

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1. Funksiyaning x o‘zgaruvchisi bo‘yicha xususiy hosilalarini toping:

a) $z = \sin\sqrt{\dfrac{y}{x+y}}$ **b)** $z = xctg\left(x/y^3\right)$

2. Ikki o‘zgaruvchili funksiyaning y o‘zgaruvchi bo‘yicha xususiy orttirmasini yozing:

a) $z = \arccos\dfrac{y}{x}$ **b)** $z = arctg\left(x^2 + y^2\right)$

3. Funksiyaning y o‘zgaruvchisi bo‘yicha xususiy hosilalarini toping:

a) $z = e^{-x^2+y^2}$ **b)** $z = y\ln\left(3x^2 + \sqrt{y}\right)$

4. $f(x,y) = \begin{cases} \dfrac{xy}{x^4+y}, \text{ agar } & (x,y) \neq (0,0) \text{ bo'lsa,} \\ 0 \quad , \text{ agar } & (x,y) = (0,0) \text{ bo'lsa} \end{cases}$

funksiyaning $(0,0)$ nuqtadagi xususiy hosilalarini toping.

5. $f(x,y) = \ln ctg\dfrac{x+y}{y}$ funksiyaning xususiy hosilalarini toping.

TESTLAR

1. $z=\dfrac{\cos y}{\sin x}$ funksiyaning xususiy hosilalarini toping.

A) $z'_x=-\dfrac{\cos x\cos y}{\sin^2 x}$, $z'_y=-\dfrac{\sin y}{\sin\ x}$ B) $z'_x=\dfrac{\cos x\cos y}{\sin^2 x}$, $z'_y=-\dfrac{\sin y}{\sin\ x}$

C) $z'_x=\dfrac{\cos x\cos y}{\sin^2 x}$, $z'_y=0$; D) $z'_x=-\dfrac{\cos x\cos y}{\sin^2 x}$, $z'_y=0$.

2. $z=\dfrac{x^2}{e^y}$ funksiyaning xususiy hosilalarini toping.

A) $z'_x=-\dfrac{2x}{e^y}$, $z'_y=\dfrac{x^2}{e^y}$ B) $z'_x=\dfrac{2x}{e^y}$, $z'_y=-\dfrac{x^2}{e^y}$

C) $z'_x=-\dfrac{2x}{e^y}$, $z'_y=-\dfrac{x^2}{e^y}$; D) $z'_x=\dfrac{2x}{e^y}$, $z'_y=\dfrac{x^2}{e^y}$.

3. M(0,-1,1) nuqtada $f(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2}}$ funksiyaning xususiy hosilalari qiymatlarini toping.

A) $f_x(M)=0$, $f_y(M)=1$, $f_z(M)=1$

B) f_x(M)=1, f_y(M)=0, f_y(M)=0.

C) f_x(M)=0, f_y(M)=0, f_y(M)=0.

D) f_x(M)=-1, f_y(M)=1, f_y(M)=0.

4. $F(x,y\,z)=0$ tenglama bilan berilgan $z(x,y)$oshkormas funksiyaning xususiy hosilalarini toping.

A) $z_x=-\dfrac{F_x}{F_y}$, $z_y=-\dfrac{F_y}{F_z}$ B) $z_x=-\dfrac{F_x}{F_z}$, $z_y=-\dfrac{F_y}{F_z}$

C) $z_x=\dfrac{F_x}{F_z}$, $z_y=\dfrac{F_y}{F_z}$ D) $z_x=\dfrac{F_x}{F_y}$, $z_y=\dfrac{F_y}{F_z}$.

5. $u=zx^y$ uch o‘zgaruvchili funksiyaning xususiy hosilalarini toping.

A) $u_x=yzx^{y-1}$, $u_y=zx^y\ln x$, $u_z=x^y$

B) $u_x=yzx^{y-1}$, $u_y=zx^y\ln y$, $u_z=x^y$

C) $u_x=yzx^{y-1}$, $u_y=zx^y\ln x$, $u_z=x^yz$

D) $u_x=yzx^{y-1}$, $u_y=zx^y$, $u_y=x^y$

5.3-§. Ko‘p o‘zgaruvchili funksiya to‘la differensiali. Yuqori tartibli xususiy hosilalar va differensiallar

5.3.1. Yuqori tartibli xususiy hosilalar va differensiallar

$z = f(x, y)$ funksiya va ixtiyoriy $P(x, y)$ nuqtani qaraymiz. Ma’lumki, x va y o‘zgaruvchilar mos ravishda Δx va Δy orttirmalar olsa, $P_1(x+\Delta x, y+\Delta y)$ nuqtaga ega bo‘lamiz. Bu nuqtalar orasidagi masofa $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ bo‘lishi ravshan. $z = f(x, y)$ funksiya

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$ to‘liq orttirma oladi.

Agar $z = f(x, y)$ funksiyaning $P(x, y)$ nuqtadagi to‘liq orttirmasini

$$\Delta z = A\cdot\Delta x + B\cdot\Delta y + o(\rho)$$

ko‘rinishda ifodalash mumkin bo‘lsa, bu funksiya $P(x, y)$ nuqtada **differensialanuvchi** deyiladi, bu yerda $A, B,$ Δx va Δy ga bog‘liq bo‘lmagan sonlar, oxirgi qo‘shiluvchi $\Delta x \to 0,\ \Delta y \to 0\ (\rho \to 0)$ da yuqori tartibli cheksiz kichik funksiya bo‘lib, u nolga intiladi, shuning uchun bu hadni tashlab yuborish mumkin.

Differensiallanuvchi $z = f(x, y)$ funksiyaning argumentlarning $\Delta x, \Delta y$ orttirmalariga nisbatan chiziqli ifodasi bo‘lgan bosh bo‘lagi $z = f(x, y)$ funksiyaning **to‘la differensiali** deyiladi va dz bilan belgilanadi:

$$dz = A\cdot\Delta x + B\cdot\Delta y\ .$$

5.1-teorema (funksiya differensiallanuvchi bo‘lishining zaruriy sharti). Agar x funksiya $P(x, y)$ nuqtada differensiallanuvchi bo‘lsa, u holda u shu nuqtada $f'_x(x, y)$ va $f'_y(x, y)$ hususiy hosilalarga ega bo‘ladi, bunda $A = f'_x(x, y),\ B = f'_y(x, y).$

A va B kattaliklarni xususiy hosilaga almashtirib, quyidagilarga ega bo‘lamiz:

$$\Delta z = f'_x(x, y)\cdot\Delta x + f'_y(x, y)\cdot\Delta y + o(\rho)$$

$$dz = f'_x(x, y)\cdot\Delta x + f'_y(x, y)\cdot\Delta y$$

Erkli o‘zgaruvchilarning orttirmalari ularning differensiallariga bevosita teng, ya’ni $dx = \Delta x,\ dy = \Delta y,$ shuning uchun to‘la differensial

$$dz = f'_x(x, y)dx + f'_y(x, y)dy \text{ yoki } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad (5.6)$$

formula bilan hisoblanadi. $f'_x(x, y)dx$ va $f'_y(x, y)dy$ qo‘shiluvchilarga **xususiy differensiallar** deyiladi, ular mos ravishda $d_x z$ va $d_y z$ bilan belgilanadi.

Uch o'zgaruvchili $u = F(x,y,z)$ funksiyaning to'la differensiali

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \qquad (5.7)$$

formula bilan hisoblanadi.

5.2-teorema (funksiya differensiallanuvchi bo'lishining yetarli sharti). Agar $z = f(x,y)$ funksiya *P(x, y)* nuqtaning biror δ atrofida xususiy hosilalarga ega bo'lib, bu hosilalar nuqtaning o'zida uzluksiz bo'lsa, u holda funksiya shu nuqtada differensiallanuvchi bo'ladi.

5.3.1-misol. $z = \ln(x^2 + y^2)$ funksiyaning to'la differensialini toping.

Yechilishi: ► Xususiy hosilalarni topamiz:

$$z'_x = \frac{(x^2+y^2)'_x}{x^2+y^2} = \frac{2x}{x^2+y^2}, \quad z'_y = \frac{(x^2+y^2)'_y}{x^2+y^2} = \frac{2y}{x^2+y^2},$$

Shunda to'la differensial (5.6) formulaga ko'ra,

$$dz = \frac{2x}{x^2+y^2}dx + \frac{2y}{x^2+y^2}dy$$ bo'ladi. ◄

To'la differensialdan funksiyaning taqribiy qiymatlarini hisoblashda foydalanish mumkin, ya'ni yuqoridagilardan $\Delta z = dz + o(\rho)$ va cheksiz kichik ρ larda $\Delta z \approx \Delta z$ yoki $f(x+\Delta x,\ y+\Delta y) - f(x,y) \approx dz$ taqribiy tenglik hosil bo'ladi. Demak, **taqribiy hisoblash** formulasi quyidagiga teng:

$$f(x+\Delta x,\ y+\Delta y) \approx f(x,y) + f_x'(x,y)dx + f_y'(x,y)dy \qquad (5.8)$$

Taqribiy hisoblash formulasini uch o'zgaruvchili funksiya uchun ham umumlashtirish mumkin.

5.3.2-misol. $arcctg\left(\frac{1,97}{1,02} - 1\right)$ ni taqribiy hisoblang.

Yechilishi: ► (5.8) formuladan foydalanamiz. Berilgan funksiyaning analitik formulasi quyidagicha: $f(x,y) = arcctg(\frac{x}{y} - 1)$.

$$x = 2,\quad y = 1,\quad \Delta x = 1.97 - 2 = -0.03,\quad \Delta y = 1.02 - 1 = 0.02$$

Endi xususiy hosilalarni topamiz:

$$f_x'(x,y) = \left[arcctg\left(\frac{x}{y} - 1\right)'_x\right] = -\frac{1}{1+\left(\frac{x-y}{y}\right)^2}\frac{1}{y} = -\frac{y}{y^2+(x-y)^2};$$

$$f'_y(x,y)=\left[arcctg\left(\frac{x}{y}-1\right)'_y\right]=\frac{x}{y^2+(x-y)^2};$$

$$f'_x(2;1)=-\frac{1}{1+(2-1)^2}=-0.5; \qquad f'_y(2;1)=\frac{2}{1+(2-1)^2}=1.$$

Demak, $arcctg\left(\frac{1.97}{1.02}-1\right)\approx arcctg\left(\frac{2}{1}-1\right)+(-0.5)(-0.03)+1\cdot 0.02=0.82$ ◀

$z=f(x,y)$ funksiyaning **ikkinchi tartibli xususiy hosilalari** deb birinchi tartibli xususiy hosilalardan olingan xususiy hosilalarga aytiladi. Ikkinchi tartibli xususiy hosilalar quyidagicha belgilanadi:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=z_{xx}''=f_{xx}''(x,y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=z_{xy}''=f_{xy}''(x,y);$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=z_{yx}''=f_{yx}''(x,y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=z_{yy}''=f_{yy}''(x,y);$$

$f_{xy}''(x,y)$ va $f_{yx}''(x,y)$ xususiy hosilalar **aralash xususiy hosilalar** deyiladi. Aralash xususiy hosilalar uzluksiz bo‘lgan nuqtalarda ular o‘zaro teng bo‘ladi.

Uchinchi va undan yuqori tartibli xususiy hosilalar ham yuqoridagidek aniqlanadi. Ushbu $\frac{\partial^n z}{\partial x^m \partial y^{n-m}}$ yozuv z funksiyani m marta x o‘zgaruvchi bo‘yicha va $(n-m)$ marta y o‘zgaruvchi bo‘yicha differensiallashni bildiradi.

5.3.3-misol. $z=x^4+4x^2y^3+9xy+15$ ikkinchi tartibli xususiy hosilalarni toping.

Yechilishi: ► Birinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x}=\left(x^4+4x^2y^3+9xy+15\right)'_x=4x^3+8xy^3+9y,$$

$$\frac{\partial z}{\partial y}=\left(x^4+4x^2y^3+9xy+15\right)'_y=12x^2y^2+9x.$$

Topilgan hosilalardan yana xususiy hosilalar olamiz:

$$\frac{\partial^2 z}{dx^2}=\left(4x^3+8xy^3+9y\right)'_x=12x^2+8y^3, \quad \frac{\partial^2 z}{dx\partial y}=\left(4x^3+8xy^3+9y\right)'_y=24xy^2+9,$$

$$\frac{\partial^2 z}{\partial y\partial x}=\left(12x^2y^2+9x\right)'_x=24xy^2+9, \qquad \frac{\partial^2 z}{\partial y^2}=\left(12x^2y^2+9x\right)'_y=24x^2y. \ ◀$$

Birinchi tartibli to‘la differensialdan olingan to‘la differensial **ikkinchi tartibli to‘la differensial** deyiladi va u $d(dz)=d^2z$ kabi aniqlanib, xususiy hosilalar orqali quyidagicha topiladi.

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2 \qquad (5.9)$$

5.3.4-misol. $z=x^2y^3$ funksiyaning ikkinchi tartibli to‘la differensialini toping.

Yechilishi: ► Xususiy hosilalarni topamiz:

$$z'_x=\left(x^2y^3\right)'_x=2xy^3; \qquad z'_y=3x^2y^2;$$

$$z''_{xx}=2y^3, \quad z''_{xy}=6xy^2, \quad z''_{yx}=6xy^2, \quad z''_{yy}=6xy^2,$$

Shunda ikkinchi tartibli to‘la differensial quyidagiga teng bo‘ladi:

$$d^2z=2y^3dx^2+12xy^2dxdy+6x^2ydy^2. ◄$$

5.3.2. Ikki o‘zgaruvchili murakkab va oshkormas funksiyalarning hosilalari

$z=z(u,v)$ ikki o‘zgaruvchili funksiya bo‘lsin. u va v argumentlar ham x erkli o‘zgaruvchining differensiallanuvchi funksiyalari bo‘lsin, ya’ni $u=u(x)$, $v=v(x)$.

Murakkab funksiyaning hosilasi quyidagicha hisoblanadi:

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx} \qquad (5.10)$$

Ikki o‘zgaruvchining differensiallanuvchi $z=z(u,v)$ funksiyasi berilgan bo‘lsin. u va v argumentlar ham x va y erkli o‘zgaruvchilarning differensiallanuvchi funksiyalari bo‘lsin, ya’ni

$u=u(x,y)$, $v=v(x,y)$. Bu holda berilgan ikki o‘zgaruvchili **murakkab funksiyaning xususiy hosilalari** quyidagi formulalar bilan hisoblanadi:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \qquad (5.11)$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

5.3.5-misol. Agar $f(x,y)$ funksiya R^2 da differensiallanuvchi bo‘lib, $\begin{cases}x=r\cos\varphi\\ y=r\sin\varphi\end{cases}$ bo‘lsa, $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \varphi}$ larni toping.

Yechilishi: ► $f(x,y)=f(r\cos\varphi, r\sin\varphi)$ murakkab funksiyaning xususiy hosilalarini topish (5.11) qoidasiga ko‘ra

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}=\cos\varphi\frac{\partial f}{\partial x}+\sin\varphi\frac{\partial f}{\partial y}=\frac{1}{\sqrt{x^2+y^2}}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f}{\partial \varphi}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \varphi}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \varphi}=-r\sin\varphi\frac{\partial f}{\partial x}+r\cos\varphi\frac{\partial f}{\partial y}=-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}. \blacktriangleleft$$

Oshkormas $F(x,y(x))=0$ funksiyaning hosilasi

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0, \qquad y_x'=-\frac{F_x'(x,y)}{F_y'(x,y)}$$

formula orqali hisoblanadi.

Uchta o‘zgaruvchini bog‘laydigan $F(x,y,z)=0$ tenglamani qaraymiz. Bu yerda $z=z(x,y)$ funksiya bo‘ladi. Bunday ko‘rinishdagi **oshkormas funksiyaning xususiy hosilalari** quyidagi formulalardan topiladi:

$$z_x'=-\frac{F_x'(x,y,z)}{F_z'(x,y,z)}, \qquad z_y'=-\frac{F_y'(x,y,z)}{F_z'(x,y,z)} \qquad (5.12)$$

5.3.6-misol. Quyidagi funksiyaning xususiy hosilalarini toping:

$$z=u^v, \text{ bu yerda } u=y\sin x,\ v=y\cos x.$$

Yechilishi: ►Berilgan funksiya ikki o‘zgaruvchili murakkab funksiyadir. Dastlab z dan u va v o‘zgaruvchilar bo‘yicha xususiy hosila olamiz:

$$\frac{\partial z}{\partial u}=v\cdot u^{v-1},\quad \frac{\partial z}{\partial v}=u^v\cdot\ln u,$$

so‘ngra u va v funksiyalardan x va y o‘zgaruvchilar bo‘yicha xususiy hosila hisoblaymiz:

$$\frac{\partial u}{\partial x}=y\cos x,\quad \frac{\partial u}{\partial y}=\sin x;$$

$$\frac{\partial v}{\partial x}=-y\sin x,\quad \frac{\partial v}{\partial y}=\cos x.$$

Endi (5.11) formulalardan foydalanamiz:

$$\frac{\partial z}{\partial x}=v\cdot u^{v-1}\cdot y\cos x+u^v\cdot\ln u\cdot(-y\sin x)=$$

$$=(y\sin x)^{y\cos x}\left[\frac{y\cos^2 x}{\sin x}-y\sin x\ln(y\sin x)\right],$$

$$\frac{\partial z}{\partial y} = v \cdot u^{v-1} \cdot \sin x + u^v \cdot \ln u \cdot \cos x = (y\sin x)^{y\cos x}\left[\cos x + \cos x \ln(y\sin x)\right]. ◀$$

5.3.7-misol. $z = \frac{v}{2u+1}$, $u = 2x,\ v = 1 + arctgx,\ x_0 = 0$ murakkab funksiyaning nuqtadagi hosilasini toping.

Yechilishi: ►Avval funksiyaning to‘la differensialini topamiz:

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx} = -\frac{2v}{(2u+1)^2}\cdot 2 + \frac{1}{2u+1}\cdot\frac{1}{1+x^2} = -\frac{4(1+arctgx)}{(4x+1)^2} + \frac{1}{4x+1}\cdot\frac{1}{1+x^2};$$

Endi berilgan nuqtani o‘zgaruvchining o‘rniga qo‘yamiz:

$$\frac{dz}{dx_0} = -\frac{4(1+arctg0)}{(4\cdot 0+1)^2} + \frac{1}{4\cdot 0+1}\cdot\frac{1}{1+0^2} = 4+1=5. ◀$$

5.3.8-misol. $x^2 - 2y^2 + 3z^2 - yz + y = 0$ oshkormas funksiya uchun $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ larni toping.

Yechilishi: ► Tenglamaning chap tomonini *F(x,y,z)* deb belgilab, xususiy hosilalarini topamiz:

$$F_x'(x,y,z) = 2x,\ F_y'(x,y,z) = -4y - z + 1,\ F_z'(x,y,z) = 6z - y.$$

Oshkormas funksiyaning xususiy hosilalari formulalaridan foydalanib, quyidagi yechimlarga ega bo‘lamiz:

$$\frac{\partial z}{\partial x} = -\frac{F_x'(x,y,z)}{F_z'(x,y,z)} = -\frac{2x}{6z-y},\quad \frac{\partial z}{\partial y} = -\frac{F_y'(x,y,z)}{F_z'(x,y,z)} = -\frac{1-4y-z}{6z-y}. ◀$$

5.3.9-misol. $x^3 - 2y^2z + 3z^2 - yz + xyz = 0$ oshkormas funksiya uchun $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ xususiy hosilalarni toping.

Yechilishi: ► Tenglamaning chap tomonini *F(x,y,z)* deb belgilab, xususiy hosilalarini topamiz:

$$F_x'(x,y,z) = 3x^2 + yz,\ F_y'(x,y,z) = -4yz - z + xz,\ F_z'(x,y,z) = 2y^2 + 6z - y + xy$$

Oshkormas funksiyaning xususiy hosilalari uchun (5.12) formulalardan foydalanib quyidagi yechimlarga ega bo‘lamiz:

$$\frac{\partial z}{\partial x} = -\frac{F_x'(x,y,z)}{F_z'(x,y,z)} = -\frac{3x^2+yz}{2y^2+6z-y+xy};\quad \frac{\partial z}{\partial y} = -\frac{F_y'(x,y,z)}{F_z'(x,y,z)} = -\frac{xz-4yz-z}{2y^2+6z-y+xy}. ◀$$

Mavzu yuzasidan savollar:

1. Ikki o‘zgaruvchili funksiyaning to‘la differensiali deb nimaga aytiladi?
2. Differensiallanuvchalikning zaruriy sharti qanday?
3. Differensiallanuvchalikning yetarli sharti qanday?
4. Xususiy differensiallar nima?
5. To‘la differensialdan taqribiy hisoblashlarda foydalanish mumkinmi?
6. Ikkinchi tartibli xususiy hosila qanday topiladi?
7. Murakkab funksiyaning hosilasi qanday topiladi?
8. Oshkormas funksiyaning hosilasi qanday topladi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Funksiyaning to‘la differensialini toping:

a) $z=\sqrt{x}e^{-x^2+y^2}$ **b)** $z=\ln\left(y^2-e^{-2x}\right)$

c) $z=arcctg(x\sqrt{y})$ **d)** $z=ye^{-\sqrt{x^2+y^2}}$

e) $z=\arcsin\left(2x^2\sqrt{y}\right)$ **f)** $z=\cos\dfrac{x-y}{x^2+y^2}$

2. Murakkab funksiyaning nuqtadagi hosilasini toping:

a) $z=\dfrac{v}{2u+1},\ u=2x,\ v=1+arctgx,\ x_0=0$

b) $z=e^{v-2u-1},\ u=\cos x,\ v=\sin x,\ x_0=\pi/2.$

c) $z=\ln\left(e^u+e^{-v}\right),\ u=x^2, v=x^3, x_0=-1.$

d) $z=u^2e^v,\ u=\cos x,\ v=\sin x,\ x_0=\pi.$

e) $z=u^v,\ u=\ln(x-1),\ v=e^{x/2},\ x_0=2.$

f) $z=\sqrt{u+v^2+3},\ u=\ln x,\ v=x^2,\ x_0=1.$

3. Oshkormas funksiya uchun $\dfrac{\partial z}{\partial x}$ va $\dfrac{\partial z}{\partial y}$ xususiy hosilalarni toping.

a) $x^3-2y^2z+3z^2-yz+xyz=0$ **b)** $\cos^2 x+\cos^2 y+\cos^2 z=3/2.$

c) $e^{z-1}=\cos x\cos y+z.$ **d)** $\ln z=x+2y-z+\ln 3.$

e) $\sqrt{x^2+y^2}+z^2-3z=5.$ **f)** $x^3 3xyz-z^2=27.$

4. $z=f(x,y)$ funksiya berilgan tenglikni qanoatlantirishini ko‘rsating:

a) $z=e^{xy}$, $x^2\dfrac{\partial^2 z}{\partial x^2}-y^2\dfrac{\partial^2 z}{\partial y^2}=0$.

b) $z=\ln\left(x^2+y^2+2y+1\right)$, $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

c) $z=\sin^2\left(y-ax\right)$ $\frac{\partial^2 z}{\partial x^2}=a^2\frac{\partial^2 z}{\partial y^2}$.

d) $z=\frac{y}{x}$ $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0$.

e) $z=\sqrt{\frac{y}{x}}$ $x^2\frac{\partial^2 z}{\partial x^2}+3y^2\frac{\partial^2 z}{\partial y^2}=0$.

f) $z=arctg\left(\frac{y}{x}\right)$ $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

TESTLAR

1. $z=2arctg\frac{y}{x}$ funksiyaning $\frac{\partial z}{\partial y}$ xususiy hosilasini toping.

A) $\frac{y}{x^2+y^2}$ **B)** $\frac{2xy}{x^2+y^2}$ **C)** $\frac{x}{x^2+y^2}$ **D)** $\frac{2x}{x^2+y^2}$

2. $z=e^{xy}$ funksiyaning $\frac{\partial^2 z}{\partial x^2}$ xususiy hosilasini toping.

A) y^2e^{xy} **B)** x^2e^{xy} **C)** e^{xy} **D)** xye^{xy}

3. $z=\sin x+\cos y^2$ funksiyaning to‘la differensialini toping.

A) $dz=\cos x\,dx+2y\sin y^2dy$ **B)** $dz=\left(\cos x-2y\sin y^2\right)dx\,dy$

C) $dz=\cos x\,dx-2y\sin y^2dy$ **D)** $dz=\cos x\,dx-y\sin y^2dy$

4. M(1,2,1) nuqtada $x^2+y^2+z^2-6x=0$ oshkormas $z(x,y)$ funksiyaning xususiy hosilalari qiymatlarini toping.

A) $z_x(M)=2$; $z_y(M)=-2$ **B)** $z_x(M)=2$; $z_x(M)=0$

C) $z_x(M)=0$; $z_y(M)=-2$ **D)** $z_x(M)=1$ $z_y(M)=-2$

5. $f(x,y)=x^2-xy+y^2$ funksiyaning $x=2.15$; $y=1.25$ nuqtadagi taqribiy qiymatini toping.

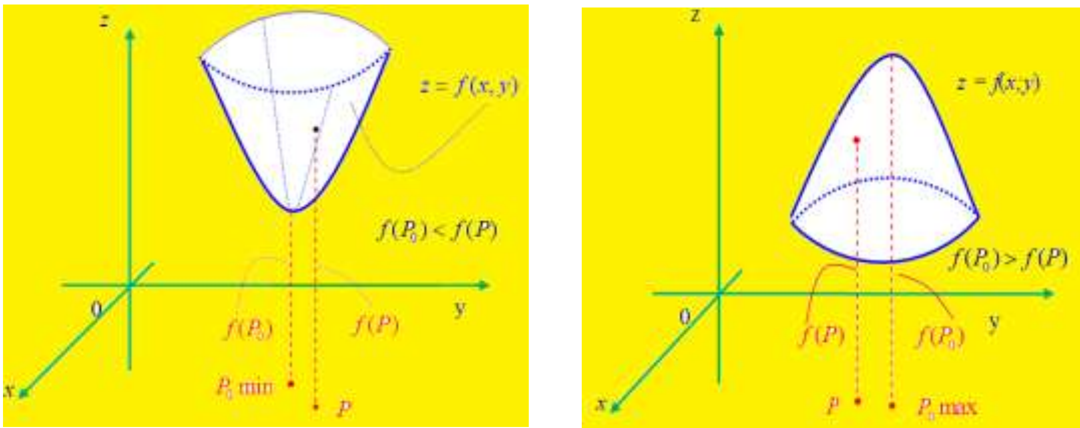
A) 4.21 **B)** 3.45 **C)** 3.05 **D)** 4.5

5.4-§. Ikki o‘zgaruvchili funksiya ekstremumlari va eng katta, eng kichik qiymatlarini topish. Shartli ekstremumlar

5.4.1. Ikki o‘zgaruvchili funksiya ekstremumi

$z = f(x, y)$ funksiyaning $P_0(x_0, y_0)$ nuqtadagi qiymati uning bu nuqtaning biror atrofidagi istalgan $P(x, y)$ nuqtalaridagi qiymatlaridan katta, ya’ni $f(x_0, y_0) > f(x, y)$ bo‘lsa, $z = f(x, y)$ **funksiya** $P_0(x_0, y_0)$ **nuqtada maksimumga ega** deyiladi.

$z = f(x, y)$ funksiyaning $P_1(x_1, y_1)$ nuqtadagi qiymati uning bu nuqtaning biror atrofidagi istalgan $P(x, y)$ nuqtalaridagi qiymatlaridan kichik bo‘lsa, ya’ni $f(x_1, y_1) < f(x, y)$ bo‘lsa, $z = f(x, y)$ **funksiya** $P_1(x_1, y_1)$ **nuqtada minimumga ega** deyiladi.



5.7-rasm. $z = f(x, y)$ funksiyaning ekstremumlari

Ta’rifga ko‘ra, funksiya maksimumga erishganda $f(P_0) > f(P)$ yoki $f(x_0, y_0) > f(x, y)$ tengsizlik o‘rinli bo‘ladi (5.7-rasm). Minimumga erishganda esa $f(P_0) < f(P)$ yoki $f(x_0, y_0) < f(x, y)$ tengsizlik o‘rinli bo‘ladi. Agar tengsizlikning chap qismidagi ifodani o‘ng qismiga o‘tkazsak, u holda **maksimum bo‘lgan holda**

$$\Delta z = f(P) - f(P_0) = f(x, y) - f(x_0, y_0) < 0 \text{ ga,}$$

minimum bo‘lgan holda

$$\Delta z = f(P) - f(P_0) = f(x, y) - f(x_0, y_0) > 0 \text{ ga ega bo‘lamiz.}$$

Xulosa: maksimumda funksiyaning to‘liq orttirmasi manfiy $\Delta z < 0$, minimumda esa funksiyaning to‘liq orttirmasi musbat $\Delta z > 0$ bo‘ladi. Teskari da’vo ham o‘rinli.

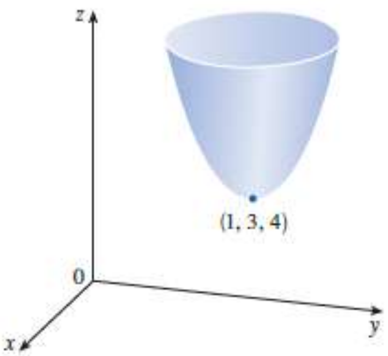
Funksiyaning maksimum va minimum nuqtalari **ekstremum nuqtalar** deyiladi.

5.3-teorema: (Ekstremum mavjudligining zaruriylik sharti). Agar differensiallanuvchi $z = f(x,y)$ funksiya $P_0(x_0, y_0)$ nuqtada ekstremumga ega bo‘lsa, u holda uning shu nuqtadagi xususiy hosilalari nolga teng bo‘lishi zarur:

$$\begin{cases} f'_x(x_0, y_0) = 0 \\ f'_y(x_0, y_0) = 0 \end{cases} \tag{5.13}$$

5.4.1-misol. $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ funksiyani ekstremumga tekshiring.

Yechilishi: ► Funksiyaning xususiy hosilalarini topamiz va nolga tenglaymiz: $f'_x = 2x - 2$, $f'_y = 2y - 6$. Bundan $x = 1$, $y = 3$ ni topamiz. $f(x,y) = (x-1)^2 + (y-3)^2 + 4$ funksiya grafigi elliptik paraboloid bo‘lib, $z(1,3) = 4$ ga teng (5.8-rasm). Bu nuqta funksiyaning minimum nuqtasi bo‘ladi.



5.8-rasm. $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ funksiya grafigi ◄

Xususiy hosilalar nolga teng bo‘ladigan, xususiy hosilalar mavjud bo‘lmaydigan yoki cheksizlikka teng bo‘ladigan nuqtalari **kritik (statsionar) nuqtalar** deyiladi.

Funksiyaning kritik nuqtalarini topish uchun uning ikkala xususiy hosilasini nolga tenglash va hosil bo‘lgan ikki o‘zgaruvchili ikkita tenglamalar sistemasini yechish kerak bo‘ladi. Bundan tashqari, xususiy hosilalari mavjud bo‘lmaydigan nuqtalarni topish kerak.

5.4-teorema: (Ekstremum mavjudligining yetarlilik sharti).

Agar $z = f(x,y)$ funksiya $P_0(x_0, y_0)$ kritik nuqtada va uning biror atrofida 2-tartibli xususiy hosilalarga ega bo‘lib, bu nuqtadagi xususiy hosilalari nolga teng bo‘lsa $\begin{cases} f'_x(x_0, y_0) = 0 \\ f'_y(x_0, y_0) = 0 \end{cases}$, u holda $P_0(x_0, y_0)$ nuqtada

1) $\Delta = A \cdot C - B^2 > 0$ bo‘lsa, ekstremum mavjud, bunda

a) agar $A>0$ bo‘lsa, minimum;

b) agar $A<0$ bo‘lsa, maksimum;

2) $\Delta = A \cdot C - B^2 < 0$ bo‘lsa, ekstremum yo‘q;

3) $\Delta = A \cdot C - B^2 = 0$ bo‘lsa, ekstremum bo‘lishi ham, bo‘lmasligi ham mumkin (qo‘shimcha shrtlar bilan tekshiriladi).

Bunda

$$A = f''_{xx}(x_0, y_0),\ \ B = f''_{xy}(x_0, y_0),\ C = f''_{yy}(x_0, y_0).$$

5.4.2-misol. $f(x,y) = x^4 + y^4 - 4xy + 1$ funksiyaning ekstremumlarini toping.

Yechilishi: ► Funksiyaning xususiy hosilalarini topamiz va nolga tenglaymiz: $f'_x = 4x^3 - 4y$, $f'_y = 4y^3 - 4x$. Bu hosilalarni nolga tenglab,

$$\begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 \\ x^9 - x = 0 \end{cases} \Rightarrow$$

$$x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1),$$

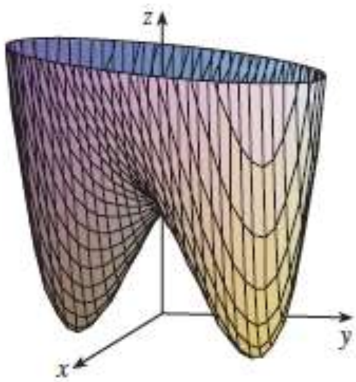
$x = 0,\ x = 1,\ x = -1$ qiymatlarni topamiz. Demak, (0,0), (1,1) va (-1,-1) uchta haqiqiy nuqta hosil bo‘ldi, tenglikning kompleks ildizlari ham mavjud (kompleks ildizlarga to‘xtalib o‘tirmaymiz). Endi 2-tartibli xususiy hosilalarni topamiz:

$$A = f''_{xx}(x_0, y_0) = 12x^2,\ \ B = f''_{xy}(x_0, y_0) = -4,\ C = f''_{yy}(x_0, y_0) = 12y^2$$

P(0,0) nuqtada $\Delta = AC - B^2 = 12x^2 \cdot 12y^2 - 16 = -16 < 0$, $A = 0$, bu nuqtada funksiyaning ekstremumi mavjud emas.

P_1(1,1) nuqtada $\Delta = AC - B^2 = 12x^2 \cdot 12y^2 - 16 = 128 > 0$, $A = 12$. Funksiyaning bu nuqtadagi qiymati $f(1,1) = -1$ bo‘lib, lokal minimum bo‘ladi (5.9-rasm).

P_2(-1,-1) nuqtada $\Delta = AC - B^2 = 12x^2 \cdot 12y^2 - 16 = 128 > 0$, $A = 12$. Funksiyaning bu nuqtadagi qiymati $f(-1,-1) = -1$ bo‘lib, lokal minimum bo‘ladi.



5.9-rasm. $f(x,y) = x^4 + y^4 - 4xy + 1$ funksiya grafigi ◄

5.4.3-misol. $z = f(x,y) = x^3 + y^2 + 2xy$ ikki o‘zgaruvchili funksiyani ekstremumga tekshiring.

Yechilishi: ► Bu funksiya butun Oxy tekislikda aniqlangan. Birinchi tartibli xususiy hosilalarini topamiz: $f'_x = 3x^2 + 2y$; $f'_y = 2y + 2x$. Ekstremumga ega bo‘lishning zaruriylik shartidan quyidagi sistemani tuzamiz:

$$\begin{cases} 3x^2 + 2y = 0 \\ 2y + 2x = 0 \end{cases} \Rightarrow, \quad \begin{cases} 3x^2 - 2x = 0 \\ y = -x \end{cases}$$

Demak, ikkita $P_1(0;0)$ va $P_2\left(\frac{2}{3}, -\frac{2}{3}\right)$ kritik nuqtalarga ega bo‘lamiz, boshqa kritik nuqtalar yo‘q, chunki $f'_x(x,y)$, $f'_y(x,y)$ xususiy hosilalar Oxy tekislikning boshqa hamma nuqtalarida mavjud va noldan farqli.

Ikkinchi tartibli xususiy hosilalarni topamiz:

$$A = f''_{xx}(x,y) = 6x;\ B = f''_{xy}(x,y) = 2;\ C = f''_{yy}(x,y) = 2.$$

Har bir kritik nuqtada Δ ni hisoblaymiz:

a) $P_1(0;0)$ nuqtada: $\Delta = AC - B^2 = 6\cdot 0\cdot 2 - 2^2 = -4 < 0,$ demak bu nuqtada ekstremum mavjud emas;

b) $P_2\left(\frac{2}{3}, -\frac{2}{3}\right)$ nuqtada: $\Delta = AC - B^2 = 6\cdot\frac{2}{3}\cdot 2 - 2^2 = 4 > 0,\ A > 0$ demak, bu nuqtada ekstremum mavjud va $P_2\left(\frac{2}{3}, -\frac{2}{3}\right)$ nuqta funksiyaning minimum nuqtasi hamda $z_{\min} = -\frac{4}{27}$ bo‘ladi. ◄

Ko‘p o‘zgaruvchili funksiya ekstremumga ega bo‘lishining zaruriylik sharti: $f(x_i)$ funksiya 1-tartibli hosilalari bilan birgalikda uzluksiz bo‘lsa, uning ekstremumi quyidagi tenglamalar sistemasini qanoatlantiradi:

$$\frac{\partial f(X)}{\partial x_j} = 0,\ j = \overline{1,n} \qquad (5.14)$$

Demak, berilgan $f(x_i)$ funksiya X_0 nuqtada ekstremumga ega bo‘lishi uchun bu X_0 nuqta (5.14) sistemaning yechimi bo‘lishi kerak:

$$\frac{\partial f(X_0)}{\partial x_j} = 0,\ j = \overline{1,n}\,.$$

Oxirgi tengliklar X_0 nuqtada $f(X_0)$ funksiya maksimum yoki minimumga ega bo‘lganda, shu nuqtada undan n ta $x_1, x_2, ..., x_n$

noma’lumlar bo‘yicha olingan xususiy hosilalar 0 ga teng bo‘lishi kerakligini ko‘rsatadi. Lekin bundan zaruriylik shartini qanoatlantiruvchi har qanday nuqta ham funksiyaga lokal maksimum yoki minimum qiymat beradi degan xulosa kelib chiqmaydi.

Ko‘p o‘zgaruvchili funksiya ekstremumga ega bo‘lishining yetarlilik sharti: n o‘zgaruvchili uzluksiz $f(x_i)=f(x_1,x_2,...,x_n)$ funksiyaning ekstremal nuqtasi bo‘lishi uchun kritik nuqtaning 1- va 2-tartibli xususiy hosilalari quyidagi shartni qanoatlantirishi kerak:

5.5-teorema. X_0 statsionar nuqta ekstremum nuqta bo‘lishi uchun shu nuqtada **Gesse matritsasi** musbat aniqlangan yoki manfiy aniqlangan bo‘lishi yetarlidir.

$$H[X_0]=\begin{pmatrix} \frac{\partial^2 f(X_0)}{\partial x_1^2} & \frac{\partial^2 f(X_0)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(X_0)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(X_0)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(X_0)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(X_0)}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f(X_0)}{\partial x_n \partial x_1} & \frac{\partial^2 f(X_0)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(X_0)}{\partial x_n^2} \end{pmatrix} \tag{5.15}$$

Xususan, ikki o‘zgaruvchili funksiya uchun Gesse matritsasi quyidagicha bo‘ladi:

$$H[X_0]=\begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} \tag{5.16}$$

Gesse matritsasi musbat aniqlanganda X_0 minimum nuqta, manfiy aniqlanganda esa X_0 maksimum nuqta bo‘ladi.

5.4.4-misol. $f(x_1,x_2,x_3)=x_1+2x_3+x_2x_3-x_1^2-x_2^2-x_3^2$ funksiyaning ekstremum nuqtalarini toping.

Yechilishi: ► Funksiya ekstremumi mavjudligining zaruriy sharti, ya’ni (5.14) formuladan foydlanamiz:

$$\frac{\partial f}{\partial x_1}=1-2x_1=0,$$

$$\frac{\partial f}{\partial x_2}=x_3-2x_2=0,$$

$$\frac{\partial f}{\partial x_3}=2+x_2-2x_3=0$$

Bu tenglamalardan tuzilgan sistemaning yechimi $X_0=\left(\frac{1}{2},\frac{2}{3},\frac{4}{3}\right)$ statsionar nuqta bo‘ladi. Yetarlilik shartining bajarilishini tekshirish uchun X_0 nuqtada Gesse (5.15) matrisasini tuzamiz:

$$\frac{\partial^2 f}{\partial x_1^{\ 2}}=-2,\qquad \frac{\partial^2 f}{\partial x_2^{\ 2}}=-2,\qquad \frac{\partial^2 f}{\partial x_3^{\ 2}}=-2$$

$$\frac{\partial^2 f}{\partial x_1\partial x_2}=0,\qquad \frac{\partial^2 f}{\partial x_1\partial x_3}=0,\qquad \frac{\partial^2 f}{\partial x_2\partial x_3}=1;$$

$$\frac{\partial^2 f}{\partial x_2\partial x_1}=0,\qquad \frac{\partial^2 f}{\partial x_3\partial x_1}=0,\qquad \frac{\partial^2 f}{\partial x_3\partial x_2}=1$$

$$H[X_0]=\begin{pmatrix}-2 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -2\end{pmatrix}$$

Bu matrisaning bosh minorlari mos ravishda –2, 4, –6. Ma’lumki, agar matrisaning bosh minorlaridan tuzilgan sonlar ketma-ketligida ishora almashinuvchi bo‘lsa, berilgan matritsa manfiy aniqlangan bo‘ladi. Demak, X_0 nuqtada $f(x_1,x_2,x_3)$ funksiya maksimumga erishadi. ◄

Agar $H[X_0]$ ishorasi aniqlanmagan (noaniq) matritsa bo‘lsa, X_0 nuqta **egilish nuqtasi** bo‘ladi, ya’ni bu nuqtada funksiya ekstremumga erishmaydi.

5.4.5-misol. $f(x_1,x_2)=8x_1x_2+x_2^2$ funksiyaning ekstremumini toping.

Yechilishi: ► (5.14) va (5.16) formulalardan foydalanamiz. Ekstremum mavjudligining zaruriylik shartiga ko‘ra:

$$\frac{\partial f}{\partial x_1}=8x_2=0,\qquad \frac{\partial f}{\partial x_2}=8x_1+2x_2=0.$$

Bu tenglamalardan tuzilgan sistemani yechib, $X_0=(0{,}0)$ statsionar nuqtani hosil qilamiz. Endi statsionar nuqtaning ekstremum nuqta bo‘lishlik shartini tekshirish uchun Gesse matritsasini tuzamiz:

$$H=\begin{pmatrix}0 & 8\\ 8 & 2\end{pmatrix}$$

Bu matritsaning bosh minorlari: $M_{11}=2>0,\ \ M_{22}=0.$ Matritsa determenanti esa -64<0**.** Bundan Gesse matritsasining ishorasi aniqlanmaganligi ko‘rinadi. Bu holda $X_0=(0{,}0)$ nuqta egilish nuqtasi bo‘ladi. ◄

5.4.2. Ikki o‘zgaruvchili funksiyaning yopiq sohadagi eng katta va eng kichik qiymatlarini topish

Yopiq, chegaralangan ***D*** sohada uzluksiz $z = f(x, y)$ funksiya bu sohada hech bo‘lmaganda 1 marta o‘zining eng katta qiymati ***M*** va eng kichik qiymati ***m*** ni qabul qiladi. Agar funksiya bu qiymatlarning birortasiga ***D*** sohaning ichida erishsa, ular ekstremal qiymatlar bilan bir xil bo‘ladi. Agar funksiya bu qiymatlarni soha chegarasi ***L*** ga tegishli ba’zi nuqtalarda qabul qilsa, ular ekstremal qiymatlar bilan bir xil bo‘lmaydi.

Demak, sohada uzluksiz funksiyaning eng katta va eng kichik qiymatlarini topish uchun:

1) Soha ichida joylashgan kritik nuqtalarni topish va funksiyaning bu nuqtadagi qiymatlarini hisoblash kerak;

2) Soha chegarasida joylashgan kritik nuqtalarni topish va funksiyaning bu nuqtadagi qiymatlarini hisoblash kerak;

3) Funksiyaning soha chegarasining turli qismlari tutashgan nuqtalardagi qiymatlarini hisoblash kerak;

4) Topilgan barcha qiymatlar ichidan eng kattasi ***M*** va eng kichigi ***m*** ni tanlash kerak.

5.4.6-misol. $z = x^2 + y^2 - xy + x + y$ funksiyaning $x \le 0, y \le 0, x + y \ge -3$ sohadagi eng katta va eng kichik qiymatlarini toping.

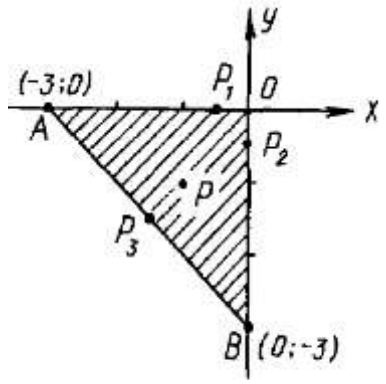
Yechilishi: ► Soha AOB uchburchakdan iborat. Soha ichidagi kritik nuqtalarni topamiz (5.10-rasm):

$$\begin{cases} \dfrac{\partial z}{\partial x} = 2x - y + 1 = 0 \\ \dfrac{\partial z}{\partial y} = 2y - x + 1 = 0 \end{cases}$$

bundan $x = -1, y = -1$ bo‘lib, P_0(-1,-1) kritik nuqtaga ega bo‘lamiz. Funksiyani soha chegarasida tekshiramiz: AO chegarada $y = 0$ bo‘lib, $z = x^2 + x$ funksiya hosil bo‘ladi. Bu funksiyaning ekstremumi: $z'_x = 2x + 1 = 0$ va $x = -\dfrac{1}{2} = -0{,}5$ bo‘ladi.

Soha AOB uchburchakdan iborat. Soha ichidagi kritik nuqtalarni topamiz:

$$\begin{cases} \dfrac{\partial z}{\partial x} = 2x - y + 1 = 0 \\ \dfrac{\partial z}{\partial y} = 2y - x + 1 = 0 \end{cases}$$



5.10-rasm. $x \le 0, y \le 0, x + y \ge -3$ chiziqlar bilan chegaralangan soha

bundan $x = -1, y = -1$ bo'lib, P_0(-1,-1) kritik nuqtaga ega bo'lamiz. Funksiyani soha chegarasida tekshiramiz: AO chegarada $y = 0$ bo'lib, $z = x^2 + x$ funksiya hosil bo'ladi. Bu funksiyaning ekstremumi: $z'_x = 2x + 1 = 0$va $x = -\frac{1}{2} = -0,5$ bo'ladi.

Demak, P_1 (-0,5, 0) AO chegaradagi kritik nuqta. Tenglamasi $x = 0$, BO chegarada $z = y^2 + y$ funksiya hosil bo'lib, $z'_y = 2y + 1 = 0$, $y = -1/2$.

Demak, $P_2\left(0,-\frac{1}{2}\right)$ BO chegaradagi kritik nuqta bo'ladi. Tenglamasi $y = -3 - x$ bo'lgan AB chegarada $z = 3x^2 + 9x + 6$ funksiya hosil bo'lib, $z'_x = 6x + 9 = 0$ $x = -\frac{3}{2}$. AB ning tenglamasidan $y = -3 + \frac{3}{2} = -\frac{3}{2}$, demak, AB chegaradagi kritik nuqta $P_3\left(-\frac{3}{2},-\frac{3}{2}\right)$ bo'ladi.

Berilgan funksiyaning P_0, P_1, P_2, P_3 kritik nuqtalardagi, hamda A, B, O nuqtalardagi qiymatlarni hisoblaymiz:

$$z_0 = f(P_0) = f(-1,-1) = -1 \ ; \qquad z_1 = f(P_1) = f\left(-\frac{1}{2},-0\right) = -\frac{1}{4};$$

$$z_2 = f(P_2) = f\left(0,-\frac{1}{2}\right) = -\frac{1}{4} \ ; \qquad z_3 = f(P_3) = f\left(-\frac{3}{2},-\frac{1}{2}\right) = -\frac{3}{4};$$

$$z_4 = f(O) = f(0,0) = 0 \ ; \qquad z_5 = f(A) = f(-3,0) = 6; \qquad z_6 = f(B) = f(0,-3) = 6.$$

Funksiyaning topilgan barcha qiymatlarini taqqoslab,

$$z_{eng kat.} = f(A) = f(B) = 6 \quad va \quad z_{eng kich.} = f(P_0) = -1$$

yopiq sohadagi eng katta va eng kichik qiymatlarini topamiz.◄

5.4.3. Shartli ekstremumlar. Lagranj ko‘paytuvchilar usuli

Ko‘p o‘zgaruvchili funksiyaning ekstremumlarini topishda shartli ekstremumdan ham foydalaniladi.

$z = f(x, y)$ funksiyaning **shartli ekstremumi** deb, bu funksiyaning x va y o‘zgaruvchilarning

$$\varphi(x, y)=0 \tag{5.17}$$

tenglik bilan bog‘langanlik shartida erishadigan ekstremum qiymatiga aytiladi.

Agar $z = f(x, y)$ funksiya va xOy tekislikda L chiziq $\varphi(x, y)=0$ tenglama bilan berilgan bo‘lib, $z = f(x, y)$ funksiyaning qiymati L chiziqning $P_0(x_0,y_0)$ nuqtasiga yaqin nuqtalarida $\varphi(x, y)=0$ funksiyaning qiymatlariga nisbatan eng katta yoki eng kichik bo‘lsa, u holda $P_0(x_0,y_0)$ nuqta **shartli ekstremum nuqtasi** deyiladi.

Odatdagi ekstremum nuqtasi (uni shartsiz ekstremum ham deyiladi) shu nuqtadan o‘tuvchi ixtiyoriy chiziq uchun shartli ekstremum nuqtasi bo‘lishi mumkin. Lekin, teskari da’vo o‘rinli emas, ya’ni shartli ekstremum nuqtasi shartsiz ekstremum nuqtasi bo‘lmaydi.

Amaliy masalalarni yechishda quyidagi algoritm bo‘yicha ish ko‘riladi:

1–qadam: $\varphi(x, y)=0$ tenglikni y ga nisbatan yechib, uni oshkor ko‘rinishga keltiramiz: $y = y(x)$.

2–qadam: Topilgan $y(x)$ ni $z = f(x, y)$ funksiyadagi y ni o‘rniga qo‘yib, uni bir o‘zgaruvchili funksiyaga keltiramiz: $z = f\big(x, y(x)\big)$.

3–qadam: Bu funksiya ekstremumga erishadigan x ning qiymatlarini aniqlaymiz.

4–qadam: Bog‘lash tenglamasi $y = y(x)$ dan y ning mos qiymatini topamiz.

5.4.7-misol. $z = \sqrt{1 - x^2 - y^2}$ funksiyaning $x + y - 1 = 0$ chiziqdagi shartli ekstremumini toping.

Yechilishi: ►

1) $x + y - 1 = 0$ ni oshkor ko‘rinishga keltiramiz: $y = 1 - x$.
2) $z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - x^2 - (1 - x)^2} = \sqrt{2x - 2x^2}$.
3) $z' = \left(\sqrt{2x - 2x^2}\right)' = 0$ desak, $x = \frac{1}{2}$ maksimum nuqtani topamiz.

4) Bog‘lash tenglamasidan $y = 1 - x = \frac{1}{2}$ ni topamiz. Demak, shartli ekstremum nuqtasi $P_0\left(\frac{1}{2}, \frac{1}{2}\right)$ ekan. ◄

Bog‘lash tenglamasi parametrik shaklda $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ berilgan bo‘lsa ham shartli ekstremum masalasini yechish uchun, bu ifodalarni $z = f(x, y)$ funksiyaga qo‘yib, bir o‘zgaruvchili funksiyaga keltiramiz va ekstremumni oson topamiz.

Agar bog‘lash tenglamasi ancha murakkab ko‘rinishda bo‘lsa, ya’ni uni oshkor ko‘rinishda ifodalashning iloji bo‘lmasa, u holda shartli ekstremumni topish uchun Lagranj ko‘paytuvchilar usulidan foydalanish mumkin.

Lagranj ko‘paytuvchilar usuli. x va y lar $\varphi(x, y) = 0$ tenglama bilan bog‘langanlik shartida $z = f(x, y)$ funksiyaning ekstremumini topish kerak bo‘lsin.

Shartli ekstremum nuqtalarida ekstremumning zaruriy sharti bajarilishi kerak, ya’ni $z = f(x, y)$ funksiyaning to‘la differensiali nolga teng bo‘lishi kerak:

$$\frac{dz}{dx} = f'_x(x, y) + f'_y(x, y) \cdot \frac{dy}{dx} = 0 \qquad (5.18)$$

(5.17) tenglikdan $\frac{dy}{dx}$ ni topamiz: $\frac{dy}{dx} = -\frac{\varphi'_x(x,y)}{\varphi'_y(x,y)}$. Hosilaning topilgan qiymatini (5.18) ga qo‘yamiz va quyidagiga ega bo‘lamiz:

$$f'_x(x, y) - f'_y(x, y) \cdot \frac{\varphi'_x(x,y)}{\varphi'_y(x,y)} = 0.$$

Bu tenglamada yangi λ o‘zgaruvchi kiritamiz va tenglamani proporsiya qilib, qulay ko‘rinishga keltiramiz:

$$\frac{f'_x(x,y)}{\varphi'_x(x,y)} = \frac{f'_y(x,y)}{\varphi'_y(x,y)} = -\lambda,$$

Bu yerda “minus” ishorasi qulaylik uchun qo‘yilgan. Proporsiyadan quyidagi sistemani hosil qilamiz:

$$\begin{cases} f'_x(x, y) + \lambda\varphi'_x(x, y) = 0 \\ f'_y(x, y) + \lambda\varphi'_y(x, y) = 0 \end{cases} \qquad (5.19)$$

x va y lar bog‘lanish tenglamasini qanoatlantirishi kerak, shuning uchun (5.19) tenglamalar sistemasi (5.17) bog‘lanish tenglamasi bilan birgalikda uchta: $x,\ y,\ \lambda$ noma’lumli uchta tenglamalar sistemasini hosil qiladi. Bu sistemani quyidagi qoida yordamida eslab qolish qulay:

$z = f(x, y)$ funksiyaning $\varphi(x, y) = 0$ bog‘lanish tenglamasi o‘rinli bo‘lganda shartli ekstremumi bo‘lishi mumkin bo‘lgan nuqtalarini topish uchun quyidagi Lagranj funksiyasi deb ataluvchi yordamchi funksiyani kiritish kerak:

$$\Phi(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y), \qquad (5.20)$$

Bunda $\lambda -$ biror o'zgarmas. So'ngra $\Phi(x,y,\lambda)$ funksiyaning x,y,λ lar bo'yicha xususiy hosilalarini nolga tenglab, hosil bo'lgan (5.19) va (5.17) tenglamalardan x,y noma'lumlarni va λ yordamchi ko'paytuvchini topish kerak.

Shunday qilib, shartli ekstremumni topishni Lagranj funksiyasi $\Phi(x,y,\lambda)$ning oddiy ekstremumga tekshirishga keltirish mumkin. (5.19) va (5.17) tenglamalar $\Phi(x,y,\lambda)$ funksiya ekstremumi mavjud bo'lishining zaruriy sharti bo'lib xizmat qiladi.

5.4.8-misol. $z=xy$ funksiyaning x va y lar $2x+3y-5=0$ tenglama bilan bog'langan shartidagi ekstremumini toping.

Yechilishi: ►

1-qadam: (5.20) ga ko'ra Lagranj funksiyasini tuzamiz:

$$\Phi(x,y,\lambda)=xy+\lambda(2x+3y-5).$$

2-qadam: x,y,λ lar bo'yicha xususiy hosilalarni topamiz:

$$\Phi'_x(x,y,\lambda)=y+2\lambda,$$
$$\Phi'_y(x,y,\lambda)=x+3\lambda,$$
$$\Phi'_\lambda(x,y,\lambda)=2x+3y-5.$$

3-qadam: Sistema tuzamiz: $\begin{cases} y+2\lambda=0 \\ x+3\lambda=0 \\ 2x+3y-5=0 \end{cases}$

Sistemani yechib, $\lambda=-\frac{5}{12}$, $x=\frac{5}{4}$, $y=\frac{5}{6}$ qiymatlarni topamiz.

4-qadam: Demak, bizda $P_1(\frac{5}{4},\frac{5}{6})$ nuqta hamda $2x+3y-5=0$ to'g'ri chiziqning nuqtalari $P_2(0,\frac{5}{3})$ va $P_3(\frac{5}{2},0)$ uchta nuqta topildi.

Topilgan nuqtalarni $z=xy$ ga qo'yamiz va taqqoslaymiz.

$$z_1\left(\frac{5}{4},\frac{5}{6}\right)=\frac{5}{4}\cdot\frac{5}{6}=\frac{25}{24}; \qquad z_2\left(0,\frac{5}{3}\right)=0; \qquad z_3\left(\frac{5}{2},0\right)=0.$$

Shunday qilib, $z=xy$ funksiya eng katta qiymatga $P_1(\frac{5}{4},\frac{5}{6})$ nuqtada erishadi: $z_{max}\left(\frac{5}{4},\frac{5}{6}\right)=\frac{25}{24}$. ◄

Lagranj ko'paytuvchilar usulini ikkitadan ko'p o'zgaruvchili funksiyalarning ham shartli ekstremumlarini topishga tatbiq qilish mumkin.

Aytaylik, bizga $u=f(x_1,x_2,\dots,x_n)$ funksiya ushbu m ta $(m<n)$

$$\varphi_1(x_1, x_2, \dots, x_n) = 0,$$
$$\varphi_2(x_1, x_2, \dots, x_n) = 0,$$
$$\dots..$$
$$\varphi_m(x_1, x_2, \dots, x_n) = 0$$

tenglamalar bilan bog‘langan degan shart ostidagi ekstremuminni topish talab etilgan bo‘lsin. U holda Lagrang funksiyasini quyidagicha tuzamiz:

$$\Phi(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = f(x_1, x_2, \dots, x_n) +$$
$$+\lambda_1\varphi_1(x_1, x_2, \dots, x_n) + \lambda_2\varphi_2(x_1, x_2, \dots, x_n) + \dots + \lambda_m\varphi_m(x_1, x_2, \dots, x_n).$$

So‘ngra bu Lagranj funksiyasining $x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m$ lar bo‘yicha xususiy hosilalarini topib, nolga tenglaymiz. Hosil bo‘lgan $m + n$ ta tenglamadan yordamchi o‘zgaruvchi $\lambda_1, \lambda_2, \dots, \lambda_m$ lar va $x_1, x_2, \dots, x_n$ larni topamiz.

Mavzu yuzasidan savollar

1. Ikki o‘zgaruvchili funksiyaning maksimum nuqtasi deb qanday nuqtaga aytiladi?
2. Ikki o‘zgaruvchili funksiyaning minimum nuqtasi nima?
3. Ikki o‘zgaruvchili funksiyaning ekstremum nuqtalari deb nimaga aytiladi?
4. Ikki o‘zgaruvchili funksiya ekstremumga ega bo‘lishining zaruriy shartini ayting.
5. Ikki o‘zgaruvchili funksiya ekstremumga ega bo‘lishining yetarli sharti qanday?
6. Ko‘p o‘zgaruvchili funksiya ekstremumga ega bo‘lishining zaruriylik sharti qanday?
7. Gesse matritsasi deb nimaga aytiladi?
8. Shartli va shartsiz ekstremum deganda nimani tushunasiz?
9. Ikki o‘zgaruvchili funksiyaning biror yopiq sohadagi eng katta va eng kichik qiymatlari qanday topiladi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Ikki o‘zgaruvchili funksiyaning ekstremumini toping:

a) $z = 3x^2 - 2xy + y^2 - 2x - 2y + 3;$

b) $z = 2x^2 + xy - y^2 - 7x + 5y + 2;$

c) $z = x^2 - 3xy - y^2 - 4x + 6y + 1;$

d) $z = 3x^2 + xy - 6y^2 - 6x - y + 9;$

e) $z = 10xy - 3x^2 - 2y^2 - 26x + 18y - 1;$

f) $z = 5 - 7x^2 - 5y^2 + 2xy - 34x + 34y;$

2. Funksiyaning sohadagi eng katta va eng kichik qiymatlarini toping:

a) $z = 3xy - x^2y - xy^2 + 6,$ $\overline{D}: 0 \le x \le 2, 0 \le y \le 3.$

b) $z = x^2 - xy + y^2 - 4x,$ $\overline{D}: x \ge 0, y \ge 0, 3x + 2y - 12 \le 0$

c) $z = x + 3y + 4,$ $\overline{D}: x \ge 0, y \ge 0, \quad x^2 + y^2 - 1 \le 0.$

d) $z = x^2(y+1) - 2y,$ $\overline{D}: \sqrt{1+x^2} \le y \le 2.$

e) $z = x^2 + y^2 - 2x - 2y + 8,$ $\overline{D}: x \ge 0, y \ge 0, x + y \le 1.$

f) $z = 3x + 6y - x^2 - xy - y^2,$ $\overline{D}: 0 \le x \le 1, 0 \le y \le 1.$

3. Uch o‘zgaruvchili $f(x_1, x_2, x_3) = x_1 + 3x_3 + x_2x_3 - 2x_1^2 - x_2^2 - 4x_3^2$ funksiyaning ekstremum nuqtalarini toping.

4. $f(x_1, x_2, x_3, x_4) = x_1 + 5x_3x_4 + 3x_2x_3 - 2x_1^2 - x_2^2 - 3x_3^2 + x_4^3$ to‘rt o‘zgaruvchili funksiyaning ekstremum nuqtalarini toping.

5. $f(x_1, x_2, x_3) = -2x_1 + 3x_3 + x_2x_3 - 2x_1^2 + x_2^2 - x_3^2$ funksiyaning ekstremum nuqtalarini toping.

TESTLAR

1. $z = \frac{1}{x} + \frac{1}{y}, \; x + y = 2$ bo‘lganda funksiyaning ekstremumini toping.

A) $z_{\min} = z(1;1) = 2$ **B)** $z_{\min} = z(-1;1) = 2$

C) $z_{\min} = z(-1;0) = 2$ **D)** $z_{\min} = z(1;0) = 2$

2. $z = x^2 + y^2 + xy - 3x - 6y$ funksiya ekstremumini toping

A) $z_{max} = z(0,1) = 1$ **B)** $z_{min} = z(0,3) = -9$

C) $z_{\min} = z(-1;0) = 2$ **D)** Ekstremum mavjud emas

3. Funksiyaning eng katta va eng kichik qiymatlarini toping:

$$z = x^2 - y^2 + 4xy - 6x + 5; x \ge 0,$$
$$y \ge 0, x + y \le 3;$$

A) $z_{eng\,kichik} = -4;$ $z_{eng\,katta} = 5$ **B)** $z_{eng\,kichik} = -5;$ $z_{eng\,katta} = 5$

C) $z_{eng\,kichik} = -5;$ $z_{eng\,katta} = 16$ **D)** $z_{eng\,kichik} = 0;$ $z_{eng\,katta} = 16$

4. $z = x^2 + y^2; \frac{x}{4} + \frac{y}{3} = 1$ shartda funksiyaning shartli ekstremumini toping:

A) $x=\frac{5}{4}, y=\frac{5}{6}$ da $z_{max}=\frac{25}{24}$ **B)** $x=\frac{36}{25}, y=\frac{48}{25}$ da $z_{min}=\frac{144}{25}$

C) $x=\frac{7}{4}, y=\frac{-1}{6}$ da $z_{min}=\frac{25}{24}$ **D)** $x=\frac{1}{4}, y=\frac{2}{3}$ da $z_{max}=\frac{22}{25}$

5. To‘la sirti $S=6\pi\ dm^2$ bo‘lgan eng katta hajmli silindrning o‘lchovlarini aniqlang.

A) $R=2,\ H=2$ **B)** $R=1,\ H=3$

C) $R=4,\ H=3$ **D)** $R=1,\ H=2$

5.5-§. Optimallashtirish usullari

5.5.1. Masalaning qo‘yilishi. Optimallashtirish masalalari

Insonlar o‘z faoliyati davomida har bir ishning mumkin bo‘lgan variantlaridan eng maqbulini tanlashga harakat qiladi. Agar bu ish harajatlar bilan bog‘liq bo‘lsa, harajatlarni kamaytirish, agar ish daromad bilan bog‘liq bo‘lsa, daromadlarni ko‘paytirish maqsad qilinadi. Demak, masala shartidan kelib chiqib, harajat yoki daromadni ifodalovchi maqsad funksiyasi tuziladi. So‘ngra shu maqsad funksiyasining eng katta yoki eng kichik qiymatlarini topish kerak bo‘ladi. Bunday masalalarga **optimallashtirish masalalari** deyiladi.

Optimallashtirish masalalarini insonlar faoliyatining istalgan doirasida, shaxsiy ishlardan tortib umumdavlat ishlarigacha bo‘lgan darajada oshkor yoki oshkormas shaklda uchratamiz. Iqtisodiy rejalashtirish, boshqarish, chegaralangan resurslarni taqsimlash, ishlab chiqarish jarayonini tahlil qilish, murakkab ob’yektlarni loyihalash doim mo‘ljallangan maqsad nuqtai nazaridan eng maqbul variantni izlashga qaratilgan bo‘lishi lozim.

Matematik nuqtai nazardan maqsad funksiyasi oshkor formula bilan berilgan va differensiallanuvchi funksiyadan iborat bo‘lgan hol **eng sodda optimallashtirish masalasi** hisoblanadi. Bu holda funksiyaning xossalarini tekshirish, uning o‘sish va kamayish oraliqlarini aniqlash, lokal ekstremum nuqtalarini izlashda hosiladan foydalanish mumkin. Optimallashtirish masalalarini hal qilishning 3 ta usuli keng qo‘llaniladi:

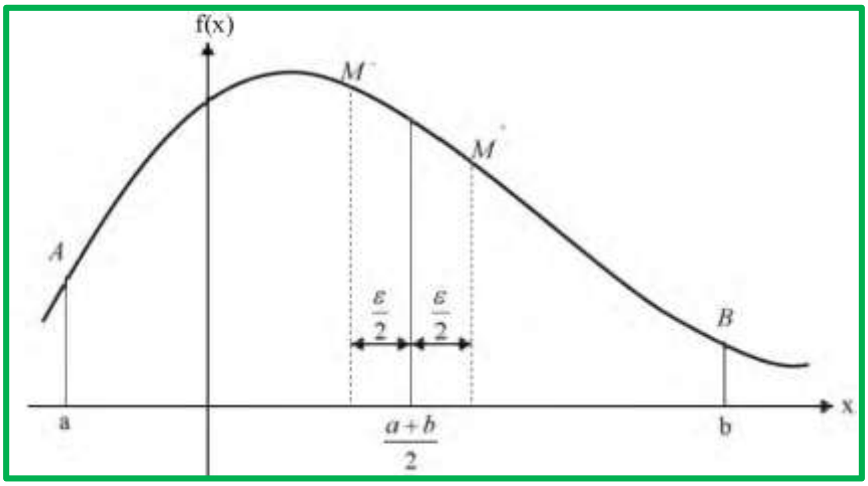
1. Oraliqni teng ikkiga bo‘lish usuli;
2. Oltin kesim usuli;
3. Nyuton usuli;
4. Eng kichik kvadratlar usuki.

Oraliqni teng ikkiga bo‘lish usuli

Aytaylik, $[a,b]$ oraliqda berilgan $y = f(x)$ funksiyaning maksimum qiymatini aniqlash kerak bo‘lsin. Faraz qilaylik, funksiya o‘zining maksimum qiymatiga ε oraliqda erishsin.

Dastlab oraliq o‘rtasini aniqlab olamiz (5.11-rasm): $x_0 = \dfrac{a+b}{2}$.

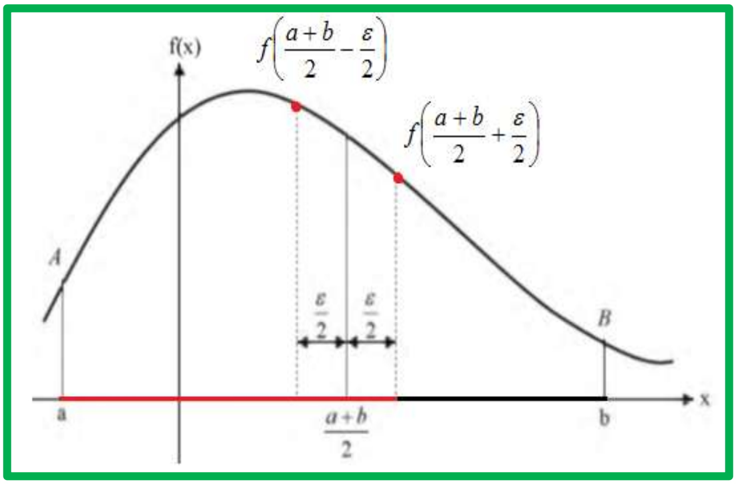
So‘ngra funksiya $f\left(\dfrac{a+b}{2}+\dfrac{\varepsilon}{2}\right)$ va $f\left(\dfrac{a+b}{2}-\dfrac{\varepsilon}{2}\right)$ qiymatlarini hisoblaymiz.



5.11-rasm. Oraliqni teng ikkiga bo‘lish usuli

Agar $f\left(\dfrac{a+b}{2}+\dfrac{\varepsilon}{2}\right) \geq f\left(\dfrac{a+b}{2}-\dfrac{\varepsilon}{2}\right)$ tengsizlik o‘rinli bo‘lsa, funksiyaning maksimumi (minimumi) $\left[\dfrac{a+b}{2}-\dfrac{\varepsilon}{2};b\right]$ oraliqda yotadi.

Agar $f\left(\dfrac{a+b}{2}+\dfrac{\varepsilon}{2}\right) \leq f\left(\dfrac{a+b}{2}-\dfrac{\varepsilon}{2}\right)$ tengsizlik o‘rinli bo‘lsa, funksiyaning maksimumi (minimumi) $\left[a;\dfrac{a+b}{2}+\dfrac{\varepsilon}{2}\right]$ oraliqda yotgan bo‘ladi (5.12-rasm).



5.12-rasm. Oraliqni teng ikkiga bo‘lish usuli

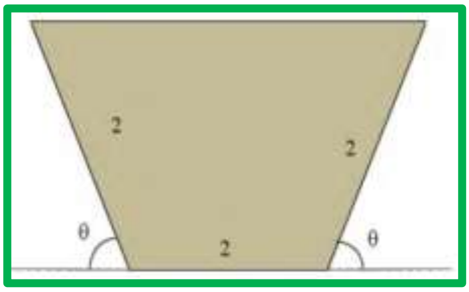
Hosil bo‘lgan oraliq o‘rtasini topib olib, shu tariqa davom qilib, oraliqlarni kichraytirib boraveramiz va lokal maksimumga erishamiz.

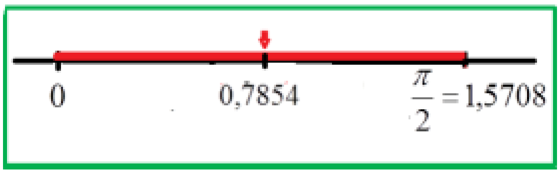
Oraliqni teng ikkiga bo‘lish usulida optimal yechim yotgan oraliqning chegaralari berilgan bo‘lishi kerak.

Nyuton usulida $y = f(x)$ funksiya berilishining o‘zi yetarli, unda yechim izlanishi kerak bo‘lgan oraliq talab qilinmaydi. Boshlang‘ich optimal qiymat faraz qilinadi. Nyuton usulida boshlang‘ich qiymat noto‘g‘ri tanlansa, optimal yechimga yaqinlashmasligi ham mumkin.

5.5.1-misol. Ma’lumot uzatish tezligi optik tolaning ko‘ndalang kesimiga bog‘liq. Optik tola ko‘ndalang kesimining yuzasi $S = 4\sin\theta(1+\cos\theta)$ bo‘lib, asosi va yon tomonlari uzunliklari 2 birlikka teng. θ burchak qanday bo‘lganda kesim yuzasi eng katta bo‘ladi? $\varepsilon = 0{,}2$ deb oling (5.13-rasm).

Yechilishi: ► Oraliqni teng ikkiga bo‘lish usulidan foydalanib yechamiz:





5.13-rasm. Optik tolaning ko‘ndalang kesimi

$$\left[0; \frac{\pi}{2}\right] = [0;\ 1{,}5708]$$

$$f(\theta) = 4\sin\theta(1+\cos\theta)$$

1-iteratsiya:

$$f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right) = f\left(\frac{0+1{,}5708}{2}+\frac{0{,}2}{2}\right) = f(0{,}8854) = 5{,}0568$$

$$f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right) = f\left(\frac{0+1{,}5708}{2}-\frac{0{,}2}{2}\right) = f(0{,}6854) = 4{,}4921$$

$$f(0{,}8854) > f(0{,}6854)$$

Qiymatlarni taqqoslab, keyingi oraliqni aniqlab olamiz:

$$\left[\frac{a+b}{2}-\frac{\varepsilon}{2}; b\right] = [0{,}6854;\ 1{,}5708]$$

2-iteratsiya:

$$f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right) = f\left(\frac{0{,}6854+1{,}5708}{2}+\frac{0{,}2}{2}\right) = f(1{,}2281) = 5{,}0334$$

$$f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right)=f\left(\frac{0{,}6854+1{,}5708}{2}-\frac{0{,}2}{2}\right)=f(1{,}0281)=5{,}1942$$

$$f(1{,}2281)<f(1{,}0281)$$

Qiymatlarni taqqoslab, keyingi oraliqni aniqlab olamiz:

$$\left[a;\frac{a+b}{2}+\frac{\varepsilon}{2}\right]=[0{,}6854;\ 1{,}2281]$$

3-iteratsiya:

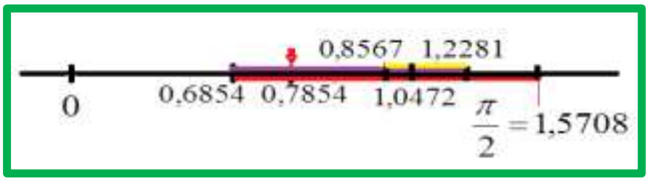
$$f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right)=f\left(\frac{0{,}6854+1{,}2281}{2}+\frac{0{,}2}{2}\right)=f(1{,}0567)=5{,}1957$$

$$f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right)=f\left(\frac{0{,}6854+1{,}2281}{2}-\frac{0{,}2}{2}\right)=f(0{,}8567)=5{,}0025$$

$$f(1{,}0567)>f(1{,}8567)$$

Qiymatlarni taqqoslab, keyingi oraliqni aniqlab olamiz:

$$\left[\frac{a+b}{2}-\frac{\varepsilon}{2};b\right]=[0{,}8567;\ 1{,}2281]$$



Va hokazo, shu tariqa davom etib, 16-iteratsiyadan keyin quyidagiga ega bo‘lamiz: $f(\theta)=5{,}1962$, bundan $\theta=1{,}0472$ ni radian qiymatini aniqlaymiz, uni gradusga o‘tkazsak $\theta=60^0$ ekanligi kelib chiqadi. Demak, $\theta=60^0$ ga teng bo‘lganda kesim yuzasi eng katta bo‘ladi: $f(1{,}0472)=5{,}1962$. ◄

5.5.2. Optimal yechim topishning Nyuton usuli

Nyuton usuli $y=f(x)$ funksiyaning ekstremum qiymatlarini topishda muhim ahamiyatga ega. Bu usulda $[a,b]$ oraliqda funksiyaning silliq bo‘lishi talab qilinadi, ya’ni $f(x)$ funksiyaning ixtiyoriy $x\in[a,b]$ qiymatlari uchun $f'(x)$ va $f''(x)$ hosilalari mavjud bo‘lishi va ular noldan farqli bo‘lishi kerak.

Nyuton usuli $f(x)=0$ tenglamani yechishda qo‘llaniladigan urinmalar (Nyuton-Rapson) usuliga o‘xshaydi.

Urinmalar usuli: Argument orttirmasi nolga intilganda *C* nuqtadan o‘tuvchi CE urinmaning burchak koeffitsiyentini aniqlaymiz (5.14-rasm):

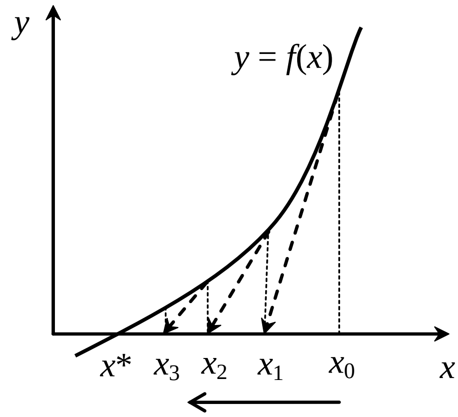
$$k\approx\frac{F(X_i)-F(X_{i+1})}{X_i-X_{i+1}}.$$

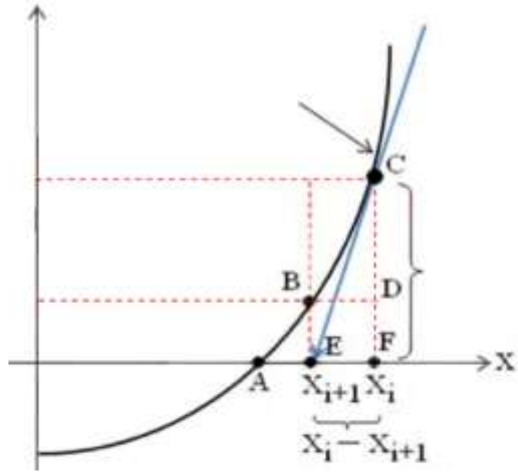
Agar keyingi X_{i+1} yaqinlashish yechim bo‘lsa yoki $F(X_{i+1})=0$ tenglik o‘rinli bo‘lsa, u holda

$$k=\frac{F(X_i)-0}{X_i-X_{i+1}}; \qquad \boldsymbol{k=F'(X_i);} \qquad F'(X_i)=\frac{F(X_i)}{X_i-X_{i+1}}$$

tenglikni hosil qilamiz. Bundan $X_i-X_{i+1}=\frac{F(X_i)}{F'(X_i)}$ kelib chiqadi. X_{i+1} ni topib olsak, hosil bo‘lgan tenglik urinmalar usuli formulasi bo‘ladi:

$$X_{i+1}=X_i-\frac{F(X_i)}{F'(X_i)}$$





5.14-rasm. Urinmalar usuli grafigi

Agar ushbu tenglamada $F(X)\equiv f'(X)$ deb qabul qilsak, u holda $F'(X)\equiv f''(X)$ ham o‘rinli bo‘lib,

$$X_{i+1}=X_i-\frac{f'(X_i)}{f''(X_i)}$$

optimal yechimga yaqinlashishning Nyuton usuli formulasi hosil bo‘ladi. Bu yerda x_i - maksimum (minimum) nuqtaga i–yaqinlashish bo‘lsa, x_{i+1} - yaqinlashish quyidagi formuladan aniqlanadi:

$$x_{i+1}=x_i-\frac{f'(x_i)}{f''(x_i)}. \qquad (5.17)$$

Nyuton usulida boshqa usullarga qaraganda ekstremum qiymatga tezroq yaqinlashiladi.

Qachon hisoblash to‘xtatiladi? Hisoblash ishlari $|x_{i+1}-x_i|<\varepsilon$ tengsizlik bajarilguncha davom ettiriladi.

5.5.1-misolni Nyuton usulida yechamiz.

Yechilishi: ► Ushbu misol Nyuton usulida yechamiz:

$$f(\theta)=4\sin\theta(1+\cos\theta)$$

Dastlab 1- va 2-tartibli hosilalarni topib olamiz:

$$f'(\theta) = 4\cos\theta(1+\cos\theta) + 4\sin\theta\cdot(-\sin\theta) = 4(\cos\theta + \cos^2\theta - \sin^2\theta);$$
$$f''(\theta) = -4\sin\theta\cdot(1+4\cos\theta).$$

Iteratsiya formulasini yozib olamiz va dastlabki yaqinlashishni aniqlab olamiz:

$$\theta_{i+1} = \theta_i - \frac{f'(\theta_i)}{f''(\theta_i)} \qquad \theta_1 = \theta_0 - \frac{f'(\theta_0)}{f''(\theta_0)} \qquad \left[0;\frac{\pi}{2}\right]$$

1-iteratsiya:

$$\theta_1 = \frac{\pi}{4} - \frac{f'\left(\frac{\pi}{4}\right)}{f''\left(\frac{\pi}{4}\right)} = \frac{\pi}{4} - \frac{4(\cos\frac{\pi}{4} + \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4})}{-4\sin\frac{\pi}{4}(1+4\cos\frac{\pi}{4})} = 1.0466$$

Topilgan burchakda funksiya qiymatini hisoblaymiz:

$$f(\theta_1) = f(1.0466) = 4\sin 1.0466\cdot(1+\cos 1.0466) = 5.1962$$

2-iteratsiya: $\theta_2 = \theta_1 - \frac{f'(\theta_1)}{f''(\theta_1)} = 1.0466 - \frac{f'(1.0466)}{f''(1.0466)} =$

$$= 1.0466 - \frac{4(\cos 1.0466 + \cos^2 1.0466 - \sin^2 1.0466)}{-4\sin 1.0466(1+4\cos 1.0466)} = 1.0472$$

Topilgan burchakda funksiya qiymatini hisoblaymiz:

$$f(\theta_2) = f(1.0472) = 4\sin 1.0472\cdot(1+\cos 1.0472) = 5.1962$$

Barcha hisoblab topilgan qiymatlarni jadvalga belgilaymiz:

Iteratsiya	θ_i	$f'(\theta_i)$	$f''(\theta_i)$	θ_{i+1}	$f(\theta_{i+1})$
1	0.78540	2.8284	-10.828	1.0466	5.1962
2	1.0466	0.0061898	-10.396	1.0472	5.1962
3	1.0472	1.0613E-06	-10.392	1.0472	5.1962
4	1.0472	3.0642E-14	-10.392	1.0472	5.1962
5	1.0472	1.3323E-15	-10.392	1.0472	5.1962

Agar 1-tartibli hosila nolga teng chiqsa, ko‘zlangan natijaga erishgan bo‘lamiz. Ish shu joyda to‘xtatiladi. $\theta = 1.0472$ radianni gradusga o‘tkazamiz:

$$\theta = 1.0472\,\text{rad} = 1.0472\cdot\frac{180^0}{\pi} = 60^0$$

Demak, $\theta = 60^0$ ga teng bo‘lganda kesim yuzasi eng katta bo‘ladi: $f(1.0472) = 5.1962$ ◄

5.5.2-misol. $f(x) = -2x^4 + 7x^3 + 5x^2 - x + 3 \to \min$ bir o‘zgaruvchili funksiyani $\varepsilon = 0.001$ xatolik bilan Nyuton usulida optimal yechimini toping.

Yechilishi: ► Dastlab 1- va 2-tartibli hosilalarni topib olamiz:

$$f'(x) = -8x^3 + 21x^2 + 10x - 1; \qquad f''(x) = -24x^2 + 42x + 10.$$

(5.17) iteratsiya formulasiga qo‘yamiz: $x_0 = 0$ deb olamiz.

1-iteratsiya: $x_1 = x_0 - \dfrac{f'(x_0)}{f''(x_0)} = 1 - \dfrac{-8+21+10-1}{-24+42+10} = 1 - \dfrac{22}{28} = 1 - 0.7857 = 0.2143.$

$$|x_1 - x_0| = |1 - 0.2143| = 0.7857 > 0.001$$

2-iteratsiya:

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = 0.2143 - \frac{-8\cdot 0.2143^3 + 21\cdot 0.2143^2 + 10\cdot 0.2143 - 1}{-24\cdot 0.2143^2 + 42\cdot 0.2143 + 10} = 0.2143 - \frac{2{,}0287}{17{,}8984} = 0.101$$

$$|x_2 - x_1| = |0.2143 - 0.101| = 0.1133 > 0.001;$$

3-iteratsiya:

$$x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)} = 0.101 - \frac{-8\cdot 0.101^3 + 21\cdot 0.101^2 + 10\cdot 0.101 - 1}{-24\cdot 0.101^2 + 42\cdot 0.101 + 10} = 0.101 - \frac{0{,}216}{13.9972} =$$

$$= 0.101 - 0.0154 = 0.0856$$

$$|x_3 - x_2| = |0.101 - 0.0856| = 0.0154 > 0.001$$

4-iteratsiya:

$$x_4 = x_3 - \frac{f'(x_3)}{f''(x_3)} = 0.0856 - \frac{-8\cdot 0.0856^3 + 21\cdot 0.0856^2 + 10\cdot 0.0856 - 1}{-24\cdot 0.0856^2 + 42\cdot 0.0856 + 10} = 0.0856 - \frac{0{,}0049}{13.4193} =$$

$$= 0.0856 - 0.00037 = 0.0852$$

$$|x_4 - x_3| = |0.0856 - 0.0852| = 0.0004 < 0.001.$$

Ruxsat etilgan xatolikdan kichik chiqquncha iteratsiyani davom qildik, yechim $x = 0.0852$ ekan. Demak, $x = 0.0852$ nuqtada funksiya minimumga erishadi:

$$f_{\min}(0.0852) = -2\cdot 0.0852^4 + 7\cdot 0.0852^3 + 5\cdot 0.0852^2 - 0.0852 + 3 = 2.9553$$ ◄

7.8. Shartsiz optimallashtirish usullari

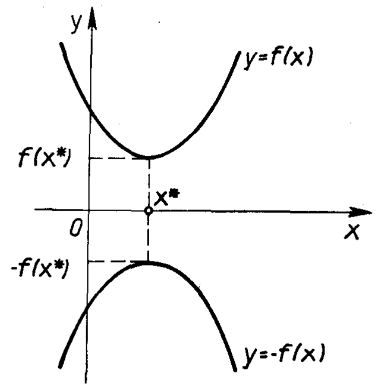
Ko‘pgina nazariy va amaliy masalalarni yechish n-o‘lchovli vektor argumentli $f(x)$ skalyar funksiya ekstremumi (eng katta yoki eng kichik qiymati) ni izlashga keltiriladi. Bundan keyin x deganda n–o‘lchovli fazodagi nuqtani, ya’ni vektor-ustunni tushunamiz: $x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$

Vektor-satr esa vektor-ustunni transponirlash bilan hosil qilinadi:

$$x^T = (x_1, \quad x_2, \quad \dots \quad x_n).$$

Optimallanuvchi $f(x)$ funksiyaga **maqsad funksiyasi** yoki **optimallashtirish kriteriysi** deyiladi. Maqsad funksiyasining minimumini aniqlovchi x^* vektorga **optimal vektor** deyiladi.

Ta’kidlash kerakki, $f(x)$ funksiyani maksimallash masalasini unga ekvivalent bo‘lgan minimallash masalasiga almashtirish mumkin yoki aksincha (5.15-rasm). Buni bir o‘zgaruvchili funksiya misolida qarab chiqamiz.



5.15-rasm. Funksiyaning maksimum va minimum qiymatlari

Agar x^* nuqta $y = f(x)$ funksiyaning minimumi bo‘lsa, u holda $f(x)$ va $-f(x)$ funksiyalarning grafiklari abssisa o‘qiga nisbatan simmetrik bo‘lganligi sababli $y = -f(x)$ funksiya uchun bu nuqta maksimum nuqtasi bo‘ladi. Demak, o‘zgaruvchining bitta qiymatida $f(x)$ funksiya minimumga; $-f(x)$ funksiya esa maksimumga erishadi, ya’ni

$$\min f(x) = -\max f(x).$$

Bir o‘zgaruvchili funksiyalar uchun o‘rinli bo‘lgan ushbu holatni ko‘p o‘zgaruvchili funksiyalar uchun ham qo‘llash mumkin. Agar $f(x_1, x_2, \dots, x_n)$ funksiyani minimallash masalasini $-f(x_1, x_2, \dots, x_n)$ funksiyani maksimallash masalasi bilan almashtirishga to‘g‘ri kelsa, u holda maksimumni topish o‘rniga $f(x_1, x_2, \dots, x_n)$ funksiya minimumini topish yetarli bo‘ladi, ya’ni

$$\min f(x_1, x_2, \dots, x_n) = -\max f(x_1, x_2, \dots, x_n).$$

Ushbu tasdiqdan bundan keyin faqat minimallash masalalari haqida gapirish mumkin degan xulosa kelib chiqadi.

Haqiqiy amaliy masalalarda $x_i, i=1,2,...,n$ o‘zgaruvchiga va ob’yekt, tizim, jarayonlar sifat xossalarini xarakterlovchi ba’zi funksiyalar $g_i(x)$ ga quyidagicha chegaralar (shartlar) qo‘yilishi mumkin:

$$g_i(x)=0, \quad i=1,2,...,n; \qquad a \le x \le b, \qquad \text{bunda}$$

$$a=\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}; \qquad b=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}.$$

Bunday masalalarga **shartli optimallashtirish masalalari** deyiladi.

Cheklovlari bo‘lmagan masalalar **shartsiz optimallashtirish masalalari** deyiladi.

Shartsiz ekstremum masalasining yechimini topish talab qilingan bo‘lsin, ya’ni $f(x_i)=f(x_1,x_2,...,x_n)$ funksiyaning maksimumi (minimumi)ni $x_i, i=1,2,...,n$ nuqtalarda izlash mumkin bo‘lsin.

Ko‘p o‘lchovli optimallashtirish masalasini Nyuton usulida yechish uchun quyidagi iteratsion formuladan foydalanamiz:

$$x_{k+1}=x_k-\mathcal{H}^{-1}(x_k)\cdot\nabla f(x_k) \qquad (5.18)$$

Bunda H – Gesse matritsasi bo‘lib, (5.15) formuladan topiladi.

5.5.3-misol. Berilgan funksiyani Nyuton usulida minimuminin toping: $z=3x^2+2y^2+z^2+2xy+xz+yz-2x-3y-4z\to min.$

Yechilishi: ►

Optimallashtirish masalasida $f(x)=f(x_1,x_2,\dots,x_n)$ funksiya kvadratik funksiya bo‘lsa, boshlang‘ich yaqinlashishni qanday tanlanishining ahamiyati yo‘q, chunki Gesse matritsasi o‘zgaruvchilarga bog‘liq bo‘lmaydi.

Mavzu yuzasidan savollar:

1. Optimallashtirish masalalari deganda nimani tushunasiz?
2. Optimallashtirish masalalarini yechishning qanday usullarini bilasiz?
3. Oraliqni teng ikkiga bo‘lish usulini tushuntiring.
4. Bir o‘zgaruvchili funksiya uchun Nyuton usulini tushuntiring.
5. Shartsiz optimallash usullariga qaysi usullar kiradi?
6. Maqsad funksiyasi nima?
7. Ko‘p o‘zgaruvchili funksiya uchun Nyuton usulini tushuntiring.
8. Funksiyani optimallashtirishda gradiyent vektori deb nimaga aytiladi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $f(x)=7x^3+5x^2-x+3\to\min$ funksiyani $\varepsilon=0.001$ xatolik bilan oraliqni teng ikkiga bo‘lish usulida optimal yechimini toping.

2. $f(x)=-2x^3-5x^2-x+1\to\min$ funksiyani $\varepsilon=0.001$ xatolik bilan Nyuton usulida optimal yechimini toping.

3. $z=x^2+xy+y^2-3x-6y\to min$ funksiyani optimallashtirishda boshlang‘ich yaqinlashish sifatida (2;1) nuqta olingan bo‘lsa, gradiyent vektorini aniqlang?

4. $z=3x^2+2y^2+z^2+2xy+xz+yz-2x-3y-4z\to min$ funksiyani optimallashtirishda boshlang‘ich yaqinlashish sifatida (1;2;3) nuqta olingan bo‘lsa, gradiyent vektorni aniqlang?

5. $z=3x^2+2y^2+z^2+2xy+xz+yz-2x-3y-4z\to min$ funksiyani optimallashtirishda boshlang‘ich yaqinlashish sifatida (1;2;3) nuqta olingan bo‘lsa, iteratsiya jarayonida birinchi yaqinlashish natijasini toping?

TESTLAR

1. Optimallashtirish masalasida k+1–qadamda iteratsion jarayon aynan minimumga qarab ketayotganlik sharti buzilsa qanday yo‘l tutiladi?

A) x_{k+1} ni o‘rniga $x'_{k+1}=\frac{x_{k+1}+x_k}{2}$ olinadi;

B) x_{k+1} ni o‘rniga $x'_{k+1}=\frac{x_{k+1}-x_k}{2}$ olinadi;

C) x_{k+1} ni o‘rniga $x'_{k+1}=\frac{x_{k+1}+x_k}{3}$ olinadi;

D) x_{k+1} ni o‘rniga $x'_{k+1}=\frac{x_{k+1}+x_k}{4}$ olinadi.

2. Optimallashtirish masalasida iteratsiya jarayonini to‘xtatish kriteriysi qanday bo‘ladi?

A) $\left|\frac{f'(x_k)}{f(x_{k+1})}\right|<\varepsilon;$ **B)** $\left|\frac{f'(x_{k+1})}{f(x_k)}\right|<\varepsilon;$

C) $\left|\frac{f'(x_{k+1})}{f(x_k)}\right| > \varepsilon;$ D) $\left|\frac{f'(x_{k+1})}{f''(x_k)}\right| < \varepsilon.$

3. $f(x) = -x^3 + 2x^2 + x \to min$, $\varepsilon = 0.001$ xatolik bilan yechimni Nyuton usulida topish talab qilingan bo‘lsin. $x_1 = -0.5$ bo‘lsa, agarda x_2 topilgandan keyin iteratsiya jarayoni minimumga qarab ketmayotganligi aniq bo‘lsa, x_2 qanday qiymat bilan almashtirilardi?

A) $x_2^* = -0.175$ **B)** $x_2^* = -0.375$

C) $x_2^* = -0.275$ **D)** $x_2^* = -0.375$

4. Uch o‘zgaruvchili *f(x,y,z)* fuksiya uchun Gesse matritsasi qanday ko‘rinishni oladi?

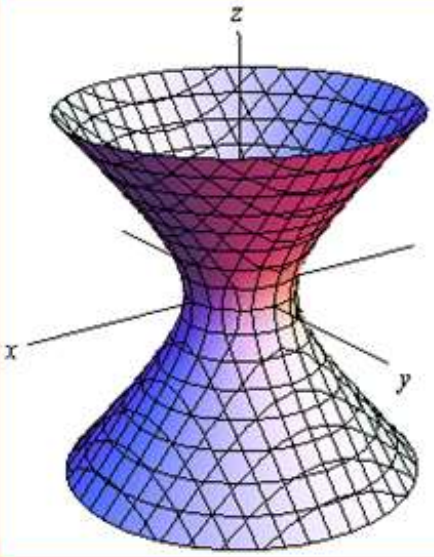
A) $\begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & 1 & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{pmatrix}$ **B)** $\begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{pmatrix}$

C) $\begin{pmatrix} 1 & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{pmatrix}$ **D)** $\begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & 1 \end{pmatrix}$

5. Ko‘p o‘lchovli optimallashtirish masalasida Nyuton usuli qanday ko‘rinishni oladi?

A) $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)^{\mathrm{T}}$ **B)** $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n-1}}\right)$

C) $\left(\frac{\partial^2 f}{\partial x_1^2}, \dots, \frac{\partial^2 f}{\partial x_n^2}\right)$ **D)** $\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 f}{\partial x_1 \partial x_n}\right).$



VI BOB. IKKI VA UCH KARRALI INTEGRALLAR

6.1-§. Ikki karrali integral

6.1.1. Ikki karrali integral ta'rifi

Ikki va uch karrali integral tushunchalari "Hisob" fanining asosiy tushunchalaridan bo'lib, ularning yordamida yassi shakl yuzalarini, jismning massasi, hajmi, og'irlik markazi, inersiya momenti va shunga o'xshash kattaliklarni aniqlash mumkin.

Bir o'zgaruvchili funksiya integralining asosiy g'oyalarini ko'p o'zgaruvchili funksiyalarga ham tatbiq qilish mumkin, ya'ni integral – bu aniq turdagi yig'indidan olingan limit g'oyasiga asoslangandir.

Shuning uchun Oxy tekisligida L chiziq bilan (yoki bir necha chiziq bilan) chegaralangan yopiq

$$R=[a,b]\times[c,d]=\{(x,y)\in R^2 \mid a\le x\le b, \quad c\le y\le d\}$$

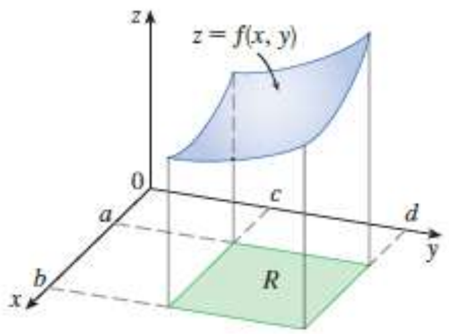
sohani qaraymiz, bu sohada $f(x,y)\ge 0$ bo'lsin va uning grafigi fazoda $z=f(x,y)$ uzluksiz funksiyani tasvirlasin. U biror sirt bo'lib, $(\boldsymbol{R})$ soha tepasida joylashgan bo'ladi (6.1-rasm):

$$\{(x,y,z)\in R^3 \mid 0\le z\le f(x,y), \quad (x,y)\in R\}.$$

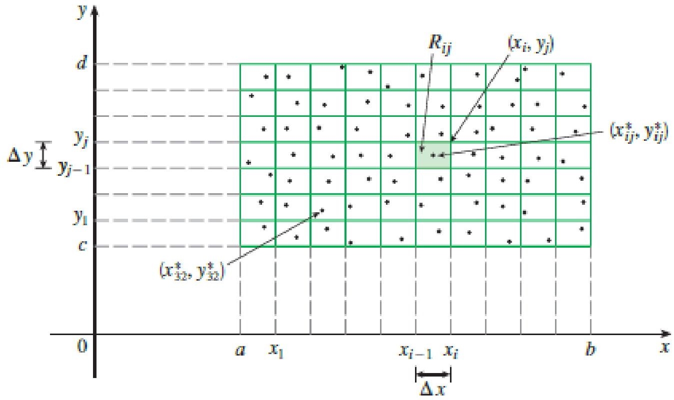
Quyidagi amallarni bajaramiz:

1) $(\boldsymbol{R})$ sohani Ox va Oy o'qlariga parallel to'g'ri chiziqlar bilan to'g'ri to'rtburchaklarga ajratamiz. Umuman olganda sohani teng yuzalarga ajratish shart emas. $[a,b]$ kesmani m ta $[x_{i-1},x_i]$ bo'lakka, $[c,d]$ kesmani esa n ta $[y_{j-1},y_j]$ bo'lakka ajratamiz, har bir to'g'ri to'rtburchak enining uzunligi Δx_i va bo'yining uzunligi Δy_j ga teng bo'ladi. Shunda

(*R*) soha yuzasi $\Delta S = \Delta x_i \Delta y_j$ bo‘lgan yuzachalardan iborat bo‘ladi (6.2-rasm): $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i,\ \ y_{j-1} \le y \le y_j\}$.



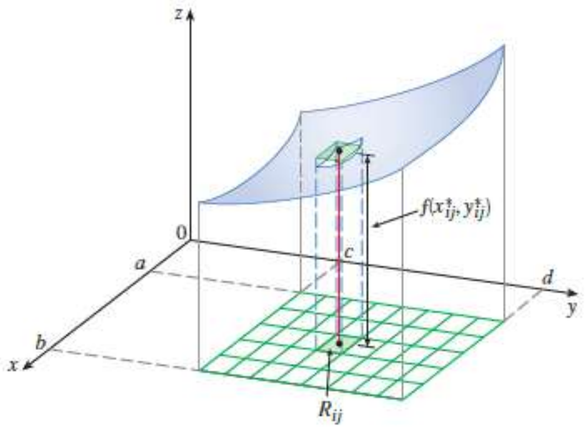
6.1-rasm. Ikki o‘zgaruvchili funksiya



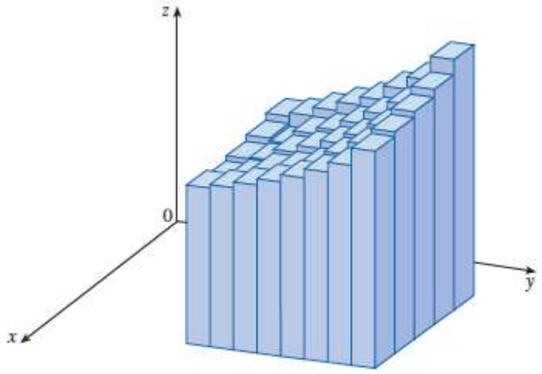
6.2-rasm. Bo‘laklarga ajratish

2) Har bir R_{ij} yuzachadan bittadan (x_{ij}^*, y_{ij}^*) nuqtani tanlab olamiz va bu nuqtada $z = f(x, y)$ funksiyaning qiymatlarini hisoblaymiz: $f(x_{ij}^*, y_{ij}^*)$;

3) Topilgan funksiya qiymatlarini elementar yuzalarga mos ravishda ko‘paytiramiz $f(x_{ij}^*, y_{ij}^*) \cdot \Delta S_{ij}$, natijada to‘g‘ri to‘rtburchakli prizma hajmi hosil bo‘ladi (6.3-rasm):



6.3-rasm. Bo‘laklarga ajratish



6.4-rasm. Bo‘laklarga ajratish

4) Bu hajmlar yig‘indisini topamiz (6.4-rasm):

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta S_{ij}$$

Bu yig‘indiga $z = f(x, y)$ funksiyaning **(*R*)** sohadagi integral yig‘indisi deyiladi. Bu integral yig‘indi turli *n, m* larda **(*R*)** sohani qismlarga bo‘lish usuliga va har bir qismda nuqtalarni tanlanishiga ham bog‘liq.

Shunday qilib, tayinlangan *n, m* larda integral yig‘indilar ketma-ketligiga ega bo‘lamiz. $m, n \to \infty$ da $\max \Delta S_{ij} \to 0$ deb faraz qilamiz.

Quyidagi tasdiq o‘rinli:

6.1-teorema(Ikki karrali integralning mavjudligi haqida). Agar chegaralangan yopiq **(*R*)** sohada $z = f(x,y)$ funksiya uzluksiz bo‘lsa, u holda bu sohani qismiy sohalarga bo‘lishlar sonini $\max \Delta S_{ij} \to 0$ qilib $(m,n \to \infty)$ bo‘laklar sonini cheksiz orrtirganda

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \cdot \Delta S_{ij}$$

ko‘rinishdagi integral yig‘indining limiti mavjud bo‘ladi.

Bu limit **(*R*)** sohani ΔS_{ij} yuzalarga bo‘lish usuliga ham va har bir qism ichida (x_{ij}^{*}, y_{ij}^{*}) nuqtani tanlash usuliga ham bog‘liq bo‘lmaydi. Bu limit qiymatga $z = f(x,y)$ funksiyadan **(*R*)** soha bo‘yicha olingan **ikki karrali integral** deyiladi va quyidagicha belgilanadi:

$$\iint_{(R)} f(x,y)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \cdot \Delta S_{ij} \qquad (6.1)$$

bu yerda **(*R*)** - integrallash sohasi, $f(x;y)$ - integral ostidagi funksiya, x, y - lar integrallash o‘zgaruvchilari, dS yuza elementi deyiladi.

Har bir R_{ij} yuzadan olingan (x_{ij}^{*}, y_{ij}^{*}) nuqtalar uchun barcha *m, n* butun sonlarda har bir $\varepsilon > 0$ son uchun quyidagi tengsizlik o‘rinli bo‘ladi:

$$\left| \iint_{(R)} f(x,y)dS - \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \cdot \Delta S_{ij} \right| < \varepsilon .$$

6.1.2. Ikki karrali integralning geometrik va mexanik ma’nosi

(*R*) soha, tenglamasi $z = f(x,y)$ dan iborat σ sirt, yo‘naltiruvchisi z hamda yasovchilari OZ o‘qqa parallel bo‘lgan silindrik sirt bilan chegaralangan jism **silidrik jism** deyiladi.

Agar **(*R*)** sohada $f(x;y) \geq 0$ bo‘lsa, u holda har bir $f(x_{ij}^{*}, y_{ij}^{*}) \cdot \Delta S_{ij}$ qo‘shiluvchini asosi ΔS_{i} dan, balandligi esa $f(x_{ij}^{*}, y_{ij}^{*})$ dan iborat kichkina silindrik jismning hajmi sifatida geometrik tasvirlash mumkin. Bu holda $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \cdot \Delta S_{ij}$ integral yig‘indi ko‘rsatilgan silindrik jismninig hajmlari yig‘indisidan, boshqacha aytganda, biror zinapoyasimon silindrik jismninig hajmidan iborat bo‘ladi. **(*R*)** soha bo‘yicha $z = f(x,y)$ funksiyadan olingan ikki karrali integral quyidan **(*R*)** soha, yuqoridan esa

$z = f(x, y)$ sirt bilan chegaralangan silindrik jismning V hajmiga teng bo‘ladi:

$$\mathrm{V} = \iint\limits_{(R)} f(x, y)dxdy$$

bunda **(*R*)** soha $z = f(x, y)$ sirtning Oxy tekislikdagi proeksiyasidir.

Bu ikki karrali integralning **geometrik ma’nosidan** iborat.

Agar **(*R*)** sohada integral osti funksiya $f(x\,;\,y) \equiv 1$ bo‘lsa, u holda ikki karrali integralning qiymati son jihatdan integrallash sohasi **(*R*)** ning S yuziga teng bo‘ladi:

$$\mathrm{S} = \iint\limits_{(R)} dxdy \;.$$

Agar integral ostidagi funksiya biror *S* yuzali bir jinsli plastinaning $\rho = f(x, y)$ zichligi bo‘lsa, u holda ikki karrali integral plastinkaning massasi m ga teng bo‘ladi:

$$\mathrm{m} = \iint\limits_{(R)} f(x, y)dS \;.$$

Bu ikki karrali integralning **mexanik** ma’nosidan iborat.

6.1.3. Ikki karrali integralning xossalari:

1⁰. $R = \{(x, y) \mid a \le x \le b, \;\; c \le y \le d\}$ sohada ikki karrali integral uchun quyidagi tenglik o‘rinli:

$$\iint\limits_{(R)} f(x, y)dS = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$

2⁰. O‘zgarmas ko‘paytuvchini ikki karrali integral belgisidan tashqariga chiqarish mumkin:

$$\iint\limits_{(R)} k \cdot f(x, y)dxdy = k \cdot \iint\limits_{(R)} f(x, y)dxdy$$

3⁰. Berilgan sohada funksiyalar yig‘indisidan olingan ikki karrali integral shu funksiyalardan alohida hisoblangan ikki karrali integrallar yig‘indisiga teng:

$$\iint\limits_{(R)} [f(x, y) \pm \varphi(x, y)]dxdy = \iint\limits_{(R)} f(x, y)dxdy \pm \iint\limits_{(R)} \varphi(x, y)dxdy$$

4⁰. Agar 1) berilgan *D* soha o‘zaro kesishmaydigan $D = D_1 + D_2$ sohalar yig‘indisidan iborat bo‘lsa, u holda quyidagi tenglik o‘rinli:

$$\iint\limits_{(D)} f(x, y)\,dS = \iint\limits_{(D_1)} f(x, y)\,dS + \iint\limits_{(D_2)} f(x, y)\,dS$$

5⁰. **a)** Agar berilgan funksiya nomanfiy bo‘lsa, u holda

$$f(x;y)\geq 0 \to \iint_{(D)} f(x;y)dS \geq 0$$ o‘rinli;

b) Agar berilgan funksiya nomusbat bo‘lsa, u holda

$$f(x;y)\leq 0 \to \iint_{(D)} f(x;y)ds \leq 0$$ o‘rinli.

6⁰. Ushbu tengsizlik o‘rinli:

$$f(x,y)\geq \varphi(x;y) \to \iint_{(D)} f(x,y)dS \geq \iint_{(D)} \varphi(x;y)dS.$$

7⁰. (O‘rta qiymat haqidagi teorema) Agar $z=f(x,y)$ funksiya yopiq chegaralangan **(*D*)** sohada uzluksiz bo‘lsa, u holda bu **(*D*)** sohada shunday $P_0(x_0,y_0)$ nuqta topiladiki, bu nuqta uchun quyidagi tenglik o‘rinli bo‘ladi:

$$\iint_{(D)} f(x,y)dS = f(x_0,y_0)\cdot S.$$

$f(x_0,y_0)$ qiymatga $z=f(x,y)$ funksiyaning **(*D*)** sohadagi **o‘rta qiymati** deyiladi:

$$f(x_0;y_0)=\frac{\iint_{(D)} f(x,y)dS}{\iint_{(D)} dxdy}$$

8⁰. (Integralning chegaralanganligi haqidaghi teorema). Agar $z=f(x,y)$ funksiya yopiq chegaralangan **(*R*)** sohada uzluksiz hamda ***M*** va ***m*** lar funksiyaning shu sohadagi eng katta va eng kichik qiymatlari bo‘lsa, u holda ikki karrali integral, funksiyaning eng kichik qiymatining integrallash sohasi **(*R*)** ning ***S*** yuzasiga ko‘paytmasi bilan eng katta qiymati ***M*** ning o‘sha yuzaga ko‘paytmasi orasida yotadi:

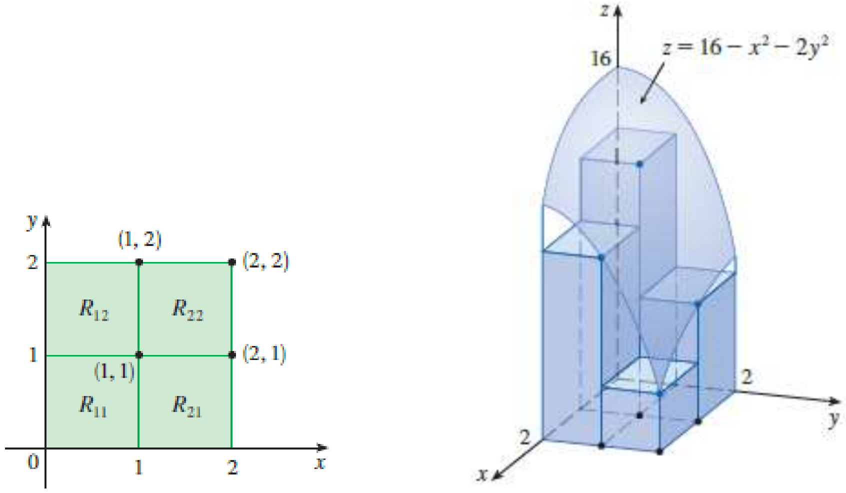
$$m\cdot S \leq \iint_{(R)} f(x,y)dS \leq M\cdot S.$$

6.1.1-misol. Aniqlanish sohasi $R=[0,2]\times[0,2]$ dan iborat $z=16-x^2-2y^2$ elliptik paraboloid ostidagi hajmni taqribiy hisoblang.

Yechilishi: ► **(*R*)** sohani yuzasi 1 birlikka teng bo‘lgan 4 ta kvadratga ajratamiz (6.5-rasm).

Ajratilgan sohalarda hajmlarni topib, ularning yig‘indisini olamiz:

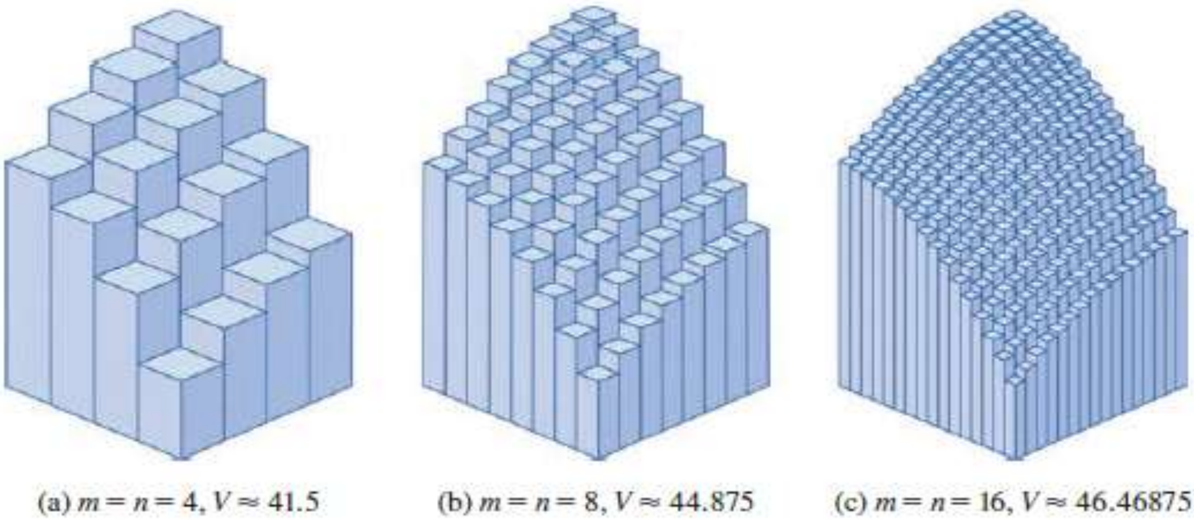
$$V\approx\sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i,y_j)\cdot\Delta S_{ij}$$



6.5-shakl. $z = 16 - x^2 - 2y^2$ funksiyani bo‘laklarga ajratish

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2}(16 - x^2 - 2y^2)\cdot 1 = f(1{,}1)\cdot 1 + f(1{,}2)\cdot 1 + f(2{,}1)\cdot 1 + f(2{,}2)\cdot 1 =$$

$$= (16 - 1^2 - 2\cdot 1^2) + (16 - 1^2 - 2\cdot 2^2) + (16 - 2^2 - 2\cdot 1^2) + (16 - 2^2 - 2\cdot 2^2) = 13 + 7 + 10 + 4 = 34.$$



6.6-rasm. $z = 16 - x^2 - 2y^2$ funksiyani bo‘laklarga ajratish

Agar $(\boldsymbol{R})$ sohani 16, 64, 256 ta kvadratchalarga ajratsak, u holda jismning hajmi mos holda 6.6-rasmda ko‘rsatilgan qiymatlarga teng bo‘ladi. ◄

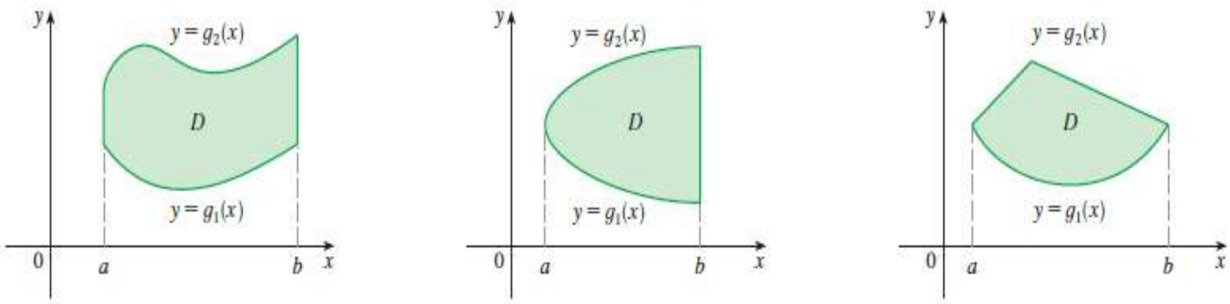
6.1.4. Dekart koordinatalar sistemasida ikki karrali integralni hisoblash

Agar (D) sohani Oy o‘qiga parallel ixtiyoriy to‘g‘ri chiziq faqat ikkita nuqtada kesib o‘tsa, hamda sohaga kirish va chiqish konturlari faqat bittadan tenglama bilan berilgan bo‘lsa, bu soha Oy o‘qi bo‘yicha **muntazam soha** deyiladi.

Oxy tekisligida yotuvchi (D) soha x o‘zgaruvchining 2 ta uzluksiz funksiyalari grafiklari bilan chegaralangan bo‘lsin:

$$D=\{(x,y)|\ a\le x\le b,\quad g_1(x)\le y\le g_2(x)\}.$$

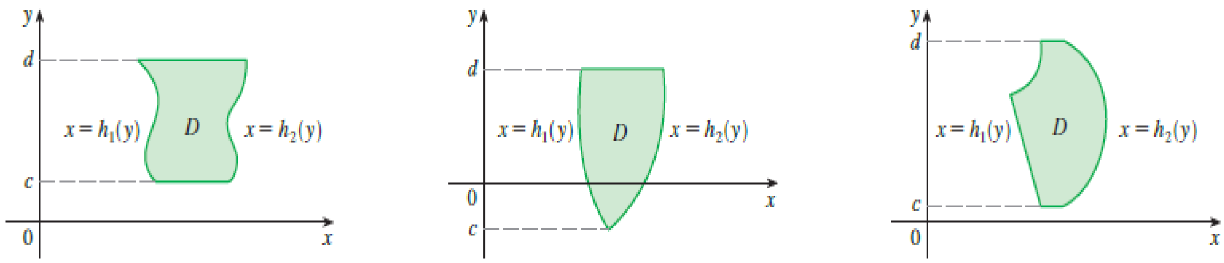
Bunda $g_1(x)$, $g_2(x)$ funksiyalarning $[a,b]$ kesmadagi bo‘lagi olinadi, Oy o‘q yo‘nalishida muntazam soha chizmada quyidagicha tasvirlanadi (6.7-rasm):



6.7-rasm. Oy o‘q yo‘nalishida muntazam soha

Ox o‘q yo‘nalishida muntazam soha ham shu singari aniqlanadi (6.8-rasm):

$$D=\{(x,y)|\ h_1(x)\le x\le h_2(x),\ \ c\le y\le d\}$$



6.8-rasm. Ox o‘q yo‘nalishida muntazam soha

6.2-teorema. $z=f(x,y)$ uzluksiz funksiyaning (D) to‘g‘ri soha bo‘yicha olingan ikki karrali integrali quyidagiga teng:

$$I_{(D)}=\iint\limits_{(D)} f(x;y)dxdy=\int_a^b dx\int_{y_1(x)}^{y_2(x)} f(x;y)dy=\int_a^b\left(\int_{y_1(x)}^{y_2(x)} f(x;y)dy\right)dx\ .$$

Bu integralni hisoblash uchun dastlab qavs ichidagi integralda x ni o‘zgarmas deb qarab, uni y bo‘yicha integrallaymiz:

$$\Phi(x)=\int_{y_1(x)}^{y_2(x)} f(x;y)dy\ .$$

Integrallash natijasida x ning uzluksiz funksiyasi hosil bo‘ladi. So‘ngra bu integralni x bo‘yicha a dan b gacha chegarada integrallaymiz:

$$I_{(D)} = \int_a^b \Phi(x)dx$$

Natijada biror o‘zgarmas son hosil bo‘ladi.

Natija: 1) Agar (D) soha Oy o‘qi bo‘yicha muntazam bo‘lsa,

$$(D):\begin{cases} a \le x \le b \\ y_1(x) \le y \le y_2(x) \end{cases} \quad \text{bo‘lganda} \quad \iint_{(D)} f(x;y)dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x;y)dy;$$

2) Agar (D) soha Ox o‘qi bo‘yicha muntazam bo‘lsa,

$$(D):\begin{cases} c \le y \le d \\ x_1(y) \le x \le x_2(y) \end{cases} \quad \text{bo‘lganda} \quad \iint_{(D)} f(x,y)dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)dx$$

3) Tashqi integralning chegaralari doim o‘zgarmas sondan iborat bo‘ladi. Agar soha nomuntazam bo‘lsa, uni muntazam sohalarga ajratiladi va integral har bir soha bo‘yicha hisoblanadi, keyin esa ularni jamlash kerak.

4) Agar soha to‘g‘ri to‘rtburchakdan iborat bo‘lsa,

$$(D):\begin{cases} a \le x \le b \\ c \le y \le d \end{cases} \quad \text{bo‘lganda}$$

$$\iint_{(D)} f(x\,;\,y)dxdy = \int_a^b dx \int_c^d f(x;y)dy = \int_c^d dy \int_a^b f(x;y)dx \quad \text{bo‘ladi.}$$

6.1.2-misol. $\int_1^2 dx \int_0^3 (1+8xy)\,dy$ hisoblang.

Yechilishi: ► Ushbu intagralni hisoblash uchun dastlab ichki integralni y o‘zgaruvchi bo‘yicha boshlang‘ichini topamiz, so‘ngra integral chegaralarini qo‘yib hisoblaymiz. 2-qadamda tashqaridagi integralni hisoblaymiz:

$$\int_1^2 dx \int_0^3 (1+8xy)dy = \int_1^2 (y+4xy^2)\Big|_0^3 dx = \int_1^2 (3+4x\cdot 3^2-0)dx = \int_1^2 (3+36x)dx = (3x+18x^2)\Big|_1^2 = \quad ◄$$

$$= 6+72-3-18 = 57.$$

6.1.3-misol. $\iint_D x^2 ydxdy$ integralni hisoblang, bunda (D) soha quyidagicha: $D = \{(x,y) \in R^2 : 2 \le x \le 5, 1 \le y \le 3\}$.

Yechilishi: ► Integrallash chegaralarini mos ravishda joylashtiramiz:

$$\iint_D x^2 ydxdy = \int_1^3 [\int_2^5 x^2 ydx]dy = \int_1^3 (\frac{x^3 y}{3})_{x=2}^{x=5} dy = \frac{1}{3}\int_1^3 (125y - 8y)dy = \frac{117}{3}(\frac{y^2}{2})_1^3 = 156.$$

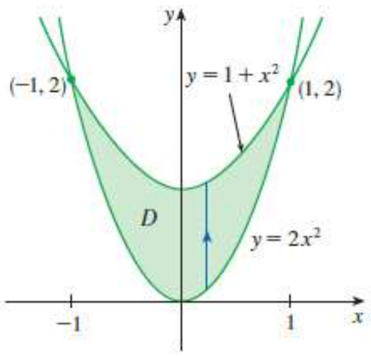
Shuningdek, integrallar tartibini (o‘rnini) almashtirib ham hisoblash mumkin, natija bir xil chiqadi:

$$\iint_D x^2 ydxdy = \int_2^5 [\int_1^3 x^2 ydy]dx = \int_2^5 (\frac{x^2 y^2}{2})_{y=1}^{y=3} dx = \frac{1}{2}\int_2^5 (9x^2 - x^2)dx = 4(\frac{x^3}{3})_2^5 = 156. \blacktriangleleft$$

6.1.4.-misol. $\iint_{(D)} (x+2y)ds$ hisoblang, bunda (D) soha $y = 2x^2$, $y = 1 + x^2$ parabolalar bilan chegaralangan soha.

Yechilishi: ► (D) sohadagi $y = 2x^2$, $y = 1 + x^2$ parabolalarning kesishish nuqtalarini topamiz (6.9-rasm).

$$2x^2 = 1 + x^2,\ x^2 = 1,\ \ x = \pm 1. \qquad D = \{(x,y) \mid -1 \le x \le 1,\ \ 2x^2 \le y \le 1 + x^2\}.$$



6.9-rasm. $y = 2x^2$, $y = 1 + x^2$ parabolalarning kesishish nuqtalari

$y = 1 + x^2$ funksiya grafigi $y = 2x^2$ funksiya grafigidan tepada joylashgan, shuning uchun $y = 1 + x^2$ yuqori chegaraga, $y = 2x^2$ esa quyi chegaraga yoziladi:

$$\iint_{(D)} (x+2y)ds = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)dydx = \int_{-1}^{1}\left[xy + y^2\right]_{2x^2}^{1+x^2} dx = \int_{-1}^{1}\left[x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2\right]dx =$$

$$= \int_{-1}^{1}\left[x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4\right]dx = \int_{-1}^{1}\left[-3x^4 - x^3 + 2x^2 + x + 1\right]dx =$$

$$= \left[-\frac{3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} = \frac{32}{15}. \quad \blacktriangleleft$$

6.1.5.-misol. $\iint_D dxdy$ integralni hisoblang, bunda (D): $y^2=2x$ parabola va uning $(2;-2)$ va $(8,\ 4)$ nuqtalarini birlashtiruvchi vatar bilan chegaralangan soha.

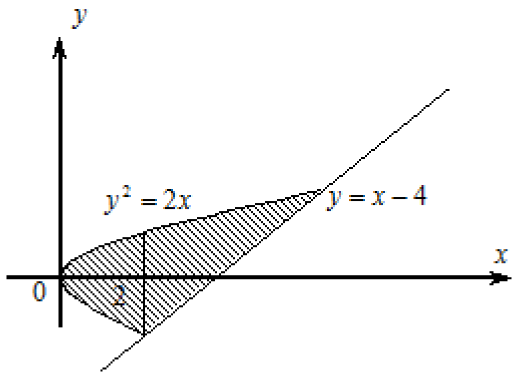
Yechilishi: ► $y^2=2x$ parabolaning $(2;-2)$ va $(8,\ 4)$ nuqtalarini birlashtiruvchi vatar tenglamasini tuzamiz: $y=x-4$. Shunda $y^2=2x$ va $y=x-4$ chiziqlar bilan chegaralangan (D) soha 6.10-rasmda tasvirlangan.

$x=2$ to‘g‘ri chiziq yordamida (D) sohani ikkita D_1 va D_2 sohalarga ajratamiz. Bunda

$$D_1=\{(x,\ y)\in R^2 : 0\le x\le 2, -\sqrt{2x}\le y\le\sqrt{2x}\},$$
$$D_2=\{(x,\ y)\in R^2 : 2\le x\le 8,\ \ x-4\le y\le\sqrt{2x}\}$$

bo‘ladi. Ikki karrrali integralning xossalaridan foydalanamiz:

$$\iint_D dxdy=\iint_{D_1} dxdy+\iint_{D_2} dxdy\,.$$



6.10-rasm. $y^2=2x$ parabola hamda vatar bilan chegaralangan soha

Endi hosil bo‘lgan integralni hisoblaymiz:

$$\iint_D dxdy=\iint_{D_1} dxdy+\iint_{D_2} dxdy=\int_0^2\left[\int_{-\sqrt{2x}}^{\sqrt{2x}} dy\right]dx+\int_2^8\left[\int_{x-4}^{\sqrt{2x}} dy\right]dx=$$
$$=\int_0^2 2\sqrt{2x}dx+\int_2^8\left(\sqrt{2x}-x+4\right)dx=18\,.$$ ◄

6.1.6.-misol. Berilgan sirtlar bilan chegaralangan Q jismning hajmini hisoblang: $z=0,\ z=x^2+y^2,\ y=x^2,\ y=1$.

Yechilishi: ► Berilgan jismni quyidagi ko‘rinishda tasvirlash kerak:

$$Q=\{(x,y,z)\ :(x,y)\in D,\quad 0\le z\le x^2+y^2\},$$

bunda D — soha *Oxy* tekislikning $y=x^2$ va $\mathbf{y=1}$ egri chiziqlari bilan chegaralangan qismi, ya’ni $D=\{(x,y):\ -1\le x\le 1,\quad x^2\le y\le 1\}$.

Ikki o‘lchovli integralning geometrik ma’nosiga ko‘ra, Q jismning hajmi quyidagicha topiladi:

$$V=\iint\limits_{D}\left(x^2+y^2\right)dx\,dy=\int\limits_{-1}^{1}dx\int\limits_{x^2}^{1}\left(x^2+y^2\right)dy=\int\limits_{-1}^{1}\left(x^2\left(1-x^2\right)+\frac{1}{3}\left(1-x^6\right)\right)dx=\frac{88}{105}.$$ ◄

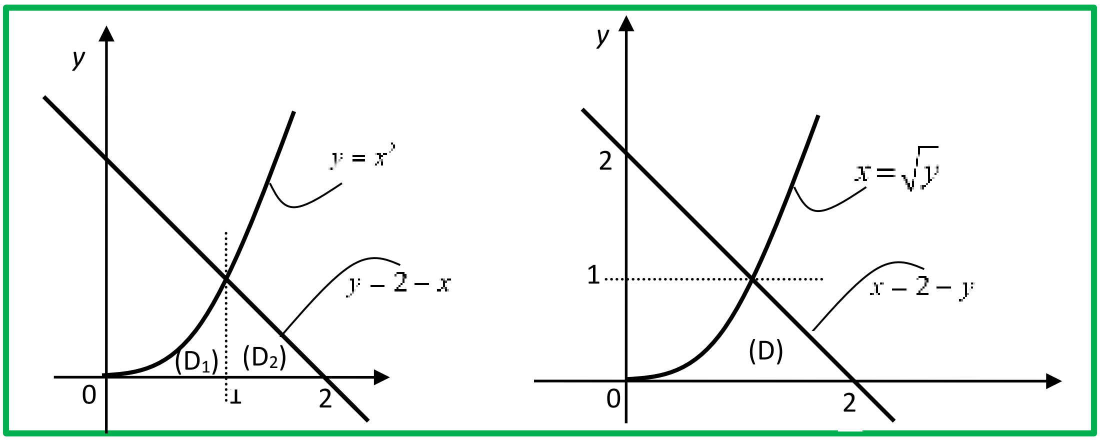
6.1.5. Ikki karrali integralda integrallash tartibini o‘zgartirish

Har bir masala yechimi (D) sohaga bog‘liq bo‘ladi. (D) sohani muntazam sohalarga ajratish usuliga ko‘ra integrallash chegarasini o‘zgartirish mumkin:

$$\iint\limits_{(D)} f(x;y)\,dxdy=\int\limits_{a}^{b}dx\int\limits_{f_1(x)}^{f_2(x)} f(x;y)\,dy=\int\limits_{c}^{d}dy\int\limits_{\varphi_1(y)}^{\varphi_2(y)} f(x;y)\,dx$$

6.1.7. -misol. Berilgan (D) soha $\begin{cases} D:\ 0\le x\le 2 \\ y=x^2,\ x+y=2 \end{cases}$ bo‘yicha integrallash chegarasini qo‘ying va integrallash tartibini almashtiring.

Yechilishi: ► Grafikni chizib olamiz (6.11-rasm):



6.11-rasm. *D* sohani *Ox* va *Oy* o‘qlari bo‘yicha ajratish

$$(D)=(D_1)+(D_2):\begin{cases}(D_1):0\le x\le 1,\quad 0\le y\le x^2\\(D_2):1\le x\le 2,\quad 0\le y\le 2-x\end{cases}$$

$$\iint\limits_{(D)} f(x;y)\,dxdy=\int\limits_{0}^{1}dx\int\limits_{0}^{x^2} f(x;y)\,dy+\int\limits_{1}^{2}dx\int\limits_{0}^{2-x} f(x;y)\,dy=\int\limits_{0}^{1}dy\int\limits_{\sqrt{y}}^{2-y} f(x;y)\,dx.$$

Tekshirish: Integrallash tartibini almashtirni to‘g‘ri bajarganimizni $f(x,y)=1$ deb faraz qilib, tekshirib ko‘rishimiz mumkin. Integrallarni hisoblaymiz.

1) $$\int_0^1 dx\int_0^{x^2}dy+\int_1^2 dx\int_0^{2-x}dy=\int_0^1 x^2dx+\int_1^2(2-x)dx=\frac{x^3}{3}\bigg|_0^1+\left(2x-\frac{x^2}{2}\right)\bigg|_1^2=\frac{1}{3}+2-\frac{3}{2}=\frac{5}{6};$$

2) $$\int_0^1 dy\int_{\sqrt{y}}^{2-y}dy=\int_0^1(2-y-\sqrt{y})dy=\left(2y-\frac{y^2}{2}-\frac{2\sqrt{y^3}}{3}\right)\bigg|_0^1=2-\frac{1}{2}-\frac{2}{3}=\frac{5}{6}.\ \Rightarrow\ \frac{5}{6}=\frac{5}{6}.$$ ◀

6.1.8. -misol. Tepadan $z=x^2+y^2$ paraboloid bilan quyidan $y=2x$ hamda $y=x^2$ chiziqlar bilan chegaralangan (D) yopiq sohadan iborat shaklni integrallang.

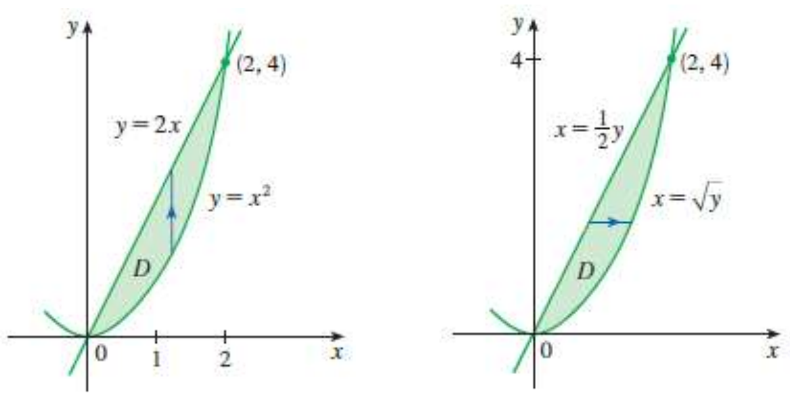
Yechilishi: ▶

I usul. 6.12, a - rasmdan foydalanamiz. Unga ko‘ra, (D) soha

$$D=\{(x,y):0\le x\le 2,\quad x^2\le y\le 2x\}$$

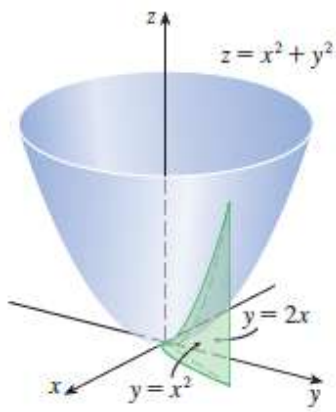
$$V=\int_0^2 dx\int_{x^2}^{2x}(x^2+y^2)dy=\int_0^2\left(x^2y+\frac{y^3}{3}\right)\bigg|_{x^2}^{2x}dx=\int_0^2\left(2x^3+\frac{8x^3}{3}-x^4-\frac{x^6}{3}\right)dx=$$

$$=\int_0^2\left(-\frac{x^6}{3}-x^4+\frac{14x^3}{3}\right)dx=\left(-\frac{x^7}{21}-\frac{x^5}{5}+\frac{7x^4}{6}\right)\bigg|_0^2=\frac{216}{35}.$$



a) b)

6.12-rasm. D sohani Ox va Oy o‘qlari bo‘yicha ajratish



6.13-rasm.

II usul. 6.12, b - rasmdan foydalanamiz.

Unga ko‘ra, (D) soha quyidagicha: $D=\left\{(x,y):\ 0\le y\le 4,\ \ \frac{y}{2}\le x\le\sqrt{y}\right\}.$

$$V=\int_0^4 dy\int_{\frac{y}{2}}^{\sqrt{y}}(x^2+y^2)dx=\int_0^4\left(\frac{x^3}{3}+xy^2\right)\bigg|_{\frac{y}{2}}^{\sqrt{y}}dy=\int_0^4\left(\frac{\sqrt{y^3}}{3}+y^2\sqrt{y}-\frac{y^3}{24}-\frac{y^3}{2}\right)dy=\frac{216}{35}.$$

Biz izlayotgan hajm 6.13-rasmda tasvirlangan jism hajmi bo‘ladi. ◀

Mavzu yuzasidan savollar:

1. Ikki o‘zgaruvchili funksiyaning integral yig‘indisi qanday ko‘rinisda bo‘ladi?
2. Ikki karrali integralning mavjudligi haqidagi teoremani ayting.
3. Ikki karrali integralning geometrik ma’nosi nima?
4. Ikki karrali integralning mexanik ma’nosini ayting
5. Ikki karrali integralning xossalarini ayting.
6. O‘rta qiymatdagi haqidagi teoremani ayting.
7. Muntazam soha nima?
8. $0x$ o‘qi bo‘yicha muntazam sohaning ikki karrali integrali qanday hisoblanadi?
9. $0y$ o‘qi bo‘yicha muntazam sohaning ikki karrali integrali qanday hisoblanadi?
10. Nomuntazam sohalar bo‘yicha integral qanday hisoblanadi?

6.2-§. Ikki karrali integralda o‘zgaruvchilarni almashtirish

Oxy tekisligida yotuvchi (D) sohada $z = f\left(x;y\right)$ uzluksiz funksiya berilgan bo‘lsin. Bu funksiya uchun $\iint\limits_{(D)} f(x,y)dxdy$ mavjud.

Faraz qilaylik, x va y koordinatalar yangi u va v o‘zgaruvchilarning funksiyalari bo‘lsin:

$$\begin{cases} x = x\left(u,v\right) = \varphi(u,v) \\ y = y\left(u,v\right) = \psi(u,v) \end{cases}$$

Bu integralda u,v o‘zgaruvchilarga o‘tamiz:

$$\iint\limits_{(D)} f\left(x;y\right)dxdy = \iint\limits_{(D0} f\left[x\left(u,v\right),\ y\left(u,v\right)\right]\cdot\left|I\right|du\,dv \qquad (6.2)$$

formula o‘rinli bo‘ladi. Bu yerda $I = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0$. Bu deteminant nemis matematigi Yakobi nomi bilan **yakobian** deb yuritiladi.

6.2.1.-misol. $\iint_D (x+y)^3(x-y)^2\,dxdy$ ikki karrali integralni hisoblang. Bu yerda (D) soha $x+y=1,\ x-y=1,\ x+y=3,\ x-y=-1$ to'g'ri chiziqlar bilan chegaralangan.

Yechilishi: ► $x+y=u,\quad x-y=v$ almashtirishni bajaramiz, bu yerdan $x=\left(\frac{1}{2}\right)(u+v),\ y=\left(\frac{1}{2}\right)(u-v)$. U holda almashtirishning

Yakobiani $J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\frac{1}{2}.$

Demak $|J|=\frac{1}{2}$. Bundan, $\iint_D (x+y)^3(x-y)^2\,dxdy=\frac{1}{2}\iint_{D'} u^3v^2dudv.$

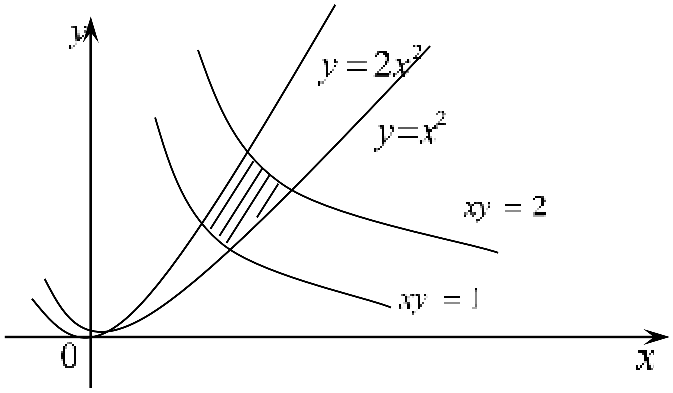
D soha ham kvadrat, u holda

$$\iint_D (x+y)^3(x-y)^2\,dxdy=\frac{1}{2}\int_1^3 u^3du\int_{-1}^1 v^2dv=\frac{1}{2}\int_1^3 u^3\left[\frac{1}{3}v^3\right]_{-1}^1 du=$$ ◄

$$=\frac{1}{6}\int_1^3 u^3(1+1)du=\frac{1}{12}u^4\Big|_1^3=\frac{20}{3}.$$

6.2.2.-misol. $\iint_D y^3dxdy$ integralni hisoblang, bunda (D) soha $y=x^2,\ y=2x^2,\ xy=1\ ,\ xy=2$ chiziqlar bilan chegaralangan.

Yechilishi: ► Berilgan chiziqlar bilan chegaralangan soha 6.14-rasmda tasvirlangan. Yangi o'zgaruvchilarni kiritamiz:



6.14-rasm. $y=x^2,\ y=2x^2,\ xy=1\ ,\ xy=2$ chiziqlar bilan chegaralangan soha

$$\begin{cases}u=\frac{y}{x^2}\\ v=xy\end{cases}\quad \text{va } (x>0). \qquad \text{Soha: } \Delta=\{(u,v)\in R^2 : 1\le u\le 2, 1\le v\le 2\}.$$

Endi yakobianni topamiz:

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}-\frac{1}{3}u^{-\frac{4}{3}}v^{\frac{1}{3}} & \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}}\\ \frac{2}{3}u^{\frac{1}{3}}v^{-\frac{2}{3}} & \frac{2}{3}u^{\frac{1}{3}}v^{-\frac{1}{3}}\end{vmatrix}=-\frac{1}{3u}, \qquad |J(u,v)|=\frac{1}{3|u|}.$$

$y^3=uv^2$ ekanini e'tiborga olib, $\begin{cases}x=u^{-\frac{1}{3}}v^{\frac{1}{3}},\\ y=u^{\frac{1}{3}}v^{\frac{2}{3}}\end{cases}$ almashtirishni bajaramiz: $\iint\limits_D y^3dxdy=\iint\limits_D uv^2|J(u,v)|dudv$.

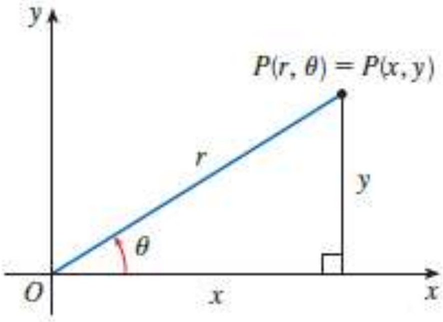
Hosil bo'lgan integralni hisoblaymiz:

$$\iint\limits_D uv^2|J(u,v)|dudv=\frac{1}{3}\iint\limits_D v^2dudv=\frac{1}{3}\int\limits_1^2(\int\limits_1^2 v^2dv)du=\frac{1}{3}(\frac{8}{3}-\frac{1}{3})=\frac{7}{9}.$$

Demak, izlanayotgan natija quyidagiga teng bo'ladi: $\iint\limits_D y^3dxdy=\frac{7}{9}$. ◄

6.2.1. Qutb koordinatalar sistemasida ikki karrali integral

$\iint\limits_{(D)} f(x,y)dxdy$ integralni hisoblashda dekart koordinatalari sistemasidan qutb koordinatasiga o'tish formulasini keltirib chiqaramiz.



6.15-rasm. Qutb koordinatalar sistemasi

Buning uchun x , y o'zgaruvchilarni

$$\begin{cases}x=r\cdot\cos\varphi\\ y=r\cdot\sin\varphi\end{cases} \tag{6.3}$$

formulalar yordamida qutb koordinatalari r va φ ga almashtiramiz (6.15-rasm). Yuqoridagi sistemadan

$$\begin{cases}r=\sqrt{x^2+y^2}\\ \varphi=arctg\,\frac{y}{x}\end{cases} \tag{6.4}$$

formulalarni topamiz. Endi yakobianni hisoblaymiz. Buning uchun dastlab r va φ o'zgaruvchilar bo'yicha xususiy hosilalarni topamiz:

$$\frac{\partial x}{\partial r} = cos\varphi \ , \quad \frac{\partial x}{\partial \varphi} = -r\cdot\sin\varphi \ , \quad \frac{\partial y}{\partial r} = \sin\varphi, \ \frac{\partial y}{\partial \varphi} = r\cos\varphi$$

$$I = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\cdot\sin\varphi \\ \sin\varphi & r\cdot cos\varphi \end{vmatrix} = r\cdot\cos^2\varphi + r\cdot\sin^2\varphi = r. \qquad (6.5)$$

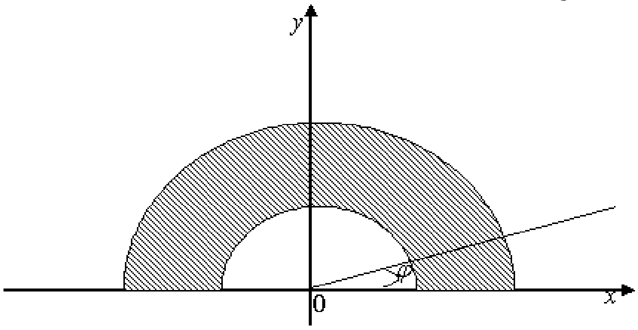
Topilgan kattaliklarni hammasini izlanayotgan integralga qo'yamiz:

$$\iint\limits_{(D)} f(x;y)dxdy = \iint\limits_{(D)} f(r\cos\varphi \ r\sin\varphi)rdrd\varphi \qquad (6.6)$$

Ushbu formula ko'pincha (D) soha markazi koordinatalar boshida bo'lgan $x^2+y^2=a^2$ doiradan iborat bo'lganda qo'llaniladi. Bu holda $(\overline{D})$ soha quyidagi $\begin{cases} 0\le r\le a \\ 0\le\varphi\le 2\pi \end{cases}$ tengsizliklar bilan aniqlanadi.

6.2.3-misol. $J=\iint\limits_{D}(x+y)dxdy$ integralni hisoblang, bunda (D) soha $x^2+y^2=1$, $x^2+y^2=4$, $y=0$ (D da $y>0$) chiziqlar bilan chegaralangan soha (6.16-rasm).

Yechilishi: ► (D) sohani chizmada tasvirlaymiz:



6.16-rasm. D sohaning tasviri

$$\iint\limits_{(D)} f(x;y)dxdy = \iint\limits_{(D)} f(r\cos\varphi \ r\sin\varphi)rdrd\varphi$$

(6.6)-formuladan foydalanamiz. Ravshanki, qutb koordinatasida soha quyidagicha bo'ladi: $\Delta=\{(r,\varphi)\in R^2 : 1\le r\le 2, 0\le\varphi\le\pi\}$.

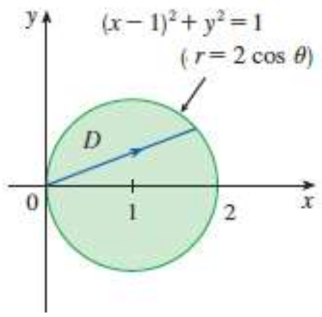
Natijada quyidagi qiymatni olamiz:

$$J=\iint\limits_{D} r^2(\cos\varphi+\sin\varphi)drd\varphi=$$

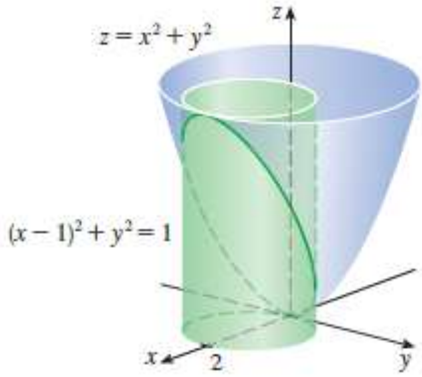
$$=\int_0^{\pi}\left(\int_1^2 r^2(\cos\varphi+\sin\varphi)dr\right)d\varphi=\frac{7}{3}\int_0^{\pi}(\cos\varphi+\sin\varphi)d\varphi=\frac{14}{3}$$ ◄

6.2.4-misol. $z = x^2 + y^2$ paraboloid, Oxy tekisligi va $x^2 + y^2 = 2x$ silindr bilan chegaralangan jism hajmini toping.

Yechilishi: ► (D) sohani chizmada tasvirlaymiz (6.17 va 6.18-rasmlar):



6.17-rasm. D sohaning tasviri



6.18-rasm. Berilgan jism tasviri

$x^2 + y^2 = 2x$ ni shakl almashtirib, aylana tenglamasiga keltiramiz:

$$x^2 - 2x + y^2 = 0 \quad \Rightarrow \quad (x-1)^2 + y^2 = 1.$$

$\iint\limits_{(D)} \left(x^2 + y^2\right) dxdy$ integralni qutb koordinatasiga o‘tib hisoblaymiz, bunda (6.3) kattaliklarni $x^2 + y^2 = 2x$ tenglamaga qo‘yamiz:

$$(r\cos\varphi)^2 + (r\sin\varphi)^2 = 2r\cos\varphi \quad \Rightarrow \quad r^2 = 2r\cos\varphi \quad \Rightarrow \quad r = 2\cos\varphi.$$

Shunda (D) soha quyidagidan iborat bo‘ladi:

$$D = \left\{(r,\varphi):\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}, \quad 0 \le r \le 2\cos\varphi\right\}.$$

Izlanayotgan hajm esa (6.6) formulaga ko‘ra quyidagiga teng:

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 r dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_0^{2\cos\varphi} d\varphi = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16\cos^4\varphi d\varphi = 8\int_0^{\frac{\pi}{2}} \cos^4\varphi d\varphi = \frac{3\pi}{2}.$$ ◄

Mavzu yuzasidan savollar:

1. Karrali integrallarda o‘zgaruvchini almashtirish formulasini ayting.
2. Yakobian nima?
3. Ikki o‘zgaruvchili funksiya uchun yakobian qanday topiladi?
4. Uch o‘zgaruvchili funksiya uchun yakobian qanday topiladi?
5. *n* o‘zgaruvchili funksiya uchun yakobian qanday topiladi?
6. Qutb koordinatasi deb nimaga aytiladi?
7. Qutb koordinatasiga o‘tish qachon maqbul hisoblanadi?
8. Qutb koordinatalar sistemasiga o‘tish formulasini ayting.

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Ikki karrali integrallarni hisoblang:

a) $\int_1^2 dx \int_0^1 (x \cdot y)\, dy$; **b)** $\int_{-2}^{1} \left(\int_x^{2-x^2} dy \right) dx$;

c) $\int_{-5}^{4} dx \int_0^2 (2x - 4y^3)\, dy$; **d)** $\int_0^1 \left(\int_0^{x^2} (x^2 + y^2)\, dy \right) dx$;

e) $\int_1^3 dx \int_{x^3}^{x} (x - y)\, dy$ **f)** $\int_0^1 \left(\int_0^{1-x} (1 - x - y)\, dy \right) dx$;

2. Ikki karrali integrallarning integrallash tartibini o‘zgartiring:

a) $\int_0^1 dy \int_y^{\sqrt{2-y^2}} f(x,y)dx.$ **b)** $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)dx.$

c) $\int_1^2 dx \int_x^{2x} f(x,y)dy.$ **d)** $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dx.$

e) $\int_0^1 dx \int_{2x^2}^{3-x} f(x,y)\, dy$. **f)** $\int_0^{3/2} dy \int_{2y^2}^{y+3} f(x,y)\, dx$.

3. $\iint\limits_D (x + y^2)dxdy$, D: $y = x$ va $y = x^2$ egri chiziqlar bilan chegeralangan soha.

4. $\iint\limits_D (2x - y)dxdy$, D: $y = 2x - 1$, $y = 2x - 3$, $y = 1 - x$, $y = 2 - x$ to‘g‘ri chiziqlar bilan chegaralangan soha.

5. $\iint_D ylnx\; dx\, dy$, bu yerda D soha $xy = 1, y = \sqrt{x}, x = 2$ chiziqlar bilan chegaralangan.

TESTLAR

1. $\int_1^3 dx \int_1^x (x - y)\, dy$ integralni hisoblang.

A) $\frac{4}{3}$ **B)** $\frac{5}{3}$ **C)** 1.5 **D)** $\frac{1}{22}$

2. $\iint\limits_{(D)} \frac{4}{x^2 + y^2} dxdy$ hisoblang, bunda (D): $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.

A) $V = 8\pi \ln 2$ **B)** $V = 8\pi \ln 5$

C) $V = 8\pi \ln 3$ **D)** $V = 4\pi \ln 2$

3. $\iint\limits_{(D)} arctg\frac{y}{x}dxdy$ hisoblang, bunda $(D)=\left\{\begin{matrix} x^2+y^2=1, x^2+y^2=9, \\ y=\frac{1}{\sqrt{3}}x, y=\sqrt{3}x \end{matrix}\right\}$

A) $\frac{1}{6}\pi^2$ **B)** $\frac{1}{4}\pi^2$ **C)** $\frac{1}{3}\pi^2$ **D)** $\frac{1}{5}\pi^2$

4. $\iint\limits_{(D)} \frac{y^2}{x}dxdy$ hisoblang, bunda $(D)=\left\{\begin{matrix} x^2=y, \quad x^2=3y, \\ y^2=x, \quad y^2=2x \end{matrix}\right\}$

A) 1/4 **B)** 1/2 **C)** 1 **D)** 2

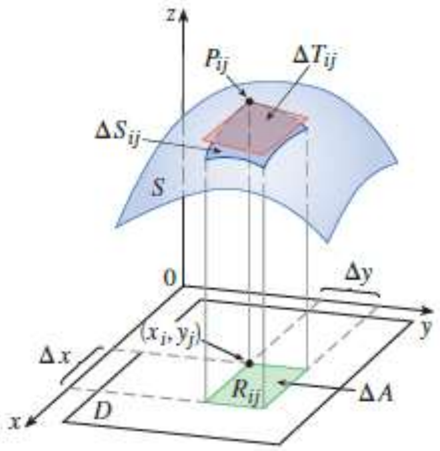
5. $r=4(1-\cos\varphi)$ va $r=4$ chiziqlar bilan chegaralanib, doira tashqarisida joylashgan soha yuzini toping.

A) $4\pi+32$ **B)** $16\pi+8$ **C)** $\pi+4$ **D)** $2\pi+16$

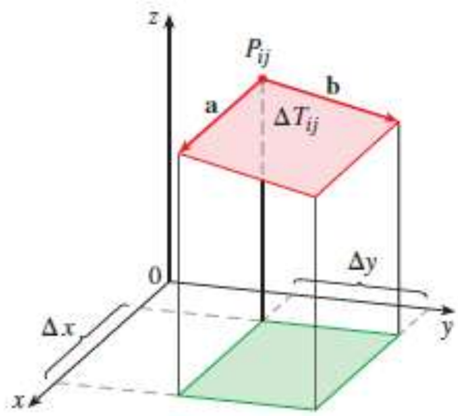
6.3-§. Ikki karrali integralning tatbiqlari

6.3.1. Ikki karrali integralning geometrik tatbiqlari

Oxy tekislikdagi proyeksiyasi R_{xy} bo‘lgan $f(x,y)\geq 0$ uzluksiz xususiy hosilalarga ega va uning grafigi fazoda $z=f(x,y)$ sirtni tasvirlasin (6.19-rasm). Bu sirt aylanma sirt hisoblanadi. Soddalik uchun sirt formulasini keltirib chiqarishda R sohani to‘g‘ri to‘rtburchak deb faraz qilamiz.



6.19-rasm. $z=f(x,y)$ sirt tasviri



6.20-rasm. ΔT_{ij} tekislik yuzasi

(*R*) sohani yuzasi $\Delta s=\Delta x\Delta y$ bo‘lgan R_{ij} yuzachalarga ajratamiz. Agar (x_i, y_j) nuqta R_{ij} bo‘lakka tegishli bo‘lsa, u holda $P_{ij}(x_i, y_j, f(x_i, y_j))$

nuqta S sirtning nuqtasi bo‘ladi. P_{ij} nuqtada S sirtga o‘tkazilgan urinma tekislik ΔT_{ij} yuzasi ΔS_{ij} yuzaga yaqinlashadi.

Umumiy sirt yuzasi quyidagiga teng bo‘ladi: $S = \lim\limits_{m,n\to\infty}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\Delta T_{ij}$. Ushbu formulani hisoblash uchun qulay ko‘rinishga keltirish uchun $\vec{a}$ va $\vec{b}$ vektorlarni kiritamiz (6.20-rasm). Bu vektorlar P_{ij} nuqtadan chiqib, ΔT_{ij} parallelogram tekisligining tomonlari bo‘ylab yo‘nalgan. Shunday qilib, ΔT_{ij} tekislikning yuzi $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko‘paytmasining son qiymatiga teng bo‘ladi:

$$\Delta T_{ij} = \left|\vec{a}\times\vec{b}\right|.$$

$$\vec{a} = \Delta x\cdot\vec{i} + f_x(x_i, y_j)\Delta x\cdot\vec{k};$$

$$\vec{b} = \Delta y\cdot\vec{j} + f_y(x_i, y_j)\Delta y\cdot\vec{k};$$

$$\vec{a}\times\vec{b} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \Delta x & 0 & f_x(x_i, y_j)\Delta x\\ 0 & \Delta y & f_x(x_i, y_j)\Delta y\end{vmatrix} = \left[-f_x(x_i, y_j)\vec{i} - f_y(x_i, y_j)\vec{j} + \vec{k}\right]\Delta x\Delta y.$$

Bundan $\Delta T_{ij} = \left|\vec{a}\times\vec{b}\right| = \sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\cdot\Delta x\Delta y$ ni topamiz.

Uni $S = \lim\limits_{m,n\to\infty}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\Delta T_{ij}$ limitga olib borib qo‘yamiz va quyidagini hosil qilamiz:

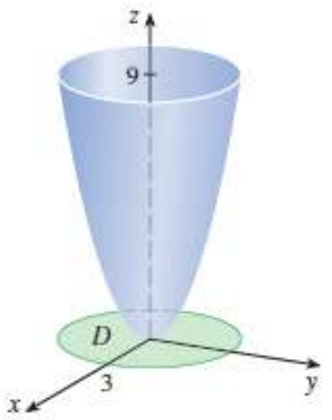
$$S = \iint\limits_{R}\sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\cdot dxdy.$$

Shunday qilib, $z = f(x, y)$ silliq sirt qismining Oxy tekislikdagi proyeksiyasi R_{xy} bo‘lsa, u holda bu sirt yuzi quyidagi formula bilan hisoblanadi:

$$S = \iint\limits_{R_{xy}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy. \qquad (6.7)$$

6.3.1.-misol. $z = x^2 + y^2$ paraboloidning $z = 9$ tekislik ostida yotgan qismining sirtini toping.

Yechilishi: ► 6.21-rasmda berilgan shaklni tasavvur qildik. Uning sirtini topish uchun (6.7) formuladan foydalanamiz.



6.21-rasm. $z = x^2 + y^2$ paraboloid

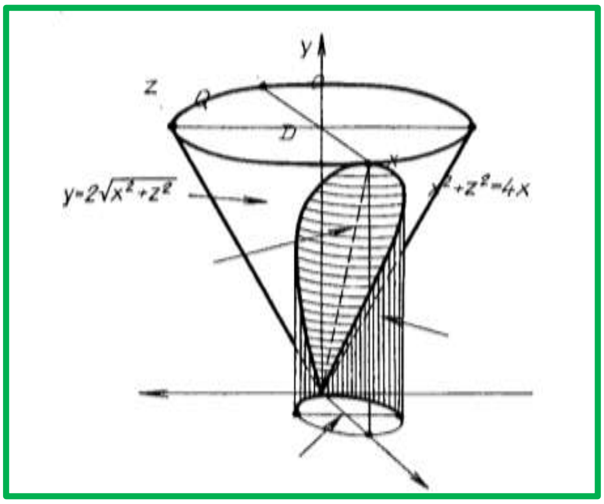
$$S = \iint_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy = \iint_{R_{xy}} \sqrt{1 + (2x)^2 + (2y)^2}\, dxdy = \iint_{R_{xy}} \sqrt{1 + 4(x^2 + y^2)}\, dxdy$$

Berilgan misolni qutb koordinatasiga o‘tib hisoblash qulay:

$$S = \int_0^{2\pi} d\varphi \int_0^3 \sqrt{1 + 4r^2}\, rdr = \frac{\pi}{6}(37\sqrt{37} - 1). \quad ◀$$

6.3.2-misol. $y = 2\sqrt{x^2 + z^2}$ konusning $x^2 + z^2 = 4x$ silindr ichidagi qismi yuzini hisoblang.

Yechilishi: ▶ Berilgan $y = 2\sqrt{x^2 + z^2}$ konus sirti qismining proyeksiyasi D_{xz} soha silindr asosi bo‘lib, $(x-2)^2 + z^2 = 4$ aylana cizig‘i bilan chegaralangan sohadir (6.22-rasm):



6.22-rasm. Konus va silindrik sirtlar kesishmasi

Yuqoridagi formulani $y = f(x, z)$ funksiya uchun qo‘llaymiz:

$$S=\iint_{D_{xz}}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dxdz. \qquad (6.8)$$

Buning uchun dastlab, xususiy hosilalarni topamiz:

$$\frac{\partial y}{\partial x}=\frac{2x}{\sqrt{x^2+z^2}};\quad \frac{\partial y}{\partial z}=\frac{2z}{\sqrt{x^2+z^2}},$$

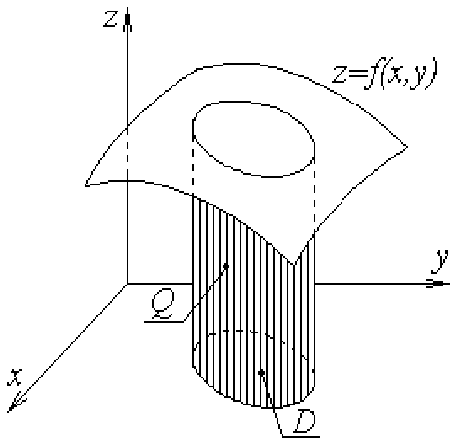
shunda izlangan yuza quyidagiga teng bo‘ladi:

$$S=\iint_{D_{xz}}\sqrt{1+\frac{4x^2}{x^2+y^2}+\frac{4z^2}{x^2+z^2}}\,dxdz=\sqrt{5}\iint_{D_{xz}}dxdz=\begin{vmatrix} z=r\cos\varphi & dxdz=rdrd\varphi \\ x=r\sin\varphi & r=4\sin\varphi \end{vmatrix}=$$

$$=\sqrt{5}\int_0^{\pi}d\varphi\int_0^{4\sin\varphi}rdr=8\sqrt{5}\int_0^{\pi}\sin^2\varphi d\varphi=4\sqrt{5}\int_0^{\pi}(1-\cos 2\varphi)d\varphi=4\pi\sqrt{5}. \blacktriangleleft$$

Agar D sohada $f(x,y)\geq 0$ bo‘lsa, u holda ikki karrali integral son jihatidan asosi D bo‘lgan yasovchilari Oz o‘qiga parallel bo‘lgan, yuqoridan $z=f(x,y)$ sirt bilan chegeralangan **silindrik jismning hajmiga** teng bo‘ladi (6.23-rasm):

$$V=\iint_D f(x,y)dxdy \qquad (6.9)$$



6.23-rasm. $z=f(x,y)$ sirt bilan chegeralangan silindrik jismning

6.3.3-misol. $z=0, z=x^2+y^2,\ y=x^2,\ y=1$ sirtlar bilan chegaralangan jism hajmini hisoblang.

Yechilishi: ► Berilgan jismni quyidagi ko‘rinishda tasvirlash kerak:

$$Q=\{(x,y,z):(x,y)\in D,\quad 0\leq z\leq x^2+y^2\},$$

bunda D — soha Oxy tekislikning $y=x^2$ va $y=1$ egri chiziqlari bilan chegaralangan qismi, ya’ni $D=\{(x,y):\ -1\leq x\leq 1,\ \ x^2\leq y\leq 1\}.$

Shunda jismning hajmi quyidagicha topiladi:

$$V=\iint_D\left(x^2+y^2\right)dx\,dy=\int_{-1}^{1}dx\int_{x^2}^{1}\left(x^2+y^2\right)dy=\int_{-1}^{1}\left(x^2\left(1-x^2\right)+\frac{1}{3}\left(1-x^6\right)\right)dx=\frac{88}{105}\text{ (kub birlik).}$$

◀

Xususan, $f(x,y)\equiv 1$ bo‘lganda, ikki karrali integral D sohaning $S(D)$ yuziga teng, ya’ni

$$S(D)=\iint_D dxdy. \tag{6.10}$$

Agar D sohani aniqlaydigan funksiyalar qutb koordinatalar sistemasida berilgan bo‘lsa, D sohaning $S(D)$ yuzi

$$S(D)=\iint_D r dr\, d\varphi \tag{6.11}$$

formula bilan hisoblanadi.

6.3.2. Ikki karrali integralning fizik tatbiqlari

Agar integral ostidagi funksiya biror S yuzali bir jinsli plastinaning $\rho=f(x,y)$ zichligi bo‘lsa, u holda ikki karrali integral **plastinkaning massasi** m ga teng bo‘ladi:

$$m=\iint_D \rho(x,y)dxdy. \tag{6.12}$$

Jismning Ox va Oy o‘qlariga nisbatan **statik momentlari** quyidagi formulalar bo‘yicha topiladi:

$$M_x=\int_D y\rho(x,y)dxdy,\qquad M_y=\int_D x\rho(x,y)dxdy. \tag{6.13}$$

Jismning **og‘irlik markazi** koordinatalari:

$$x_c=\frac{M_y}{m}=\frac{1}{m}\int_D x\rho(x,y)dxdy,\qquad y_c=\frac{M_x}{m}=\frac{1}{m}\int_D y\rho(x,y)dxdy. \tag{6.14}$$

D yassi jismning koordinata o‘qlariga va koordinata boshiga nisbatan **inersiya momentlari**:

$$\begin{gathered}I_x=\iint_D y^2\rho(x,y)dxdy,\\ I_y=\iint_D x^2\rho(x,y)dxdy,\\ I_0=I_x+I_y=\iint_D\left(x^2+y^2\right)\rho(x,y)dxdy\end{gathered} \tag{6.15}$$

formulalar bilan hisoblanadi.

6.3.4-misol. Sirt zichligi $\rho(x,y)=xy^2$ ga teng bo‘lgan, Ox o‘qi, $y=x^2$ parabola va $x+y=6$ to‘g‘ri chiziq bilan chegaralangan egri chiziqli uchburchakdan iborat D yupqa plastinkaning massasini hisoblang.

Yechilishi: ► Plastinkaning massasini hisoblash uchun dastlab D sohani aniqlaymiz: $D:\sqrt{y}\le x\le 6-y;\ 4\le y\le 9.$

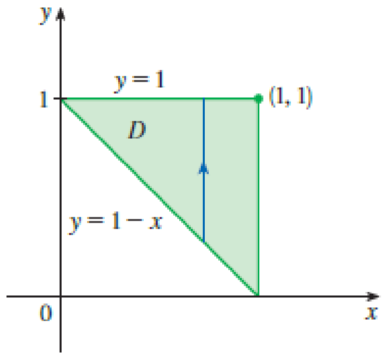
$$m=\int_4^9 dy\int_{\sqrt{y}}^{6-y} xy^2dx=\int_4^9 \frac{x^2y^2}{2}\Bigg|_{\sqrt{y}}^{6-y} dy=$$

$$=\int_4^9\left(\frac{(6-y)^2y^2}{2}-\frac{y^3}{2}\right)dy=\int_4^9\left(18y^2-\frac{13y^3}{2}-\frac{y^4}{2}\right)dy=453\frac{1}{8}\ ◄$$

Fiziklar boshqa turdagi zichlikni ham ko‘rib chiqishadi: agar elektr zaryad biror S yuzali bir jinsli plastina bo‘ylab bir xil taqsimlangan bo‘lsa, ya’ni $\sigma=f(x,y)$ bo‘lsa, u holda ikki karrali integral plastinkadagi umumiy elektr zaryad miqdorini Q ifodalaydi:

$$Q=\iint_D \sigma(x,y)dS. \qquad (6.16)$$

6.3.5-misol. Agar elektr zaryadi 6.24-rasmdagi S uchburchak yuza bo‘ylab $\sigma=xy$ tenglama bo‘yicha taqsimlangan bo‘lsa, umumiy zaryad miqdorini toping.



6.24-rasm. Elektr zaryadi taqsimlangan S uchburchak yuza

Yechilishi: ►

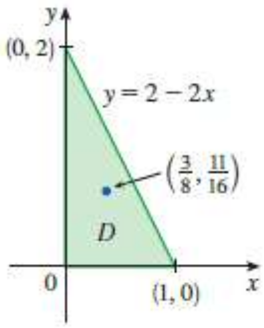
$$Q=\iint_D \sigma(x,y)dS=\int_0^1 dx\int_{1-x}^1 xydy=\int_0^1 x\left(\frac{y^2}{2}\right)\Bigg|_{1-x}^1 dx=\frac{1}{2}\int_0^1 (x-x(1-x)^2)dx=$$

$$=\frac{1}{2}\int_0^1(2x^2-x^3)=\frac{1}{2}\left(\frac{2x^3}{3}-\frac{x^4}{4}\right)\Bigg|_0^1=\frac{1}{2}\cdot\frac{8-3}{12}=\frac{5}{24}.\ ◄$$

6.3.6-misol. Uchlari (0,0), (0,1) va (0,2) nuqtalarda bo‘lgan plastinaning zichligi $\rho = 1 + 3x + y$ tenglama bilan berilgan bo‘lsa, uning massasini va og‘irlik markazini toping (6.25-rasm).

Yechilishi: ►

$$m = \iint_D \rho(x,y)dS = \int_0^1 \int_0^{2-2x} (1+3x+y)dydx = \int_0^1 \left(y + 3xy + \frac{y^2}{2} \right)\Bigg|_0^{2-2x} dx = 4\int_0^1 (1-x^2)dx = \frac{8}{3}.$$



6.25-rasm. Uchburchak shaklidagi plastina

Endi plastinaning og‘irlik markazini topamiz:

$$x_c = \frac{1}{m}\int_D x\rho(x,y)dxdy = \frac{3}{8}\int_0^1 xdx \int_0^{2-2x} (1+3x+y)dy = \frac{3}{8}\int_0^1 x\left(y + 3xy + \frac{y^2}{2} \right)\Bigg|_0^{2-2x} dx =$$

$$= \frac{3}{8}\cdot 4\int_0^1 x(1-x^2)dx = \frac{3}{2}\left(\frac{x^2}{2} - \frac{x^4}{4} \right)\Bigg|_0^1 = \frac{3}{8}.$$

$$y_c = \frac{1}{m}\int_D y\rho(x,y)dxdy = \frac{3}{8}\int_0^1 dx \int_0^{2-2x} y(1+3x+y)dy = \frac{11}{16}.$$

Demak, plastinkaning massa markazi $\left(\frac{3}{8}, \frac{11}{16}\right)$ nuqtada ekan. ◄

6.3.7-misol. $\rho(x,y) = 1$ zichlikka ega bo‘lgan, egri chiziqlar $xy = 1$, $xy = 2$, $y = 2x$, $x = 2y$ bilan chegaralangan va I chorakda joylashgan yassi jismning koordinata o‘qlariga nisbatan inersiya momentlarini toping.

Yechilishi: ► Berilgan D yassi jism 6.26-rasmda tasvirlangan.

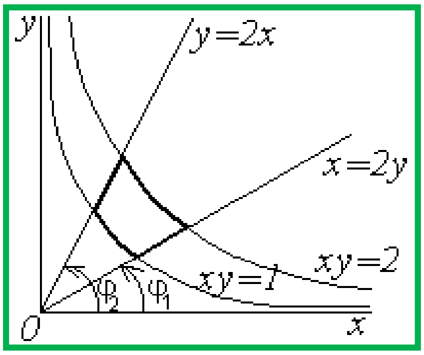
Koordinata o‘qlariga nisbatan inersiya momentlarini (6.15) formulalardan topamiz. Bu integrallarni qutb koordinatalariga o‘tib hisoblash qulay: $x = r\cos\varphi$, $y = r\sin\varphi$.

$xy = 1$ egri chiziqning qutb koordinatalaridagi tenglamasini keltirib chiqaramiz:

$$y=\frac{1}{x}\ \Rightarrow\ r\sin\varphi=\frac{1}{r\cos\varphi}\ \Rightarrow\ r^2=\frac{1}{\sin\varphi\cos\varphi}\ \Rightarrow\ r_1(\varphi)=1\big/\sqrt{\sin\varphi\cos\varphi},$$

$xy=2$ ning qutb koordinatalaridagi tenglamasi quyidagicha:

$$r_2(\varphi)=\sqrt{2}\big/\sqrt{\sin\varphi\cos\varphi}\,.$$



6.26-rasm. $xy=1,\quad xy=2,\quad y=2x,\quad x=2y$ chiziqlar bilan chegaralangan yassi jism

U holda φ burchak $\varphi_1=\operatorname{arctg}\frac{1}{2}$ dan $\varphi_2=\operatorname{arctg}2$ gacha o‘zgaradi, $[\varphi_1;\varphi_2]$ kesmadan olingan φ ning har bir qiymatida r o‘zgaruvchi $r_1(\varphi)$ dan $r_2(\varphi)$ gacha o‘zgaradi. Shunda Ox o‘qiga nisbatan inersiya momenti quyidagiga teng bo‘ladi:

$$I_x=\int_{\varphi_1}^{\varphi_2}d\varphi\int_{r_1(\varphi)}^{r_2(\varphi)}r^3\sin^2\varphi\,dr=\frac{1}{4}\int_{\varphi_1}^{\varphi_2}\sin^2\varphi\left(r_2^4(\varphi)-r_1^4(\varphi)\right)d\varphi=$$

$$=\frac{3}{4}\int_{\varphi_1}^{\varphi_2}\frac{d\varphi}{\cos^2\varphi}=\frac{3}{4}\operatorname{tg}\varphi\Big|_{\varphi_1}^{\varphi_2}=\frac{3}{4}(\operatorname{tg}\varphi_2-\operatorname{tg}\varphi_1)=\frac{3}{4}\left(2-\frac{1}{2}\right)=\frac{9}{8}.$$

Xuddi shuningdek, Oy o‘qiga nisbatan inersiya momentini topamiz:

$$I_y=\frac{9}{8}.$$ ◀

6.3.8-misol. $y^2=4x+4,\ y^2=-2x+4$ chiziqlar bilan chegaralangan sohaning o‘girlik markazini toping.

Yechilishi: ► Berilgan soha Ox o‘qiga simmetrik bo‘lganligi sababli, $y_c=0$ bo‘ladi. x_c ni topamiz. Buning uchun dastlab, berilgan soha yuzini hisoblaymiz (6.27-rasm).

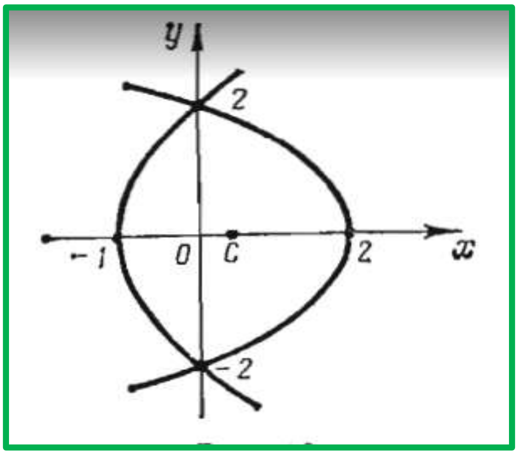
$$S=\iint_D dxdy=2\int_0^2 dy\int_{\frac{(y^2-4)}{4}}^{\frac{(4-y^2)}{2}}dx=2\int_0^2\left(\frac{4-y^2}{2}-\frac{y^2-4}{4}\right)dy=$$

$$=2\int_0^2\left(3-\frac{3y^2}{4}\right)dy=6\left[y-\frac{1}{12}y^3\right]_0^2=8.$$

U holda,

$$x_c=\frac{1}{8}\iint_D xdxdy=\frac{1}{8}2\int_0^2 dy\int_{\frac{(y^2-4)}{4}}^{\frac{(4-y^2)}{2}} xdx=\frac{1}{8}\int_0^2\left[\frac{1}{4}(4-y^2)^2-\frac{1}{16}(y^2-4)^2\right]dy=$$

$$=\frac{1}{8}\int_0^2\left(3-\frac{3}{2}y^2+\frac{3}{16}y^4\right)dy=\frac{1}{8}\left[3y-\frac{y^3}{2}+\frac{3y^5}{80}\right]_0^2=\frac{2}{5}.$$



6.27-rasm. $y^2=4x+4, y^2=-2x+4$ chiziqlar bilan chegaralangan soha ◄

6.3.9-misol. $\frac{x^2}{25}+\frac{y^2}{9}=1$ ellips va uning xordasi $\frac{x}{5}+\frac{y}{3}=1$ bilan chegaralangan sohaning og‘irlik markazini toping.

Yechilishi: ► Buning uchun dastlab yuzani hisoblaymiz:

$$S=\iint_D dxdy=\int_0^5 dx\int_{3\left(1-\frac{x}{5}\right)}^{\left(\frac{3}{5}\right)\sqrt{25-x^2}} dy=\int_0^5\left(\frac{3}{5}\sqrt{25-x^2}-3+\frac{3x}{5}\right)dx=\frac{15}{4}(\pi-2).$$

Endi og‘irlik markazi koordinatalarini hisoblaymiz:

$$x_c=\frac{1}{S}\iint_D xdxdy=\frac{4}{15(\pi-2)}\int_0^5 xdx\int_{3\left(1-\frac{x}{5}\right)}^{\left(\frac{3}{5}\right)\sqrt{25-x^2}} dy=\frac{4}{15(\pi-2)}\int_0^5\left[\frac{3}{5}x\sqrt{25-x^2}-3x\left(1-\frac{x}{5}\right)\right]dx=$$

$$=\frac{4}{15(\pi-2)}\left[-\frac{3}{5}\cdot\frac{1}{2}\cdot\frac{2}{3}(25-x^2)^{\frac{3}{2}}-\frac{3x^2}{2}+\frac{x^3}{5}\right]_0^5=\frac{4}{15(\pi-2)}\left(25-\frac{75}{2}+25\right)=\frac{10}{3(\pi-2)};$$

$$y_c=\frac{1}{S}\iint_D ydxdy=\frac{4}{15(\pi-2)}\int_0^5 dx\int_{3\left(1-\frac{x}{5}\right)}^{\left(\frac{3}{5}\right)\sqrt{25-x^2}} ydy=\frac{4}{15(\pi-2)}\cdot\frac{1}{2}\int_0^5\left[\frac{9}{25}(25-x^2)-9\left(1-\frac{x}{5}\right)^2\right]dx=$$

$$=\frac{2\cdot9\cdot2}{15(\pi-2)\cdot25}\int_0^5(5x-x^2)dx=\frac{12}{125(\pi-2)}\left[\frac{5x^2}{2}-\frac{1}{3}x^3\right]_0^5=\frac{12}{125(\pi-2)}\left(\frac{125}{2}-\frac{125}{3}\right)=\frac{2}{\pi-2}$$

◄

Mavzu yuzasidan savollar:

1. Ikki karrali integral yordamida soha yuza qanday hisoblanadi?
2. Ikki karrali integral yordamida jism hajmi qaysi formula bilan hisoblanadi?
3. Ikki karrali integral yordamida sirt yuzi qanday hisoblanadi?
4. Plastinka massasini hisoblash formulasini keltiring.
5. Yassi jismning og‘irlik markazi qanday hisoblanadi?
6. Yassi jismning koordinata o‘qlariga va koordinata boshiga nisbatan inersiya momentlari qanday aniqlanadi?
7. Plastinkadagi umumiy elektr zaryad miqdori ikki karrali integral yordamida qanday topiladi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Berilgan sohada jism hajmini hisoblang:

$$\iint_D y\left(1+x^2\right)dxdy,\quad D:\ y=x^3,\ \ y=3x.$$

2. Zichligi integral ostidagi funksiyaga teng bo‘lgan bir jinsli plastinaning massasini toping: $\iint_D \left(xy-4x^3y^3\right)dxdy,\ D: x=1,\ y=x^2,\ y=-\sqrt{x},$

3. Zichligi integral ostidagi funksiyaga teng bo‘lgan bir jinsli plastinadagi umumiy zaryad miqdorini toping:

$$\iint_D \sqrt{1-x^2-y^2}\,dxdy,\ D: x^2+y^2=4.$$

4. Zichligi integral ostidagi funksiyaga teng bo‘lgan bir jinsli plastinaning og‘irlik markazini toping: $\iint_D y\sin xy\,dxdy,\ \ D: y=\dfrac{\pi}{2},\ y=\pi,\ x=1,\ x=2,$

5. Zichligi integral ostidagi funksiya bilan berilgan plastinaning inersiya momentini toping: $\iint_D \dfrac{dxdy}{1+x^2+y^2},\ D: x^2+y^2=9$.

TESTLAR

1. $xy=1,\ xy=4,$ $x=y,\ x=9y$ chiziqlar bilan chegaralangan soha yuzasini hisoblang

A) 3 **B)** $3\ln 3$ **C)** $\ln 3$ **D)** $3\ln 2$

2. $y^2 = 1 - x,\ x = 0,\ y = 0$ chiziqlar bilan chegaralangan D moddiy yassi shaklning har bir nuqtasidagi sirt zichligi y ga teng bo‘lsa, Oy o‘qqa nisbatan inersiya momenti hisoblpansin.

A) $I_{Oy} = \frac{1}{24}$ **B)** $I_{Oy} = \frac{5}{24}$ **C)** $I_{Oy} = \frac{1}{12}$ **D)** $I_{Oy} = \frac{7}{24}$

3. $x^2 + y^2 = a^2$ silindrning $x^2 + z^2 = a^2$ silindr bilan kesishganidan hosil bo‘lgan qismining sirtini hisoblang.

A) $\sigma = 8a^2$ **B)** $\sigma = 2a^2$ **C)** $\sigma = 4a^2$ **D)** $\sigma = 6a^2$

4. $z = 4 - y^2,\ z = 0, y = \frac{x^2}{2}$ sirtlar bilan chegaralangan jismning hajmini hisoblang.

A) $\frac{255}{21}$ **B)** $\frac{254}{21}$ **C)** $\frac{253}{21}$ **D)** $\frac{256}{21}$

5. $x^2 + y^2 = a^2,\ \ y = 0$ chiziqlar bilan chegaralangan yuzaning og‘irlik markazi koordinatalarini toping.

A) $\left(0; \frac{3a}{4\pi}\right)$ **B)** $\left(0; \frac{4a}{3\pi}\right)$ **C)** $\left(0; \frac{4\pi}{3a}\right)$ **D)** $\left(0; \frac{3\pi}{4a}\right)$

6.4-§. Uch karrali integral

6.4.1. Dekart koordinatalarida uch o‘lchovli integrallarni hisoblash

Uch o‘lchovli integral ham ikki o‘lchovli integralga o‘xshash aniqlanadi. Fazoning biror (ω) sohasida va shu sohaning σ chegarasida aniqlangan uchta o‘zgaruvchining uzluksiz funksiyasi $u = f(x\,;\,y\,;\,z)$ ni qaraymiz. Quyidagi amallarni bajaramiz:

1) (ω) sohani har xil sirtlar (xususiy holda bu sirtlar koordinata tekisliklariga parallel bo‘lishi mumkin) bilan n ta ixtiyoriy jismga bo‘lamiz: $\Delta\omega_1,\ \Delta\omega_2,\ \Delta\omega_3, ..., \Delta\omega_i\ ,..., \Delta\omega_n$.
Bu qismlarni elementar hajmlar deb ataymiz va tegishli jismlarning hajmlarini ham xuddi shunday belgilaymiz.

2) Har bir elementar hajmdan bittadan $P_i(x_i; y_i; z_i)$ nuqta olamiz, natijada ***n*** nuqtaga ega bo‘lamiz:

$$P_1(x_1; y_1; z_1),\quad P_2(x_2; y_2; z_2),\quad ...,\quad P_i(x_i; y_i; z_i),\quad ...,\quad P_n(x_n; y_n; z_n)$$

3) Tanlab olingan $P_i(x_i;y_i;z_i)$ nuqtalarda $u=f(P)=f(x;y;z)$ funksiyaning qiymatlarini hisoblab quyidagiga kelamiz:

$f(P_1)=f(x_1;y_1;z_1)$, $f(P_2)=f(x_2;y_2;z_2)$,..., $f(P_i)=f(x_i;y_i;z_i)$,..., $f(P_n)=f(x_n;y_n;z_n)$.

4) Ushbu ko‘rinishdagi ko‘paytmani tuzamiz:

$$f(P_i)\cdot\Delta\omega_i=f(x_i;y_i;z_i)\cdot\Delta\omega_i$$

5) Bu ko‘paytmalardan tuzilgan yig‘indi:

$$\sum_{i=1}^{n} f(P_i)\cdot\Delta\omega_i=\sum_{i=1}^{n} f(x_i;y_i;z_i)\cdot\Delta\omega_i \qquad (6.17)$$

6.3-teorema. Agar $u=f(x,y,z)$ funksiya yopiq chegaralangan (ω) sohada uzluksiz bo‘lsa, u holda sohani $\Delta\omega_i$ qismlarga bo‘lish sonining ortishi bilan $(n\to\infty)$ elementar hajmlar diametrining eng kattasi nolga intilganda (6.17) ko‘rinishdagi integral yig‘indining limiti mavjud bo‘ladi. Bu limit (ω) sohani $\Delta\omega_i$ qismlarga bo‘lish usuliga va ulardan olingan $P_i(x_i;y_i;z_i)$ nuqtalarni tanlash usuliga bog‘liq emas.

Bu limit $u=f(x,y,z)$ funksiyadan soha (ω) bo‘yicha olingan **uch o‘lchovli integral** deyiladi va quyidagicha belgilanadi:

$$\lim_{\max d\Delta\omega_i\to 0}\sum_{i=1}^{\infty} f(x_i,y_i,z_i)\cdot\Delta\omega_i=\iiint_{(\omega)} f(x,y,z)\,d\omega \qquad (6.18)$$

bu yerda (ω) - integrallash sohasi, $f(x;y;z)$ - integral ostidagi funksiya, x,y,z - lar integrallash o‘zgaruvchilari, $d\omega$ **hajm elementi** deyiladi.

Uch o‘lchovli integral sohani qismlarga bo‘lish usuliga bog‘liq bo‘lmaganligi uchun uni koordinatalar tekisliklariga parallel tekisliklar bilan tomonlari Δx_i, Δy_i, Δz_i ga teng bo‘lgan to‘g‘ri burchakli prizmalarga bo‘lish mumkin, bunda $\Delta\omega_i=\Delta x_i\cdot\Delta y_i\cdot\Delta z_i$.

Uch o‘lchovli integral ta’rifiga ko‘ra:

$$\lim_{\max d\Delta\omega_i\to 0}\sum_{i=1}^{\infty} f(x_i,y_i,z_i)\cdot\Delta\omega_i=\iiint_{(\omega)} f(x,y,z)\,d\omega=\iiint_{(\omega)} f(x,y,z)\,dxdydz \qquad (6.19)$$

Bu yerda $dx\,dy\,dz$ - dekart koordinata sistemasidagi hajm elementi deyiladi.

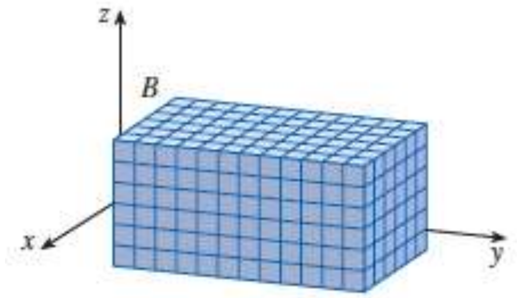
Uch karrali integrallarni hisoblash formulalari integrallash sohasining berilishiga qarab, turlicha bo‘ladi.

a) Aytaylik, $f(x,y,z)$ funksiya R^3 fazodagi

$$V=\{(x,y,z)\in R^3 : a\le x\le b, c\le y\le d, p\le z\le q\}$$

to‘plam(parallelepiped)da uzluksiz bo‘lsin (6.28-rasm). U holda quyidagi o‘rinli:

$$\iiint_V f(x,y,z)dxdydz = \int_a^b [\int_c^d (\int_p^q f(x,y,z)\,dz)dy]dx\,. \qquad (6.20)$$

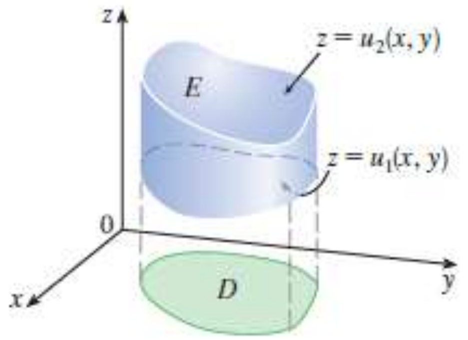


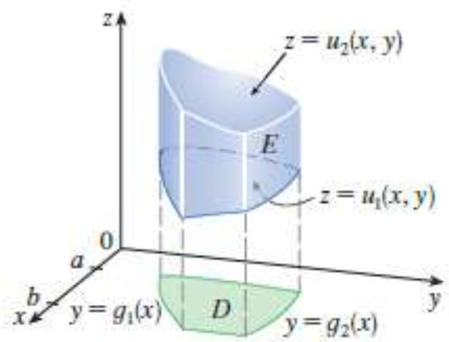
6.28-rasm. To‘g‘ri parallelepiped shaklidagi integrallash sohasi

b) Aytaylik, R^3 fazodagi to‘plam V – quyidan $z = u_1(x,y)$, yuqoridan $z = u_2(x,y)$ sirt (bunda $D \subset R^2$ to‘plam V jismning Oxy tekisligidagi proyeksiyasi) bilan chegaralangan to‘plam bo‘lsin (6.29, a va b-rasmlar). Agar bu V da $f(x,y,z)$ uzluksiz, $u_1(x,y)$ va $u_2(x,y)$ funksiyalar D da uzluksiz bo‘lsa, u holda

$$\iiint_V f(x,y,z)dxdydz = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dxdy \qquad (6.21)$$

o‘rinli bo‘ladi.





a) b)

6.29-rasm. Integrallash D sohalari

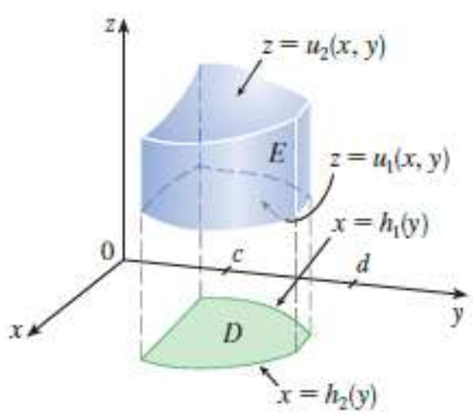
v) Agar b) holdagi D to‘plam quyidagicha (6.30-rasm).

$$D = \{(x,y) \in R^2 : c \le y \le d,\;\; h_1(x) \le x \le h_2(x)\}$$

bo‘lib, h_1 va h_2 funksiyalar $[a,b]$ da uzluksiz bo‘lsa, u holda

$$\iiint_V f(x,y,z)dxdydz = \int_c^d [\int_{h_1(x)}^{h_2(x)} (\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz)dx]dy \qquad (6.22)$$

tenglik o‘rinli bo‘ladi.



6.30-rasm. Integrallash D sohasi

6.4.1-misol. $\int_0^3\int_{-1}^2\int_0^1 xyz^2dxdydz$ integralni hisoblang.

Yechilishi: ▶

$$\int_0^3\int_{-1}^2\int_0^1 xyz^2dxdydz = \int_0^3 z^2\int_{-1}^2 y\left(\frac{x^2}{2}\right)\Bigg|_0^1 dy = \frac{1}{2}\int_0^3 z^2\int_{-1}^2 ydydz = \frac{1}{2}\int_0^3 z^2\left(\frac{y^2}{2}\right)\Bigg|_{-1}^2 dz = \frac{3}{4}\cdot\left(\frac{z^3}{3}\right)\Bigg|_0^3 = \frac{27}{4}. ◀$$

6.4.2-misol. $J = \iiint\limits_V (x+y+z)dxdydz$ integralni hisoblang, bunda

$$V = \{(x,y,z)\in R^3 : 0\le x\le 1, 0\le y\le 3, 0\le z\le 2\}.$$

Yechilishi: ▶ Mos formulani tanlab olamiz va quyidagini hisoblaymiz:

$$\int_0^1[\int_0^3(\int_0^2(x+y+z)dz)dy]dx = \int_0^1[\int_0^3(xz+yz+\frac{z^2}{2})\Big|_{z=0}^{z=2}dy]dx = \int_0^1[\int_0^3 2(x+y+1)dy]dx =$$

$$= \int_0^1 2(xy+\frac{y^2}{2}+y)\Big|_{y=0}^{y=3}dx = \int_0^1(6x+15)dx = 18. ◀$$

6.4.3-misol. $\iiint\limits_{(V)} z^2dxdydz$ integralni hisoblang, bunda V –quyidagi $z=\sqrt{x^2+y^2}$ konus va $z=h$ tekisliklar bilan chegaralangan to‘plam.

Yechilishi: ▶ V ning Oxy tekislikdagi proyeksiyasi

$$D = \{(x,y)\in R^2 : x^2+y^2\le h^2\}$$

bo‘ladi. Yuqoridagi mos formuladan foydalanib topamiz:

$$J = \iint\limits_D\left(\int_{\sqrt{x^2+y^2}}^h z^2dz\right)dxdy = \iint\limits_D\left[\frac{h^3}{3}-\frac{1}{3}\left(x^2+y^2\right)^{\frac{3}{2}}\right]dxdy.$$

Bu integralda (6.3) qutb koordinatasiga o‘tamiz va hisoblaymiz:

$$J = \int_0^{2\pi}\left[\int_0^h\left(\frac{h^3}{3}-\frac{1}{3}r^3\right)rdr\right]d\varphi = \frac{1}{5}\pi h^5. ◀$$

6.4.2. Uch karrali integralda o‘zgaruvchilarni almashtirish. Silindrik va sferik koordinatalar sistemasida uch karrali integral

$f(x, y, z)$ funksiya $V \subset R^3$ to‘plamda berilgan va uzluksiz bo‘lsin. Bu funksiyaning argumentlari ham biror u, v, w o‘zgaruvchilarning funksiyasi bo‘lsin: $\begin{cases} x = \varphi(u, v, w), \\ y = \psi(u, v, w), \\ z = \chi(u, v, w) \end{cases}$

x, y, z larni mos ravishda u, v, w o‘zgaruvchilar bilan almashtiramiz:

$$\iiint\limits_V f(x, y, z)dzdydz = \iiint\limits_\Delta f(\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w))|J|dudvdw$$

Bunda x, y, z o‘zgaruvchilarni u, v, w ga o‘tishdagi yakobianni topamiz:

$$J = \begin{vmatrix} \dfrac{dx}{du} & \dfrac{dx}{dv} & \dfrac{dx}{dw} \\ \dfrac{dy}{du} & \dfrac{dy}{dv} & \dfrac{dy}{dw} \\ \dfrac{dz}{du} & \dfrac{dz}{dv} & \dfrac{dz}{dw} \end{vmatrix}$$

Ko‘p hollarda uch karrali integrallar dekart koordinatalaridan silindrik yoki sferik koordinatalarga o‘tish orqali hisoblanadi.

a) Dekart koordinatalari x, y, z dan silindrik koordinatalar r, θ, z ga o‘tish quyidagi formulalar yordamida amalga oshiriladi (6.31, a va b-rasm):

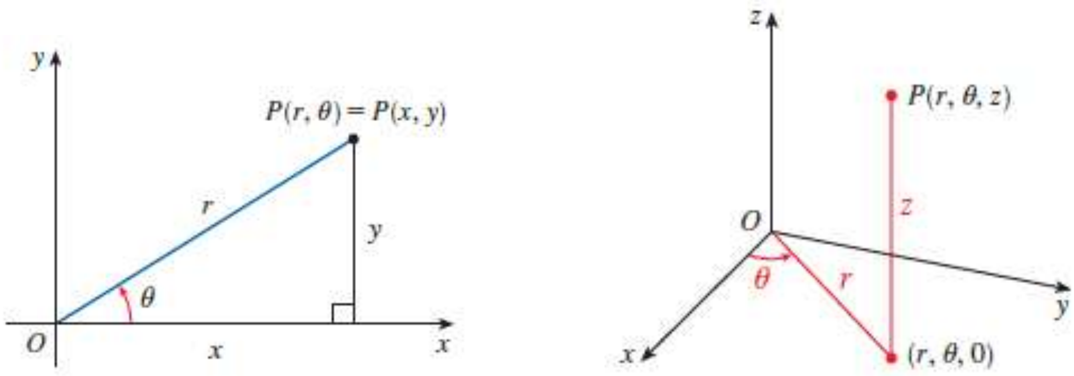
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \quad (0 \le r \le +\infty, \ 0 \le \theta \le 2\pi, \ -\infty < z < +\infty)$$

Bu almashtirishning yakobianini topamiz:

$$J = \begin{vmatrix} \dfrac{dx}{dr} & \dfrac{dx}{d\theta} & \dfrac{dx}{dz} \\ \dfrac{dy}{dr} & \dfrac{dy}{d\theta} & \dfrac{dy}{dz} \\ \dfrac{dz}{dr} & \dfrac{dz}{d\theta} & \dfrac{dz}{dz} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Shunda silindrik koordinatalar sistemasida uch karrali integral quyidagiga teng bo‘ladi:

$$\iiint\limits_V f(x, y, z)dxdydz = \iiint\limits_\Delta f(r\cos\theta, r\sin\theta, z)rdr\,d\theta\,dz \qquad (6.23)$$



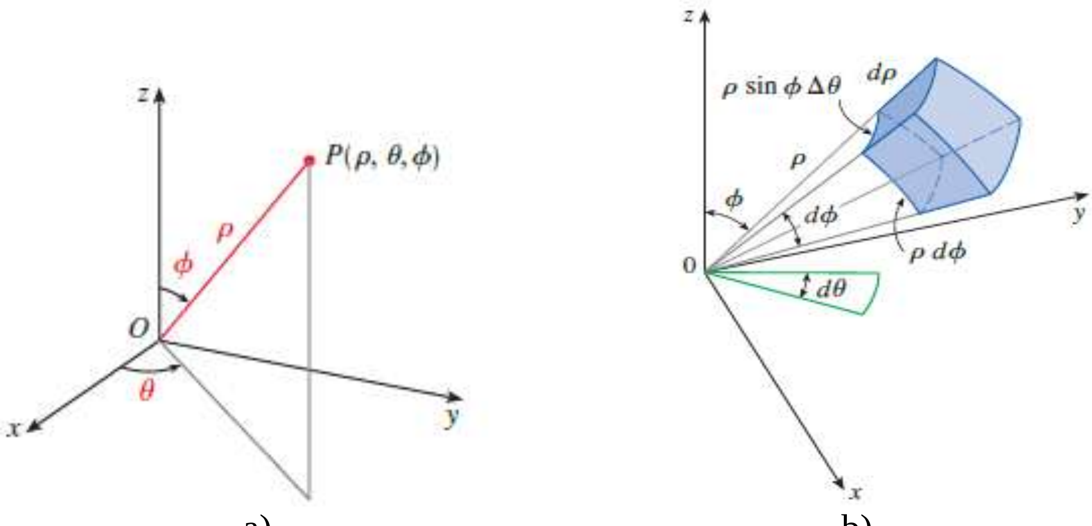
a) b)

6.31-rasm. Silindrik koordinata sistemasi

b) $x,\ y,\ z$ dekart koordinatalaridan $\rho,\ \theta, \phi$ sferik koordinatalariga o‘tish (6.32, a va b-rasm):

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}, \qquad \left(0 \le \rho \le +\infty,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\right)$$

formulalar yordamida amalga oshiriladi.



a) b)

6.32-rasm. Sferik koordinata sistemasi

Almashtirish yakobiani $J = \rho^2 \sin\phi$ bo‘lib, sferik koordinatalar sistemasida uch karrali integral quyidagiga teng bo‘ladi:

$$\iiint\limits_V f(x,\ y,\ z)dxdydz = \iiint\limits_V f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

Sferik koordinatalardan dekart koordinatalariga o‘tish ham mumkin:

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \varphi = arctg\, \dfrac{y}{x} \\ \theta = \arccos \dfrac{z}{r} \end{cases}$$

d) x, y, z dekart koordinatalaridan ρ, θ, ϕ umumlashgan sferik koordinatalariga o‘tish mumkin. Bu

$$\begin{cases} x = a\rho \sin\phi \cos\theta \\ y = b\rho \sin\phi \sin\theta \\ z = c\rho \cos\phi \end{cases}, \qquad (0 \le \rho \le +\infty,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi)$$

formulalar yordamida amalga oshiriladi. Almashtirish yakobiani $J = abc\rho^2 \sin\phi$ bo‘lib, sferik koordinatalar sistemasida uch karrali integral quyidagiga teng bo‘ladi:

$$\iiint\limits_V f(x, y, z) dxdydz =$$

$$= \iiint\limits_V f(a\rho \sin\phi \cos\theta, b\rho \sin\phi \sin\theta, c\rho \cos\phi) abc\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi. \tag{6.24}$$

6.4.4-misol. $\iiint\limits_V zdzdydz$ integralni hisoblang, bunda

$V: \ \dfrac{x^2 + y^2}{R^2} = \dfrac{z^2}{h^2} \qquad (h > 0)$ konusning yuqori qismi va $z = h$ tekislik bilan chegaralangan to‘plam.

Yechilishi: ► Berilgan integralda silindrik koordinataga o‘tamiz.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad (0 \le r \le R,\ \ 0 \le \theta \le 2\pi)$$

$$\frac{x^2 + y^2}{R^2} = \frac{z^2}{h^2} \quad \Rightarrow \frac{r^2}{R^2} = \frac{z^2}{h^2} \quad \Rightarrow \quad z = \pm \frac{h}{R} r,$$

Natijada $\iiint\limits_V zdxdydz = \int\limits_0^R \left[\int\limits_0^{2\pi} \left(\int\limits_{\frac{h}{r}p}^{h} rzdz \right) d\theta \right] dr$ bo‘ladi. Uni hisoblaymiz:

$$\int\limits_0^R \left[\int\limits_0^{2\pi} \left(r\frac{z^2}{2} \right)_{z = \frac{h}{R}r}^{z = h} d\theta \right] dr = \int\limits_0^R rdr \int\limits_0^{2\pi} \left(\frac{h^2}{2} - \frac{h^2}{2R^2} r^2 \right) d\theta = \frac{h^2}{2} \int\limits_0^R rdr \int\limits_0^{2\pi} \left(1 - \frac{r^2}{R^2} \right) d\theta =$$

$$=\frac{h^2}{2}\int_0^R r\left(\theta-\frac{r^2}{R^2}\theta\right)\Bigg|_0^{2\pi} dr=\frac{2\pi h^2}{2}\int_0^R\left(r-\frac{r^3}{R^2}\right)dr=\pi h^2\left(\frac{r^2}{2}-\frac{r^4}{4R^2}\right)\Bigg|_0^R=\frac{\pi h^2R^2}{4}. \blacktriangleleft$$

6.4.5-misol. $J=\iiint\limits_V\left(x^2+y^2+z^2\right)dxdydz$ integralni hisoblang, bunda V – to‘plam $x^2+y^2+z^2\le r^2$ shardan iborat.

Yechilishi: ▶ Berilgan integralda sferik koordinataga o‘tamiz.

$$\begin{cases} x=\rho\sin\phi\cos\theta \\ y=\rho\sin\phi\sin\theta \\ z=\rho\cos\phi \end{cases} \qquad \left(0\le\rho\le+\infty,\ 0\le\theta\le2\pi,\ 0\le\phi\le\pi\right),\ J=\rho^2\sin\phi.$$

Natijada berilgan integral quyidagi ko‘rinishga keladi:

$$J=\iiint\limits_V\left(x^2+y^2+z^2\right)dxdydz=\int_0^r\left[\int_0^\pi\left(\int_0^{2\pi}\rho^2\rho^2\sin\phi d\theta\right)d\phi\right]d\rho$$

Oxirgi integralni hisoblaymiz:

$$\int_0^r\left[\int_0^\pi\left(\int_0^{2\pi}p^4\sin\phi d\theta\right)d\phi\right]d\rho=\int_0^r\left[\int_0^\pi\left(p^4\sin\phi\cdot2\pi\right)d\phi\right]d\rho=4\pi\int_0^r\rho^4d\rho=\frac{4\pi r^5}{5}. \blacktriangleleft$$

6.4.3. Uch karrali integralning tatbiqlari

Agar (ω) sohada $f(x,y,z)\equiv1$ bo‘lsa, u holda uch o‘lchovli integralning qiymati (ω) sohaning V **hajmiga teng bo‘ladi** (uch o‘lchovli integralning geometrik ma’nosi).

$$V=\iiint\limits_{(\omega)}d\omega=\iiint\limits_{(\omega)}dxdydz.$$

Agar $f(P)=f(x,y,z)$ funksiya (ω) sohada massa taqsimlanishining zichligi bo‘lsa, u holda uch o‘lchovli integralning qiymati V hajmdagi **modda massasini** beradi (uch o‘lchovli integralning fizik ma’nosi).

$$m=\iiint\limits_{(\omega)}f(P)d\omega=\iiint\limits_{(\omega)}f(x,y,z)dxdydz.$$

Uch o‘lchovli integral yordamida, shuningdek, quyidagilarni hisoblash mumkin:

a) Jismning Oxy,Oxz va Oyz koordinata tekisliklariga nisbatan **statik momentlari:**

$$M_{xy}=\iiint\limits_G z\rho(x,y,z)dxdydz,$$

$$M_{xz}=\iiint\limits_G y\rho(x,y,z)dxdydz,$$

$$M_{yz}=\iiint\limits_G x\rho(x,y,z)dxdydz,$$

bunda $\rho(x,y,z)$ — moddaning solishtirma zichligi;

b) Jismning **og‘irlik markazi** koordinatalarini topish formulalari:

$$x_c=\frac{M_{yz}}{m}=\frac{\iiint\limits_G x\rho(x,y,z)dxdydz}{m};$$

$$y_c=\frac{M_{xz}}{m}=\frac{\iiint\limits_G y\rho(x,y,z)dxdydz}{m};$$

$$z_c=\frac{M_{xy}}{m}=\frac{\iiint\limits_G z\rho(x,y,z)dxdydz,}{m}$$

bunda m — jism massasi;

v) Jismning koordinata tekisliklari, koordinata o‘qlari va koordinata boshiga nisbatan **inersiya momentlari**:

$$I_{xy}=\iiint\limits_G z^2\rho(x,y,z)dxdydz,$$

$$I_{xz}=\iiint\limits_G y^2\rho(x,y,z)dxdydz,$$

$$I_{yz}=\iiint\limits_G x^2\rho(x,y,z)dxdydz,$$

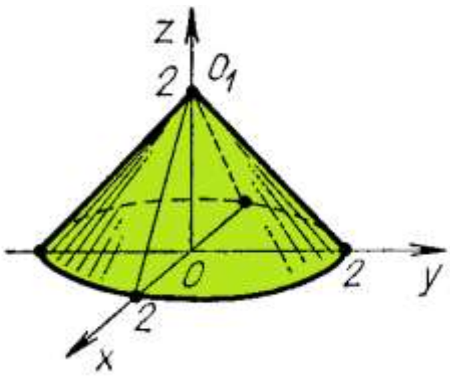
$$I_x=I_{xy}+I_{xz},\quad I_y=I_{xy}+I_{yz},\quad I_z=I_{xz}+I_{xz},\quad I_0=I_{xy}+I_{xz}+I_{yz}.$$

6.4.6-misol. Zichligi $\delta(x,y,z)=z$ bo‘lgan $(z-2)^2=x^2+y^2$ konus sirt va $z=0$ tekislik bilan chegaralangan jism massasini toping.

Yechilishi: ►Konusning uchi $O_1(0,0,2)$ nuqtada, konusni $z=0$ tekislik bilan kesimida

$$(z-2)^2=x^2+y^2 \quad\Rightarrow\quad (0-2)^2=x^2+y^2 \quad\Rightarrow\quad x^2+y^2=4$$

aylana hosil bo‘ladi (6.33-rasm). Qaralayotgan jismning sirt tenglamasi quyidagiga teng: $z=2-\sqrt{x^2+y^2}$.



6.33-rasm. Konus sirt bilan chegaralangan jism

Endi silindrik koordinataga o‘tib hisoblaymiz, chunki aniqlanish sohasining proyeksiyasi doiradan iborat.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{bunda} \quad r^2 = x^2 + y^2, \quad \begin{matrix} 0 \le r \le 2, \\ 0 \le \theta \le 2\pi, \\ 0 < z < 2 - r \end{matrix}$$

Bularni formulaga qo‘yamiz va jism massasini hisoblaymiz:

$$m = \iiint\limits_{(\omega)} \delta(x,y,z)dxdydz = \iiint\limits_{(\omega)} zdxdydz = \int_0^{2\pi} d\theta \int_0^2 rdr \int_0^{2-r} zdz =$$

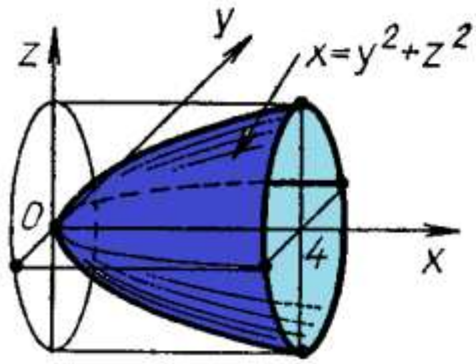
$$= \int_0^{2\pi} d\theta \int_0^2 r\left(\frac{z^2}{2}\right)_0^{2-r} dr = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^2 \left(4r - 4r^2 + r^3\right)dr = \frac{1}{2}\int_0^{2\pi}\left(2r^2 - \frac{4r^3}{3} + \frac{r^4}{4}\right)_0^2 d\theta =$$

$$= \frac{1}{2}\int_0^{2\pi}\left(8 - \frac{24}{3} + 4\right)d\theta = 2\int_0^{2\pi} d\theta = 4\pi.$$

Demak, jismning massasi 4π ga teng. ◄

6.4.7-misol. $x = y^2 + z^2$ va $x = 4$ sirtlar bilan chegaralangan jismning og‘irlik markazini toping.

Yechilishi: ► Jismni chizmada tasavvur qilamiz (6.34-rasm).



6.34-rasm. $x = y^2 + z^2$ va $x = 4$ sirtlar bilan chegaralangan jism

Jism zichligi $\delta(x,y,z)=1$ ga teng. Jismning Oyz tekislikka proyeksiyasi markazi koordinata boshida, radiusi 2 ga teng bo‘lgan doiradan iborat. Silindrik koordinataga o‘tamiz:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{bunda} \quad r^2 = x^2 + y^2, \quad \begin{matrix} 0 \le r \le 2, \\ 0 \le \theta \le 2\pi, \\ r^2 < x < 4 \end{matrix}$$

topilgan kattaliklarni formulaga qo‘yamiz va jism massasini hisoblaymiz:

$$m = \iiint\limits_{(\omega)} dxdydz = \int_0^{2\pi} d\theta \int_0^2 rdr \int_{r^2}^4 dx = \int_0^{2\pi} d\theta \int_0^2 (4r - r^3)dr = 8\pi.$$

Endi og‘irlik markazinining x koordinatasini topamiz:

$$x_c = \frac{\iiint\limits_G x\rho(x,y,z)dxdydz}{m} = \frac{1}{8\pi}\iiint\limits_G xdxdydz = \frac{1}{8\pi}\int_0^{2\pi} d\theta \int_0^2 rdr \int_{r^2}^4 xdx = \frac{16}{5};$$

$$y_c == \frac{\iiint\limits_G ydxdydz}{m} = 0; \qquad z_c = \frac{\iiint\limits_G zdxdydz,}{m} = 0.$$

Demak, jismning og‘irlik markazi $\left(\frac{16}{5},0,0\right)$ koordinatali nuqtada ekan.◄

Mavzu yuzasidan savollar:

1. Uch o‘zgaruvchili funksiyaning integral yig‘indisi qanday ko‘rinishda bo‘ladi?
2. Uch o‘lchovli integralning mavjudligi haqidagi teoremani ayting.
3. Uch o‘lchovli integralning geometrik ma’nosi nima?
4. Uch o‘lchovli integralning mexanik ma’nosini ayting
5. Uch o‘lchovli integralning xossalarini ayting.
6. O‘rta qiymat haqidagi teoremani ayting.
7. Integralning chegaralanganligi haqidagi teoremani ayting.
8. Uch o‘lchovli integral qanday hisoblanadi?
9. Uch karrali integral nima?
10. Uch karrali integralda Dekart koordinatalaridan silindrik koordinatasiga o‘tish formulasini ayting.
11. Uch karrali integralda Dekart koordinatalaridan sferik koordinatasiga o‘tish formulasini ayting.
12. Uch karrali integralda Dekart koordinatalaridan silindrik koordinatasiga o‘tishda almashtirish yakobiani nimaga teng?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Uch karrali integrallarni hisoblang.

a) $\int_0^1 dx\int_{-2}^0 dy\int_0^3 dz$; **b)** $\int_0^1 xdx\int_0^2 ydy\int_0^3 zdz$;

c) $\int_0^1 xdx\int_0^x ydy\int_0^{xy} zdz$; **d)** $\int_0^1 dx\int_0^x xydy\int_0^{xy} xyz^2dz$;

e) $\int_1^3 zdz\int_0^2 xdx\int_2^5 ydy$; **f)** $\int_0^1 (1+x)dx\int_0^x xydy\int_0^{xy} xyz^2dz$;

2. Hisoblang: $\iiint\limits_V xy^2zdxdydz,\quad V: -2\le x\le 1,\ \ 0\le y\le 2,\ \ 0\le z\le 3.$

3. Hisoblang: $\iiint\limits_V \left(x^2+y^2+z^2\right)dxdydz, V: x^2+y^2+z^2=4,\ \ x\ge 0,\ y\ge 0,\ z\ge 0.$

4. Hisoblang: $\iiint\limits_V 21xzdxdydz,\ \ V: y=x,\ y=0,\ x=2,\ z=xy,\ z=0.$

5. Hisoblang: $\iiint\limits_V \left(xy-z^2\right)dxdydz,\ \ V: 0\le x\le 1, -1\le y\le 2,\ 0\le z\le 3.$

TESTLAR

1. Uch karrali integral yordamida $x=0,\ z=0,\ y=1,$ $y=3,\ x+2z=3$ tekisliklar bilan chegaralangan jism hajmini toping.

A) 4.5 **B)** 3.5 **C)** 4 **D)** 0.5

2. Hisoblang: $\iiint\limits_V xyzdxdydz$, agar $V:\begin{cases} x=0, y=0, \ \ z=0,\\ x^2+y^2+z^2=1\end{cases}$

A) 1/18 **B)** 1/48 **C)** 1/81 **D)** 1/84

3. Uch karrali integral yordamida $(z-2)^2=x^2+y^2$ va $z=0$ sirtlar bilan chegaralangan, zichligi $\gamma(x,y,z)=z$ bo‘lgan jism massasini toping

A) $\frac{3\pi}{8}$ **B)** $\frac{8\pi}{3}$ **C)** $\frac{\pi}{8}$ **D)** $\frac{4\pi}{3}$

4. Uch karrali integral yordamida $z=1$ va $z=5-x^2-y^2$ sirtlar bilan chegaralangan jism hajmini toping.

A) $8+\pi$ **B)** 8π **C)** π **D)** 8

5. Agar V: $x^2+y^2+z^2\le R^2$ shar bo‘lsa, $\iiint\limits_V x^2dxdydz,$ hisoblang.

A) $\frac{4\pi R^5}{15}$ **B)** $\frac{4\pi R^5}{5}$ **C)** $\frac{\pi R^5}{15}$ **D)** $\frac{4R^5}{15}$.

VII BOB. EGRI CHIZIQLI VA SIRT INTEGRALLARI

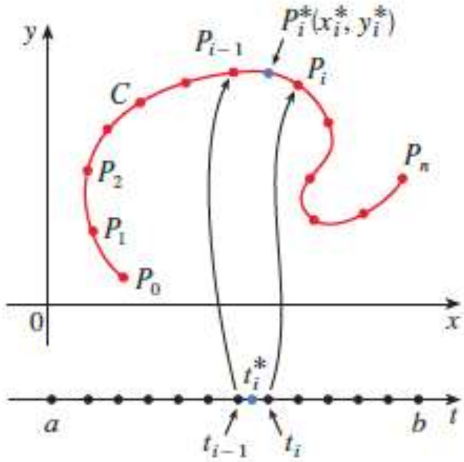
7.1-§. I va II tur egri chiziqli integrallar. Grin formulasi

Ushbu bobda vektor maydonlar nazariyasini o‘rganamiz. Xususan ob’yektni egri chiziq bo‘ylab kuch maydoni ta’sirida aylantirishda bajarilgan ishni hisoblash amaliyotini qaraymiz. Buni egri chiziqli integrallar yordamida amalga oshiramiz. Shuningdek, sirt bo‘ylab suyuqlik oqimi tezligini topish uchun sirt integrallarini o‘rganamiz.

7.1.1. I tur egri chiziqli integral va uning geometrik va fizik ma’nolari

Tekislikda sodda uzunlikka ega bo‘lgan C egri chiziqni qaraylik (7.1-rasm).

1) Egri chiziqda P_0 dan P_n ga yo‘nalishni musbat yo‘nalish deb, uni quyidagi nuqtalar bilan bo‘laklarga ajratamiz: $P_0, P_1, ..., P_{n-1}, P_n$. Natijada C egri chiziq $P_{i-1}P_i \ (i = 0,1,2,...,n-1)$ bo‘lakchalarga ajraladi.



7.1-rasm. Tekislikda sodda uzunlikka ega bo‘lgan C egri chiziq

2) Bo‘lakchalarning uzunligini Δl_k $(k=0,1,2,...,n-1)$ deb belgilaymiz. U holda P bo‘laklashning maksimal uzunligi $\lambda_P = \max\{\Delta l_k\}$ bo‘ladi.

3) Aytaylik, C egri chiziqda $f(x,y)$ funksiya aniqlangan bo‘lsin $((x,y)\in C)$. Har bir $P_{i-1}P_i$ bo‘lakchada ixtiyoriy $P_i^*(x_i^*, y_i^*)$ nuqtani tanlaymiz.

4) Tanlangan nuqtalardagi $f(x,y)$ funksiya qiymatini $f(\xi_k,\eta_k)$ hisoblaymiz.

5) Topilgan $f(\xi_k,\eta_k)$ funksiya qiymatlarni Δl_k ga ko‘paytiramiz.

6) Ko‘paytmalarning yig‘indisini hosil qilamiz: $\sigma = \sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta l_k$.

7) $\lambda_p = \max_k\{\Delta S_k\}$ nolga intilganda limitga o‘tamiz: $\lim_{\lambda_P\to 0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta l_k$.

Agar $\forall\varepsilon>0$ olinganda ham $\exists\delta>0$ son topilsaki, C egri chiziqning diametri $\lambda_p<\delta$ bo‘lgan har qanday P bo‘laklash uchun tuzilgan σ yig‘indi ixtiyoriy $P_i^*(x_i^*, y_i^*)$ nuqtalarda $|\sigma - J|<\varepsilon$ tengsizlikni qanoatlantirsa, $f(x,y)$ funksiya C egri chiziq bo‘yicha integrallanuvchi

$$\int_C f(x,y)dl = \lim_{\lambda_P\to 0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta l_k$$

deyilib, J son esa $f(x,y)$ $f(x,y)$ funksiyaning $\breve{AB}$ egri chiziq bo‘yicha **I tur egri chiziqli integrali** deyiladi va quyidagicha belgilanadi:

$$\int_C f(x,y)dl\ .$$

Ta’rifdan ko‘rinadiki, $f(x,y)$ funksiyaning I tur egri chiziqli integrali egri chiziqning yo‘nalishiga bog‘liq emas:

$$\int_{\overset{\cup}{AB}} f(x,y)dl = \int_{\overset{\cup}{BA}} f(x,y)dl\ . \tag{7.1}$$

I tur egri chiziqli integral ta’rifidan ko‘rinadiki, bu integral $f(x,y)$ funksiya va $\breve{AB}$ egri chiziqqa bog‘liq bo‘ladi.

I tur egri chiziqli integrallar yordamida egri chiziqning uzunligini, jismning massasini, og‘irlik markazini, inersiya momentini topish kabi fizikaviy masalalar hal qilinadi.

1. Tekislikda uzunlikka ega bo‘lgan $\breve{AB}$ egri chiziqning uzunligi

$$L = \int_{\overset{\cup}{AB}} dl \tag{7.2}$$

integral formula yordamida aniqlanadi.

2. Tekislikda uzunlikka ega bo‘lgan $\breve{AB}$ egri chizig‘i bo‘yicha massa tarqatilgan bo‘lib, uning zichligi $\rho = \rho(x,y)$ bo‘lsin. Bu egri chiziqning massasi ushbu

$$m = \int_{\breve{AB}} \rho(x,y)dl \qquad (7.3)$$

og‘irlik markazining koordinatalari esa quyidagi

$$x_0 = \frac{1}{m}\int_{\breve{AB}} x\rho(x,y)dl, \qquad y_0 = \frac{1}{m}\int_{\breve{AB}} y\rho(x,y)dl \qquad (7.4)$$

integrallar yordamida topiladi.

3. Tekislikda uzunlikka ega bo‘lgan $\breve{AB}$ egri chiziqning Ox ва Oy koordinata o‘qlariga nisbatan statik momentlari quyidagi formulalar bilan

$$M_x = \int_{\breve{AB}} ydl, \qquad M_y = \int_{\breve{AB}} xdl, \qquad (7.5)$$

shu o‘qlarga nisbatan inersiya momentlari esa quyidagi

$$\mathrm{I}_x = \int_{\breve{AB}} y^2dl, \qquad \mathrm{I}_y = \int_{\breve{AB}} x^2dl \qquad (7.6)$$

integrallar yordamida topiladi.

7.1.2. I tur egri chiziqli integralning xossalari. I tur egri chiziqli integralni hisoblash

Egri chiziqli integrallarning xossalari ham oddiy bir o‘zgaruvchili integral xossalariga o‘xshash bo‘ladi:

1⁰. O‘zgarmas ko‘paytuvchini integral belgisidan tashqariga chiqarish mumkin: $\int_{AB} kf(x,y)d\ell = k\int_{AB} f(x,y)d\ell,$ k – o‘zgarmas son.

2⁰. Yig‘indining integrali integrallar yig‘indisiga teng:

$$\int_{AB}\big(f(x,y) \pm \varphi(x,y)\big)d\ell = \int_{AB} f(x,y)d\ell \pm \int_{AB}\varphi(x,y)d\ell .$$

3⁰. Integrallash oralig‘ini bir nechta chekli sondagi, o‘zaro kesishmaydigan oraliqlarga ajratib, hisoblash xossasi o‘rinli:

$$\int_{AB} f(x,y)d\ell = \int_{AC} f(x,y)d\ell + \int_{CB} f(x,y)d\ell,$$

bu yerda $AC+CB=AB$.

Faraz qilaylik, $\breve{AB}$ sodda silliq chiziq tekislikda parametrik tenglamalar bilan berilgan bo‘lsin:

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \quad (\alpha \le t \le \beta) \text{ va } \quad A = (x(\alpha), y(\alpha)),\ B = (x(\beta), y(\beta)) .$$

Shu egri chiziqda $f(x,y)$ funksiya berilgan bo‘lsin.

7.1-teorema. Agar $f(x,y)$ funksiya $\breve{AB}$ da uzluksiz bo‘lsa, u holda I tur egri chiziqli integral $\int\limits_{\breve{AB}} f(x,y)dl$ mavjud bo‘lib, u quyidagiga teng bo‘ladi:

$$\int\limits_{\breve{AB}} f(x,y)dl = \int\limits_{\alpha}^{\beta} f(x(t),y(t))\sqrt{x'^2(t)+y'^2(t)}dt \qquad (7.7)$$

1-natija. $\breve{AB}$ egri chiziq $y = y(x)\ (a \le x \le b)$ tenglama bilan aniqlangan bo‘lsin va $y(x)$ funksiya $[a,b]$ kesmada uzluksiz hamda uzluksiz $y'(x)$ hosilaga ega bo‘lsin ($y(a) = A$, $y(b) = B$).

Agar $f(x,y)$ funksiya shu $\breve{AB}$ egri chiziqda uzluksiz bo‘lsa, $\int\limits_{\breve{AB}} f(x,y)dl$ I tur egri chiziqli integral mavjud bo‘ladi:

$$\int\limits_{\breve{AB}} f(x,y)dl = \int\limits_{a}^{b} f(x,y(x))\sqrt{1+y'^2}dx\,. \qquad (7.8)$$

2-natija. $\breve{AB}$ egri chiziq $x=x(y)$ tenglama bilan berilgan bo‘lsa, $(c \le y \le d)$, integral

$$\int\limits_{\breve{AB}} f(x,y)dl = \int\limits_{c}^{d} f(x(y),y)\sqrt{1+x'^2}dy \qquad (7.9)$$

formula bilan hisoblanadi.

3-natija. $\breve{AB}$ egri chiziq qutb koordinatasida $\rho = \rho(\theta)$ $(\alpha \le \theta \le \beta)$ tenglama bilan berilgan bo‘lsin, bunda $\rho = \rho(\theta)$ funksiya $[\alpha,\beta]$ segmentda uzluksiz va ρ' uzluksiz hosilaga ega bo‘lsin. Bu egri chiziqda $f(x,y)$ funksiya aniqlangan va uzluksiz bo‘lsin. U holda I tur egri chiziqli integral mavjud bo‘ladi:

$$\int\limits_{\breve{AB}} f(x,y)dl = \int\limits_{\alpha}^{\beta} f(\rho\cos\theta, \rho\sin\theta)\sqrt{\rho^2+\rho'^2}d\theta \qquad (7.10)$$

Faraz qilaylik, $\breve{AB}$ sodda silliq chiziq fazoda parametrik tenglamalar bilan berilgan bo‘lsin:

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \text{ va } \quad A = (x(\alpha), y(\alpha), z(\alpha)),\ B = (x(\beta), y(\beta), z(\beta))$$

bo‘lsin. Shu egri chiziqda $f(x,y,z)$ funksiya berilgan bo‘lsin.

7.2-teorema. Agar $f(x,y,z)$ funksiya $\breve{AB}$ da uzluksiz bo‘lsa, u holda I tur egri chiziqli $\int_{\overset{\cup}{AB}} f(x,y,z)dl$ integral mavjud bo‘lib, u quyidagiga teng bo‘ladi:

$$\int_{\overset{\cup}{AB}} f(x,y,z)dl = \int_{\alpha}^{\beta} f(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt \qquad (7.11)$$

7.1.1-misol. $\int_{\overset{\cup}{AB}} \frac{x}{y}dl$ integralni hisoblang, bunda $\breve{AB}$ egri chiziq $y^2 = 2x$ parabolaning $\left(1, \sqrt{2}\right)$, $\left(2, 2\right)$ nuqtalari orasidagi qismi.

Yechilishi: ► $y^2 = 2x \Rightarrow y = \sqrt{2x}$ va $1 \le x \le 2$ qiymatlarni aniqlab, so‘ngra (7.8) formulaga qo‘yamiz:

$$\int_{\overset{\cup}{AB}} \frac{x}{y}dl = \int_1^2 \frac{x}{\sqrt{2x}}\sqrt{1+\left(\sqrt{2x}\right)'^2}dx = \int_1^2 \frac{x}{\sqrt{2x}}\frac{\sqrt{1+2x}}{\sqrt{2x}}dx = \frac{1}{2}\int_1^2\sqrt{1+2x}dx = \frac{1}{6}(5\sqrt{5}-3\sqrt{3}). \blacktriangleleft$$

7.1.2-misol. $\int_C \frac{1}{\rho}\cos\theta dl$ integralni hisoblang, bunda C-markazi $(a,0)$ nuqtada, radiusi a ga teng bo‘lgan aylana.

Yechilishi: ► Hisoblashni qutb koordinatasida bajaramiz. Aylananing tenglamasi qutb koordinatasida quyidagicha bo‘ladi:

$$\rho(\theta) = 2a\cos\theta \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

(7.10) formuladan foydalanamiz.

$$\int_C \frac{1}{\rho}\cos\theta dl = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\rho}\sqrt{\rho^2(\theta)+\rho'^2(\theta)}d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi. \blacktriangleleft$$

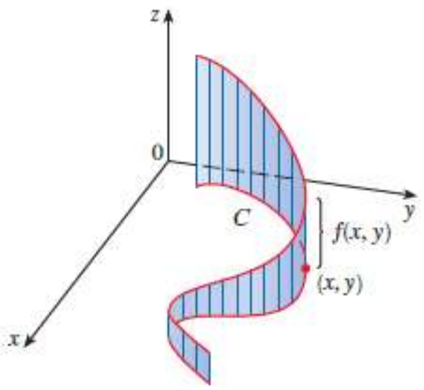
7.1.3-misol. $\int_C (2+x^2y)dl$ integralni hisoblang, bunda C: $x^2+y^2=1$ aylana (7.2 va 7.3-rasmlar).

Yechilishi: ► Hisoblashni (7.7) formuladan foydalanib bajaramiz.

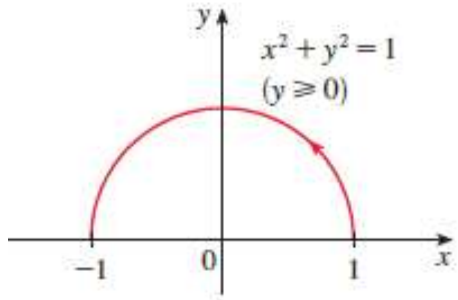
Bizning misolda $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $\quad 0 \le t \le \pi$ ekanligini aniqlaymiz.

$$\int_C (2+x^2y)dl = \int_0^{\pi}(2+\cos^2 t\sin t)\sqrt{\sin^2 t+\cos^2 t}dt = \int_0^{\pi}(2+\cos^2 t\sin t)dt =$$

$$= \left(2t - \frac{\cos^3 t}{3} \right)\Bigg|_0^{\pi} = 2\pi + \frac{2}{3}.$$



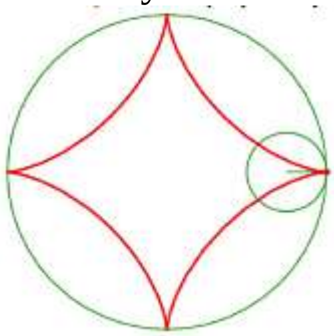
7.2-rasm. $\int_C (2+x^2y)dl$ integral egri chiziq



7.3-rasm. Integrallash chizig‘i ◄

7.1.4-misol. $\begin{cases} x(t) = a\cos^3 t, \\ y(t) = a\sin^3 t \end{cases}$ $(0 \le t \le 2\pi)$ tenglamalar sistemasi bilan aniqlangan $\breve{AB}$ egri chiziq (astroida)ning uzunligini toping (7.4-rasm).

Yechilishi: ► Astroida koordinata o‘qlarga nisbatan simmetrik bo‘ladi, shuning uchun bitta bo‘lagini uzunligini topib, uni 4 ga ko‘paytiramiz. (7.7) formuladan foydalanamiz:



7.4-rasm. Astroida chizig‘i

$$L = \int_{\breve{AB}} dl = 4\int_0^{\frac{\pi}{2}} \sqrt{x'^2(t) + y'^2(t)}dt = \int_0^{\frac{\pi}{2}} \sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}\, dt =$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{\frac{9a^2}{4}\sin^2 2t}\,dt = 6a\int_0^{\frac{\pi}{2}} \sin 2t\,dt = 6a. \ ◄$$

7.1.5-misol. Chiziqning zichligi $\rho(x,y) = |y|$ bo‘lgan $\breve{AB}$ massali egri chiziq $y^2 = 2px$ $\quad (0 \le x \le \frac{p}{2})$ parabolaing massasi hamda og‘irlik markazini toping.

Yechilishi: ► (7.3) va (7.4) formulalardan foydalanamiz. Parabolaning massasi $m=\int\limits_{\breve{AB}}|y|dl$ ga teng. Endi I tur egri chiziqli integralni aniq integralga keltirib, hisoblaymiz:

$$m=\int\limits_{-p}^{p}|y|\sqrt{1+\frac{y^2}{p^2}}dy=2\frac{1}{p}\int\limits_0^p y\sqrt{p^2+y^2}dy=\frac{1}{p}\int\limits_0^p\sqrt{p^2+y^2}d(p^2+y^2)=$$

$$=\frac{1}{p}((p^2+y^2)^{\frac{3}{2}}\frac{2}{3})_0^p=\frac{2}{3}p^2(2\sqrt{2}-1).$$

Qaralayorgam massali parabolaning og'irlik markazining koordinatalarini tegishli formuladan foydalanib topamiz:

$$x_0=\frac{1}{m}\int\limits_{\breve{AB}}x|y|dl=\frac{1}{m}\int\limits_0^p y^3\sqrt{p^2+y^2}dy,,$$

$$\int\limits_0^p y^3\sqrt{p^2+y^2}dy=\left[\begin{array}{l} y^2=u,\quad du=2ydy \\ y\sqrt{p^2+y^2}dy=dv,\ v=\frac{1}{3}(p^2+y^2)^{\frac{3}{2}}\end{array}\right]=$$

$$=\frac{1}{3}y^2(p^2+y^2)\Big|_0^p-\frac{1}{3}\int\limits_0^p 2y\ (p^2+y^2)^{\frac{3}{2}}dy=\frac{2\sqrt{2}p^5}{3}-\frac{1}{3}\cdot\frac{2}{5}(p^2+y^2)^{\frac{5}{2}}\Big|_0^p=\frac{2p^5(1+\sqrt{2})}{15}.$$

Demak,

$$x_0=\frac{1}{m}\frac{2p^5\left(\sqrt{2}+1\right)}{15}=\frac{3}{2}\frac{1}{p^2\left(2\sqrt{2}-1\right)}\frac{2p^5\left(\sqrt{2}+1\right)}{15}=\frac{p^3\left(3\sqrt{2}+5\right)}{35}.$$

Xuddi shunga o'xshash 2-koordinatasi ham

$$y_0=\frac{1}{m}\int\limits_{\breve{AB}}y|y|dl=\frac{3(2\sqrt{2}+p)}{28}(3\sqrt{2}+\ln(1+\sqrt{2}))$$

bo'lishi topiladi. ◄

7.1.6-misol. $x^2+y^2=a^2$ aylananing diametriga nisbatan inersiya momentini toping.

Yechilishi: ► Berilgan aylananing parametrik tenglamasi quyidagicha: $\begin{cases} x=a\cos t, \\ y=a\sin t\end{cases}$ $(0\le t\le 2\pi)$ bo'ladi. Aylana diametrini Ox o'qqa joylashtirib, so'ng (7.6) formuladan foydalanamiz:

$$\mathrm{I}_x=\int\limits_{\breve{AB}}y^2dl=\int\limits_0^{2\pi}a^2\sin^2 t\sqrt{(a\cos t)'^2+(a\sin t)'^2}dt=a^3\int\limits_0^{2\pi}\sin^2 tdt=\pi a^3.\ ◄$$

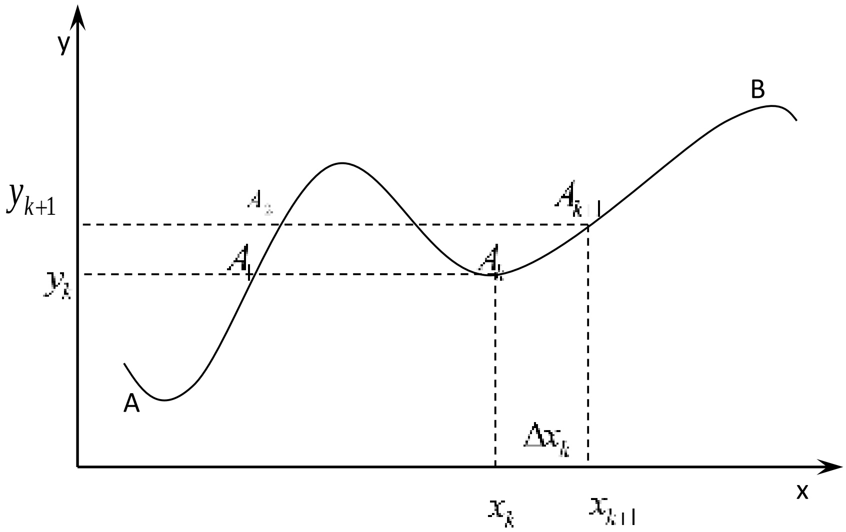
7.1.7-misol. Agar $\breve{AB}$ vint chizig'i $\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$ $0 \le t \le 2\pi$ parametrik ko'rinishda berilgan bo'lsa, $\int\limits_{\breve{AB}} y \sin z dl$ integralni hisoblang,

Yechilishi: ► (7.5) formuladan foydalanamiz:

$$\int\limits_{\breve{AB}} y \sin z dl = \int\limits_0^{2\pi} \sin t \cdot \sin t \cdot \sqrt{(\cos t)'^2 + (\sin t)'^2 + t'^2} dt = \int\limits_0^{2\pi} \sin^2 t \sqrt{\sin^2 + \cos^2 + 1} dt = \pi\sqrt{2}. \quad ◄$$

7.1.3. II tur egri chiziqli integral va uning geometrik va fizik ma'nolari, xossalari

Tekislikda uzunlikka ega bo'lgan (sodda) $\breve{AB}$ egri chiziqni qaraylik.



7.5-rasm. Tekislikda sodda $\breve{AB}$ egri chiziq

Bu egri chiziqning biror $P = \{A_0, A_1, A_2, ..., A_n\}$ $(A_0 = A, A_n = B)$ bo'laklashlarini olamiz. Natijada $\breve{AB}$ egri chiziq $A_k\breve{A}_{k+1}$ $(k = 0,1,2,..n-1)$ bo'lakchalarga ajraladi. $A_k\breve{A}_{k+1}$ ning OX va OY koordinatalar o'qlardagi proyeksiyalari mos ravishda Δx_k va Δy_k bo'lsin:

$$np_{ox} A_k\breve{A}_{k+1} = \Delta x_k, \quad np_{oy} A_k\breve{A}_{k+1} = \Delta y_k \qquad (k = 0,1,2,..n-1).$$

Aytaylik, $\breve{AB}$ egri chiziqda $f(x, y)$ funksiya berilgan bo'lsin. Har bir $A_k\breve{A}_{k+1}$ da ixtiyoriy (ξ_k, η_k) nuqtalarni olib, so'ng bu nuqtadagi funksiyaning $f(\xi_k, \eta_k)$ qiymatini Δx_k va Δy_k larga ko'paytirib, quyidagi

$$\sigma_1 = \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta x_k ,$$

$$\sigma_2 = \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta y_k$$

yig‘indilarni hosil qilamiz. Bu yig‘indilar $f(x,y)$ funksiyaga va $\breve{AB}$ egri chiziqni bo‘laklashga hamda har bir $A_k \breve{A}_{k+1}$ da olingan (ξ_k, η_k) nuqtalarga bog‘liq bo‘ladi (7.5-rasm).

Agar $\forall \varepsilon > 0$ olinganda ham $\exists \delta > 0$ son topilsaki, $\breve{AB}$ egri chiziqning diametri $\lambda_p < \delta$ bo‘lgan har qanday P bo‘laklash uchun tuzilgan $\sigma_1(\sigma_2)$ yig‘indi $\forall (\xi_k, \eta_k) \in A_k \breve{A}_{k+1}$ nuqtalarda $|\sigma_1 - J_1| < \varepsilon$ $(|\sigma_2 - J_2| < \varepsilon)$ tengsizlik bajarilsa, ya’ni

$$\int_{\breve{AB}} f(x,y)dx = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta x_k ,$$

$$\left(\int_{\breve{AB}} f(x,y)dy = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta y_k\right).$$

limit o‘rinli bo‘lsa, $f(x,y)$ funksiya $\breve{AB}$ egri chiziq bo‘yicha integrallanuvchi va J_1 son (J_2 son) $f(x,y)$ funksiyaning **II tur egri chiziqli integrali** deyiladi va quyidagicha belgilanadi:

$$\int_{\breve{AB}} f(x,y)dx, \quad \left(\int_{\breve{AB}} f(x,y)dy\right)$$

Ushbu ta’rifdan quyidagilar kelib chiqadi:

1) $f(x,y)$ funksiyaning $\breve{AB}$ egri chiziq bo‘yicha II tur egri chiziqli integrali 2 ta bo‘ladi: $\int_{\breve{AB}} f(x,y)dx, \quad \int_{\breve{AB}} f(x,y)dy$.

Aytaylik, $\breve{AB}$ egri chiziqda $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo‘lib, $\int_{\breve{AB}} P(x,y)dx, \quad \int_{\breve{AB}} Q(x,y)dy$ lar ularning II tur egri chiziqli integrallari bo‘lsin. U holda ushbu yig‘indi $\int_{\breve{AB}} P(x,y)dx + \int_{\breve{AB}} Q(x,y)dy$ II tur egri chiziqli integralning umumiy ko‘rinishi deyiladi va quyidagicha belgilanadi:

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy$$

Demak, $$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_{\breve{AB}} P(x,y)dx + \int_{\breve{AB}} Q(x,y)dy . \qquad (7.12)$$

2) $f(x,y)$ funksiyaning II tur egri chiziqli integrallari $\breve{AB}$ egri chiziqning yo‘nalishiga bog‘liq bo‘ladi:

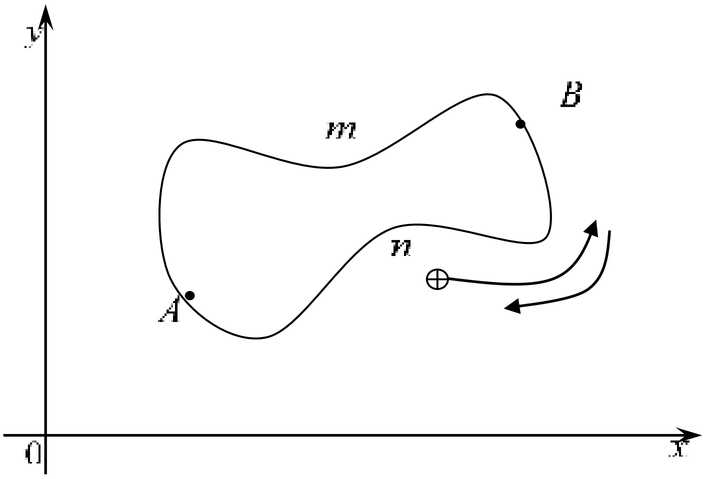
$$\int_{\breve{BA}} f(x,y)dx = -\int_{\breve{AB}} f(x,y)dx , \qquad \int_{\breve{BA}} f(x,y)dy = -\int_{\breve{AB}} f(x,y)dy \qquad (7.13)$$

3) Agar $\breve{AB}$ egri chiziq Ox (Oy) koordinatalar o‘qiga perpendikulyar bo‘lgan to‘g‘ri chiziq kesmasidan iborat bo‘lsa,

$$\int_{\breve{BA}} f(x,y)dy = 0 \qquad \left(\int_{\breve{AB}} f(x,y)dy = 0 \right)$$

tenglik o‘rinli bo‘ladi.

Aytaylik, $K = \breve{AB}$ sodda yopiq egri chiziq bo‘lsin. U holda A va B nuqtalar ustma-ust tushadi (7.6-rasm):



7.6-rasm. Tekislikda yopiq sodda $\breve{AB}$ egri chiziq

Yopiq egri chiziq K da chizmada ko‘rsatilganidek, 2 xil yo‘nalish bo‘lib, ulardan biri musbat, ikkinchisi esa manfiy bo‘ladi.

Agar kuzatuvchi K chiziq bo‘yicha harakatlanganda K bilan chegaralangan to‘plam har doim chap tomonda qolsa, bunday yo‘nalish **musbat yo‘nalish** bo‘ladi, aks holda manfiy yo‘nalish bo‘ladi.

K egri chiziqda $f(x,y)$ funksiya berilga bo‘lsin. K chiziqda ixtiyoriy ikkita A va B nuqtalarni olaylik. Bu nuqtalar K egri chiziqni ikkita $A\breve{n}B$ va $B\breve{m}A$ egri chiziqlarga ajratadi.

Faraz qilaylik, $\int_{A\breve{n}B} f(x,y)dx,$ $\int_{B\breve{m}A} f(x,y)dx$ integrallar mavjud bo‘lsin. Ushbu $\int_{A\breve{n}B} f(x,y)dx + \int_{B\breve{m}A} f(x,y)dx$ yig‘indi $f(x,y)$ funksiyaning K **yopiq egri chiziq bo‘yicha II tur egri chiziqli integrali** deyiladi va quyidagicha belgilanadi:

$$\int_K f(x,y)dx \quad \text{yoki} \quad \oint_K f(x,y)dx$$

Bu holda K yopiq chiziqning musbat yo‘nalishi olinadi:

$$\int_K f(x,y)dx = \int_{A\breve{n}B} f(x,y)dx + \int_{B\breve{m}A} f(x,y)dx . \qquad (7.14)$$

Xuddi shunga o‘xshash $\int_K f(x,y)dy$ hamda umumiy holda

$$\int_K P(x,y)dx + Q(x,y)dy$$

integrallar ham ta'riflanadi.

Aytaylik, $\breve{AB}$ fazodagi sodda uzunlikka ega bo'lgan egri chiziq bo'lib, bu egri chiziqda $f(x,y,z)$ funksiya berilgan bo'lsin. Yuqoridagidek, fazodagi $f(x,y,z)$ funksiya uchun ham II tur egri chiziqli integrallar ta'riflanadi va ular quyidagicha belgilanadi:

$$\int_{\breve{AB}} f(x,y,z)dx, \quad \int_{\breve{AB}} f(x,y,z)dy, \quad \int_{\breve{AB}} f(x,y,z)dz$$

$$\int_{\breve{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz. \qquad (7.15)$$

II tur egri chiziqli integral yordamida tekis shaklning yuzi, kuch ta'sirida bo'lgan maydonda bajarilgan ishni topish mumkin. Shuningdek, boshqa turli fizik va mexanik masalalarni hal qilish mumkin.

Tekislikda biror yuzaga ega bo'lgan D shakl berilgan bo'lib, uning chegarasi to'g'rilanuvchi yopiq ∂D chiziqdan iborat bo'lsin. Bu shaklning yuzi ushbu

$$\mu D = \int_{\partial D} xdy \ , \quad \mu D = -\int_{\partial D} ydx \qquad \mu D = \frac{1}{2}\int_{\partial D} xdy - ydx$$

formulalar yordamida topiladi.

Agar uzunlikka ega bo'lgan $\breve{AB}$ egri chiziq berilgan bo'lib, uning har bir (x,y) nuqtasi ushbu $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ kuch ta'sirida bo'lsa, u holda A nuqtani B nuqtaga o'tkazishda bajarilgan ish quyidagiga teng bo'ladi:

$$W = \int_{\breve{AB}} P(x,y)dx + Q(x,y)dy. \qquad (*)$$

Xossalari. Ikkinchi tur egri chiziqli integral birinchi tur egri chiziqli integralning hamma hossalariga egadir.

7.1.4. II tur egri chiziqli integralni hisoblash

Faraz qilaylik, $\breve{AB}$ egri chiziq parametrik tenglamalar bilan berilgan bo'lsin: $\begin{cases} x = x(t), \\ y = y(t) \end{cases}$ $(\alpha \le t \le \beta)$. $x = x(t)$ fuksiya $[\alpha, \beta]$ da uzluksiz va $x'(t)$ hosilaga ega, $y(t)$ funksiya ham $[\alpha, \beta]$ da uzluksiz hamda $\mathrm{A} = (x(\alpha), y(\alpha)),\quad \mathrm{B} = (x(\beta), y(\beta))$ bo'lsin. t parametr α dan β ga qarab o‘zgarganda $\breve{AB}$ egri chiziqning $(x,y)=(x(t),y(t))$ nuqtasi A dan B ga qarab $\breve{AB}$ yoyni chizsin.

7.3-teorema. Agar $f(x,y)$ funksiya $\breve{AB}$ da uzluksiz bo‘lsa, u holda II tur egri chiziqli integral $\int\limits_{\breve{AB}} f(x,y)dx$ mavjud bo‘ladi va quyidagi tenglik bilan hisoblanadi:

$$\int\limits_{\breve{AB}} f(x,y)dx=\int\limits_{\alpha}^{\beta} f(x(t),y(t))x'(t)dt \qquad (7.16)$$

Aytaylik, parametrik tenglama bilan berilgan $x(t)$, $y(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz bo‘lib, $y(t)$ funksiya uzluksiz $y'(t)$ hosilaga ega bo‘lsin.

7.4-teorema. Agar $f(x,y)$ funksiya $\breve{AB}$ da uzluksiz bo‘lsa, u holda II tur egri chiziqli integral $\int\limits_{\breve{AB}} f(x,y)dy$ mavjud bo‘lib, u quyidagiga teng bo‘ladi:

$$\int\limits_{\breve{AB}} f(x,y)dy=\int\limits_{\alpha}^{\beta} f(x(t),y(t))y'(t)dt \qquad (7.17)$$

Aytaylik, parametrik tenglama bilan berilgan $x(t)$, $y(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz $x'(t)$ va $y'(t)$ hosilalarga ega bo‘lsin.

7.5-teorema. Agar $P(x,y)$ va $Q(x,y)$ funksiyalar $\breve{AB}$ da uzluksiz bo‘lsa, u holda egri chiziqli integral mavjud bo‘lib, u quyidagiga teng bo‘ladi:

$$\int\limits_{\breve{AB}} P(x,y)dx+Q(x,y)dy=\int\limits_{\alpha}^{\beta}[P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)]dt\,. \qquad (7.18)$$

Bu teoremalar egri chiziqli integrallarni hisoblash imkonini beradi.

Agar $\breve{AB}$ egri chiziq $y=y(x)$ $(a\le x\le b)$, $x=x(y)$ $(c\le y\le d)$ parametrik tenglamalar bilan berilgan bo‘lsa, u holda egri chiziqli itegrallar birmuncha sodda ko‘rinishga ega bo‘ladi.

Agar $\breve{AB}$ egri chiziq $y=y(x)$ $(a\le x\le b)$ tenglama bilan berilgan bo‘lib, $y(x)$ funksiya $[a,b]$ da uzluksiz $y'(x)$ hosilaga ega bo‘lsa, u holda

$$\int\limits_{\breve{AB}} f(x,y)dx=\int\limits_{a}^{b} f(x,y(x))dx,$$

$$\int\limits_{\breve{AB}} P(x,y)dx+Q(x,y)dy=\int\limits_{a}^{b}[P(x,y(x))+Q(x,y(x))y'(x)]dx \qquad (7.19)$$

tengliklar o‘rinli bo‘ladi.

Agar $\breve{AB}$ egri chiziq $x = x(y)$ $(c \le y \le d)$ tenglama bilan berilgan bo‘lib, $x = x(y)$ funksiya $[c,d]$ da uzluksiz $x'(y)$ hosilaga ega bo‘lsa, u holda quyidagi tengliklar o‘rinli bo‘ladi:

$$\int_{\breve{AB}} f(x,y)dy = \int_c^d f(x(y),y)dy \,,$$

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_c^d [P(x(y),y)x'(y) + Q(x(y),y)]dy \,. \qquad (7.20)$$

7.1.8-misol. $J_1 = \int_{\breve{AB}} (x^2 - y^2)dx,$ $J_2 = \int_{\breve{AB}} (x^2 - y^2)dy$ integrallani hisoblang, bunda $\breve{AB}$ egri chiziq $y = x^2$ parabolaning absissalari $x = 0, x = 2$ bo‘lgan nuqtalari orasidagi qismi.

Yechilishi: ► $\breve{AB}$ egri chiziq $y = x^2$ tenglama bilan aniqlanishini e’tiborga olib, J_1 integralni hisoblashda (7.19) formuladan foydalanamiz:

$$J_1 = \int_{\breve{AB}} (x^2 - y^2)dx = \int_0^2 (x^2 - x^4)dx = \left(\frac{x^3}{3} - \frac{x^5}{5} \right)\Bigg|_0^2 = \frac{8}{3} - \frac{32}{5} = -\frac{56}{15} \,.$$

J_2 integralda integrallash egri chizig‘i $x^2 = y$ bo‘lib, (7.20) formulaga ko‘ra,

$$J_2 = \int_{\breve{AB}} (x^2 - y^2)dy = \int_0^4 (y - y^2)dy = \left(\frac{y^2}{2} - \frac{y^3}{3} \right)\Bigg|_0^4 = 8 - \frac{64}{3} = -\frac{40}{3}$$

bo‘ladi. ◄

7.1.9-misol. $\int_{\breve{AB}} y^2 dx + x^2 dy$ integralni hisoblang, bunda $\breve{AB}$ egri chiziq $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning yuqori yarim tekislikdagi qismi.

Yechilishi: ► Ushbu ellipsning parametrik tenglamasi quyidagicha:

$$\begin{cases} x = a\cos t, \\ y = b\sin t \end{cases}$$

$A = (a,0)$ nuqtaga parametrning $t = 0$ qiymati, $B = (-a,0)$ nuqtaga esa $t = \pi$ qiymati mos bo‘lib, t parametr 0 dan π gacha o‘zgarganda (x,y) nuqta A dan B ga qarab ellipsning yuqori yarim tekislikdagi qismini chizadi. $P(x,y) = y^2$, $Q(x,y) = x^2$ funksiyalar $\breve{AB}$ da uzluksiz. Berilgan integralni (7.18) formuladan foydalanib hisoblaymiz:

$$\int_{\breve{AB}} y^2dx + x^2dy = \int_0^{\pi}[b^2\sin^2 t(-a\sin t) + a^2\cos^2 t\, b\cos t]dt =$$

$$= ab\int_0^{\pi}(a\cos^3 t - b\sin^3 t)dt = -\frac{4}{3}ab^2. \blacktriangleleft$$

7.1.10-misol. $\oint_K 2xydx - x^2dy$ integralni hisoblang, bunda K yopiq chiziq $O(0,0)$ va $A(2,1)$ nuqtalarni birlashtiruvchi to‘g‘ri chiziq kesmasi hamda $y^2 = \frac{1}{2}x$ parabola yoyidan tashkil topgan yopiq egri chiziq.

Yechilishi: ► OAO yopiq egri chiziq bo‘yicha integralni quyidagicha yozamiz:

$$\oint_K 2xydx - x^2dy = \int_{\breve{OA}} 2xydx - x^2dy + \int_{\breve{AO}} 2xydx - x^2dy.$$

a) $O(0,0)$ va $A(2,1)$ nuqtalarni birlashtiruvchi to‘g‘ri chiziq tenglamasini tuzamiz:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} \Rightarrow \quad \frac{x-0}{2-0} = \frac{y-0}{1-0} \quad \Rightarrow \quad x = 2y.$$

Demak, $\breve{OA}$ kesmada $x = 2y$ bo‘lib, (7.20) formulaga ko‘ra hisoblaymiz:

$$\int_{\breve{OA}} 2xydx - x^2dy = \int_0^1 [2\cdot 2y^2\cdot 2 - 4y^2]dy = \frac{4}{3}.$$

$\breve{AO}$ yoyda $y^2 = \frac{1}{2}x \quad \Rightarrow \quad x = 2y^2$ bo‘lib, yana shu formulani qo‘llaymiz:

$$\int_{\breve{AO}} 2xydx - x^2dy = \int_0^1 [2\cdot 2y^2\cdot y\cdot 4y - 4y^4]dy = -\frac{12}{5}.$$

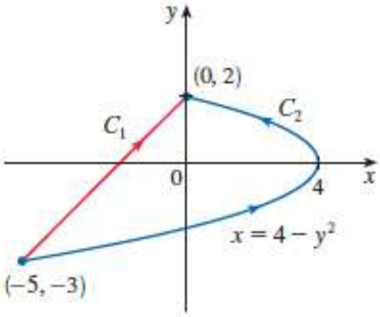
Teskari yo‘nalish bo‘lgani uchun ishorani manfiy qilib olamiz. Shunday qilib, izlanayotgan integralni topamiz: $\oint_K 2xydx - x^2dy = \frac{4}{3} - \frac{12}{5} = -\frac{16}{15}$. ◄

7.1.11-misol. $\int_C y^2dx + xdy$ integralni C bo‘ylab 2 xil yo‘nalishda hisoblang: bunda

$a)\ C = C_1 \qquad (-5,-3)$ dan $(0,2)$ gacha;

$b)\ C = C_2 \quad x = 4 - y^2$ bo'ylab $(-5,-3)$ dan $(0,2)$ gacha.

Yechilishi: ►



7.7-rasm. C_1 va C_2 integrallash yo‘nalishlari

a) $(-5,-3)$, $(0,2)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzamiz (7.7-rasm):

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0} \quad \Rightarrow \quad \frac{x-0}{-5-0}=\frac{y-2}{-3-2} \quad \Rightarrow \quad x=y-2$$

Demak, $\tilde{AB}$ kesmada $x=y-2$ bo‘lib, (7.20) formulaga ko‘ra

$$\int_{AB} y^2dx+xdy=\int_{-3}^{2}[y^2+y-2]dy=\left(\frac{y^3}{3}+\frac{y^2}{2}-2y\right)\Bigg|_{-3}^{2}=\frac{8}{3}+1-\frac{9}{2}=-\frac{5}{6}.$$

$x=4-y^2$ parabolaning $(-5,-3)$, $(0,2)$ nuqtalari orasidagi yo‘nalish bo‘ylab integralni hisoblaymiz, (7.20) formulaga ko‘ra:

$$\int_{AB}[y^2\cdot(-2y)+(4-y^2)]dy=\int_{-3}^{2}[-2y^3+4-y^2]dy=\left(-\frac{y^4}{2}+4y-\frac{y^3}{3}\right)\Bigg|_{-3}^{2}=$$

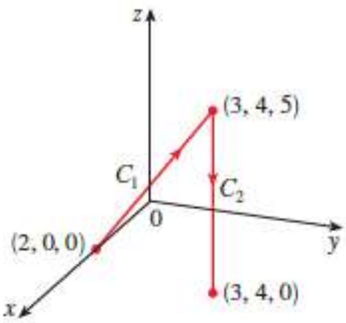
$$=-8+8-\frac{8}{3}+\frac{81}{2}+12-9=3+\frac{227}{6}=40\frac{5}{6}. \blacktriangleleft$$

7.1.12-misol. $\int_C ydx+zdy+xdz$ integralni hisoblang, bunda

C_1 $(2,0,0)$ dan $(3,4,5)$ gacha bo‘lgan yo‘nalish;

C_2 $(3,4,5)$ dan $(3,4,0)$ gacha bo‘lgan yo‘nalish.

Yechilishi: ►



7.8-rasm. C_1 va C_2 integrallash yo‘nalishlari

1) 7.8-rasmdan *C_1* $(2,0,0)$ dan $(3,4,5)$ gacha bo‘lgan yo‘nalish bo‘yicha hisoblaymiz. Dastlab bu 2 nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzib olamiz:

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}=t \quad\Rightarrow\quad \frac{x-2}{3-2}=\frac{y}{4}=\frac{z}{5}=t \quad\Rightarrow \begin{cases} x=t+2 \\ y=4t \\ z=5t \end{cases} \text{ bunda } \ 0\le t\le 1.$$

$$\int_C ydx+zdy+xdz=\int_0^1[4t+5t\cdot4+(t+2)\cdot5]dt=\int_0^1(29t+10)dt=\left(29\frac{t^2}{2}+10t\right)_0^1=24{,}5.$$

2) C_2 $(3{,}4{,}5)$ dan $(3{,}4{,}0)$ gacha bo‘lgan yo‘nalish bo‘yicha hisoblaymiz.

Dastlab bu 2 nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzamiz:

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}=t \quad\Rightarrow 0\le t\le 1,\quad \frac{z}{5}=t-1 \quad \begin{cases} x=3 \\ y=4 \\ z=5t-5 \end{cases} \text{ bunda } \ 0\le t\le 1.$$

Yo‘nalish oldingiga teskari, shuning uchun $dx=dy=0$ va

$$\int_C ydx+zdy+xdz=\int_1^0[4\cdot0+(5t-5)\cdot0+3\cdot5]dt=\int_1^0 15dt=(15t)_1^0=-15.$$

Shunday qilib, $\int_C ydx+zdy+xdz=24{,}5-15=9{,}5$ ni topamiz. ◀

7.1.13-misol. $\begin{cases} x=a\cos t, \\ y=b\sin t \end{cases}$ $(0\le t\le 2\pi)$ ellips bilan chegaralangan shakl yuzini toping.

Yechilishi: ▶ Bu shaklning yuzi $\mu D=\frac{1}{2}\oint_{\partial(D)} xdy-ydx$ formuladan topiladi. Egri chiziqli integralni hisoblaymiz:

$$\mu D=\frac{1}{2}\int_0^{2\pi}(a\cos t\cdot b\cos t+b\sin t\cdot a\sin t)dt=$$

$$=\frac{1}{2}ab\int_0^{2\pi}(\cos^2 t+\sin^2 t)dt=\pi ab\,. \ ◀$$

7.1.14-misol. $\breve{AB}$ egri chizig‘i $y=x^3$ chiziqning $(0{,}0)$ ва $(1{,}1)$ nuqtalari orasidagi qismi bo‘lib, uning har bir nuqtasi

$$\vec{F}(x,y)=4x^6\,\vec{i}+xy\,\vec{j}$$

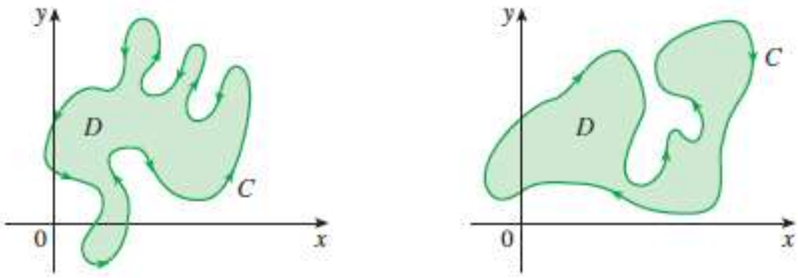
kuch ta’sirida bo‘lsin. Bu kuch ta’sirida bajarilgan ishni toping.

Yechilishi: ▶ Kuch ta’sirida bajarilgan ishni topish uchun (*) formulani qo‘llaymiz. $P(x,y)=4x^6$, $Q(x,y)=xy$ larni e’tiborga olib, bajarilgan ishni hisoblaymiz:

$$W=\int_{\overset{\cup}{AB}}4x^6dx+xydy=\int_0^1(4x^6+x\cdot x^3\cdot3x^2)dx=1. \ ◀$$

7.1.5. Grin formulasi

Grin formulasi I va II tur egri chiziqli integrallar bilan ikki karrali integral orasidagi bog‘lanishni ifodalaydi.



a) b)

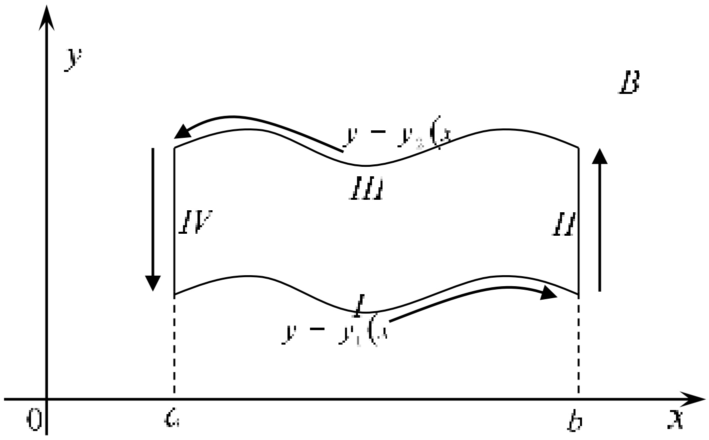
7.9-rasm. Musbat va manfiy integrallash yo‘nalishlari

7.9, a - rasmda musbat yo‘nalish, 7.9, b - rasmda manfiy yo‘nalish tasvirlangan. Agar D soha va C yopiq egri chiziq berilgan bo‘lsa, u holda **Grin formulasi** quyidagicha bo‘ladi:

$$\int_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \ . \qquad (7.21)$$

Grin formulasini keltirib chiqaramiz:

1) Tekislikda $y = y_1(x), y = y_2(x)$ $(a \le x \le b, y_1(x) \le y_2(x))$ hamda $x = a, x = b$ chiziqlar bilan chegaralangan D_1 to‘plam berilgan bo‘lsin, bunda $y_1(x)$ va $y_2(x)$ funksiyalar $[a,b]$ da uzluksiz bo‘lsin.



7.10-rasm. Integrallash yo‘nalishlari

D_1 ning chegarasi (konturi) ∂D_1 7.10-rasmdagi I, II, III, IV chiziqlarga ajraladi (bunda II va IV chiziqlar nuqtalarga aylanishi mumkin). $\overline{D} = D_1 \cup \partial D_1$ da $P(x,y)$ funksiya uzluksiz bo‘lib, u uzluksiz

$\frac{\partial P(x,y)}{\partial y}$ xususiy hosilaga ega bo‘lsin. Ushbu $\int\limits_{\partial D_1} P(x,y)dx$ egri chiziqli integralni qaraymiz. Uni quyidagicha yozib olamiz:

$$\int\limits_{\partial D_1} P(x,y)dx = \int\limits_{I} P(x,y)dx + \int\limits_{II} P(x,y)dx + \int\limits_{III} P(x,y)dx + \int\limits_{IV} P(x,y)dx$$

II va IV chiziqlar OX 0’qiga perpendikulyar bo‘lganligi sababli

$$\int\limits_{II} P(x,y)dx = \int\limits_{IV} P(x,y)dx = 0$$

Integrallar nolga teng, shunda $\int\limits_{\partial D_1} P(x,y)dx = \int\limits_{I} P(x,y)dx + \int\limits_{III} P(x,y)dx$ bo‘ladi.

Endi $\int\limits_{I} P(x,y)dx + \int\limits_{III} P(x,y)dx = \int\limits_a^b P(x,y_1(x))dx + \int\limits_a^b P(x,y_2(x))dx =$

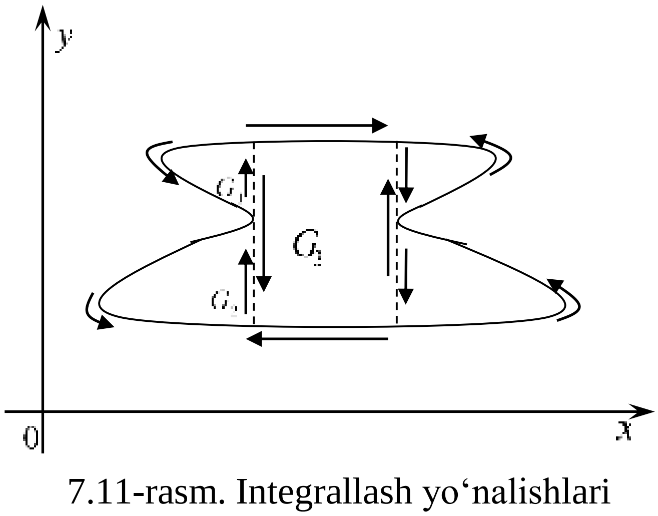
$$= \int\limits_a^b [P(x,y_1) - P(x,y_2)]dx = -\int\limits_a^b P(x,y)\ \Big|_{y=y_1}^{y=y_2} dx =$$

$$= -\int\limits_a^b \left[\int\limits_{y=y_1}^{y=y_2} \frac{\partial P(x,y)}{\partial y} dy\right] dx = -\iint\limits_{D_1} \frac{\partial P(x,y)}{\partial y} dxdy$$

bo‘lishini e’tiborga olsak, unda quyidagi tenglikka ega bo‘lamiz:

$$\int\limits_{\partial D_1} P(x,y)dx = -\iint\limits_{D_1} \frac{\partial P(x,y)}{\partial y} dxdy. \qquad (7.22)$$

2) Agar tekislikdagi G to‘plamni (vertikal chiziqlar yordamida) yuqoridagi D_1 kabi G_k $(k = 1,2,3...)$ qismlarga ajratish mumkin bo‘lsa (7.11-rasm),

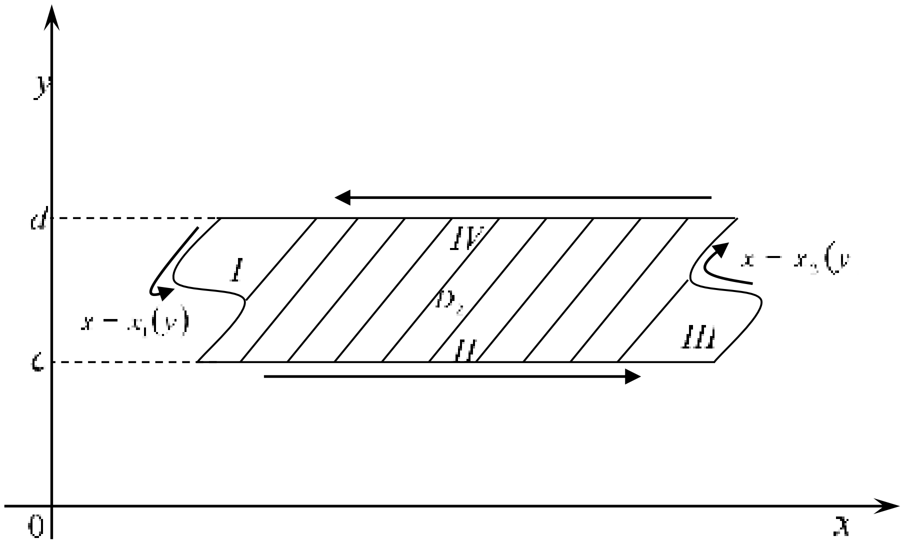


7.11-rasm. Integrallash yo‘nalishlari

bunday to‘plam uchun ushbu formula o‘rinli bo‘ladi:

$$\int\limits_{\partial G_1} P(x,y)dx = \sum_{k=1}^{n} \int\limits_{\partial G_k} P(x,y)dx = \sum_{k=1}^{n}\left(-\iint\limits_{G_k} \frac{\partial P(x,y)}{\partial y} dxdy \right) = -\iint\limits_{G} \frac{\partial P(x,y)}{\partial y} dxdy.$$

3) Endi tekislikdagi $x=x_1(y), x=x_2(y)$ $(c \le y \le d)$ hamda $y=c, y=d$ chiziqlar bilan chegaralangan D_2 to‘plamni olaylik, bunda $x_1(y)$, $x_2(y)$ funksiyalar $[c,d]$ da uzluksiz bo‘lsin (7.12-rasm).



7.12-rasm. Integrallash yo‘nalishlari

D_2 ning chegarasi (konturi) ∂D_2 quyidagi I, II, III, IV chiziqlarga ajraladi (bunda II va IV chiziqlar nuqtalarga aylanishi mumkin).

Faraz qilaylik, $\overline{D_2} = D_2 \cup \partial D_2$ da $Q(x,y)$ funksiya uzluksiz bo‘lib, u uzluksiz $\frac{\partial Q(x,y)}{\partial x}$ xususiy hosilalarga ega bo‘lsin. Ushbu $\int\limits_{\partial G_2} Q(x,y)dy$ egri chiziqli integralni qaraymiz. Uni quyidagicha yozib olamiz:

$$\int\limits_{\partial G_2} Q(x,y)dy = \int\limits_{I} Q(x,y)dy + \int\limits_{II} Q(x,y)dy + \int\limits_{III} Q(x,y)dy + \int\limits_{IV} Q(x,y)dy\,.$$

II va IV chiziqlar OY o‘qiga perpendikulyar bo‘lganligi uchun ular nolga teng: $\int\limits_{II} Q(x,y)dy = \int\limits_{IV} Q(x,y)dy = 0$ va $\int\limits_{\partial G_2} Q(x,y)dy = \int\limits_{I} Q(x,y)dy + \int\limits_{III} Q(x,y)dy$ bo‘ladi. Endi

$$\int\limits_{I} Q(x,y)dy + \int\limits_{III} Q(x,y)dy = \int\limits_{c}^{d} Q(x_1(y),y)dy +$$

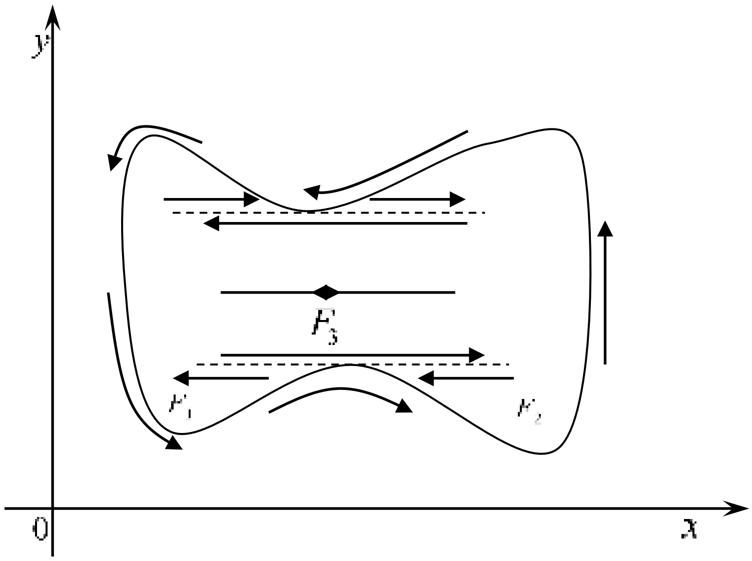
$$+\int\limits_{c}^{d} Q(x_2(y),y)dy = \int\limits_{c}^{d} \left[Q(x_1,y) - Q(x_2,y)\right]dy =$$

$$= \int\limits_{c}^{d} Q(x,y)\Big|_{x=x_1}^{x=x_2} dy = \int\limits_{c}^{d} \left[\frac{\partial Q(x,y)}{\partial x}dx\right]dy = \iint\limits_{D_2} \frac{\partial Q(x,y)}{\partial x}dxdy$$

bo‘lishini e’tiborga olib topamiz:

$$\int\limits_{\partial G_2} Q(x,y)dy = \iint\limits_{D_2} \frac{\partial Q(x,y)}{\partial x}dxdy\,. \tag{7.23}$$

4) Agar tekislikdagi F to‘plamni (gorizontal chiziqlar bilan) yuqoridagi D_2 kabi F_k $(k = 1,2,3...)$qismlarga ajratish mumkin bo‘lsa, u holda quyidagi o‘rinli bo‘ladi (7.13-rasm):



7.13-rasm. Integrallash yo‘nalishlari

$$\int\limits_{\partial F} Q(x,y)dy = \sum_{k=1}^{n} \oint\limits_{\partial F_k} Q(x,y)dy = \sum_{k=1}^{n}\left(\iint\limits_{F_k} \frac{\partial Q(x,y)}{\partial x} dxdy \right) = \iint\limits_{G} \frac{\partial Q(x,y)}{\partial x} dxdy$$

Faraz qilaylik, tekislikdagi D to‘plam yuqoridagi D_1 va D_2 lar xususiyatiga ega bo‘lib, unda $P(x,y),\ Q(x,y)$ funksiyalar uzluksiz va uzluksiz $\frac{\partial P(x,y)}{\partial y}, \frac{\partial Q(x,y)}{\partial x}$ xususiy hosilalarga ega bo‘lsin. U holda $P(x,y)$ va $Q(x,y)$ funksiyalar uchun bir yo‘la

$$\int\limits_{\partial D_1} P(x,y)dx = -\iint\limits_{D_1} \frac{\partial P(x,y)}{\partial y} dxdy \ \text{ va } \int\limits_{\partial G_2} Q(x,y)dy = \iint\limits_{D_2} \frac{\partial Q(x,y)}{\partial x} dxdy$$

formulalar o‘rinli bo‘ladi. Ularni hadlab qo‘shamiz:

$$\int\limits_{\partial D} P(x,y)dx + Q(x,y)dy = \iint\limits_{D} \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy. \qquad (7.24)$$

Shunday qilib, **Grin formulasini keltirib chiqardik[1].** Demak, Grin formulasi to‘plam bo‘yicha olingan ikki karrali integral bilan shu to‘plam chegarasi bo‘yicha olingan egri chiziqli integralning bog‘lanishini ifodalaydi.

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[1] R. W. Gatterman, “The planimeter as an example of Green’s Theorem” *Amer. Math.Monthly,* Vol. 88 (1981), pp. 701–4.

Grin formulasidan foydalanib, tekis shakl yuzining egri chiziqli integral yordamida ifodalanishini, yakobianning geometrik ma’nosini va ba’zi tasdiqlarning ekvivalentligini ko‘rsatish mumkin.

1) Tekis shakl yuzini egri chiziqli integral orqali ifodalanishi.

Faraz qilaylik, $P^*(x,y),\ Q^*(x,y)$ funksiyalar D to‘plamda uzluksiz va uzluksiz xususiy hosilalarga ega bo‘lsin va $\frac{\partial Q^*(x,y)}{\partial x}-\frac{\partial P^*(x,y)}{\partial y}\equiv 1$ shartni ham qanoatlantirsin. U holda

$$\iint_D \left(\frac{\partial Q^*(x,y)}{\partial x}-\frac{\partial P^*(x,y)}{\partial y} \right) dxdy = \mu D$$

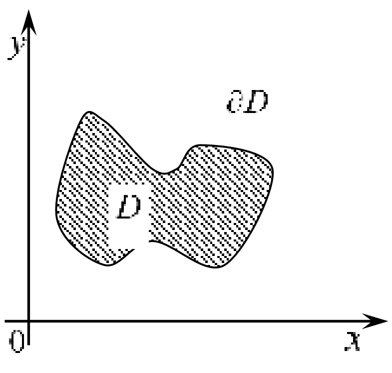
bo‘lib, Grin formulasiga ko‘ra, $\mu D=\int_{\partial D} P^*(x,y)dx + Q^*(x,y)dy$ bo‘ladi. Xususan, $P^*(x,y)=-y,\ Q(x,y)=0$ yoki $P^*(x,y)=0,\ Q(x,y)=x$ yoki $P^*(x,y)=-\frac{1}{2}y,\ Q(x,y)=\frac{1}{2}x$ bo‘lsa, $\frac{\partial Q^*(x,y)}{\partial x}-\frac{\partial P^*(x,y)}{\partial y}\equiv 1$ bo‘lib, to‘plamning yuzi quyidagiga teng bo‘ladi:

$$\mu D=-\oint_{\partial D} ydx=\oint_{\partial D} xdy=\frac{1}{2}\oint_{\partial D} xdy-ydx. \qquad (7.24)$$

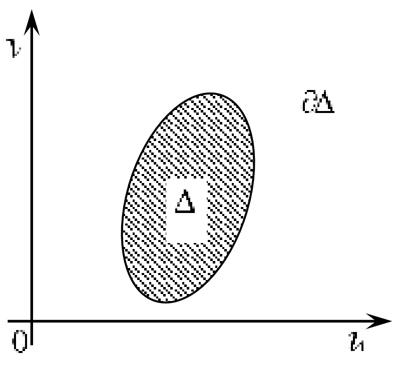
2) Yakobianning geometrik ma’nosi. Faraz qilaylik, XOY tekislikda D to‘plam berilgan bo‘lib, uning chegarasi (konturi) ∂D bo‘lsin. UOV tekislikda esa Δ to‘plam berilgan bo‘lib, uning chegarasi (konturi) $\partial\Delta$ bo‘lsin.

Aytaylik, D va Δ to‘plam nuqtalari o‘rtasida o‘zaro bir qiymatli moslik o‘rnatilgan bo‘lib, ular ushbu formula bilan ifodalansin:

$$\begin{cases} x=x(u,v), \\ y=y(u,v) \end{cases}$$



7.14-rasm. D to‘plamning ∂D konturi



7.15-rasm. Δ to‘plam chegarasi $\partial\Delta$

Bunda $x(u,v), y(u,v)$ funksiyalar yopiq Δ to'plamda uzluksiz va uzluksiz xususiy hosilalarga ega bo'lsin. Δ to'plam chegarasi $\partial\Delta$ chiziq ushbu parametrik tenglama bilan ifodalansin: $\begin{cases} u = u(t), \\ v = v(t) \end{cases}$ $(t_1 \le t \le t_2)$

Bunda $u(t), v(t)$ funksiyalar $[t_1, t_2]$ oraliqda uzluksiz va uzluksiz hosilalarga ega. Unda D to'plamning ∂D chegarasi

$$\begin{cases} x = x(u(t), v(t)) = x(t), \\ y = y(u(t), v(t)) = y(t) \end{cases} \quad (t_1 \le t \le t_2)$$

tenglamalar sistemasi bilan aniqlanadi. Bunda $\partial\Delta$ ning nuqtalariga ∂D ning nuqtalari mos keladi. Ma'lumki,

$$\mu D = \int_{\partial D} xdy. \tag{7.25}$$

Bu tenglikning o'ng tomonidagi integral uchun

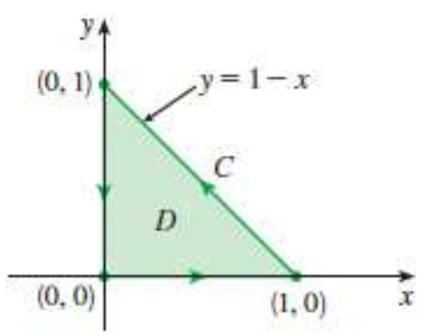
$$\oint_{\partial D} xdy = \int_{t_1}^{t_2} x\frac{dy}{dt}dt = \int_{t_1}^{t_2} x\left(\frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial t}\right)dt = \pm\int_{\partial\Delta} x\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) \tag{7.26}$$

bo'ladi. (t parametr t_1 dan t_2 ga qarab o'zgarganda ∂D egri chiziq musbat yo'nalishda bo'lsa, $\partial\Delta$ egri chiziqning yo'nalishi musbat ham, manfiy ham bo'lishi mumkin. Shuning uchun $\int_{\partial D} xdy$, $\int_{\partial\Delta}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$ mavjud.

7.1.15-misol. $\int_C x^4dx + xydy$ integralni hisoblang, bunda nuqta $(0,0)$ dan (1,0) ga, (1.0) dan (0,1) ga va (0,1)dan (0,0) ga tomon harakatlanadi (7.17-rasm).

Yechilishi: ►

$$\int_C x^4dx + xydy = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)ds = \int_0^1\int_0^{1-x}(y-0)dydx = \int_0^1\int_0^{1-x} ydydx = \frac{1}{6}.$$



7.17-rasm. Integrallash konturi ◄

7.1.16-misol. $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy$ integralni hisoblang, bunda C yopiq soha $x^2 + y^2 = 9$ aylanadan iborat.

Yechilishi: ►

$$\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy = \iint_D [(7x + \sqrt{y^4+1})'_x + (3y - e^{\sin x})'_y]ds =$$

$$= \int_0^{2\pi}\int_0^3 (7-3) r dr d\theta = 36\pi. \quad ◄$$

Mavzu yuzasidan savollar:

1. Birinchi tur egri chiziqli integral qanday hisoblanadi?
2. Birinchi tur egri chiziqli integral integrallash yo‘li parametrik tenglama bilan berilgan holda qanday hisoblanadi?
3. Birinchi tur egri chiziqli integralning xossalarini ayting.
4. I tur egri chiziqli integral yordamida nimalarni hisoblash mumkin?
5. Ikkinchi tur egri chiziqli integral deb nimaga aytiladi?
6. Ikkinchi tur egri chiziqli integral qanday hisoblanadi?
7. Ikkinchi tur egri chiziqli integral integrallash yo‘li parametrik tenglama bilan berilgan holda qanday hisoblanadi?
8. Ikkinchi tur egri chiziqli integralning xossalarini ayting.
9. II tur egri chiziqli integral yordamida nimalarni hisoblash mumkin?
10. Ikkinchi tur egri chiziqli integral integrallash yo‘liga bog‘liqmi?
11. Qanday shart bajarilganda ikkinchi tur egri chiziqli integral integrallash yo‘liga bog‘liq bo‘lmaydi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. I tur egri chiziqli integralni hisoblang: $\int_{\overset{\cup}{AB}} (x+y)zdl,$ bu yerda

$\overset{\cup}{AB}: A(1;2;-1),\ B(2;0;1)$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq kesmasi.

2. II tur egri chiziqli integralni hisoblang: $\int_{AB} \frac{y}{x}dx + xdy, \quad AB:\ y = \ln x$

egri chiziqning $A(1;0)$ va $B(\ell;1)$ nuqtalari orasidari yoyi.

3. Agar egri chiziqning har bir nuqtasida massa taqsimotining zichligi nuqta ordinatasi kvadratiga teng bo‘lsa, I chorakda joylashgan $x^2 + y^2 = 9$ aylana yoyi massasini toping.

4. Egri chiziqli integralni hisoblang: $\int_L (x^2 + 2xy)dx + (x^2 + 3x)dy$, bu yerda

L: uchlari $A(4;2)$, $B(1;-1)$ va $C(-1;1)$ nuqtada bo‘lgan $ABCA$ yopiq siniq chiziq konturi.

5. $\vec{F} = y\vec{i} + z\vec{j} - x\vec{k}$ kuchning L: $x = 2\cos t,\ y = 2\sin t, z = 3t$ fazoviy vint chizig‘i bo‘ylab $A(2;0;6\pi)$ nuqtadan $B(2;0;0)$ nuqtaga ko‘chishda bajargan ishini toping.

TESTLAR

1. Egri chiziqli integralni hisoblang: $\int\limits_{AB}\sqrt{1+x^2}\,dl$, bu yerda $AB:\quad y=x^2/2$ egri chiziq yoyi, $0\le x\le 3$.

A) 12 B) 32/3 C) 14/3 D) 9

2. Egri chiziqli integralni hisoblang: $\int\limits_{L}(xy-1)dx-x^2ydy$, bu yerda $L:AB$ to‘g‘ri chiziq bo‘ylab $A(1;2)$ dan $B(2;4)$ gacha kesmasi

A) -34/3 B) 46/3 C) 56/3 D) -28/3

3. Egri chiziqli integralni hisoblang: $\int\limits_{L}xdx+ydy+zdz$, bu yerda $L:\ x=t^3$, $y=\sqrt{t}$, $z=t$, $0\le t\le 1$.

A) 4/3 B) 3/2 C) 1/2 D) ¾

4. Egri chiziqli integralni $y=2x$ to‘g‘ri chiziqni kesib o‘tmaydigan yo‘l bo‘yicha hisoblang: $\int\limits_{(3;1)}^{(4;6)}\frac{xdy-ydx}{(2x-y)^2}$

A) 1.25 B) 2.6 C) 1.4 D) 1.6

5. $\vec{F}=(x^2-2y)\vec{i}+(y^3-2x)\vec{j}$ kuchning, $y=3-x$ to‘g‘ri chiziq bo‘ylab, $A(1;2)$ nuqtadan $B(4;-1)$ nuqtaga ko‘chishda bajargan ishini toping.

A) 27.25 B) 25.75 C) 29.25 D) 30

7.2-§. I va II tur sirt integrallari

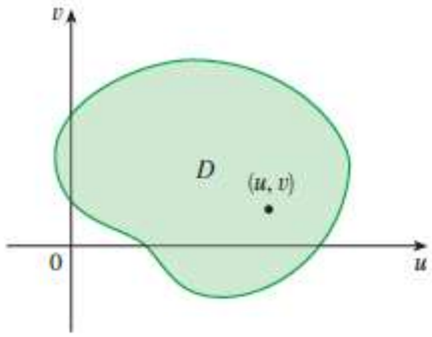
7.2.1. Sirt va sirt yuzi tushunchalari

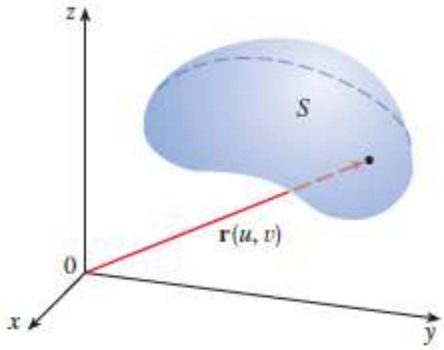
Aytaylik, Ouv tekislikdagi Δ to‘plamda

$$x=x(u,v),\quad y=y(u,v),\quad z=z(u,v)$$

funksiyalar berilgan bo‘lib, ular Δ da uzluksiz bo‘lsin. $(u_0,v_0)\in\Delta$ nuqtani olib, yuqoridagi funksiyalarning shu nuqtadagi qiymatlarini topamiz:

$$x_0=x(u_0,v_0),\ y_0=y(u_0,v_0),\ z_0=z(u_0,v_0).$$





9.18-rasm. *Ouv* tekislikdagi Δ to‘plam; 9.19-chizma. R^3 fazoda biror S to‘plam

Hosil bo‘lgan (x_0, y_0, z_0) ni R^3 fazoda M_0 nuqtaning koordinatalari deb qaraymiz: $M_0 = M(x_0, y_0, z_0)$. Ravshanki (u, v) nuqta Δ to‘plamda o‘zgarganda (x, y, z) lar R^3 fazoda biror S to‘plamni hosil qiladi. Demak, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ munosabatni Δ to‘plamni S to‘plamga uzluksiz akslantirish deb qarash mumkin.

Agar bu akslantirish o‘zaro bir qiymatli akslantirish bo‘lsa, ya’ni Δ to‘plamning turli $(u, v), (\bar{u}, \bar{v})$ nuqtalarini S to‘plamning turli $(x, y, z), (\bar{x}, \bar{y}, \bar{z})$ nuqtalariga akslantirsa, S to‘plamni R^3 fazoda sirt deb qarash mumkin. Odatda, bunday sirtlar **sodda sirtlar** deyiladi. Shu sababli

$$\begin{cases} x = x(u, v), \\ y = y(u, v), \\ z = z(u, v) \end{cases} \qquad ((u, v) \in \Delta)$$

tenglamalar sistemasi **sirtning parametrik tenglamasi** deyiladi, bunda u va v lar parametrlar deyiladi.

Xususan, $u = x, v = y$ bo‘lganda ushbu $\begin{cases} x = x, \\ y = y, \\ z = f(x, y) \end{cases}$ tenglamalar sistemasi aniqlaydigan sirt $z = f(x, y)$ tenglama bilan ifodalanadi.

Faraz qilaylik, sirtning parametrik tenglamasi Δ to‘plamda uzluksiz xususiy hosilalarga ega bo‘lib, ular yordamida quyidagi

$$F = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

funksiaonal matritsa tuzilgan bo‘lsin.

1) Agar $(u_0, v_0) \in \Delta$ nuqta uchun (bu nuqtaning S sirtdagi aksi $M_0(x_0, y_0, z_0)$) funksional F matritsaning 2-tartibli determinantidan kamida bittasi noldan farqli bo‘lsa, masalan

$$\begin{vmatrix} x_u'(u_0, v_0) & y_u'(u_0, v_0) \\ x_v'(u_0, v_0) & y_v'(u_0, v_0) \end{vmatrix} \neq 0.$$

U holda S sirtning M_0 nuqta atrofidagi qismi quyidagi

$$z = z(u(x, y), v(x, y)) = f(x, y)$$

tenglama bilan ifodalanadi, bunda $f(x, y)$ funksiya uzluksiz va uzluksiz xususiy hosilalarga ega bo‘ladi.

2) Agar (u_0, v_0) uchun F matritsaning barcha 2-tartibli determinantlari nolga teng bo‘lsa, S sirtning $M_0(x_0, y_0, z_0)$ nuqtasi **uning maxsus nuqtasi** deyiladi. Sirtning maxsus nuqta atrofidagi qismini $z = z(u(x, y), v(x, y)) = f(x, y)$ ko‘rinishda ifodalab bo‘lmaydi.

S sirt $z = f(x, y)$ tenglama bilan aniqlangan bo‘lsin. Bunda $f(x, y)$ funksiya XOY tekislikdagi D to‘plamda uzluksiz va uzluksiz $f_x'(x, y), f_y'(x, y)$ xususiy hosilalarga ega. Bunday S sirt har bir $M(x_0, y_0, z_0) \in S$ nuqtada urinma tekislikka ega. Urinma tekislikning tenglamasi ushbu

$$z - z_0 = f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0) \qquad (7.27)$$

ko‘rinishda bo‘ladi, bunda x, y, z lar urinma tekislikdagi o‘zgaruvchi nuqtaning koordinatalari. S sirt $M(x_0, y_0, z_0)$ nuqtada urinma tekislikka ega bo‘lsa, u holda shu nuqtada sirtning normalini aniqlash mumkin. Sirtning $M(x_0, y_0, z_0)$ nuqtasidan o‘tuvchi va shu nuqtadagi urinma tekislikka perpendikulyar bo‘lgan to‘g‘ri chiziq **sirt normali** deyiladi. Bu holda normalning tenglamasi

$$\frac{x - x_0}{f_x'(x_0, y_0)} = \frac{y - y_0}{f_y'(x_0, y_0)} = \frac{z - z_0}{-1} \qquad (7.28)$$

bo‘ladi. Bu yerda x, y, z lar normaldagi o‘zgaruvchi nuqtaning koordinatalari.

7.2.1-misol. $z = f(x, y) = x^2 + 2y^2$ paraboloidga (1,1,3) nuqtada o‘tkazilgan urinma va normal tenglamasini tuzing.

Yechilishi: ► Urinma tenglamasini tuzish uchun (7.27) formuladan foydalanamiz. Dastlab xususiy hosilalarni olamiz:

$$f_x'(x_0, y_0) = 2x = 2, \quad f_y'(x_0, y_0) = 2y = 2.$$

Shunda $z=f(x,y)=x^2+2y^2$ sirtga (1,1,3) nuqtada o‘tkazilgan urinma tenglamasi quyidagicha bo‘ladi: $z-3=2(x-1)+4(y-1)$.

Normal to‘g‘ri chiziq tenglamasini tuzish uchun (7.28) formulani qo‘llaymiz, shunda $z=f(x,y)=x^2+2y^2$ sirtga (1,1,3) nuqtada o‘tkazilgan normal tenglamasi quyidagicha bo‘ladi:

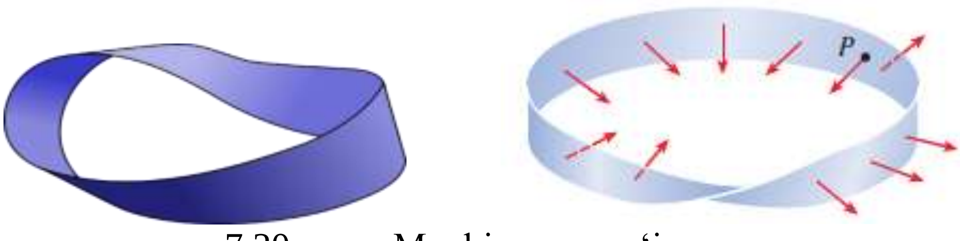
$$\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-3}{-1}. \quad \blacktriangleleft$$

Normalning OX, OY, OZ koordinata o‘qlarining muabat yo‘nalishi bilan tashkil etgan burchaklarini mos ravishda α, β, γ deb belhilasak, unda $\cos\alpha, \cos\beta, \cos\gamma$ lar **normalning yo‘naltiruvchi kosinuslari** deyiladi.

Fazoda S sodda sirt berilgan bo‘lsin. Bu sirtning har bir nuqtasida urinma tekislik mavjud bo‘lib, uning urinish nuqtasi sirt bo‘ylab uzluksiz o‘zgarib borsa, mos urinma tekislik ham o‘z holatini uzluksiz o‘zgartirib boradi.

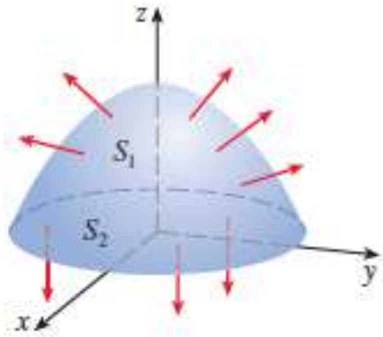
S sirtda biror M_0 nuqtani olaylik. Bu nuqta orqali o‘tkazilgan sirt normali 2 yo‘nalishga ega bo‘ladi, ulardan birini tanlaymiz. So‘ng M_0 nuqtadan chiqib, yana M_0 nuqtaga qaytadigan konturni qaraylik. Bu kontur S sirtga tegishli bo‘lib, uning chegarasini kesib o‘tmasin. M_0 nuqtada sirt normalining ma’lum yo‘nalishi olinganligini e’tiborga olib, o‘zgaruvchi M nuqtani M_0 dan boshlab, kontur bo‘yicha harakatlantirib, yana M_0 nuqtaga qaytganda (M nuqta kontur bo‘ylab o‘zgarganda mos nuqtadagi sirt normali ham o‘zgarib boradi) 2 xil holat yuz beradi:

1. M_0 nuqtadagi sirt normali kontur bo‘ylab harakatlanib, qaytib shu nuqtaga kelganda yo‘nalishi teskarisiga o‘zgaradi. Bunday sirtga **bir tomonli sirt** deyiladi. Bunga Myobius yaprog‘i misol bo‘ladi (7.20-rasm).



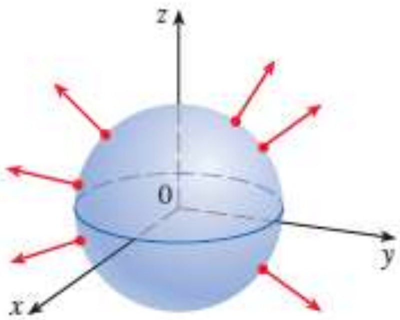
7.20-rasm. Myobius yaprog‘i

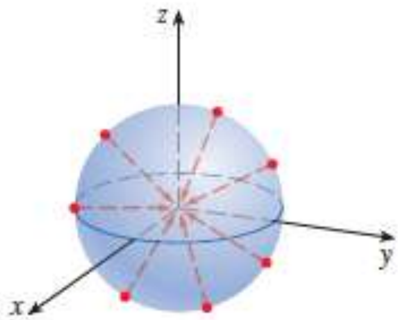
2. M_0 nuqtadagi sirt normali kontur bo‘ylab harakatlanib, qaytib shu nuqtaga kelganda ham yo‘nalishi o‘zgarmaydi. Bunday sirtlarga **2 tomonli sirt** deyiladi. Bunga $z=-(x^2+y^2)$ tenglama bilan aniqlanadigan giperboloid misol bo‘ladi (7.21-rasm). Bu sirt ustki (musbat yo‘nalishli) va ostki (manfiy yo‘nalishli) tomonlarga ega.



7.21-rasm. $z = -(x^2 + y^2)$ tenglama bilan aniqlanadigan giperboloid

$z^2 = 1 - x^2 - y^2$ tenglama bilan aniqlanadigan sirt (markazi koordinata boshida, radiusi 1 ga teng bo‘lgan sfera) ham 2 tomonli sirt bo‘lib, uning tashqi va ichki tomonlari bo‘ladi (7.22, a-rasmda musbat yo‘nalish, 7.22, b-rasmda manfiy yo‘nalish).





7.22-rasm. a) sferaning musbat yo‘nalishi; a) sferaning manfiy yo‘nalishi

$z = f(x, y)$ tenglama bilan aniqlanadigan sirt normalining yo‘naltiruvchi kosinuslari quyidagi formulalar bilan topiladi:

$$\cos\alpha = \frac{-f_x'(x,y)}{\pm\sqrt{1 + f_x'^2(x,y) + f_y'^2(x,y)}}, \qquad (7.29,a)$$

$$\cos\beta = \frac{-f_y'(x,y)}{\pm\sqrt{1 + f_x'^2(x,y) + f_y'^2(x,y)}} \qquad (7.29, b)$$

$$\cos\gamma = \frac{1}{\pm\sqrt{1 + f_x'^2(x,y) + f_y'^2(x,y)}}. \qquad (7.29, c)$$

Ushbu formulalarda $\cos\alpha, \cos\beta, \cos\gamma$ larning kvadrat ildizlari oldidagi ishoralarni tanlash bilan sirt tomonini aniqlaymiz. Agar musbat ishora olinsa, ustki tomon olinganini bildiradi.

Sirt yuzini hisoblash

$z = f(x,y)$ tenglama bilan aniqlanadigan S sirt yuzini hisoblash kerak bo‘lsin. $f(x,y)$ funksiya tekislikda yuzaga ega bo‘lgan D sohada uzluksiz va uzluksiz $f_x'(x,y), f_y'(x,y)$ xususiy hosilalarga ega bo‘lsin. D sohaning biror $P_D = \{D_1, D_2, ..., D_n\}$ bo‘laklashini olaylik. Bu bo‘laklashning bo‘lakchalari $D_1, D_2, ..., D_n$ bo‘ladi. Olingan bo‘laklashning bo‘luvchi chiziqlarini yo‘naltiruvchi sifatida qarab, ular orqali yasovchilari OZ o‘qiga parallel bo‘lgan silindrik sirtlar o‘tkazamiz. Bu silindrik sirtlar S sirtning ushbu $P_S = \{S_1, S_2, ..., S_n\}$ bo‘laklashnini hosil qiladi. Uning bo‘lakchalari $S_1, S_2, ..., S_n$ bo‘ladi.

Endi har bir D_k $(k = 1,2,...,n)$ da ixtiyoriy (ξ_k, η_k) nuqta olib, S sirtda ung mos nuqta (ξ_k, η_k, z_k) $(z_k = f(\xi_k, \eta_k))$ ni topamiz. Bu nuqta $(\xi_k, \eta_k, z_k) \in S_k$ $(k = 1,2,...,n)$ bo‘ladi. So‘ngra S sirtga shu (ξ_k, η_k, z_k) nuqtada urinma tekislik o‘tkazamiz. Bu urinma tekislik bilan silindrik sirtning kesishishidan hosil bo‘lgan urinma tekislik qismini T_k bilan, uning yuzini esa μT_k bilan belgilaymiz.

D_k to‘plam T_k ning orthogonal proyeksiyasi bo‘lganligi uchun

$$\mu D_k = \mu T_k \cdot |\cos\gamma_k|$$

bo‘ladi, bunda γ_k - S sirtga (ξ_k, η_k, z_k) nuqtada o‘tkazilgan urinma tekislik normalining OZ o‘qi bilan tashkil qilgan burchagi.

Ko‘rinib turibdiki, $\lambda_{P_D} \to 0$ da $\lambda_{P_S} \to 0$ bo‘ladi. Agar $\lambda_{P_D} \to 0$ da $\sum_{k=1}^{n} \mu T_k$ yig‘indi chekli limitga ega bo‘lsa, S **sirt yuzaga ega** deyiladi, limitning qiymati esa S sirtning yuzi deyiladi: $\mu S = \lim_{\lambda_{P_D} \to 0} \sum_{k=1}^{n} \mu T_k$.

Ma’lumki, $\cos\gamma_k = \dfrac{1}{\sqrt{1 + f_x'^2(\xi_k, \eta_k) + f_y'^2(\xi_k, \eta_k)}}$ bo‘lib, $\mu T_k = \dfrac{1}{\cos\gamma_k} \cdot \mu D_k$ tenglikdan $\sum_{k=1}^{n} \mu T_k = \sum_{k=1}^{n} \sqrt{1 + f_x'^2(\xi_k, \eta_k) + f_y'^2(\xi_k, \eta_k)} \cdot \mu D_k$ bo‘lishi kelib chiqadi. Bu tenglikning o‘ng tomonidagi yig‘indi $\sqrt{1 + f_x'^2(x,y) + f_y'^2(x,y)}$ funksiyaning integral yig‘indisi bo‘ladi. Bu funksiya D sohada uzluksiz va integrallanuvchi. Shunga ko‘ra,

$$\lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n}\sqrt{1+f_x'^2(\xi_k,\eta_k)+f_y'^2(\xi_k,\eta_k)}\cdot\mu D_k =$$

$$=\iint_D\sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}dxdy.$$

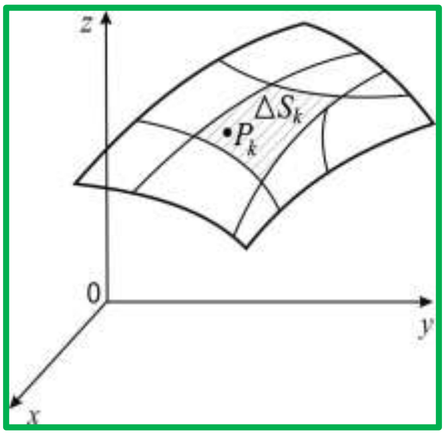
Demak, S **sirtning yuzi** quyidagiga teng bo'ladi:

$$\mu S == \iint_D\sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}dxdy. \qquad (7.30)$$

7.2.2. I tur sirt integrali va uning tatbiqlari

Fazoda bo'lakli silliq L yopiq chiziq bilan chegaralangan S silliq sirtni qaraymiz (7.23-rasm). Bu sirtni $S_1,S_2,...,S_n$ bo'laklarga bo'lamiz va bu bo'laklarning yuzalarini ham ΔS_k deb belgilaymiz. S sirtning har bir nuqtasida $f(x,y,z)$ uzluksiz funksiya berilgan bo'lsin. Sirtning har bir S_k bo'lagidan $P_k(x_k,y_k,z_k)$ nuqtalarni tanlaymiz va yig'indi tuzamiz:

$$\sigma=\sum_{i=1}^{n}F(P_k)\cdot\Delta S_k=\sum_{k=1}^{n}f(x_k,y_k,z_k)\cdot\Delta S_k$$



7.23-rasm. S silliq sirt

Bu yig'indi **I tur sirt integralining integral yig'indisi** deb ataladi. ΔS_k bo'laklarning diametrini d_k bilan belgilaymiz. Agar integral yig'indining $\max d_k\to 0$ dagi chekli limiti, S sirtni $S_1,S_2,...,S_n$ bo'laklarga bo'linish usuliga va har bir bo'lakdan $P_k(x_k,y_k,z_k)$ nuqtalarni tanlash usuliga bog'liq bo'lmagan holda mavjud bo'lsa, bu limit $f(x,y,z)$ funksiyadan S sirt yuzi bo'yicha olingan integral yoki I **tur sirt integrali** deyiladi:

$$\iint_S f(x,y,z)dS=\lim_{\max d_k\to 0}\sum_{k=1}^{n}f(x_k,y_k,z_k)\cdot\Delta S_k$$

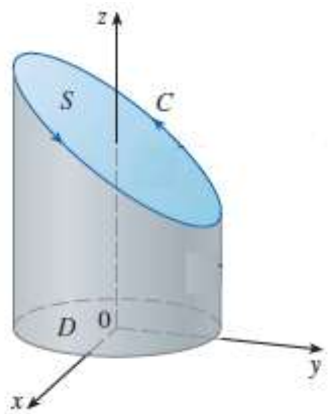
7.6-teorema (I tur sirt integralining mavjudligi haqida). Agar $f(x,y,z)$ funksiya S sirtda uzluksiz bo‘lsa, u holda bu funksiyaning sirt bo‘yicha I tur sirt integrali mavjud va quyidagicha bo‘ladi:

$$\iint_S f(x,y,z)ds = \iint_D f(x,y,z(x,y))\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}dxdy\,.$$

Agar S sirt oshkor ko‘rinishda $z=z(x,y)$ tenglama bilan berilgan bo‘lib, bu funksiya o‘zining $z'_x(x,y),\ z'_y(x,y)$ xususiy hosilalari bilan D_{xy} sohada uzluksiz bo‘lsa, u holda I tur sirt integralni hisoblash uni ikki karrali integralga keltirish bilan amalga oshiriladi:

$$\iint_S f(x,y,z)dS = \iint_{D_{xy}} f(x,y,z(x,y))\sqrt{1+(z'_x)^2+(z'_y)^2}\,dxdy. \qquad (7.31)$$

Bu yerda D_{xy} soha S sirtning Oxy tekislikdagi proyeksiyasidir (7.24-rasm).



7.24-rasm. S sirtning proyeksiyasi

Agar S sirt tenglamasi $x=x(y,z)$ yoki $y=y(x,z)$ tenglamalar bilan berilgan bo‘lsa, I tur sirt integralini hisoblash mos ravishda quyidagi formulalar bilan amalga oshiriladi:

$$\iint_S f(x,y,z)dS = \iint_{D_{yz}} f(x(y,z),y,z)\sqrt{1+(x'_y)^2+(x'_z)^2}\,dydz. \qquad (7.32)$$

$$\iint_S f(x,y,z)dS = \iint_{D_{xz}} f(x,y(x,z),z)\sqrt{1+(y'_x)^2+(y'_z)^2}\,dxdz. \qquad (7.33)$$

Agar integral ostidagi funksiya $f(x,y,z)\equiv 1$ bo‘lsa, I tur sirt integrali

$$S=\iint_S dS = \iint_{D_{xy}} \sqrt{1+(z'_x)^2+(z'_y)^2}\,dxdy \qquad (7.34)$$

sirt yuzini aniqlaydi.

Agar integral ostidagi funksiya S moddiy sirt bo‘yicha massa taqsimlanishining har bir nuqtasidagi $\rho(x,y,z)$ zichligini bildirsa, u holda I tur sirt integrali S sirtning massasini aniqlaydi:

$$m=\iint_S \rho(x,y,z)dS. \tag{7.35}$$

Moddiy sirtning koordinata tekisliklariga nisbatan statik momentlari quyidagi formulalar bilan hisoblanadi:

$$M_{xy}=\iint_S z\rho(x,y,z)dS,\quad M_{yz}=\iint_S x\rho(x,y,z)dS,\quad M_{xz}=\iint_S y\rho(x,y,z)dS, \tag{7.36}$$

Moddiy sirtning (x_c,y_c,z_c) og‘irlik markazi quyidagi formulalar bilan hisoblanadi:

$$x_c=\frac{1}{m}\iint_S x\rho(x,y,z)dS,\quad y_c=\frac{1}{m}\iint_S y\rho(x,y,z)dS,\quad z_c=\frac{1}{m}\iint_S z\rho(x,y,z)dS, \tag{7.37}$$

Moddiy sirtning Ox, Oy, Oz koordinata o‘qlariga va koordinata boshiga nisbatan inersiya momentlari

$$I_x=\iint_S\left(y^2+z^2\right)\rho(x,y,z)dS;\quad I_y=\iint_S\left(x^2+z^2\right)\rho(x,y,z)dS;$$

$$I_z=\iint_S (x^2+y^2)\rho(x,y,z)dS; \tag{7.38}$$

$$I_0=\iint_S\left(x^2+y^2+z^2\right)\rho(x,y,z)dS$$

I tur sirt integralining xossalari:

1⁰. $\iint_S kf(x,y,z)dS=k\iint_S f(x,y,z)dS$, k – o‘zgarmas son.

2⁰. $\iint_S\left(f(x,y,z)+\varphi(x,y,z)\right)dS=\iint_S f(x,y,z)dS+\iint_S \varphi(x,y,z)dS$.

3⁰. $\iint_S f(x,y,z)dS=\iint_{S_1} f(x,y,z)dS+\iint_{S_2} f(x,y,z)dS$, bu yerda $S=S_1\cup S_2, S_1\cap S_2=\varnothing$.

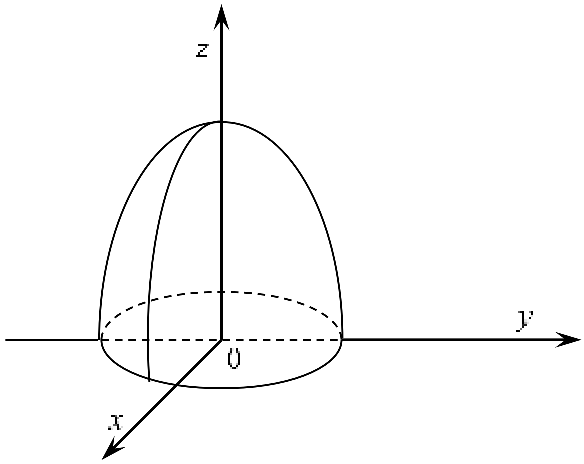
4⁰. Agar berilgan sirt ustida $f(x,y,z)\le\varphi(x,y,z)$ tengsizlik o‘rinli bo‘lsa, quyidagi tengsizlik o‘rinli bo‘ladi: $\iint_S f(x,y,z)dS\le\iint_S \varphi(x,y,z)dS$.

5⁰. $\left|\iint_S f(x,y,z)dS\right|\le\iint_S |f(x,y,z)|dS$.

6⁰. O‘rta qiymat haqidagi teorema. Agar $f(x,y,z)$ funksiya S sirtda uzluksiz bo‘lsa, u holda bu sirtda shunday $M_0(x_0,y_0,z_0)$ nuqta topiladiki, $\iint_S f(x,y,z)dS=f(x_0,y_0,z_0)\cdot S$ tenglik o‘rinli bo‘ladi, bu yerda S -sirt yuzi.

7.2.2-misol. $\iint\limits_S \sqrt{1+4x^2+4y^2}\,dS$ integralni hisoblang, bunda S sirt $z=1-x^2-y^2$ sirtning $z=0$ tekislik bilan kesilgan chekli qismi.

Yechilishi: ► Ushbu misolda S sirt tenglamasi $z=z(x,\ y)$ ko‘rinishdagi sirt $S: z=1-x^2-y^2$ (7.25-rasm).



7.25-rasm. $S: z=1-x^2-y^2$ sirt

(7.31) formuladan foydalanamiz. $z'_x(x,\ y)=-2x,\quad z'_y(x,\ y)=-2y$ bo‘lib, bundan $\sqrt{1+z'^2_x(x,\ y)+z'^2_y(x,\ y)}=\sqrt{1+4x^2+4y^2}$ bo‘ladi. S sirtning $X0Y$ tekislikdagi proyeksiyasi $D=\{(x,\ y)\in R^2 : x^2+y^2\le 1\}$. Shunda topilganlarni formulaga qo‘ysak, quyidagini hosil qilamiz:

$$\iint\limits_S \sqrt{1+4x^2+4y^2}\,dS=\iint\limits_D \sqrt{1+4x^2+4y^2}\cdot\sqrt{1+4x^2+4y^2}\,dxdy=\iint\limits_D \left(1+4x^2+4y^2\right)dxdy.$$

Endi ikki karrali integralni hisoblaymiz:

$$\iint\limits_D \left(1+4x^2+4y^2\right)dxdy=\left[\begin{array}{c} x=r\sin\varphi,\ 0\le\varphi\le 2\pi \\ y=r\cos\varphi,\ 0\le r\le 1\end{array}\right]=$$

$$=\int\limits_0^{2\varphi}\left[\int\limits_0^1\left(1+4r^2\right)rdr\right]d\varphi=\int\limits_0^{2\pi}\left(\frac{r^2}{2}+r^4\right)\Bigg|_0^1 d\varphi=3\pi .$$

Demak, I tur sirt integrali quyidagiga teng ekan:

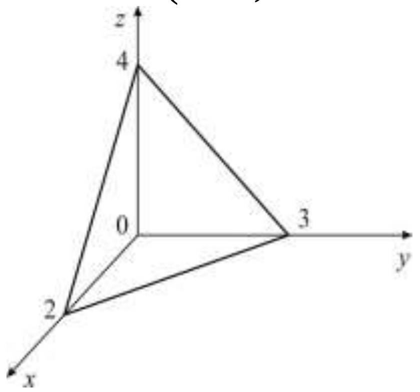
$$\iint\limits_S \sqrt{\left(1+4x^2+4y^2\right)}ds=3\pi .\ ◄$$

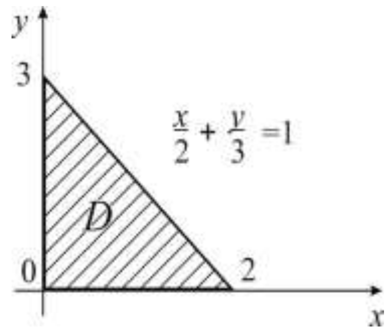
7.2.3-misol. $\iint\limits_S (2x+\frac{4}{3}y+z)dS$ integralini hisoblang, bu yerda S - $6x+4y+3z-12=0$ tekislikning birinchi oktantdagi qismi (7.26,a-rasm).

Yechilishi: ► Avval S sirtning oshkor tenglamasini tuzib olamiz:

$$z=4\left(1-\frac{x}{2}-\frac{y}{3}\right),\ (x,y)\in D_{xy}$$

bu yerda D_{xy} berilgan S sirtning $Oxy(z=0)$ tekislikdagi proyeksiyasini aniqlaydi, ya'ni $x=0$, $y=0$ va $\frac{x}{2}+\frac{y}{3}=1$ chiziqlar bilan chegaralangan uchburchak sohasi (9.26,b-chizma).





7.26-rasm. a) $6x+4y+3z-12=0$ tekislik; b) Tekislikning 1-oktantdagi qismi

I tur sirt integralini ikki karrali integralga keltirib hisoblaymiz. $z'_x=-2,\quad z'_y=-\frac{4}{3}$ bo'lgani uchun

$$dS=\sqrt{1+z_x'^2+z_y'^2}\,dxdy=\sqrt{1+4+\frac{16}{9}}\,dxdy=\frac{\sqrt{61}}{3}\,dxdy$$

bo'ladi. Bundan,

$$\iint_S (2x+\frac{4}{3}y+z)dS=\iint_{D_{xy}}\left(2x+\frac{4}{3}y+4\left(1-\frac{x}{2}-\frac{y}{3}\right)\right)\frac{\sqrt{61}}{3}dxdy=\frac{4\sqrt{61}}{3}\iint_{D_{xy}}dxdy$$

Oxirgi integral D_{xy} uchburchak soha yuzini beradi va u 3 ga teng.

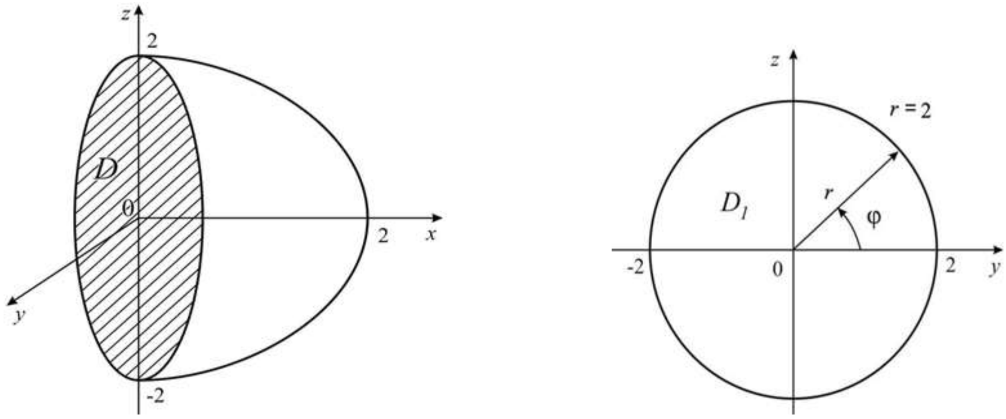
Demak, $\iint_S (2x+\frac{4}{3}y+z)dS=4\sqrt{61}.$ ◀

7.2.4-misol. $\iint_S (x+y^2+z^2-\frac{3}{2})dS$ integralini hisoblang, bu yerda S - $2x+y^2+z^2-4=0$ paraboloid sirtining $x=0$ tekislik bilan kesilgan qismi, $x\geq 0$ (7.27-rasm).

Yechilishi: ▶ Avval S sirtning oshkor tenglamasini tuzib olamiz. Bu yerda x ni y va z o'zgaruvchilar orqali ifodalash qulay $x=2-\frac{y^2+z^2}{2}$.

Shuning uchun sirt integrali quyidagi formuladan topamiz:

$$\iint_S f(x,y,z)dS=\iint_{D_{yz}} f(x(y,z),y,z)\sqrt{1+(x'_y)^2+(x'_z)^2}\,dydz.$$



7.27-rasm. a) Paraboloid; b) Paraboloidning aniqlanish sohasi

S sirtning Oyz tekislikdagi proyeksiyasini topamiz, buning uchun quyidagi sistemani yechamiz: $\begin{cases} x = 2 - \dfrac{y^2 + z^2}{2} \\ x = 0 \end{cases}$

Paraboloid $x = 0$ tekislik bilan $y^2 + z^2 = 4$ aylanada kesishadi. Demak, S paraboloid sirtining Oyz tekislikdagi proyeksiyasi $y^2 + z^2 \le 4$ doira sohasi bo‘ladi. $x'_y = -y,\ x'_z = -z$ va $dS = \sqrt{1 + y^2 + z^2}\,dydz$ ekanini e’tiborga olgan holda

$$\iint\limits_S \left(x + y^2 + z^2 - \frac{3}{2} \right) dS = \iint\limits_{D_{yz}} \left(2 - \frac{y^2 + z^2}{2} + y^2 + z^2 - \frac{3}{2} \right) \sqrt{1 + y^2 + z^2}\,dydz =$$

$$= \frac{1}{2} \iint\limits_S \left(1 + y^2 + z^2\right) \sqrt{1 + y^2 + z^2}\,dydz$$

ikki karrali integralga ega bo‘lamiz. Bu integralda $y = r\cos\varphi,\ z = r\sin\varphi$ va $dxdy = rdrd\varphi$ qutb koordinatalar sistemasiga o‘tish qulay. Qutb koordinatalar sistemasida D_{yz} soha $0 \le \varphi \le 2\pi,\ 0 \le r \le 2$ tengsizliklar bilan aniqlanadi (7.30-rasm). Natijada quyidagiga ega bo‘lamiz:

$$\frac{1}{2} \iint\limits_{D_{yz}} \left(1 + r^2\right) \sqrt{1 + r^2}\,rdrd\varphi = \frac{1}{4} \int_0^{2\pi} d\varphi \int_0^2 \left(1 + r^2\right)^{\frac{3}{2}} d\left(1 + r^2\right) = \frac{1}{4} \int_0^{2\pi} \frac{2}{5} \left(1 + r^2\right)^{5/2} \Bigg|_0^2 d\varphi = \frac{1}{10} \int_0^{2\pi} \left(25\sqrt{5} - 1\right) d\varphi =$$

$$= \frac{\pi}{5}\left(25\sqrt{5} - 1\right). \quad ◄$$

7.2.5-misol. $x = \sqrt{R^2 - y^2 - z^2}$ yarim sfera bo‘yicha massa tarqalgan bo‘lib, har bir nuqtadagi zichlik shu nuqtadan koordinata boshigacha bo‘lgan masofaga proporsional. Massani toping.

Yechilishi: ► Shartga ko‘ra $\gamma(x, y, z) = k\cdot(x^2+y^2+z^2)$ bo‘ladi, bunda k-proporsionallik koeffitsiyenti. (7.35) formulaga ko‘ra $m = \iint\limits_S k\cdot(x^2+y^2+z^2)dS$ bo‘ladi, bunda S-yuqori yarim sfera. Shunda

$$x'_y(y, z) = -\frac{y}{\sqrt{R^2-y^2-z^2}},\ x'_z(y, z) = -\frac{z}{\sqrt{R^2-y^2-z^2}}$$

bo‘lib, $\sqrt{1+x'^2_y(y, z)+x'^2_z(y, z)} = \frac{R}{x}$ ga egamiz. Natijada

$$m = \iint\limits_S k\cdot(x^2+y^2+z^2)dS = k\iint\limits_S R^2\frac{R}{x}dydz = kR^3\iint\limits_D \frac{1}{\sqrt{R^2-y^2-z^2}}dydz$$

tenglikni hosil qilamiz, bunda $D = \{(y,z): y^2+z^2 \le R^2\}$.

Endi ikki karrali integralni hisoblaymiz:

$$\iint\limits_D \frac{dydz}{\sqrt{R^2-y^2-z^2}} = \begin{bmatrix} y = r\cos\varphi,\ 0\le\varphi\le 2\pi \\ z = r\sin\varphi,\ 0\le r\le R \end{bmatrix} = \int_0^{2\pi}\left[\int_0^R \frac{rdr}{\sqrt{R^2-r^2}}\right]d\varphi =$$

$$= -\int_0^{2\pi}\left[\int_0^R (R^2-r^2)^{-\frac{1}{2}}\frac{1}{2}d(R^2-r^2)\right]d\varphi = -\frac{1}{2}\frac{(R^2-r^2)^{\frac{1}{2}}}{\frac{1}{2}}2\pi\Bigg|_0^R = 2\pi R.$$

Shunday qilib, izlanayotgan massa quyidagiga teng bo‘ladi:

$$m = 2kR^4\pi.\ ◄$$

7.2.3. II tur sirt integrali va uning tatbiqlari

R^3 fazoda D kontur bilan chegaralangan S silliq sirt berilgan bo‘lsin. S silliq sirt tenglamasi $F(x,y,z)=0$ oshkormas tenglama bilan berilgan bo‘lsin. Bunda $F(x,y,z)$ funksiya V sohada uzluksiz va $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ uzluksiz xususiy hosilalarga ega bo‘lsin. Bu holda normalning yo‘naltiruvchi kosinuslari

$$cos\alpha = \frac{F'_x}{\pm\sqrt{F'^2_x+F'^2_y+F'^2_z}},\quad cos\beta = \frac{F'_y}{\pm\sqrt{F'^2_x+F'^2_y+F'^2_z}},\quad cos\gamma = \frac{F'_z}{\pm\sqrt{F'^2_x+F'^2_y+F'^2_z}},$$

formulalar yordamida topiladi.

Ikki tomonli silliq yoki bo‘lakli silliq sirt berilgan bo‘lib, uning tomonidan biri tanlangan, ya’ni sirtning “yuqori” - S^+ musbat tomoni tanlangan bo‘lsin. Bu sirtning har bir nuqtasida $R = R(x,y,z)$ uzluksiz funksiya aniqlangan bo‘lsin. Sirtni ixtiyoriy chiziqlar bilan n ta ixtiyoriy $S_1, S_2, ..., S_n$ bo‘laklarga bo‘lamiz. Bu bo‘laklardan ixtiyoriy $M_i(x_i, y_i, z_i)$

nuqtalar tanlaymiz va bu nuqtalardagi funksiya qiymati $R(x_i, y_i, z_i)$ni hisoblaymiz. Sirtning $S_1, S_2, ..., S_n$ bo‘laklarining *Oxy* tekislikdagi proyeksiyalarini ΔS_k deb belgilaymiz va $\sum_{i=1}^{n} R(x_i, y_i, z_i) \cdot \Delta S_i$ integral yig‘indi tuzamiz. Bu integral yig‘indining $\max d_k \to 0$ dagi chekli limitiga **II tur sirt integrali** deyiladi va quyidagicha belgilanadi:

$$\iint_{S^+} R(x, y, z)\, dxdy = \lim_{\max d_k \to 0} \sum_{i=1}^{n} R(x_i, y_i, z_i) \cdot \Delta S_i$$

Xuddi shu kabi $\iint_{S^+} Q(x, y, z)\, dxdz$ va $\iint_{S^+} P(x, y, z)\, dydz$ integrallarni ta’riflash mumkin, bu yerda soha mos ravishda *Oxz* va *Oyz* tekisliklarga proyeksiyalanadi.

7.7-teorema (II tur sirt integralining mavjudligi haqida). Agar $f(x, y, z)$ funksiya S sirtda uzluksiz bo‘lsa, u holda bu funksiyaning S sirt bo‘yicha II tur integrali mavjud va quyidagicha bo‘ladi:

$$\oiint_{S} f(x, y, z) dxdy = \iint_{S} f(x, y, z(x, y)) dxdy. \qquad (**)$$

Agar silliq yoki bo‘lakli silliq *S* sirt tenglamasi $z = z(x, y)$ oshkor funksiya bilan berilgan bo‘lsa va sirtning yuqori ($\cos\gamma > 0$) tomoni tanlangan bo‘lsa, hamda $R = R(x, y, z)$ funksiya *S* da uzluksiz funksiya bo‘lsa, u holda $\iint_{S^+} R(x, y, z) dxdy = \iint_{D_{xy}} R(x, y, z(x, y)) dxdy$ tenglik o‘rinli. Bu yerda D_{xy} - *S* sirtning *Oxy* tekislikdagi proyeksiyasi. O‘ng tomondagi ikki karrali integral mavjud bo‘lsa, ikkinchi tur sirt integrali ham mavjud bo‘ladi. Agar *S* sirtning quyi(ya’ni $\cos\gamma < 0$) tomoni bo‘yicha integral hisoblansa, $\iint_{S^-} R(x, y, z) dxdy = -\iint_{D_{xy}} R(x, y, z(x, y)) dxdy$ formula o‘rinli bo‘ladi.

Xuddi shu kabi, *S* sirt tenglamasi $x = x(y, z)$ yoki $y = y(x, z)$ tenlamalar bilan berilsa, mos ravishda quyidagi tengliklar o‘rinli:

$$\iint_{S} P(x, y, z) dydz = \pm \iint_{D_{yz}} P(x(y, z), y, z) dydz$$

$$\iint_{S} Q(x, y, z) dxdz = \pm \iint_{D_{xz}} P(x, y(x, z), z) dxdz .$$

Bu yerdagi “+” ishora $\cos\alpha > 0$ (mos holda $\cos\beta > 0$) bo‘lganda, “-” ishora esa $\cos\alpha < 0$ (mos holda $\cos\beta < 0$) bo‘lganda olinadi.

Umumiy II tur sirt integrali deb quyidagi integralga aytiladi:

$$\iint_{S^+} Pdydz + Qdxdz + Rdxdy.$$

Uchta $P(x,y,z)$, $Q = Q(x,y,z)$, $R = R(x,y,z)$ uzluksiz funksiyalar va S^+ $\vec{n}(\cos\alpha;\cos\beta;\cos\gamma)$ normal bilan xarakterlanuvchi silliq S sirtning tomoni bo‘lsa, u holda **II tur sirt integrali I tur sirt integrali orqali quyidagi formula bilan ifodalanadi:**

$$\iint_{S^+} Pdydz + Qdxdz + Rdxdy = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS. \qquad (7.39)$$

II tur sirt integrali I tur sirt integralining hamma xossalariga ega, faqat sohaning tomoni o‘zgarganda integral ishorasi qarama-qarshisiga o‘zgaradi.

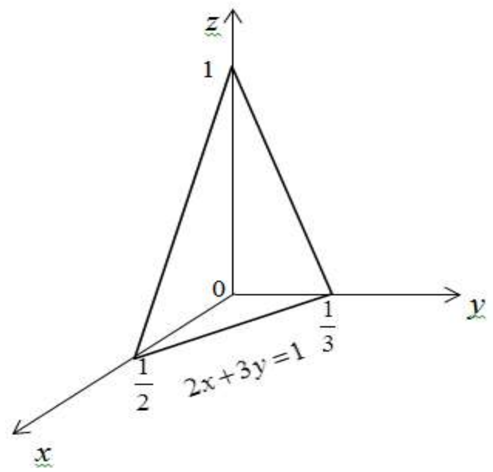
7.2.6-misol. $\iint_S 2xdydz + 3ydzdx + 4zdxdy$ integralini hisoblang, bu yerda S - $2x + 3y + z = 1$ tekislikning birinchi oktantdagi qismi(normal Oz o‘qi bilan o‘tkir burchak tashkil etadi).

Yechilishi: ► Berilishiga ko‘ra, S sirt musbat oriyentirlangan (7.28-rasm), $\cos\gamma > 0$, normal $\vec{n}(2,3,1)$ va $|\vec{n}| = \sqrt{4+9+1} = \sqrt{14}$,

$$\cos\alpha = \frac{2}{\sqrt{14}}, \cos\beta = \frac{3}{\sqrt{14}}, \cos\gamma = \frac{1}{\sqrt{14}}.$$

Shuningdek, $z = 1 - 2x - 3y$, $z'_x = -2$, $z'_y = -3$ bo‘lgani uchun,

$$dS = \sqrt{1 + (z'_x)^2 + (z'_y)^2}dxdy = \sqrt{1+4+9}dxdy = \sqrt{14}dxdy.$$



7.28-rasm. $2x + 3y + z = 1$ tekislik

Demak,

$$\iint_S 2xdydz + 3ydzdx + 4zdxdy = \iint_S (2x\cos\alpha + 3y\cos\beta + 4z\cos\gamma)dS =$$

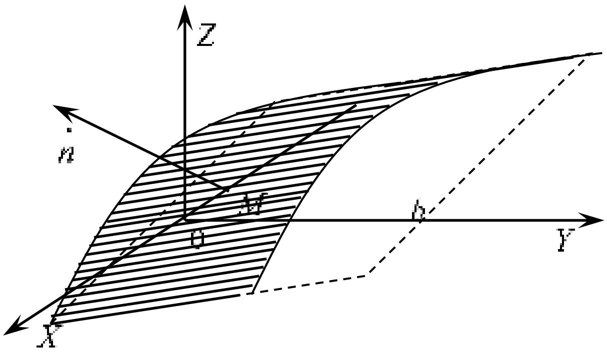
$$= \iint_{D_{xy}} \left(\frac{2\cdot 2x}{\sqrt{14}} + \frac{3\cdot 3y}{\sqrt{14}} + \frac{4z}{\sqrt{14}} \right)\sqrt{14}dxdy = \int_0^{1/2} dx \int_0^{\frac{1}{3}-\frac{2}{3}x} (4x + 9y + 4(1-2x-3y))dy =$$

$$= \int_0^{1/2} dx \int_0^{\frac{1}{3}-\frac{2}{3}x} (4-4x-3y)dy = \frac{1}{3}\int_0^{1/2}\left(\frac{7}{2} - 10x + 6x^2\right)dx = \frac{1}{3}\left(\frac{7}{2}x - 5x^2 + 2x^3\right)\Big|_0^{1/2} = \frac{1}{4}$$

◄

7.2.7-misol. $\iint_S (y^2 + z^2)dxdy$ integralni hisoblang, bunda S sirt $z = \sqrt{a^2 - x^2}$ ning $y = 0$, $y = b$ tekisliklar orasidagi qismining ustki tomoni (7.29-rasm).

Yechilishi: ► Sirtning M nuqtadagi normali OZ o‘qi bilan o‘tkir



7.29-rasm. $z = \sqrt{a^2 - x^2}$ sirt

burchak tashkil qiladi. Shuning uchun berilgan integralni (**) formulaga ko‘ra hisoblashda musbat ishora bilan olamiz. S sirtning XOY tekisligidagi proyeksiyasi ushbu $D = \{(x, y) \in R^2 : -a \le x \le a,\ 0 \le y \le b\}$ to‘rtburchakdan iborat bo‘ladi. Shunda so‘ralgan integral quyidagiga teng bo‘ladi:

$$\iint_S (y^2 + z^2)dxdy = \iint_D \left[y^2 + \sqrt{a^2 - x^2}\right]dxdy =$$

$$= \int_{-a}^{a}\left[\int_0^b (y^2 + a^2 - x^2)dy\right]dx = \int_{-a}^{a}\left(\frac{y^3}{3} + a^2y - x^2y\right)\Bigg|_{y=0}^{y=b} dx =$$

$$= \int_{-b}^{a}\left(\frac{b^3}{3} + a^2b - x^2b\right)dx = \left(\frac{b^3}{3}x + a^2bx - \frac{x^3}{3}b\right)\Bigg|_{-a}^{a} = \frac{2}{3}ab(b^2 + 2a^2).$$ ◄

7.2.8-misol. $\iint\limits_S \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+kz\right)dxdy$ integralni hisoblang, bunda S sirt $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoidning pastki yarim qismining tashqi tomoni.

Yechilishi: ► Ellipsning pastki yarim qismi $z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ ga teng bo'lib, uning XOY tekislikdagi proyeksiyasi quyidagicha

$$D=\left\{(x,\ y)\in R^2:\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$$

bo'ladi. S sirtning tashqi tomonidagi normali OZ o'qi bilan o'tmas burchak tashkil etadi. Shuning uchun berilgan integralni hisoblashda manfiy ishora olinadi (**):

$$\iint\limits_S \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+kz\right)dxdy=-\iint\limits_D \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dxdy=$$

$$=\iint\limits_D \left(kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy\,.$$

Bu integralni hisoblash uchun $x=ar\cos\varphi$, $y=br\sin\varphi$ almashtirish bajaramiz:

$$\iint\limits_D \left(kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy=\int_0^{2\pi}\left[\int_0^1\left(kc\sqrt{1-r^2}-r^2\right)abrdr\right]d\varphi=$$

$$=ab\int_0^{2\pi}\left[\int_0^1\left(kcr\sqrt{1-r^2}-r^3\right)dr\right]=2\pi ab\left[-\frac{kc}{2}\frac{\left(1-r^2\right)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{r^4}{4}\right]\Bigg|_0^1=2\pi ab\left(\frac{kc}{3}-\frac{1}{4}\right).$$

Shunday qilib, izlanayotgan integral quyidagiga teng bo'ladi:

$$\iint\limits_S \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+kz\right)dxdy=2\pi ab\left(\frac{kc}{3}-\frac{1}{4}\right).$$ ◄

Mavzu yuzasidan savollar:

1. I tur sirt integrali deb nimaga aytiladi?
2. I tur sirt integrali qanday hisoblanadi?
3. I tur sirt integralining geometrik tatbig'ini ayting.
4. I tur sirt integralining fizik tatbiqlari deb nimaga aytiladi?
5. I tur sirt integralining xossalarini ayting.
6. Ikki tomonlama sirt nima?
7. Bir tomonlama sirtga misol keltiring.

8. II tur sirt integrali deb nimaga aytiladi?
9. II tur sirt integrali qanday hisoblanadi?
10. I va II tur sirt integrallari qanday formula bilan bog‘langan?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Birinchi tur sirt integralini hisoblang: $\iint\limits_{\sigma}(x^2+y^2)d\sigma$, bu yerda $\sigma: z^2=x^2+y^2$ konus sirtining $z=0$ va $z=1$ tekisliklar orasidagi qismi.

2. Agar moddiy $S:z=\sqrt{4-x^2-y^2}$ yarim sfera sirti bo‘yicha massa taqsimlanishining har bir nuqtasidagi zichligi $\rho=x^2y^2$ bo‘lsa, uning massasini hisoblang.

3. Ikkinchi tur sirt integralini hisoblang: $\iint\limits_{S} xdydz$, bu yerda $S: x+2y+z=6$ tekislikning birinchi oktantdagi qismining yuqori tomoni.

4. $\iint\limits_{\sigma} xdydz+ydzdx+zdxdy$ ikkinchi tur sirt integralini hisoblang, bunda $\sigma-$sirt $x+z-1=0$ tekislikning $y=0,\ y=4$ tekisliklar bilan kesib olingan va birinchi oktantda yotgan qismining ustki tomoni.

5. $\iint\limits_{\sigma}\frac{d\sigma}{(x+z+1)^2}$ birinchi tur sirt integralini hisoblang, bunda $\sigma-$sirt $x+y+z=1$ tekislikning birinchi oktantda yotgan qismi.

TESTLAR

1. Birinchi tur sirt integralini hisoblang: $\iint\limits_{S} x^2y^2dS$, bu yerda $S: z=\sqrt{9-x^2-y^2}$ yarim sfera.

A) $\frac{282\pi}{5}$ B) $\frac{182\pi}{5}$ C) $\frac{82\pi}{5}$ D) $\frac{486\pi}{5}$

2. Agar moddiy $S:z=\sqrt{4-x^2-y^2}$ yarim sfera sirti bo‘yicha massa taqsimlanishining har bir nuqtasidagi zichligi $\rho=x^2+y^2$ bo‘lsa, uning massasini hisoblang.

A) $\frac{128\pi}{15}$ B) $\frac{64\pi}{15}$ C) $\frac{64\pi}{3}$ D) $\frac{128\pi}{3}$

3. Ikkinchi tur sirt integralini hisoblang: $\iint\limits_S xdydz + ydxdz$, bu yerda $S: x+2y+z=6$ tekislikning birinchi oktantdagi qismining yuqori tomoni.

A) 36 B) 18 C) 9 D) 32

4. Agar $f(x,y,z)=1$ bo‘lsa, I tur sirt integral nimani aniqlaydi?

A) Integrallash sirtining yuzini
B) Sirt massasini
C) Sirt bilan chegaralangan jism hajmini
D) $\vec{F}$ kuch bajargan ishni

5. Quyidagilardan qaysi biri II tur sirt integralning tatbig‘i bo‘ladi?

A) Vektor maydon oqimini hisoblash
B) Integrallash sirtining yuzini
C) Sirt bilan chegaralangan jism hajmini
D) Sirt massasini

7.3-§. Vektor va skalyar maydonlar

7.3.1. Skalyar maydon. Sath sirti va sath chizig‘i

Fazoning har bir M nuqtasida u skalyar kattalikning son qiymati aniqlangan qismiga (yoki butun fazoga) **skalyar maydon** deyiladi.

Agar u kattalik t vaqtga bog‘liq bo‘lmasa, bu kattalik bilan aniqlangan maydonga **statsionar maydon**, aks holda **nostatsionar maydon** deyiladi.

Statsionar maydonda u kattalik faqat M nuqtaning fazodagi o‘rniga bog‘liq bo‘ladi va $u=u(x,y,z)$ kabi belgilanadi, bu funksiyaga **maydon funksiyasi** deyiladi.

Skalyar maydonning geometrik tasviri sath sirtlari hisoblanadi. Fazoning $u=u(x,y,z)$ maydon funksiyasi o‘zgarmas C qiymatga teng bo‘ladigan nuqtalari to‘plamiga skalyar maydonning **sath sirti** deyiladi. Sath sirti $u(x,y,z)=C$ tenglama bilan aniqlanadi. Bunday sirtlar to‘plami qaralayotgan sohani to‘ldiradi, ayni paytda sohaning har bir nuqtasidan bitta va faqat bitta sath sirti o‘tadi. Ravshanki, bunday sirtlar o‘zaro kesishmaydi.

Tekislikning har bir M nuqtasida z skalyar kattalik aniqlangan qismiga (yoki butun tekislikka) **yassi skalyar maydon** deyiladi. Yassi skalyar maydon funksiyasi $z = z(x, y)$ ko‘rinishida bo‘ladi. Yassi skalyar maydonning geometrik tasviri sath chizig‘i bo‘ladi va u $z(x, y) = C$ tenglik bilan aniqlanadi.

7.3.1-misol. $u = \ln(x^2 + y^2 + z^2)$ funksiya bilan aniqlanadigan skalyar maydonning sath sirtini toping.

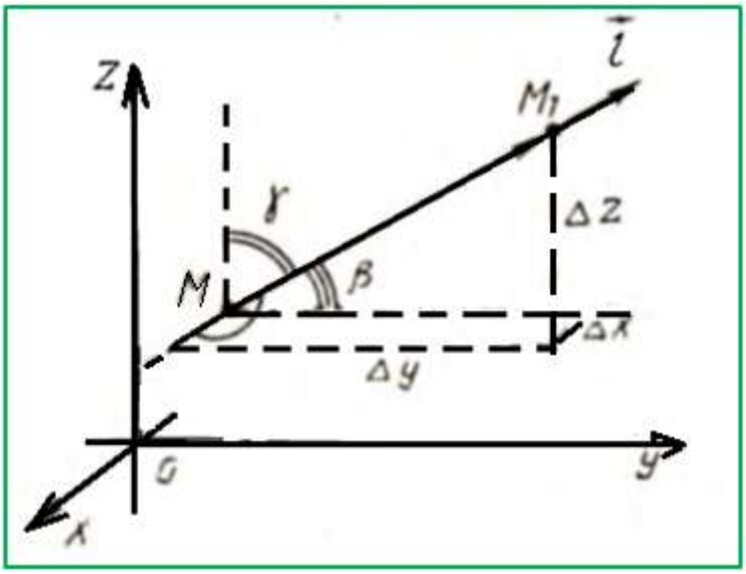
Yechilishi: ► $\ln\left(x^2 + y^2 + z^2\right) = C$, $x^2 + y^2 + z^2 = e^C$ bo‘lgani uchun berilgan skalyar maydon sath sirtlari radiusi $R = \sqrt{e^C}$ bo‘lgan sfera sirtidan iborat.◄

7.3.2. Yo‘nalish bo‘yicha hosila. Skalyar maydon gradiyenti

Skalyar maydonnning muhim tushunchasi – bu berilgan yo‘nalish bo‘yicha hosiladir.

Faraz qilaylik, skalyar maydonning differensiallanuvchi funksiyasi $u = u(x, y, z)$ bo‘lsin. Bu maydondagi biror $M(x, y, z)$ nuqtani va shu nuqtadan chiquvchi biror $\vec{l}$ nurni qaraymiz. Bu nurning *Ox*, *Oy*, *Oz* o‘qlari bilan tashkil qilgan burchaklarini α, β, γ orqali belgilaymiz. Agar birlik vektor $\vec{l}_0$ bu nur bo‘yicha yo‘nalgan bo‘lsa, u holda quyidagiga ega bo‘lamiz (7.30-rasm):

$$\vec{l}_0 = \cos\alpha \cdot \vec{i} + \cos\beta \cdot \vec{j} + \cos\gamma \cdot \vec{k}$$



7.30-rasm. $\vec{l}$ nur yo‘nalishi

Skalyar maydonning differensiallanuvchi $u = u(x, y, z)$ funksiyasining $\vec{l}$ yo‘nalish bo‘yicha hosilasi $\frac{\partial u}{\partial \vec{l}}$ quyidagi

$$\frac{\partial u}{\partial \vec{l}} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma \qquad (7.40)$$

formula bilan aniqlanadi.

Agar M nuqta tayinlangan bo‘lsa, u holda hosilaning kattaligi faqat $\vec{l}$ nurning yo‘nalishigagina bog‘liq bo‘ladi. $\vec{l}$ yo‘nalish bo‘yicha hosila xususiy hosilalarga o‘xshash u funksiyaning mazkur yo‘nalishdagi o‘zgarish tezligini xarakterlaydi. Hosilaning $\vec{l}$ yo‘nalish bo‘yicha absolyut miqdori $\left|\frac{\partial u}{\partial \vec{l}}\right|$ tezlik kattaligini aniqlaydi, hosilaning ishorasi esa u funksiya o‘zgarishini xarakterlaydi, ya’ni

agar $\frac{\partial u}{\partial \vec{l}} > 0$ bo‘lsa, u holda funksiya bu yo‘nalishda o‘sadi,

agar $\frac{\partial u}{\partial \vec{l}} < 0$ bo‘lsa, u holda funksiya bu yo‘nalishda kamayadi.

Agar yo‘nalish koordinatalar o‘qining yo‘nalishlaridan biri bilan bir xil bo‘lsa, u holda bu yo‘nalish bo‘yicha hosila tegishli xususiy hosilaga teng, masalan, $\cos\alpha = 1,\ \cos\beta = 0,\ \cos\gamma = 0$ va $\frac{\partial u}{\partial \vec{l}} = \frac{\partial u}{\partial x}$.

7.3.2-misol. $u = xyz$ funksiyaning $M(-1;\ 2;\ 4)$ nuqtada, shu nuqtadan $M_1(-3;\ 4;\ 5)$ nuqtaga tomon yo‘nalishdagi hosilasini toping.

Yechilishi: ► Dastlab $\overrightarrow{MM_1}$ vektorni topamiz:

$$\overrightarrow{MM_1} = (-3+1)\vec{i} + (4-2)\vec{j} + (5-4)\vec{k} = -2\vec{i} + 2\vec{j} + \vec{k}$$

va unga mos birlik vektorni topamiz: $\vec{l}_0 = -\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$.

Shunday qilib, $\vec{l}_0$ vektor quyidagi yo‘naltiruvchi kosinuslarga ega:

$$\cos\alpha = -\frac{2}{3},\ \cos\beta = \frac{2}{3},\ \cos\gamma = \frac{1}{3}.$$

Endi $u = xyz$ funksiyaning xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x} = yz,\ \frac{\partial u}{\partial y} = xz,\ \frac{\partial u}{\partial z} = xy.$$

va ularni (7.40) formulaga ko‘ra, $M(-1;\ 2;\ 4)$ nuqtada hisoblaymiz:

$$\frac{\partial u}{\partial \vec{l}} = 8\cdot\left(-\frac{2}{3}\right) + (-4)\cdot\frac{2}{3} + (-2)\cdot\frac{1}{3} = -\frac{26}{3}.$$

Yechimdagi minus ishora berilgan yo‘nalishda $u = xyz$ funksiyaning kamayishini ko‘rsatadi. ◀

$u = f(x,y,z)$ **skalyar maydonning gradiyenti** deb, quyidagi tenglik bilan aniqlanadigan vektorga aytiladi:

$$grad\, u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k} \text{ yoki } \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}. \qquad (7.41)$$

Binda $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ - differensiallovchi operator.

$u = f(x,y,z)$ funksiyaning berilgan M nuqtadagi gradiyenti bilan bu nuqtadagi yo‘nalish bo‘yicha hosila orasidagi bog‘lanishni ifodalovchi quyidagi munosabat o‘rinli: $\frac{\partial u}{\partial \vec{l}} = grad\, u \cdot \vec{l}$.

7.3.3-misol. $u = \sqrt{x^2 + y^2 + z^2}$ skalyar maydonning $M(x,y,z)$ nuqtadagi gradiyentini toping.

Yechilishi: ▶ Avval xususiy hosilalarni hisoblaymiz:

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{u}, \quad \frac{\partial u}{\partial x} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{u}, \quad \frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{u}.$$

Demak (7.41) formulaga ko‘ra, $gradu = \frac{x}{u}\vec{i} + \frac{y}{u}\vec{j} + \frac{z}{u}\vec{k}$. ◀

Skalyar maydonning sath sirtlari konsentrik sferalardan iborat bo‘lgani uchun *gradu* uning radiusi bo‘ylab yo‘nalgan bo‘ladi, shu bilan birga $|gradu| = \sqrt{\frac{x^2}{u^2} + \frac{y^2}{u^2} + \frac{z^2}{u^2}} = \frac{u}{u} = 1$, ya’ni funksiya o‘sishining eng katta tezligi 1 ga teng bo‘ladi.

7.3.3. Vektor maydon. Vektor maydon oqimi

R^2 fazoda (tekislikda) D to‘plamdan olingan har bir (x,y) nuqtaga biror qoida asosida faqat bitta $\vec{F}(x,y)$ vektor mos qo‘yilgan bo‘lsa, bunday fazo **2 o‘lchovli vektor maydon** deyiladi va quyidagicha yoziladi:

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = \langle P, Q \rangle.$$

R^3 fazoda E to‘plamdan olingan har bir (x,y,z) nuqtaga biror qoida asosida faqat bitta $\vec{F}(x,y,z)$ vektor mos qo‘yilgan bo‘lsa, unga **uch o‘lchovli vektor maydon** deyiladi va quyidagicha yoziladi:

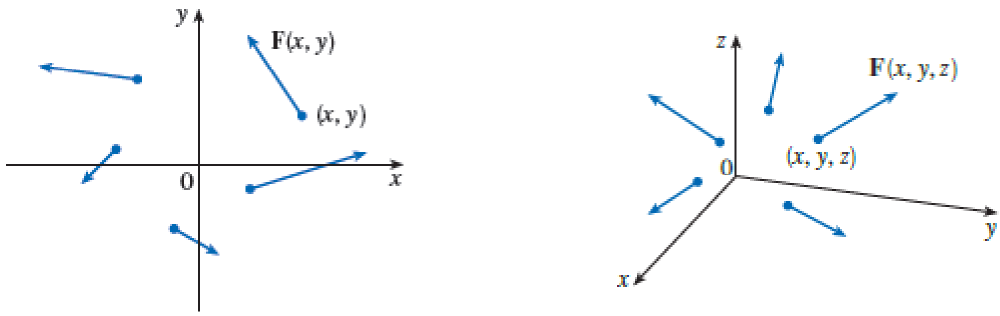
$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}. \qquad (7.42)$$

$\vec{F}(x,y,z)$ vektor maydonning berilishi uchta $P=P(x,y,z)$, $Q=Q(x,y,z)$, $R=R(x,y,z)$ skalyar maydonning berilishiga teng kuchli bo‘ladi. Agar P, Q, R o‘zgarmas kattaliklar bo‘lsa, vektor maydon **bir jinsli maydon** deyiladi.

Kuch maydoni (og‘irlik kuchi maydoni), elektr maydoni, elektromagnit maydon, oqayotgan suyuqlikning tezliklari maydoni vektor maydonga misol bo‘ladi.

Biz $\vec{F}$ vektor faqat nuqtaning vaziyatiga bog‘liq bo‘ladigan, lekin vaqtga bog‘liq bo‘lmaydigan statsionar maydonlarini qarab chiqamiz.

Vektor maydonni o‘rganishda vektor chiziqlari muhim rol o‘ynaydi.

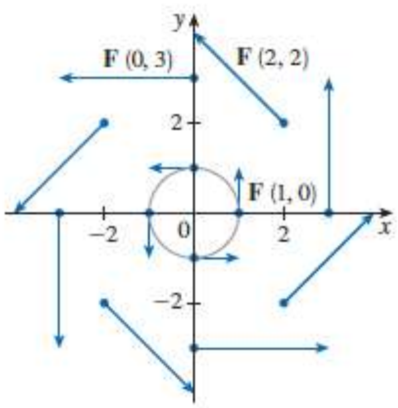


7.31-rasm. a) Ikki o‘lchovli vektor maydon; b) Uch o‘lchovli vektor maydon

7.31,a-rasmda ikki o‘lchovli vektor maydon, 7.31,b-rasmda esa 3 o‘lchovli vektor maydon tasvirlangan.

7.3.4-misol. $\vec{F}(x,y)=-y\vec{i}+x\vec{j}$ funksiya bilan berilgan vektor maydon komponentlarini ba’zilarini aniqlang.

Yechilishi: ► $\vec{F}(1,0)=-0\cdot\vec{i}+1\cdot\vec{j}=\vec{j}$ funksiya $\vec{j}=\langle 0,1\rangle$ vektorni aniqlaydi. Bu vektor 7.32-rasmda $(1,0)$ nuqtadan chiquvchi birlik vektorni tasvirlaydi. Jadvalda vektor funksiyaning bir nechta qiymatlari keltirilgan.



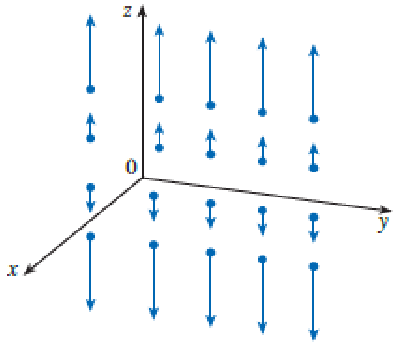
(x, y)	F(x, y)	(x, y)	F(x, y)
(1, 0)	⟨0, 1⟩	(−1, 0)	⟨0, −1⟩
(2, 2)	⟨−2, 2⟩	(−2, −2)	⟨2, −2⟩
(3, 0)	⟨0, 3⟩	(−3, 0)	⟨0, −3⟩
(0, 1)	⟨−1, 0⟩	(0, −1)	⟨1, 0⟩
(−2, 2)	⟨−2, −2⟩	(2, −2)	⟨2, 2⟩
(0, 3)	⟨−3, 0⟩	(0, −3)	⟨3, 0⟩

◄

7.32-rasm. $\vec{F}(x,y)=-y\vec{i}+x\vec{j}$ vektor maydon

7.3.5-misol. $\vec{F}(x,y,z) = z\vec{k}$ funksiya bilan berilgan vektor maydonni tasvirlang.

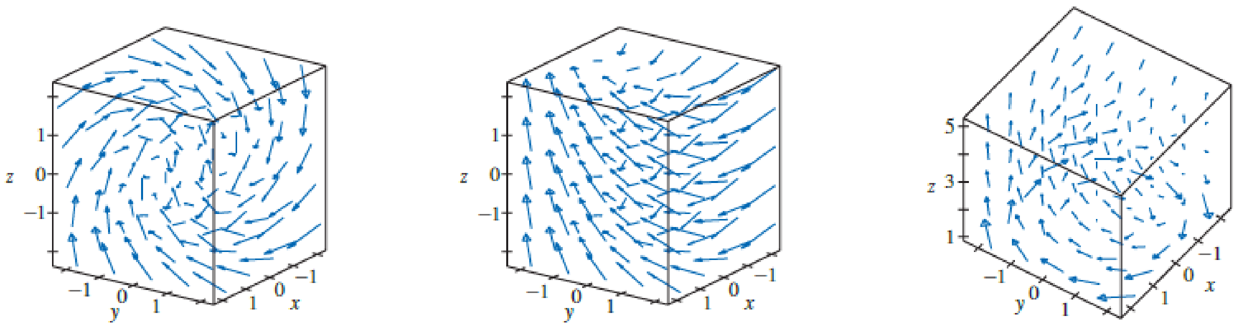
Yechilishi: ► $\vec{F}(x,y,z) = z\vec{k}$ funksiya 7.33-rasmdagidek vertikal vektorlarni chizadi.



7.33-rasm. $\vec{F}(x,y,z) = z\vec{k}$ vektor maydon ◄

Aniq maydonlarda vektor chiziqlar ma’lum fizik ma’noga ega bo‘ladi. Agar vektor chiziqlar suyuqlikning zarrachalari harakatlanayotgan yo‘nalishni bildirsa, u holda S maydonni – baliqchilarning to‘ri deb faraz qilish mumkin, chunki u suyuqlik oqimini to‘sa olmaydi, shunda $\vec{F}$ oqayotgan suyuqlik miqdoriga teng bo‘ladi.

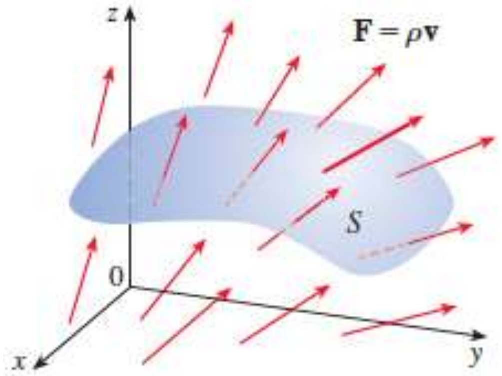
7.34-rasmda turli xil ko‘riishdagi vektor funksiyalar va ularning yo‘nalishlari tasvirlangan:



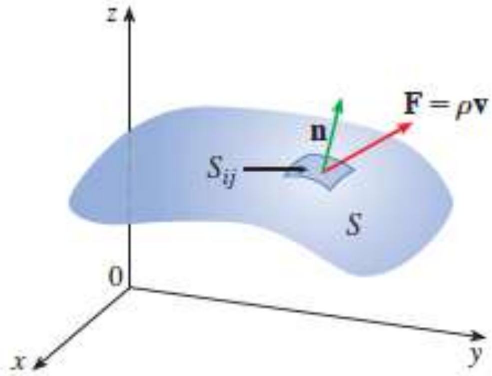
7.34-rasm.

a) $\vec{F}(x,y,z) = y\vec{i} + z\vec{j} + x\vec{k}$; b) $\vec{F}(x,y,z) = y\vec{i} - 2\vec{j} + x\vec{k}$; c) $\vec{F}(x,y,z) = \frac{y}{z}\vec{i} - \frac{x}{z}\vec{j} + \frac{z}{4}\vec{k}$

Agar $\vec{F}$ elektr maydoni bo‘lsa, u holda vektor chiziqlar bu maydonning kuch chiziqlari bo‘ladi (7.35- va 7.36-rasmlar).



7.35-rasm. Kuch chiziqlari



7.36-rasm.

Agar *l* vektor chiziqning tenglamasi parametrik ko‘rinishda berilgan bo‘lsa, ya’ni

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

u holda uning radius-vektori $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ ko‘rinishga ega bo‘ladi va $d\vec{r} = dx \cdot \vec{i} + dy \cdot \vec{j} + dz \cdot \vec{k}$ vektor *l* ga o‘tkazilgan urinma bo‘yicha yo‘naladi. Vektor chiziqning ta’rifiga asosan $\vec{F}$ va $d\vec{r}$ vektorlar kollinear bo‘lgani uchun, ushbu ifodani yozish mumkin:

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

Bu ifodani sistema ko‘rinishida yechib, vektor maydonning vektor chizig‘ini topish mumkin.

Faraz qilaylik, *Oxyz* fazoning ***V*** sohasida (7.42) vektor maydon berilgan bo‘lsin, bunda $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ - shu sohadagi uzluksiz funksiyalar. Bu sohada oriyentirlangan S sirtni olamiz, uning har bir nuqtasida normalning musbat yo‘nalishi

$$\overrightarrow{n_0} = \cos\alpha \cdot \vec{i} + \cos\beta \cdot \vec{j} + \cos\gamma \cdot \vec{k}$$

birlik vektor orqali aniqlansin, bunda α, β, γ - normal $\vec{n}_0$ ning koordinata o‘qlari bilan hosil qilgan burchaklari.

$\vec{F}$ vektorning *S* sirt orqali o‘tuvchi **oqimi** deb, quyidagi II tur sirt integraliga aytiladi:

$$\Pi = \iint_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$$

yoki
$$\iint_S \left(P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma\right)dS$$

ko‘rinishda yoki soddaroq $\iint_S \vec{F} \cdot \vec{n}_0 dS$ shaklda ifodalash mumkin.

Vektor maydon oqimining fizik ma'nosi: S sirt orqali vaqt birligi ichida sirt oriyentirlangan yo'nalishida $\vec{F}$ tezlik bilan oqib o'tgan suyuqlik miqdoridir.

Yopiq soha bo'yicha integral $\oiint\limits_S \vec{F}\cdot\vec{n}_0 dS$ kabi yoziladi.

Normal yopiq sirtning tashqi tomoniga qarab yo'nalgan va bu yo'nalish bo'yicha suyuqlik sirt tashqarisiga oqib chiqsa, qarama-qarshi harakat suyuqlik yopiq sirt ichiga oqib kirishini anglatadi. Demak, $\oiint\limits_S \vec{F}\cdot\vec{n}_0 dS$ integral yopiq sirtdan oqib chiqayotgan va oqib kirayotgan suyuqlik farqini anglatar ekan. Agar oqim nolga teng bo'lsa, sohaga undan qancha suyuqlik oqib chiqsa, shuncha oqib kirishini bildiradi.

Oqim musbat bo'lsa, sohadan unga oqib kirayotganidan ko'proq suyuqlik oqib chiqayotganini bildiradi.

7.3.4. Ostrogradskiy teoremasi. Vektor maydon divergensiyasi

Yopiq soha bo'yicha olingan sirt integrali (vektor maydon oqimi) hamda shu sirt chegaralagan fazoviy soha bo'yicha olingan uch karrali integral orasidagi bog'lanishni aniqlaymiz.

7.8-teorema (Ostrogradskiy teoremasi). Agar (7.42) vektor maydon proyeksiyalari S sohada o'zining birinchi tartibli xususiy hosilalari bilan birga uzluksiz bo'lsa, u holda S yopiq sirt orqali $\vec{a}$ vektor oqimini shu sirt bilan chegaralangan ω hajm bo'yicha uch karrali integralga quyidagi formula bilan almashtirish mumkin:

$$\oiint\limits_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy =$$

$$= \iiint\limits_\omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz\,, \qquad (7.43)$$

bu yerda integrallash S sirtning tashqi tomoni bo'yicha amalga oshiriladi.

Ushbu (7.43) formulaga **Ostrogradskiy formulasi** deyiladi.

7.3.6-misol. Ostrogradskiy formulasidan foydalanib, ushbu

$$\iint\limits_S x^2dydz + y^2dzdx + z^2dxdy$$

sirt integralini hisoblang, bunda S sirt quyidagi kubning tashqi tomoni:

$$V = \{(x,y,z) \in R^3 : 0 \le x \le a, 0 \le y \le a, 0 \le z \le a\}.$$

Yechilishi: ► (7.43) Ostrogradskiy formulasiga ko'ra

$$\iint\limits_S x^2dydz + y^2dzdx + z^2dxdy = \iiint\limits_V (2x+2y+2z)dxdydz$$

bo'ladi. Hosil qilingan uch karrali integralni hisoblaymiz:

$$\iiint\limits_V (2x+2y+2z)dxdydz = 2\int_0^a\int_0^a\int_0^a (x+y+z)dxdydz =$$

$$= 2\int_0^a\left[\int_0^a\left(a(x+y)+\frac{a^2}{2}\right)dy\right]dx = 2\int_0^a (a^2x+a^3)dx = 3a^4.$$

Demak, $\iint\limits_S x^2dydz + y^2dzdx + z^2dxdy = 3a^4$. ◄

$\vec{a}(M)$ **vektor maydonning divergensiyasi** deb

$$div\vec{a}(M) = \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} \tag{7.44}$$

tenglik bilan anilanadigan skalyar maydonga aytiladi.

Divergensiya yordamida (7.43) Ostrogradskiy formulasini vektor shaklida yozish mumkin:

$$\oiint\limits_S \vec{a}\cdot\vec{n}_0 dS = \iiint\limits_\omega div\vec{a}(M)d\omega. \tag{7.45}$$

Mavzu yuzasidan savollar:

1. Skalyar maydon deb nimaga aytiladi?
2. Sath sirti deb nimaga aytiladi?
3. Sath chizig'i nima?
4. Yo'nalish bo'yicha hosila qanday hisoblanadi?
5. Skalyar maydon gradiyenti ta'rifi va xossalarini ayting.
6. Vektor maydon deb nimaga aytiladi?
7. Vektor maydon oqimi nima?
8. Vektor chizig'i nima va u qanday topiladi?
9. Ostrogradskiy teormasini ayting.
10. Vektor maydon divergensiyasi formulasini ayting.

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $u = u(x;y;z)$ funksiyaning M_1 nuqtadagi $\overrightarrow{M_1M_2}$ vektor yo'nalishidagi hosilasini toping:

a) $u = \ln(1+x+y^2)$, $M_1(1;1;1)$, $M_2(3;-5;4)$,

b) $u = \frac{1}{2}x^2y^2z^2$, $M_1(1;-1;0)$, $M_2(2;-1;2)$,

c) $u = \ln(xy+yz+xz)$, $M_1(-2;3;-1)$, $M_2(2;1;-3)$,

d) $u = x^2y + y^3z + z^2x,\ M_1(1;-1;2),\ M_2(3;4;-1),$

e) $u = \dfrac{10}{1+x^2+y^2+z^2},\quad M_1(-1;2;-2),\ M_2(2;0;1),$

f) $u = x - 2y + e^x,\ M_1(-4;-5;0),\ M_2(2;3;4),$

2. Agar $u = x^2yz - xy^2z + xyz^2$ bo‘lsa, $M_0(1,1,1)$ nuqtadagi gradiyentini toping.

3. $M_0(1,1,1)$ nuqtadan o‘tuvchi, $xy + xz + yz = 3$ sirtga perpendikulyar birlik vektorning koordinatalarini toping.

4. $\vec{a} = 2x\vec{i} - (z-1)\vec{k}$ vektor maydonning $S: x^2 + y^2 = 4,\ z = 0,\ z = 1$ sirtning tashqi tomonidan o‘tuvchi oqimini toping.

5. $\vec{a} = (x^2 + y)\vec{i} + (y^2 + z)\vec{j} + (z^2 + x)\vec{k}$ vektor maydonning $M_0(1,-2,\ 3)$ nuqtadagi divergensiyasini hisoblang.

TESTLAR

1. Agar $u = x^2yz + xyz - xyz^2$ bo‘lsa, $M_0(1,1,1)$ nuqtadagi gradiyentini toping.

A) $grad\,u = 2i + j$ B) $grad\,u = i - 2j + 2k$

C) $grad\,u = 2i - 2j + 2k$ D) $grad\,u = 2i + j - 2k$

2. $u = x + \ln(y^2 + z^2)$ funksiyaning $M_0(2,1,1)$ nuqtada, $\vec{s} = -2i + j - k$ vektor yo‘nalishi bo‘yicha hosilasini toping.

A) $\dfrac{7\sqrt{2}}{2}$ B) 0 C) $-\dfrac{\sqrt{6}}{3}$ D) 1

3. $r = ti + t^2j + t^3k$ egri chiziqning $t = 3$ nuqtasida o‘tkazilgan urinmaning kanonik tenglamasini ko‘rsating.

A) $\dfrac{x-3}{1} = \dfrac{y-9}{6} = \dfrac{z}{27}$ B) $\dfrac{x-3}{1} = \dfrac{y-9}{6} = \dfrac{z-27}{27}$

C) $\dfrac{x-1}{11} = \dfrac{y-9}{6} = \dfrac{z-27}{27}$ D) $\dfrac{x-1}{1} = \dfrac{y-9}{9} = \dfrac{z-27}{27}$

4. Agar $u = -\dfrac{1}{2}x^2 + 3y^2 - 2z^2$ va $\upsilon = x^2yz$ funksiyalar berilgan bo‘lsa, $M_0\left(2, \dfrac{1}{3}, \dfrac{\sqrt{3}}{2}\right)$ nuqtadagi gradiyentlari orasidagi φ burchakni toping.

A) $\varphi = \dfrac{3\pi}{2}$ B) $\varphi = \dfrac{3\pi}{2}$ C) $\varphi = \dfrac{2\pi}{3}$ D) $\varphi = \dfrac{\pi}{2}$

5. $u = x^2y^2z - \ln(z-1)$ skalyar maydonning $M_0(1;1;2)$ nuqtadagi eng katta hosilasini toping.

A) $|gradu| = 4\sqrt{2}$ B) $|gradu| = 3\sqrt{2}$

C) $|gradu| = 2\sqrt{2}$ D) $|gradu| = \sqrt{2}$

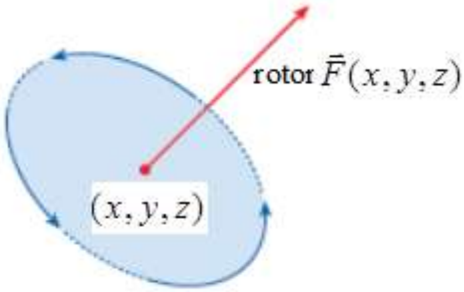
7.4-§. Vektor maydon sirkulyatsiyasi. Stoks formulasi. Vektor maydon uyurmasi

7.4.1. Vektor maydon sirkulyatsiyasi. Vektor maydon uyurmasi

Ushbu mavzuda biz vektor maydonda ro‘y beradigan va shu sababli ham vektor hisobning suyuqliklar oqimi, elektr va magnetizmdagi tatbiqlarida muhim rol o‘ynaydigan ikkita kattalikni kiritamiz. Bu kattaliklar rotor va divergensiya deyiladi, ular skalyar maydonning elementi bo‘lib turib, vektor maydon hosil qiladi.

Agar R^3 fazodagi (7.42) vektor maydon va P,Q,R funksiyalar xususiy hosilalarga ega bo‘lsa, u holda R^3 fazoda **vektor maydon rotori (uyurmasi)** deb, quyidagi tenglik bilan aniqlanadigan kattalikka aytiladi:

$$rotor\ \vec{F}(x,y,z)=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k} \quad (7.46)$$



7.37-rasm. Rotor

(7.46) rotor formulasini vektor differensial operator ko‘rinishida tasvirlaymiz: $\nabla=\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}$.

Bu operatorning o‘z vazifasi bor, agar operator f skalyar funksiyaga ta’sir qildirilsa, **funksiya gradiyenti** hosil bo‘ladi:

$$\nabla f=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}. \quad (7.47)$$

Agar ∇ operator bilan $\vec{F}$ vektor maydonni vektor ko‘paytirsak, u holda **vektor maydon uyurmasi** hosil bo‘ladi:

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=$$

$$=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k}=rotor\ \vec{F} \quad (7.48)$$

Shunday qilib, vektor maydon uyurmasi ta'rifini yodda saqlab qolishning eng oson yo'li uni quyidagi tenglik bilan yozishdir:

$$rotor\,\vec{F} = \nabla \times \vec{F}\,.$$

7.4.1-misol. $\vec{F}(x, y, z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\,\vec{k}$ vektor maydon rotorini toping.

Yechilishi: ► (7.48) formuladan foydalanamiz:

$$rotor\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} =$$

$$= \left(\frac{\partial(-y^2)}{\partial y} - \frac{\partial(xyz)}{\partial z} \right)\vec{i} + \left(\frac{\partial(xz)}{\partial z} - \frac{\partial(-y^2)}{\partial x} \right)\vec{j} + \left(\frac{\partial(xyz)}{\partial x} - \frac{\partial(xz)}{\partial y} \right)\vec{k} =$$

$$= (-2y - xy)\vec{i} + (x - 0)\vec{j} + (yz - 0)\vec{k} = -y(2 + x)\vec{i} + x\vec{j} + yz\vec{k}.$$ ◄

7.9-teorema (Klero teoremasi). Agar f uch o'zgaruvchili funksiya va 2-tartibli uzluksiz xususiy hosilalarga ega bo'lsa, u holda $rotor(\nabla f) = 0$ bo'ladi.

Isboti: ►

$$rotor(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} =$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)\vec{k} = 0.$$ ◄

Agar R^3 fazodagi (7.42) vektor maydon va P, Q, R funksiyalar xususiy hosilalarga ega bo'lsa, u holda R^3 fazoda **vektor maydon divergensiyasi** deb, quyidagi tenglik bilan aniqlanadigan skalyar maydonga aytiladi:

$$div\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \qquad (7.49)$$

Agar ∇ operator bilan $\vec{F}$ vektor maydonni skalyar ko'paytirsak, u holda vektor maydon divergensiyasi hosil bo'ladi:

$$div\,\vec{F} = \nabla \cdot \vec{F}\,.$$

7.4.2-misol. $\vec{F}(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\,\vec{k}$ vektor maydon divergensiyasini toping.

Yechilishi: ►

$$div\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial (xz)}{\partial x} + \frac{\partial (xyz)}{\partial y} + \frac{\partial (-y^2)}{\partial z} = z + xz. ◄$$

7.10-teorema. Agar R^3 fazodagi $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ vektor maydon va P,Q,R funksiyalar 2-tartibli uzluksiz xususiy hosilalarga ega bo‘lsa, u holda $div\,rotor\,\vec{F} = 0$ bo‘ladi.

Isboti: ►

$$div\,rotor\vec{F} = \nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) =$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0. \quad ◄$$

Endi gradiyentning divergensiyasini topamiz:

$$div(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$$

Bu yerdagi $\nabla^2 = \nabla \cdot \nabla$ operatorga **Laplas operatori** deyiladi, chunki u quyidagi **Laplas tenglamasidan** kelib chiqadi:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0. \tag{7.50}$$

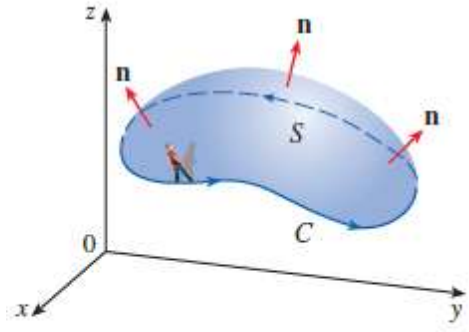
Xuddi shuningdek, Laplas operatorini $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ vektor maydonga, ya’ni uning komponentlariga ta’sir qildirish mumkin:

$$\nabla^2 \vec{F} = \nabla^2 P\vec{i} + \nabla^2 Q\vec{j} + \nabla^2 R\vec{k}.$$

7.4.2. Stoks formulasi

7.11-teorema (Stoks teoremasi). Fazoda oriyentirlangan bo‘lakli-silliq S sirtni qaraylik, bu sirt yopiq, bo‘lakli – silliq, musbat oriyentirlangan C egri chiziq bilan chegaralangan bo‘lsin. $\vec{F}$ vektor maydon bo‘lib, uning komponentlari R^3 fazoda S sirtni o‘z ichiga olgan ochiq sohada uzluksiz xususiy hosilalarga ega bo‘lsin (7.38-rasm). U holda vektor maydon uchun quyidagi tenglik o‘rinli:

$$\int_C \vec{F} \cdot dr = \iint_S rotor\,\vec{F} \cdot dS$$



7.38-rasm. S sirt

Stoks formulasini egri chiziqli integral bilan ifodalashga harakat qilamiz. Faraz qilaylik, fazoda berilgan $z = z(x, y)$ funksiya C konturda uzluksiz va uzluksiz $z'_x(x, y),\ z'_y(x, y)$ xususiy hosilalarga ega bo‘lsin va S sirtda $P = P(x, y, z)$ funksiya aniqlangan bo‘lib, u uzluksiz va uzluksiz xususiy $\frac{\partial P(x, y, z)}{\partial x}, \frac{\partial P(x, y, z)}{\partial y}, \frac{\partial P(x, y, z)}{\partial z}$ hosilalarga ega bo‘lsin. U holda ushbu $\int\limits_{\partial S} P(x, y, z)dx$ egri chiziqli integral mavjud bo‘ladi. Egri chiziqli integrallar mavzusidan ma’lumki, bunday holda ∂S kontur yo‘nalishi sirt tomoni bilan muvofiq bo‘ladi. Modomiki, ∂S kontur S sirtga tegishli ekan, unda ∂S ning nuqtalari $z = z(x, y)$ tenglamani qanoatlantiradi. Shuningdek, ∂S da $P = P(x, y, z)$ funksiya $P = P(x, y, z(x, y))$ bo‘lib, u C da berilgan ikki o‘zgaruvchili funksiyaga aylanadi va quyidagi tenglik o‘rinli bo‘ladi:

$$\int\limits_{\partial S} P(x, y, z)dx = \int\limits_{\partial S} P(x, y, z(x, y))dx.$$

Grin formulasiga ko‘ra, $\int\limits_{\partial S} P(x, y, z)dx = -\iint\limits_{C} \frac{\partial}{\partial y} P(x, y, z(x, y))dxdy$ bo‘ladi. Bu tenglikning o‘ng tomonidagi xususiy hosilani topamiz:

$$\frac{\partial}{\partial y} P(x, y, z(x, y)) = \frac{\partial P(x, y, z(x, y))}{\partial y} + \frac{\partial P(x, y, z(x, y))}{\partial z} \cdot z_y{}'(x, y).$$

Topilgan xususiy hosilani o‘rniga olib borib qo‘ysak,

$$\int\limits_{\partial S} P(x, y, z(x, y))dx = -\iint\limits_{C} \left(\frac{\partial P(x, y, z(x, y))}{\partial y} + \frac{\partial P(x, y, z(x, y))}{\partial z} \cdot z_y{}'(x, y) \right) dxdy \quad (7.51)$$

hosil bo‘ladi.

Bilamizki, agar S sirtning ustki qismi qaralsa, uning $\vec{n}$ normalining yo‘naltiruvchi kosinuslari bo‘ladi:

$$\cos\alpha=-\frac{z_x'(x,y)}{\sqrt{1+z_x'^2+z_y'^2}},\quad \cos\beta=-\frac{z_y'(x,y)}{\sqrt{1+z_x'^2+z_y'^2}},\quad \cos\gamma=\frac{1}{\sqrt{1+z_x'^2+z_y'^2}}.$$

Bu munosabatlardan $\frac{\cos\beta}{\cos\gamma}=-z_y'(x,y)$ bo‘lishi kelib chiqadi. Natijada buni (7.51) formulaga qo‘ysak,

$$\int_{\partial S} P(x,y,z(x,y))dx=-\iint_C\left(\frac{\partial P(x,y,z(x,y))}{\partial y}-\frac{\partial P(x,y,z(x,y))}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right)dxdy \quad (7.52)$$

ga ega bo‘lamiz.

Endi (7.52) tenglikdagi ikki karrali integralni (**) formuladan foydalanib, II tur sirt integrali orqali quyidagicha yozib olamiz:

$$\iint_S\left(\frac{\partial P(x,y,z(x,y))}{\partial y}-\frac{\partial P(x,y,z(x,y))}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right)dxdy=$$

$$=\iint_S\left(\frac{\partial P(x,y,z)}{\partial y}-\frac{\partial P(x,y,z)}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right)dxdy$$

So‘ngra bu II tur sirt integrali uchun I va II tur sirt integrallarini o‘zaro bog‘lovchi ushbu

$$\begin{aligned}\iint_S f(x,y,z)dydz&=\iint_S f(x,y,z)\cos\alpha dS\\ \iint_S f(x,y,z)dzdx&=\iint_S f(x,y,z)\cos\beta dS\\ \iint_S f(x,y,z)dxdy&=\iint_S f(x,y,z)\cos\gamma dS\end{aligned} \quad (7.53)$$

formulalarning 3-siga ko‘ra

$$\iint_S\left(\frac{\partial P(x,y,z)}{\partial y}-\frac{\partial P(x,y,z)}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right)dxdy=$$

$$=\iint_S\left(\frac{\partial P(x,y,z)}{\partial y}-\frac{\partial P(x,y,z)}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right)\cdot\cos\gamma dS=$$

$$=\iint_S\frac{\partial P(x,y,z)}{\partial y}\cdot\cos\gamma dS-\iint_S\frac{\partial P(x,y,z)}{\partial z}\cdot\cos\beta dS$$

bo‘lib, bu tenglikdagi I tur sirt integrallari (7.53) formulalarga ko‘ra

$$\iint_S\frac{\partial P(x,y,z)}{\partial y}\cdot\cos\gamma dS=\iint_S\frac{\partial P(x,y,z)}{\partial y}dxdy,$$

$$\iint_S\frac{\partial P(x,y,z)}{\partial z}\cdot\cos\beta dS=\iint_S\frac{\partial P(x,y,z)}{\partial z}dzdx$$

bo‘ladi. Yuqoridagi barcha munosabatlarni inobatga olsak,

$$\int_{\partial S} P(x,y,z)dx=\iint_S\frac{\partial P(x,y,z)}{\partial z}dzdx-\frac{\partial P(x,y,z)}{\partial y}dxdy \quad (7.54)$$

kelib chiqadi.

Xuddi shunga o‘xshash S sirt va unda aniqlangan $Q(x,y,z)$, $R(x,y,z)$ funksiyalar uchun tegishli shartlarda

$$\begin{aligned}&\int_{\partial S} Q(x,y,z)dy = \iint_S \frac{\partial Q(x,y,z)}{\partial x}dxdy - \frac{\partial Q(x,y,z)}{\partial z}dydz,\\ &\int_{\partial S} R(x,y,z)dz = \iint_S \frac{\partial R(x,y,z)}{\partial y}dydz - \frac{\partial R(x,y,z)}{\partial x}dzdx\end{aligned} \tag{7.55}$$

munosabatlarni yozish mumkin.

(7.54) va (7.55) tengliklarni hadlab qo‘shib, quyidagini topamiz:

$$\begin{aligned}&\int_{\partial S} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =\\ &= \iint_S \left[\frac{\partial Q(x,y,z)}{\partial x} - \frac{\partial P(x,y,z)}{\partial y}\right]dxdy +\\ &+ \left[\frac{\partial R(x,y,z)}{\partial y} - \frac{\partial Q(x,y,z)}{\partial z}\right]dydz +\\ &+ \left[\frac{\partial P(x,y,z)}{\partial z} - \frac{\partial R(x,y,z)}{\partial x}\right]dzdx\,.\end{aligned} \tag{7.56}$$

(7.56)ga **Stoks formulasi** deyiladi.

Stoks formulasi S sirt bo‘yicha olingan sirt integralini shu sirt chegarasi ∂S yopiq egri chiziq bo‘yicha olingan egri chiziqli integral bilan bog‘laydi.

Tarixiy ma’lumot. Stoks teoremasi Irlandiyalik matematik, fizik J.Stoks (1819-1903) sharafiga nomlanadi. Stoks Kembridj universiteti professori lavozimida ishlagan va suyuqlik oqimi hamda nurlarni o‘rgangan. Stoks teoremasi aslida Shotlandiyalik fizik olim U.Tomson (1824-1907) tomonidan kashf etilgan. Tomson 1850 yilda Stoksga ushbu teoremani xat orqali ma’lum qiladi. 1854 yilda Storks 1-marta Kembridj universiteti talabalaridan imtihonda bu teoremaning isbotini so‘raydi. 1-bo‘lib, teoremani e’lon qilgani uchun ham uning nomi bilan ataladi.

7.4.3-misol. $\vec{F}(x,y,z) = -y^2\vec{i} + x\vec{j} + z^2\vec{k}$ va C $x^2+y^2=1$ silindr bilan $y+z=2$ sirt bilan kesishish egri chizig‘i bo‘lsa, $\int_C \vec{F}\cdot dr$ ni hisoblang.

Yechilishi: ► C egri chiziq ellipsdan iborat, u $x^2+y^2=1$ silindr bilan $y+z=2$ tekislikning kesishishidan hosil bo‘ladi (7.39-rasm).

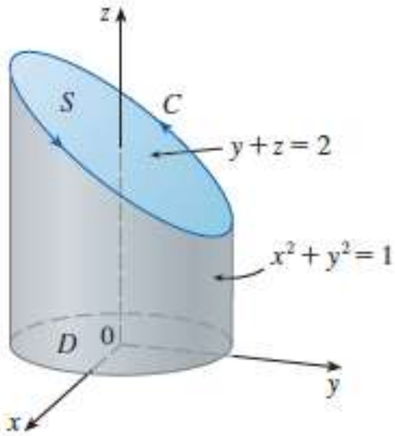
$\int_C \vec{F}\cdot dr$ integralni Stoks teoremasidan aniqlaymiz, chunki

$$\int_C \vec{F}\cdot dr = \iint_S rotor\ \vec{F}\cdot dS;$$

Bunda $rotor\ \vec{F}$ ni topish kerak.

$$rotor\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} =$$

$$= \left(\frac{\partial(z^2)}{\partial y} - \frac{\partial x}{\partial z}\right)\vec{i} + \left(\frac{\partial(-y^2)}{\partial z} - \frac{\partial(z^2)}{\partial x}\right)\vec{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial(-y^2)}{\partial y}\right)\vec{k} = (1+2y)\vec{k}.$$



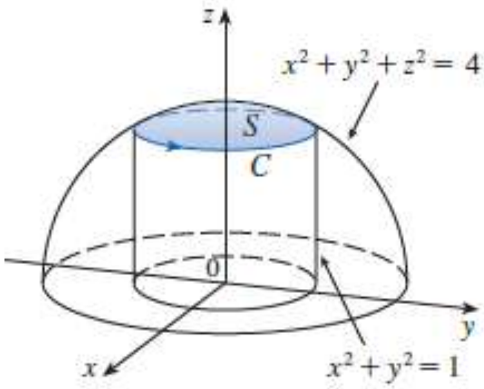
7.39-rasm. Silindr va tekislikning kesishishidan hosil bo‘lgan ellips

Stoks teoremasi bizga C egri chiziq bilan chegaralangan ixtiyoriy (oriyentirlangan, bo‘lakli-silliq) sirtni tanlashga imkon beradi. Mumkin bo‘lgan bir qancha shunday sirtlar orasidan biz eng maqbulini tanlaymiz – bu $y+z=2$ tekislikning C egri chiziq bilan chegaralangan qismi S elliptik sohadir. Agar S sohaning yuqori qismini olsak, u holda C musbat oriyentirlangan bo‘ladi. S sohaning xOy tekislikdagi D proyeksiyasi $x^2+y^2\le 1$ doiradan iborat bo‘ladi. D sohaning yuzasini A bilan belgilasak, shunda aniqlanish sohasi doiradan iborat bo‘lganligi uchun qutb koordinatasiga o‘tib ishlaymiz:

$$\int_C \vec{F}\cdot dr = \iint_S rotor\ \vec{F}\cdot dS = \iint_D (1+2y)dA = \int_0^{2\pi} d\varphi \int_0^1 (1+2r\sin\varphi)r dr = \pi. \ ◀$$

7.4.4.-misol. $\vec{F}(x,y,z) = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$ va S sirt $x^2+y^2+z^2=4$ sfera bilan $x^2+y^2=1$ silinrning kesishgan qismi bo‘lsa, $\iint_S rotor\ \vec{F}\cdot dS$ ni hisoblang.

Yechilishi: I usul. ▶ $x^2+y^2+z^2=4$ sfera bilan $x^2+y^2=1$ silinrning kesishgan qismini chizamiz (7.40-rasm).



7.40-rasm. Sfera va silindr kesishmasi

$$x^2+y^2+z^2=4 \quad \Rightarrow \quad \underbrace{x^2+y^2}_{1}+z^2=4 \quad \Rightarrow \quad z^2=3 \quad \Rightarrow \quad z=\sqrt{3} \ (z>0).$$

Shunda C egri chiziqning silindrik koordinatadagi ko‘rinishi quyidagicha:

$\vec{r}(t)=\cos t\,\vec{i}+\sin t\,\vec{j}+\sqrt{3}\vec{k}, \quad 0\le t\le 2\pi$.

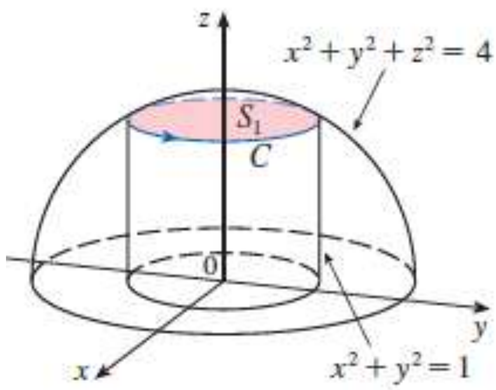
$\vec{r}'(t)=-\sin t\,\vec{i}+\cos t\,\vec{j}$.

Demak, $\vec{F}(\vec{r}'(t))=xz\,\vec{i}+yz\,\vec{j}+xy\,\vec{k}=\sqrt{3}\cos t\,\vec{i}+\sqrt{3}\sin t\,\vec{j}+\cos t\sin t\,\vec{k}$.

Topilgan ifodalarni Stoks formulasiga qo‘yamiz:

$$\iint_S rotor\,\vec{F}\cdot dS=\int_C \vec{F}\cdot dr=\int_C \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)dt=\int_0^{2\pi}(-\sqrt{3}\cos t\sin t+\sqrt{3}\sin t\cos t)dt=0.$$

II usul. ▶ $x^2+y^2+z^2=4$ sfera bilan $x^2+y^2=1$ silinrning kesishgan qismini $z=\sqrt{3}$ ekanini topdik (7.41-rasm). S_1 soha S soha bilan bir xil, u C egri chiziq bilan chegaralangan, shunga ko‘ra quyidagi tenglikni yozish mumkin: $\iint_S rotor\,\vec{F}\cdot dS=\iint_{S_1} rotor\,\vec{F}\cdot dS$. Chunki, S_1 gorizontal tekislik bo‘lib, $\vec{k}$ unga normal hisoblanadi.



7.41-rasm. Sfera va silindr kesishmasi S_1

$\vec{F}(x,y,z)=xz\,\vec{i}+yz\,\vec{j}+xy\,\vec{k}$ vektor maydon rotorini hisoblaymiz:

$$rotor\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

formulaga ko‘ra topamiz:

$$rotor\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y)\vec{i} + (x-y)\vec{j}.$$

$$\iint_S rotor\ \vec{F} \cdot dS = \iint_{S_1} rotor\ \vec{F} \cdot \vec{n} \cdot dS_1 = \iint_{S_1} [(x-y)\,\vec{i} + (x-y)\,\vec{j}\,] \cdot \vec{k} \cdot dS_1 = 0,$$

chunki perpendikulyar vekrotlarning skalyar ko‘paytmasi 0 ga teng. ◄

7.4.3. Potensial va solenoidli vektor maydonlar

Agar $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ (7.42) vektor maydonning har bir nuqtasida uyurmasi nolga teng bo‘lsa, ya’ni $rotor\ \vec{F} = 0$ bo‘lsa, bunday vektor maydonga **potensial** (yoki **gradiyentli** yoki **uyurmasiz**) **maydon** deyiladi. Nuqtaviy zaryadlar kuchlanishining elektrostatik maydoni potensial maydonga misol bo‘ladi.

Potensial maydonning shu maydondagi ixtiyoriy yopiq chiziq bo‘yicha sirkulyatsiyasi nolga teng.

Potensial maydon biror bir $u = f(x,y,z)$ skalyar funksiyaning gradiyentiga teng, ya’ni $rotor\ \vec{F} = \nabla f$. Bunday $u = f(x,y,z)$ funksiya **vektor maydon potensiali** (yoki **potensial funksiyasi**)deyiladi.

(7.42) vektor maydonning potensiali quyidagi formula yordamida topiladi:

$$f(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \int_{x_0}^{x} P(x,y_0,z_0)dx + \int_{y_0}^{y} Q(x,y,z_0)dy + \int_{z_0}^{z} R(x,y,z)dz,$$

bu yerda (x_0,y_0,z_0) tayinlangan nuqtaning koordinatalari, (x,y,z) esa ixtiyoriy nuqta koordinatasidir.

Agar $\vec{F}(x,y,z)$ vektor maydonning har bir nuqtasida divergensiyasi nolga teng bo‘lsa, ya’ni $div\ \vec{F} = \nabla \cdot \vec{F} = 0$ bo‘lsa, bunday vektor maydonga **solenoidli (**yoki **naychasimon) maydon** deyiladi.

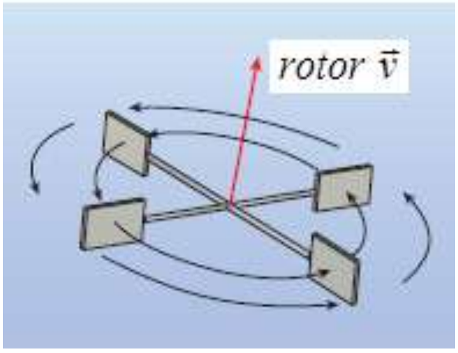
Solenoidli vektor maydonning ixtiyoriy yopiq sirt bo‘yicha oqimi nolga teng bo‘ladi. Solenoidli maydonga magnit maydoni, aylanma jism tezligi maydoni kabilar misol bo‘la oladi.

Aytaylik, $P_0(x_0, y_0, z_0)$ suyuqlikdagi biror zarracha bo‘lsin va S_a biror radiusi a ga teng, markazi P_0 nuqtada bo‘lgan kichikroq disk bo‘lsin. U holda barcha S_a diskdagi barcha P nuqtalar uchun ushbu tenglik o‘rinli bo‘ladi, chunki $rotor\,\vec{F}$ uzluksiz: $(rotor\,\vec{F})(P) \approx (rotor\,\vec{F})(P_0)$. Shunday qilib, Stoks formulasiga ko‘ra, aylana C_a chegarasi bo‘ylab sirkulyatsiyaning taqribiy qiymatini topish mumkin:

$$\int_{C_a} \vec{v} \cdot d\vec{r} = \iint_{S_a} rotor\,\vec{v} \cdot \vec{n} dS \approx \iint_{S_a} rotor\,\vec{v}(P_0) \cdot \vec{n}(P_0) dS = rotor\,\vec{v} \cdot \vec{n}(P_0)\pi a^2.$$

$a \to 0$ da limitga o‘tamiz:

$$rotor\,\vec{v} \cdot \vec{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \vec{v} \cdot d\vec{r}$$



7.42-rasm. Rotor

Ushbu tenglik rotor va sirkulyatsiya orasidagi munosabatni bildiradi, ya’ni $rotor\,\vec{v} \cdot \vec{n}$ o‘chov suyuqlikning $\vec{n}$ o‘q atrofida aylanish effektini ko‘rsatadi. Eng samarali sirkulyatsiyaga $rotor\,\vec{v}$ ga parallel o‘q bo‘yicha aylanganda erishish mumkin (7.42-rasm).

Mavzu yuzasidan savollar:

1. Vektor maydon divergensiyasi formulasini ayting.
2. Vektor maydon sirkulyatsiyasi deb nimaga aytiladi?
3. Vektor maydonning chiziqli integrali deb nimaga aytiladi?
4. Stoks formulasi deb nomlanadigan tenglikni yozing?
5. Stoks formulasining tekislikdagi xususiy holi qanday nomlanadi?
6. Vektor maydon uyurmasi(rotori) ta’rifini ayting.
7. Stoks formulasi uyurma formulasidan foydalanib qanday yoziladi?
8. Uyurmani determinant shaklini yozing.

9. Potensial vektor maydon deb nimaga aytiladi va uning potensiali qanday topiladi?
10. Ixtiyoriy yopiq chiziq bo'yicha sirkulyatsiyasi nolga teng bo'lgan vektor maydon qanday maydon bo'ladi?
11. Qanday vektor maydon solenoidli maydon deyiladi?
12. Ixtiyoriy yopiq sirt bo'yicha oqimi nolga teng bo'lgan vektor maydon qanday maydon bo'ladi?

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $\vec{a}(M)=z^2\cdot\vec{i}+x^2\cdot\vec{j}+y^2\cdot\vec{k}$ vektor maydonning uyurmasini toping.

2. Ushbu $\vec{a}(M)=xy\vec{i}+yz\vec{j}+xz\vec{k}$ vektor maydonning $2x-3y+4z-12=0$ tekislikning koordinata tekisliklari bilan kesishish chizig'i bo'yicha sirkulyatsiyasini Stoks formulasi yordamida hisoblang.

3. $\vec{a}=grad\left(arctg\,\dfrac{y}{x}\right)$ maydonning $C:\ x^2+y^2=1$ aylana bo'ylab sirkulyatsiyasini toping.

4. $\vec{a}(M)=2xyz\vec{i}++x^2z\vec{j}+x^2y\vec{k}$ vektor maydon potensialini toping.

5. $\vec{a}(M)=(4x+y)\vec{i}+(x+z)\vec{j}+y\vec{k}$ vektor maydonning $2x+y+2z-6=0$ tekislikning koordinata o'qlari bilan hosil qilgan uchburchak konturi bo'yicha sirkulyatsiyasini Stoks formulasi yordamida hisoblang.

TESTLAR

1. Potensial maydonni aniqlang.

A) $\vec{a}(M)=(2xy+z^2)\vec{i}+(2xz+x^2)\vec{j}+(2xz+y^2)\vec{k}$;

B) $\vec{a}(M)=2xyz\vec{i}+x^2z\vec{j}+x^2y\vec{k}$;

C) $\vec{a}(M)=2xyz\vec{i}+2x^2z\vec{j}+2x^2y\vec{k}$;

D) $\vec{a}(M)=(2xz+z^2)\vec{i}+(2yz+x^2)\vec{j}+(2xz+y^2)\vec{k}$.

2. $\vec{a}(M)=(2xy+z^2)\vec{i}+(2yz+x^2)\vec{j}+(2xz+y^2)\vec{k}$ vektor maydon potensialini toping.

A) $u=x^2y+xy^2+y^2z+C$; B) $u=x^2y+xz^2+y^2x+C$;

C) $u=x^2y+yz^2-y^2x+C$; D) $u=x^2y+xz^2+y^2z+C$.

3. $\vec{a}(M)=z\vec{i}+(x+y)\vec{j}+y\vec{k}$ vektor maydonning $2x+y+2z-2=0$ tekislikning koordinata o'qlari bilan hosil qilgan uchburchak konturi bo'yicha sirkulyatsiyasini Stoks formulasi yordamida hisoblang.

A) 2 B) 2.5 C) 0 D) 1

4. $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ ushbu determinant nimani hisoblaydi?

A) Maydon uyurmasini; B) Maydon divergensiyasini;
C) Maydon potensialini; D) Maydon sirkulyatsiyasini.

5. Solenoidli maydonni aniqlang.

A) $\vec{a}(M) = 2yz\vec{i} + x^2 z\vec{j} + x^2 y\vec{k}$;

B) $\vec{a}(M) = xyz\vec{i} + x^2 z\vec{j} + x^2 y\vec{k}$;

C) $\vec{a}(M) = 2xyz\vec{i} + 2x^2 z\vec{j} + 2x^2 y\vec{k}$;

D) $\vec{a}(M) = \left(2xz + z^2\right)\vec{i} + \left(2yz + x^2\right)\vec{j} + \left(2xz + y^2\right)\vec{k}$.

Lug‘atli ko‘rsatkich

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