

**UNIT I****1 INTRODUCTION TO OPERATIONS RESEARCH****LESSON****LESSON STRUCTURE**

- 1.1 Introduction
- 1.2 History of Operations Research
- 1.3 Stages of Development of Operations Research
- 1.4 Relationship Between Manager and OR Specialist
- 1.5 OR Tools and Techniques
- 1.6 Applications of Operations Research
- 1.7 Limitations of Operations Research
- 1.8 Summary
- 1.9 Key Terms
- 1.10 Self Assessment Questions
- 1.11 Further References

**Objectives**

After Studying this lesson, you should be able to:

- ❖ Understand the meaning, purpose, and tools of Operations Research
- ❖ Describe the history of Operations Research
- ❖ Describe the Stages of O.R
- ❖ Explain the Applications of Operations Research
- ❖ Describe the Limitations of Operation Research
- ❖ Understand the OR specialist and Manager relationship

## 1.1 Introduction

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" - which is the term we will use.

Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "OR/MS" or "ORMS". Yet other terms sometimes used are "industrial engineering" ("IE") and "decision science" ("DS"). In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

Operation Research is a relatively new discipline. The contents and the boundaries of the OR are not yet fixed. Therefore, to give a formal definition of the term Operations Research is a difficult task. The OR starts when mathematical and quantitative techniques are used to substantiate the decision being taken. The main activity of a manager is the decision making. In our daily life we make the decisions even without noticing them. The decisions are taken simply by common sense, judgment and expertise without using any mathematical or any other model in simple situations. But the decision we are concerned here with are complex and heavily responsible. Examples are public transportation network planning in a city having its own layout of factories, residential blocks or finding the appropriate product mix when there exists a large number of products with different profit contributions and production requirement etc.

Operations Research tools are not from any one discipline. Operations Research takes tools from different discipline such as mathematics, statistics, economics, psychology, engineering etc. and combines these tools to make a new set of knowledge for decision making. Today, O.R. became a professional discipline which deals with the application of scientific methods for making decision, and especially to the allocation of scarce resources. The main purpose of O.R. is to provide a rational basis for decisions making in the absence of complete information, because the systems composed of human, machine, and procedures may do not have complete information.

Operations Research can also be treated as science in the sense it describing, understanding and predicting the systems behaviour, especially man-machine system. Thus O.R. specialists are involved in three classical aspect of science, they are as follows:

- i) Determining the systems behaviour
- ii) Analyzing the systems behaviour by developing appropriate models
- iii) Predict the future behaviour using these models

The emphasis on analysis of operations as a whole distinguishes the O.R. from other research and engineering. O.R. is an interdisciplinary discipline which provided solutions to problems of military operations during World War II, and also successful in other operations. Today business applications are

primarily concerned with O.R. analysis for the possible alternative actions. The business and industry benefitted from O.R. in the areas of inventory, reorder policies, optimum location and size of warehouses, advertising policies, etc.

As stated earlier defining O.R. is a difficult task. The definitions stressed by various experts and Societies on the subject together enable us to know what O.R. is, and what it does. They are as follows:

1. According to the Operational Research Society of Great Britain (OPERATIONAL RESEARCH QUARTERLY, 13(3):282, 1962), Operational Research is the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defense. Its distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as change and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically.
2. Randy Robinson stresses that Operations Research is the application of scientific methods to improve the effectiveness of operations, decisions and management. By means such as analyzing data, creating mathematical models and proposing innovative approaches, Operations Research professionals develop scientifically based information that gives insight and guides decision-making. They also develop related software, systems, services and products.
3. Morse and Kimball have stressed O.R. is a quantitative approach and described it as “ a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control”.
4. Saaty considers O.R. as tool of improving quality of answers. He says, “O.R. is the art of giving bad answers to problems which otherwise have worse answers”.
5. Miller and Starr state, “O.R. is applied decision theory, which uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough-going rationality in dealing with his decision problem”.
6. Pocock stresses that O.R. is an applied Science. He states “O.R. is scientific methodology (analytical, mathematical, and quantitative) which by assessing the overall implication of various alternative courses of action in a management system provides an improved basis for management decisions”.

## **1.2 History of Operations Research**

Operation Research is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study Operation Research, indeed the term O.R. did not exist then. It was really only in the late 1930's that operational research began in a systematic fashion, and it started in the UK. As such it would be interesting to give a short history of O.R.

**1936**

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

### **1937**

The first of three major pre-war air-defence exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

### **1938**

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the outbreak of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical - aspects of the system should begin immediately. The term "*operational research*" [RESEARCH into

(military) OPERATIONS] was coined as a suitable description of this new branch of applied science.

The first team was selected from amongst the scientists of the radar research group the same day.

### **1939**

In the summer of 1939 Britain held what was to be its last pre-war air defence exercise. It involved some 33,000 men, 1,300 aircraft, 110 anti-aircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defence warning and control system. The contribution made by the OR team was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore in north London.

Initially, they were designated the "Stanmore Research Section". In 1941 they were redesignated the "Operational Research Section" when the term was formalised and officially accepted, and similar sections set up at other RAF commands.

### **1940**

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyse a French request for ten additional fighter squadrons (12 aircraft a squadron - so 120 aircraft in all) when losses were running at some three squadrons every two days (i.e. 36 aircraft every 2 days). They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

### **1941 onward**

In 1941, an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

The responsibility of Coastal Command was, to a large extent, the flying of long-range sorties by single aircraft with the object of sighting and attacking surfaced U-boats (German submarines). The technology of the time meant that (unlike modern day submarines) surfacing was necessary to recharge batteries, vent the boat of fumes and recharge air tanks. Moreover U-boats were much faster on the surface than underwater as well as being less easily detected by sonar.

Thus the Operation Research started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The *objective* was to find the most effective utilization of limited military resources by the use of quantitative techniques. Following the end of the war OR spread, although it spread in different ways in the UK and USA.

In 1951 a committee on Operations Research formed by the National Research Council of USA, and the first book on “Methods of Operations Research”, by Morse and Kimball, was published. In 1952 the Operations Research Society of America came into being.

Success of Operations Research in army attracted the attention of the industrial managers who were seeking solutions to their complex business problems. Now a days, almost every organization in all countries has staff applying operations research, and the use of operations research in government has spread from military to wide variety of departments at all levels. The growth of operations research has not limited to the U.S.A. and U.K., it has reached many countries of the world.

India was one the few first countries who started using operations research. In India, Regional Research Laboratory located at Hyderabad was the first Operations Research unit established during 1949. At the same time another unit was set up in Defense Science Laboratory to solve the Stores, Purchase and Planning Problems. In 1953, Operations Research unit was established in Indian Statistical Institute, Calcutta, with the objective of using Operations Research methods in National Planning and Survey. In 1955, Operations Research Society of India was formed, which is one of the first members of International Federation of Operations Research societies. Today Operations Research is a popular subject in management institutes and schools of mathematics.

### **1.3 Stages of Development of Operations Research**

The stages of development of O.R. are also known as phases and process of O.R, which has six important steps. These six steps are arranged in the following order:

Step I: Observe the problem environment

Step II: Analyze and define the problem

Step III: Develop a model

Step IV: Select appropriate data input

Step V: Provide a solution and test its reasonableness

Step VI: Implement the solution

**Step I: Observe the problem environment**

The first step in the process of O.R. development is the problem environment observation. This step includes different activities; they are conferences, site visit, research, observations etc. These activities provide sufficient information to the O.R. specialists to formulate the problem.

**Step II: Analyze and define the problem**

This step is analyzing and defining the problem. In this step in addition to the problem definition the objectives, uses and limitations of O.R. study of the problem also defined. The outputs of this step are clear grasp of need for a solution and its nature understanding.

**Step III: Develop a model**

This step develops a model; a model is a representation of some abstract or real situation. The models are basically mathematical models, which describes systems, processes in the form of equations, formula/relationships. The different activities in this step are variables definition, formulating equations etc. The model is tested in the field under different environmental constraints and modified in order to work. Some times the model is modified to satisfy the management with the results.

**Step IV: Select appropriate data input**

A model works appropriately when there is appropriate data input. Hence, selecting appropriate input data is important step in the O.R. development stage or process. The activities in this step include internal/external data analysis, fact analysis, and collection of opinions and use of computer data banks. The objective of this step is to provide sufficient data input to operate and test the model developed in Step\_III.

**Step V: Provide a solution and test its reasonableness**

This step is to get a solution with the help of model and input data. This solution is not implemented immediately, instead the solution is used to test the model and to find there is any limitations. Suppose if the solution is not reasonable or the behaviour of the model is not proper, the model is updated and modified at this stage. The output of this stage is the solution(s) that supports the current organizational objectives.

**Step VI: Implement the solution**

At this step the solution obtained from the previous step is implemented. The implementation of the solution involves no many behavioural issues. Therefore, before implementation the implementation authority has to resolve the issues. A properly implemented solution results in quality of work and gains the support from the management.

The process, process activities, and process output are summarized in the following Table 1-1.

Process Activities	Process	Process Output
Site visits, Conferences, Observations, Research	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Step 1: Observe the problem environment</div> <div style="text-align: center;">↓</div>	Sufficient information and support to proceed
Define: Use, Objectives, limitations	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Step 2: Analyze and define the problem</div> <div style="text-align: center;">↓</div>	Clear grasp of need for and nature of solution requested
Define interrelationships, Formulate equations, Use known O.R. Model , Search alternate Model	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Step 3: Develop a Model</div> <div style="text-align: center;">↓</div>	Models that works under stated environmental constraints
Analyze: internal-external data, facts  Collect options,  Use computer data banks	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Step 4: Select appropriate data input</div> <div style="text-align: center;">↓</div>	Sufficient inputs to operate and test model
Test the model  find limitations  update the model	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">Step 5: Provide a solution and test its reasonableness</div> <div style="text-align: center;">↓</div>	Solution(s) that support current organizational goals
<div style="border: 1px solid black; padding: 5px; margin: 0 auto; width: 200px;">Step 6: Implement the solution</div>		



Resolve behavioural issues		Improved working and
Sell the idea		Management support for longer
Give explanations		run operation of model
Management involvement		

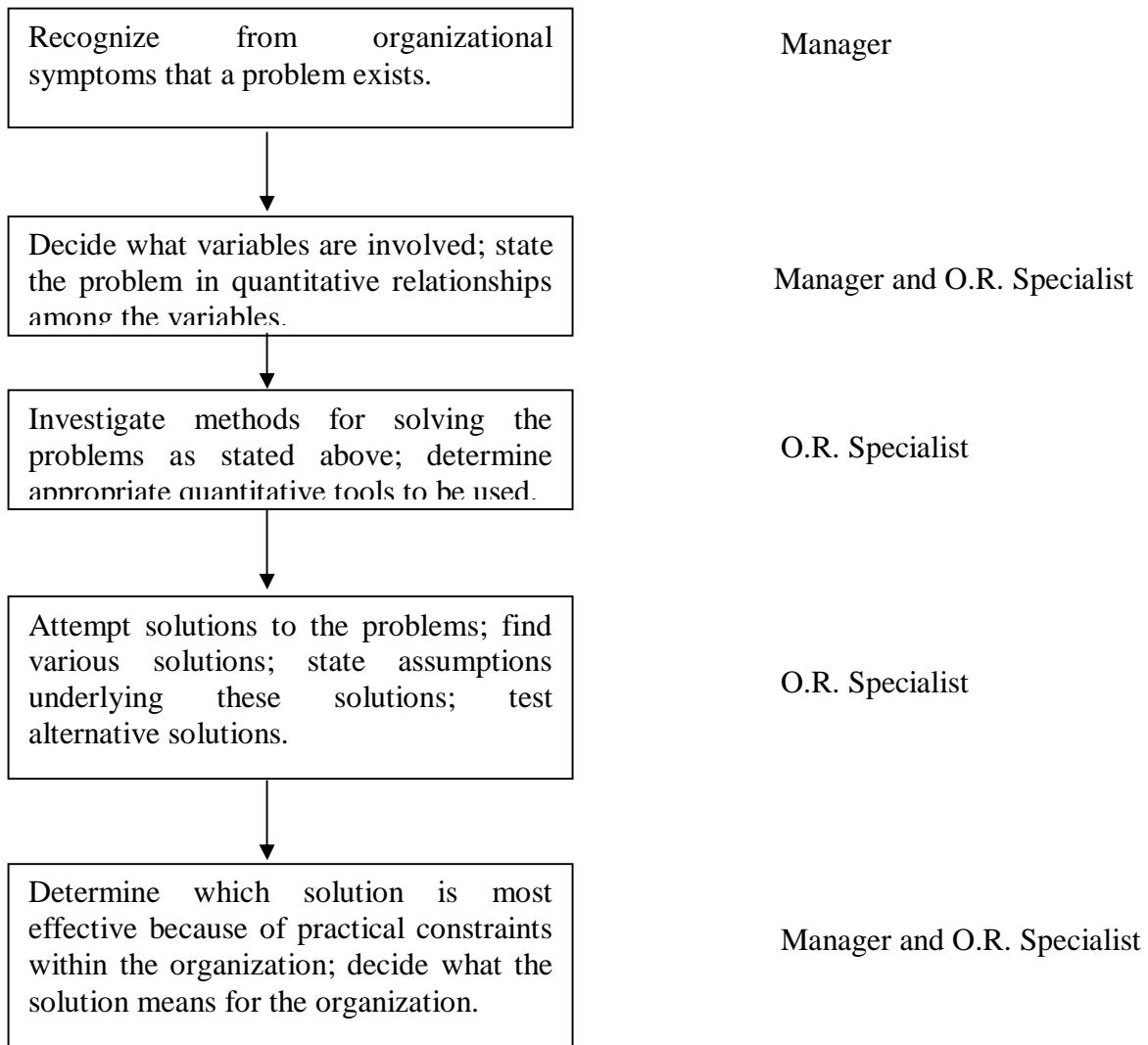
**Table 1-1: Process, Process activities, Process output of O.R. development stages**

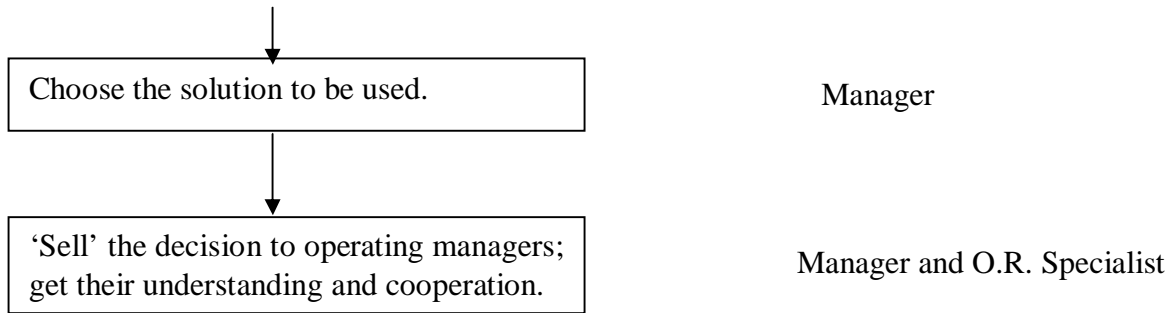
**1.4 Relationship between the Manager and O.R. Specialist**

The key responsibility of manager is decision making. The role of the O.R. specialist is to help the manager make better decisions. Figure 1-1 explains the relationship between the O.R. specialist and the manager/decision maker.

STEPS IN PROBLEM RECOGNITION,  
FORMULATION AND SOLUTION

INVOLVEMENT: O.R. SPECIALIST or  
MANAGER





**Figure 1-1 Relationship between Manager/Decision Maker and O.R. Specialists**

### 1.5 O.R. Tools and Techniques

Operations Research uses any suitable tools or techniques available. The common frequently used tools/techniques are mathematical procedures, cost analysis, electronic computation. However, operations researchers given special importance to the development and the use of techniques like linear programming, game theory, decision theory, queuing theory, inventory models and simulation. In addition to the above techniques, some other common tools are non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling (PERT/CPM), symbolic Model, information theory, and value theory. There is many other Operations Research tools/techniques also exists. The brief explanations of some of the above techniques/tools are as follows:

#### **Linear Programming:**

This is a constrained optimization technique, which optimize some criterion within some constraints. In Linear programming the objective function (profit, loss or return on investment) and constraints are linear. There are different methods available to solve linear programming.

#### **Game Theory:**

This is used for making decisions under conflicting situations where there are one or more players/opponents. In this the motive of the players are dichotomized. The success of one player tends to be at the cost of other players and hence they are in conflict.

#### **Decision Theory:**

Decision theory is concerned with making decisions under conditions of complete certainty about the future outcomes and under conditions such that we can make some probability about what will happen in future.

#### **Queuing Theory:**

This is used in situations where the queue is formed (for example customers waiting for service, aircrafts waiting for landing, jobs waiting for processing in the computer system, etc). The objective here is minimizing the cost of waiting without increasing the cost of servicing.

**Inventory Models:**

Inventory model make a decisions that minimize total inventory cost. This model successfully reduces the total cost of purchasing, carrying, and out of stock inventory.

**Simulation:**

Simulation is a procedure that studies a problem by creating a model of the process involved in the problem and then through a series of organized trials and error solutions attempt to determine the best solution. Some times this is a difficult/time consuming procedure. Simulation is used when actual experimentation is not feasible or solution of model is not possible.

**Non-linear Programming:**

This is used when the objective function and the constraints are not linear in nature. Linear relationships may be applied to approximate non-linear constraints but limited to some range, because approximation becomes poorer as the range is extended. Thus, the non-linear programming is used to determine the approximation in which a solution lies and then the solution is obtained using linear methods.

**Dynamic Programming:**

Dynamic programming is a method of analyzing multistage decision processes. In this each elementary decision depends on those preceding decisions and as well as external factors.

**Integer Programming:**

If one or more variables of the problem take integral values only then dynamic programming method is used. For example number of motor in an organization, number of passenger in an aircraft, number of generators in a power generating plant, etc.

**Markov Process:**

Markov process permits to predict changes over time information about the behavior of a system is known. This is used in decision making in situations where the various states are defined. The probability from one state to another state is known and depends on the current state and is independent of how we have arrived at that particular state.

**Network Scheduling:**

This technique is used extensively to plan, schedule, and monitor large projects (for example computer system installation, R & D design, construction, maintenance, etc.). The aim of this technique is minimize trouble spots (such as delays, interruption, production bottlenecks, etc.) by identifying the critical factors. The different activities and their relationships of the entire project are represented diagrammatically with the help of networks and arrows, which is used for identifying critical activities and path. There are two main types of technique in network scheduling, they are:

Program Evaluation and Review Technique (PERT) – is used when activities time is not known accurately/ only probabilistic estimate of time is available.

Critical Path Method (CPM) – is used when activities time is know accurately.

**Information Theory:**

This analytical process is transferred from the electrical communication field to O.R. field. The objective of this theory is to evaluate the effectiveness of flow of information with a given system. This is used mainly in communication networks but also has indirect influence in simulating the examination of business organizational structure with a view of enhancing flow of information.

**1.6 Applications of Operations Research**

Today, almost all fields of business and government utilizing the benefits of Operations Research. There are voluminous of applications of Operations Research. Although it is not feasible to cover all applications of O.R. in brief. The following are the abbreviated set of typical operations research applications to show how widely these techniques are used today:

**Accounting:**

- Assigning audit teams effectively
- Credit policy analysis
- Cash flow planning
- Developing standard costs
- Establishing costs for byproducts
- Planning of delinquent account strategy

**Construction:**

- Project scheduling, monitoring and control
- Determination of proper work force
- Deployment of work force
- Allocation of resources to projects

**Facilities Planning:**

- Factory location and size decision
- Estimation of number of facilities required
- Hospital planning

International logistic system design  
 Transportation loading and unloading  
 Warehouse location decision

**Finance:**

Building cash management models  
 Allocating capital among various alternatives  
 Building financial planning models  
 Investment analysis  
 Portfolio analysis  
 Dividend policy making

**Manufacturing:**

Inventory control  
 Marketing balance projection  
 Production scheduling  
 Production smoothing

**Marketing:**

Advertising budget allocation  
 Product introduction timing  
 Selection of Product mix  
 Deciding most effective packaging alternative

**Organizational Behavior / Human Resources:**

Personnel planning  
 Recruitment of employees  
 Skill balancing  
 Training program scheduling  
 Designing organizational structure more effectively

**Purchasing:**

Optimal buying  
 Optimal reordering  
 Materials transfer

**Research and Development:**

R & D Projects control  
 R & D Budget allocation  
 Planning of Product introduction

**1.7 Limitations of Operations Research**

Operations Research has number of applications; similarly it also has certain limitations. These limitations are mostly related to the model building and money and time factors problems involved in its application. Some of them are as given below:

- i) Distance between O.R. specialist and Manager

Operations Researchers job needs a mathematician or statistician, who might not be aware of the business problems. Similarly, a manager is unable to understand the complex nature of Operations Research. Thus there is a big gap between the two personnel.

- ii) Magnitude of Calculations

The aim of the O.R. is to find out optimal solution taking into consideration all the factors. In this modern world these factors are enormous and expressing them in quantitative model and establishing relationships among these require voluminous calculations, which can be handled only by machines.

iii) Money and Time Costs

The basic data are subjected to frequent changes, incorporating these changes into the operations research models is very expensive. However, a fairly good solution at present may be more desirable than a perfect operations research solution available in future or after some time.

iv) Non-quantifiable Factors

When all the factors related to a problem can be quantifiable only then operations research provides solution otherwise not. The non-quantifiable factors are not incorporated in O.R. models. Importantly O.R. models do not take into account emotional factors or qualitative factors.

v) Implementation

Once the decision has been taken it should be implemented. The implementation of decisions is a delicate task. This task must take into account the complexities of human relations and behavior and in some times only the psychological factors.

## 1.8 Summary

Operations Research is relatively a new discipline, which originated in World War II, and became very popular throughout the world. India is one of the few first countries in the world who started using operations research. Operations Research is used successfully not only in military/army operations but also in business, government and industry. Now a day's operations research is almost used in all the fields.

Proposing a definition to the operations research is a difficult one, because its boundary and content are not fixed. The tools for operations search is provided from the subject's viz. economics, engineering, mathematics, statistics, psychology, etc., which helps to choose possible alternative courses of action. The operations research tool/techniques include linear programming, non-linear programming, dynamic programming, integer programming, Markov process, queuing theory, etc.

Operations Research has a number of applications. Similarly it has a number of limitations, which is basically related to the time, money, and the problem involves in the model building. Day-by-day operations research gaining acceptance because it improve decision making effectiveness of the managers. Almost all the areas of business use the operations research for decision making.

## 1.9 Key Terms

**OR:** Operations Research.

**MS:** Management Science.

**Symbolic Model:** An abstract model, generally using mathematical symbols.

**Criterion:** is measurement, which is used to evaluation of the results.

**Integer Programming:** is a technique, which ensures only integral values of variables in the problem.

**Dynamic Programming:** is a technique, which is used to analyze multistage decision process.

**Linear Programming:** is a technique, which optimizes linear objective function under limited constraints.

**Inventory Model:** these are the models used to minimize total inventory costs.

**Optimization:** Means maximization or minimization.

### 1.10 Self Assessment Questions

Q1. Define Operations Research.

Q2. Describe the relationship between the manager and O.R. specialist.

Q3. Explain the various steps in the O.R. development process.

Q4. Discuss the applications of O.R.

Q5. Discuss the limitation of O.R.

Q6. Describe different techniques of O.R.

Q7. Discuss few areas of O.R. applications in your organization or organization you are familiar with.

### 1.11 Further References

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Sharma, J.K., 1989. Mathematical Models in Operations Research, Tata McGraw Hill Publishing Company Ltd., New Delhi.

Beer, Stafford, 1966. Decision and Control, John Wiley & Sons, Inc., New York.

Levin, Rubin, Stinson, Gardner, 1992. Quantitative Approaches to Management, Tata McGraw Hill Publishing Company Ltd. New Delhi.

Wagner, Harvery M., 1975. Principles of Operations Research, PHI, Egnlewood Cliffs, N.J.

**UNIT I**

**LESSON**

**2 LINEAR PROGRAMMING –GRAPHICAL METHOD**

**LESSON STRUCTURE**

- 2.1 Introduction to Linear Programming
- 2.2 Linear Programming Problem Formulation
- 2.3 Formulation with Different Types of Constraints
- 2.4 Graphical Analysis of Linear Programming
- 2.5 Graphical Linear Programming Solution
- 2.6 Multiple Optimal Solutions
- 2.7 Unbounded Solution
- 2.8 Infeasible Solution
- 2.9 Summary
- 2.10 Key Terms
- 2.11 Self Assessment Questions
- 2.12 Key Solutions
- 2.13 Further References



**Objectives**

After studying this lesson, you should be able to:

- ❖ Formulate Linear Programming Problem
- ❖ Identify the characteristics of linear programming problem
- ❖ Make a graphical analysis of the linear programming problem
- ❖ Solve the problem graphically
- ❖ Identify the various types of solutions

## 2.1 Introduction to Linear Programming

Linear Programming is a special and versatile technique which can be applied to a variety of management problems viz. Advertising, Distribution, Investment, Production, Refinery Operations, and Transportation analysis. The linear programming is useful not only in industry and business but also in non-profit sectors such as Education, Government, Hospital, and Libraries. The linear programming method is applicable in problems characterized by the presence of decision variables. The objective function and the constraints can be expressed as **linear functions** of the **decision variables**. The decision variables represent quantities that are, in some sense, controllable inputs to the system being modeled. An objective function represents some principal objective criterion or goal that measures the effectiveness of the system such as maximizing profits or productivity, or minimizing cost or consumption. There is always some practical limitation on the availability of resources viz. man, material, machine, or time for the system. These constraints are expressed as linear equations involving the decision variables. Solving a linear programming problem means determining actual values of the decision variables that optimize the objective function subject to the limitation imposed by the constraints.

The main important feature of linear programming model is the presence of linearity in the problem. The use of linear programming model arises in a wide variety of applications. Some model may not be strictly linear, but can be made linear by applying appropriate mathematical transformations. Still some applications are not at all linear, but can be effectively approximated by linear models. The ease with which linear programming models can usually be solved makes an attractive means of dealing with otherwise intractable nonlinear models.

## 2.2 Linear Programming Problem Formulation

The linear programming problem formulation is illustrated through a product mix problem. The product mix problem occurs in an industry where it is possible to manufacture a variety of products. A product has a certain margin of profit per unit, and uses a common pool of limited resources. In this case the linear programming technique identifies the products combination which will maximize the profit subject to the availability of limited resource constraints.

### Example 2.1:

Suppose an industry is manufacturing two types of products P1 and P2. The profits per Kg of the two products are Rs.30 and Rs.40 respectively. These two products require processing in three types of machines. The following table shows the available machine hours per day and the time required on each

machine to produce one Kg of P1 and P2. Formulate the problem in the form of linear programming model.

Profit/Kg	P1 Rs.30	P2 Rs.40	Total available Machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

**Solution:**

The procedure for linear programming problem formulation is as follows:

Introduce the decision variable as follows:

Let  $x_1$  = amount of P1

$x_2$  = amount of P2

In order to maximize profits, we establish the objective function as

$$30x_1 + 40x_2$$

Since one Kg of P1 requires 3 hours of processing time in machine 1 while the corresponding requirement of P2 is 2 hours. So, the first constraint can be expressed as

$$3x_1 + 2x_2 \leq 600$$

Similarly, corresponding to machine 2 and 3 the constraints are

$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

In addition to the above there is no negative production, which may be represented algebraically as

$$x_1 \geq 0 \quad ; \quad x_2 \geq 0$$

Thus, the product mix problem in the linear programming model is as follows:

Maximize

$$30x_1 + 40x_2$$

Subject to:

$$3x_1 + 2x_2 \leq 600$$

$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

$$x_1 \geq 0, x_2 \geq 0$$

**2.3 Formulation with Different Types of Constraints**

The constraints in the previous example 2.1 are of “less than or equal to” type. In this section we are going to discuss the linear programming problem with different constraints, which is illustrated in the following Example 2.2.

**Example 2.2:**

A company owns two flour mills viz. A and B, which have different production capacities for high, medium and low quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs.2000 and Rs.1500 per day to run mill A and B respectively. On a day, Mill A produces 6, 2 and 4 quintals of high, medium and low quality flour, Mill B produces 2, 4 and 12 quintals of high, medium and low quality flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically.

**Solution:**

Let us define  $x_1$  and  $x_2$  are the mills A and B. Here the objective is to minimize the cost of the machine runs and to satisfy the contract order. The linear programming problem is given by

Minimize

$$2000x_1 + 1500x_2$$

Subject to:

$$6x_1 + 2x_2 \geq 8$$

$$2x_1 + 4x_2 \geq 12$$

$$4x_1 + 12x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0$$

## 2.4 Graphical Analysis of Linear Programming

This section shows how a two-variable linear programming problem is solved graphically, which is illustrated as follows:

**Example 2.3:**

Consider the product mix problem discussed in section 2.2

Maximize

$$30x_1 + 40x_2$$

Subject to:

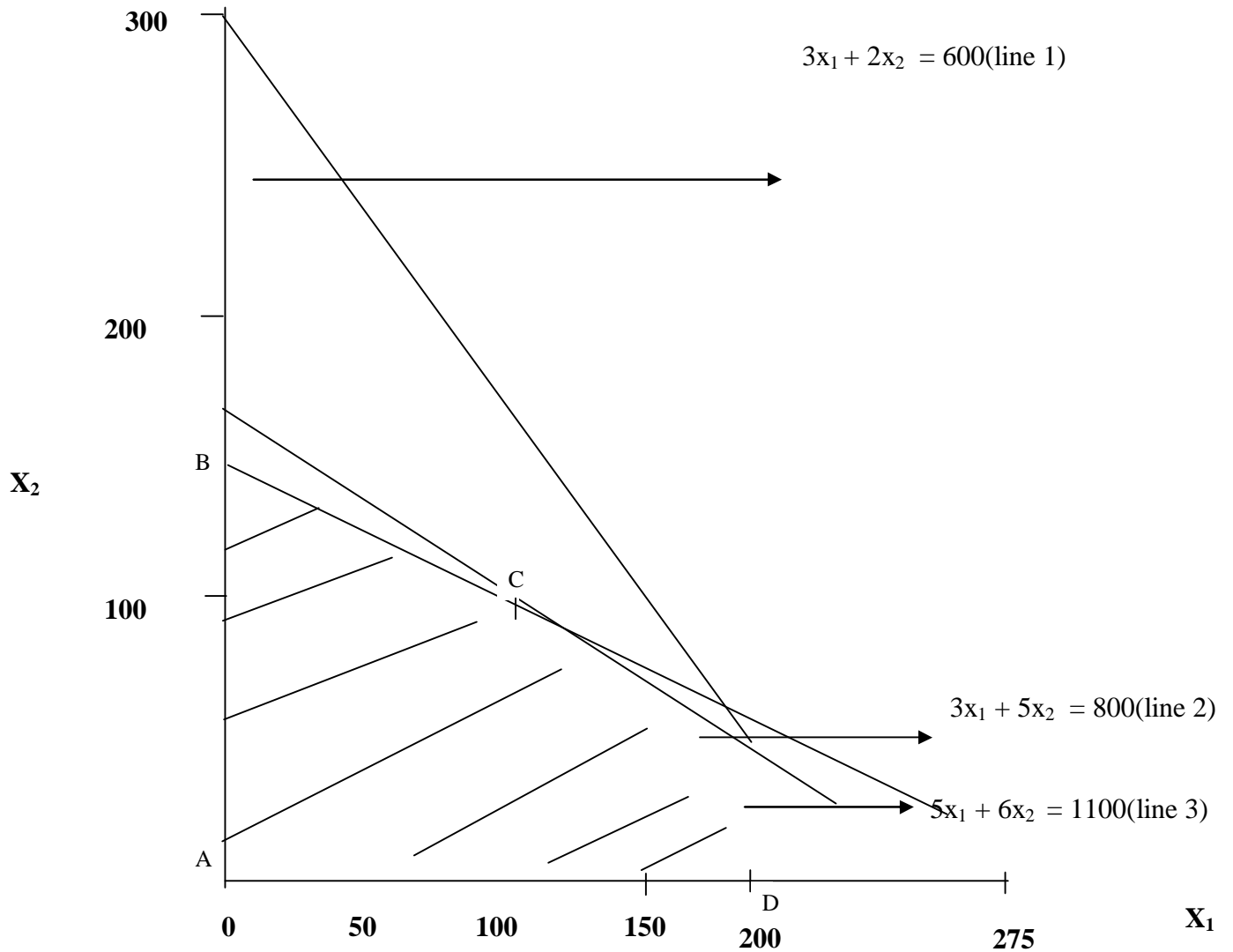
$$3x_1 + 2x_2 \leq 600$$

$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

$$x_1 \geq 0, x_2 \geq 0$$

From the first constraints  $3x_1 + 2x_2 \leq 600$ , draw the line  $3x_1 + 2x_2 = 600$  which passes through the point  $(200, 0)$  and  $(0, 300)$ . This is shown in the following graph as line 1.



**Graph 1: Three closed half planes and Feasible Region**

Half Plane - A linear inequality in two variables is called as a half plane.

Boundary - The corresponding equality (line) is called as the boundary of the half plane.

Close Half Plane – Half plane with its boundary is called as a closed half plane.

In this case we must decide in which side of the line  $3x_1 + 2x_2 = 600$  the half plane is located. The easiest way to solve the inequality for  $x_2$  is

$$3x_1 \leq 600 - 2x_2$$

And for the fixed  $x_1$ , the coordinates satisfy this inequality are smaller than the corresponding ordinate on the line and thus the inequality is satisfied for all the points below the line 1.

Similarly, we have to determine the closed half planes for the inequalities  $3x_1 + 5x_2 \leq 800$  and  $5x_1 + 6x_2 \leq 1100$  (line 2 and line 3 in the graph). Since all the three constraints must be satisfied simultaneously we have consider the intersection of these three closed half planes. The complete intersection of these three closed half planes is shown in the above graph as ABCD. The region ABCD is called the feasible region, which is shaded in the graph.

**Feasible Solution:**

Any non-negative value of  $x_1, x_2$  that is  $x_1 \geq 0$  and  $x_2 \geq 0$  is known as feasible solution of the linear programming problem if it satisfies all the existing constraints.

**Feasible Region:**

The collection of all the feasible solution is called as the feasible region.

**Example 2.4:**

In the previous example we discussed about the less than or equal to type of linear programming problem, i.e. maximization problem. Now consider a minimization (i.e. greater than or equal to type) linear programming problem formulated in Example 2.2.

Minimize

$$2000x_1 + 1500x_2$$

Subject to:

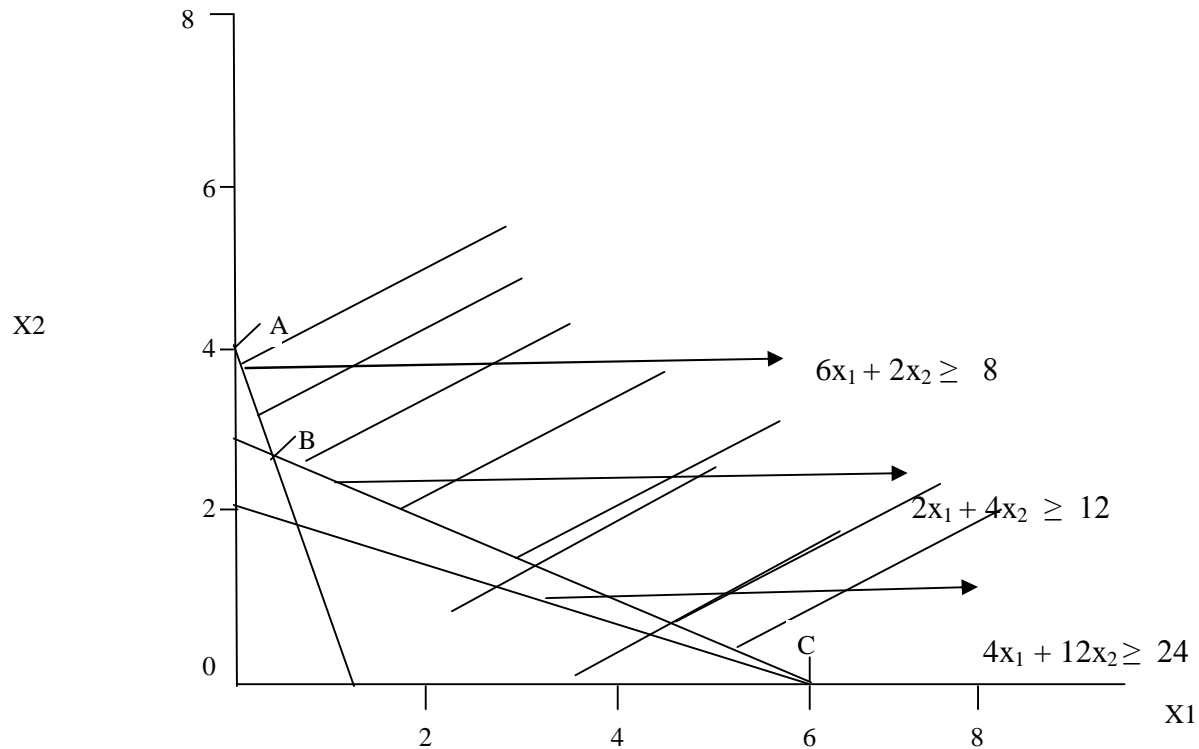
$$6x_1 + 2x_2 \geq 8$$

$$2x_1 + 4x_2 \geq 12$$

$$4x_1 + 12x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0$$

The three lines  $6x_1 + 2x_2 = 8$ ,  $2x_1 + 4x_2 = 12$ , and  $4x_1 + 12x_2 = 24$  passes through the point (1.3,0) (0,4), (6,0) (0,3) and (6,0) (0,2). The feasible region for this problem is shown in the following Graph 2. In this problem the constraints are of greater than or equal to type of feasible region, which is bounded on one side only.



**Graph 2: Feasible Region**

### 2.5 Graphical Linear Programming Solution

A two variable linear programming problem can be easily solved graphically. The method is simple but the principle of solution is depends on certain analytical concepts, they are:

#### **Convex Region:**

A region  $R$  is convex if and only if for any two points on the region  $R$  the line connecting those points lies entirely in the region  $R$ .

#### **Extreme Point:**

The extreme point  $E$  of a convex region  $R$  is a point such that it is not possible to locate two distinct points in  $R$ , so that the line joining them will include  $E$ . The extreme points are also called as corner points or vertices.

Thus, the following result provides the solution to the linear programming model:

- “If the minimum or maximum value of a linear function defined over a convex region exists, then it must be on one of the extreme points”.

In this section we are going to describe linear programming graphical solution for both the maximization and minimization problems, discussed in Example 2.3 and Example 2.4.

### Example 2. 5:

Consider the maximization problem described in Example 2.3.

Maximize

$$30x_1 + 40x_2$$

Subject to:

$$3x_1 + 2x_2 \leq 600$$

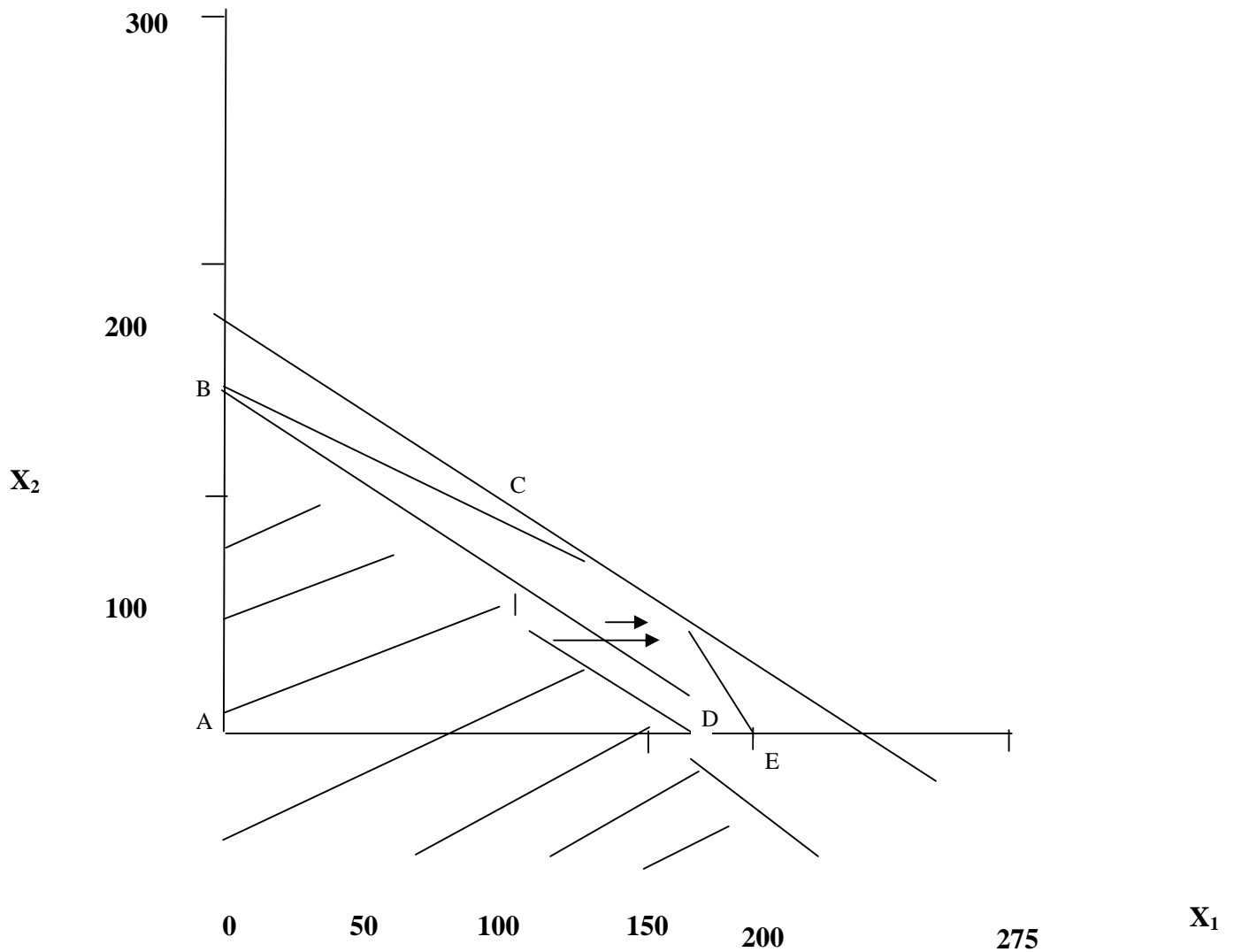
$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

$$x_1 \geq 0, x_2 \geq 0$$

$$M = 30x_1 + 40x_2$$

The feasible region identified in the Example 2.3 is a convex polygon, which is illustrated in the following Graph 3. The extreme point of this convex region are A, B, C, D and E.





### Graph 3: Graphical Linear Programming Solution

In this problem the objective function is  $30x_1 + 40x_2$ . Let be  $M$  is a parameter, the graph  $30x_1 + 40x_2 = M$  is a group of parallel lines with slope  $-30/40$ . Some of these lines intersects the feasible region and contains many feasible solutions, whereas the other lines miss and contain no feasible solution. In order to maximize the objective function, we find the line of this family that intersects the feasible region and is farthest out from the origin. Note that the farthest is the line from the origin the greater will be the value of  $M$ .

Observe that the line  $30x_1 + 40x_2 = M$  passes through the point  $D$ , which is the intersection of the lines  $3x_1 + 5x_2 = 800$  and  $5x_1 + 6x_2 = 1100$  and has the coordinates  $x_1 = 170$  and  $x_2 = 40$ . Since  $D$  is the only feasible solution on this line the solution is unique.

The value of  $M$  is 6700, which is the objective function maximum value. The optimum value variables are  $x_1 = 170$  and  $x_2 = 40$ .

The following Table 1 shows the calculation of maximum value of the objective function.

Extreme Point	Coordinates		Objective Function $30x_1 + 40x_2$
	$X_1$	$X_2$	
A	$X_1 = 0$	$X_2 = 0$	0
B	$X_1 = 0$	$X_2 = 160$	6400
C	$X_1 = 110$	$X_2 = 70$	6100
D	$X_1 = 170$	$X_2 = 40$	6700
E	$X_1 = 200$	$X_2 = 0$	6000

**Table 1: Shows the objective function Maximum value calculation**

#### Example 2.6:

Consider the minimization problem described in Example 2.4.

Minimize

$$2000x_1 + 1500x_2$$

Subject to:

$$6x_1 + 2x_2 \geq 8$$

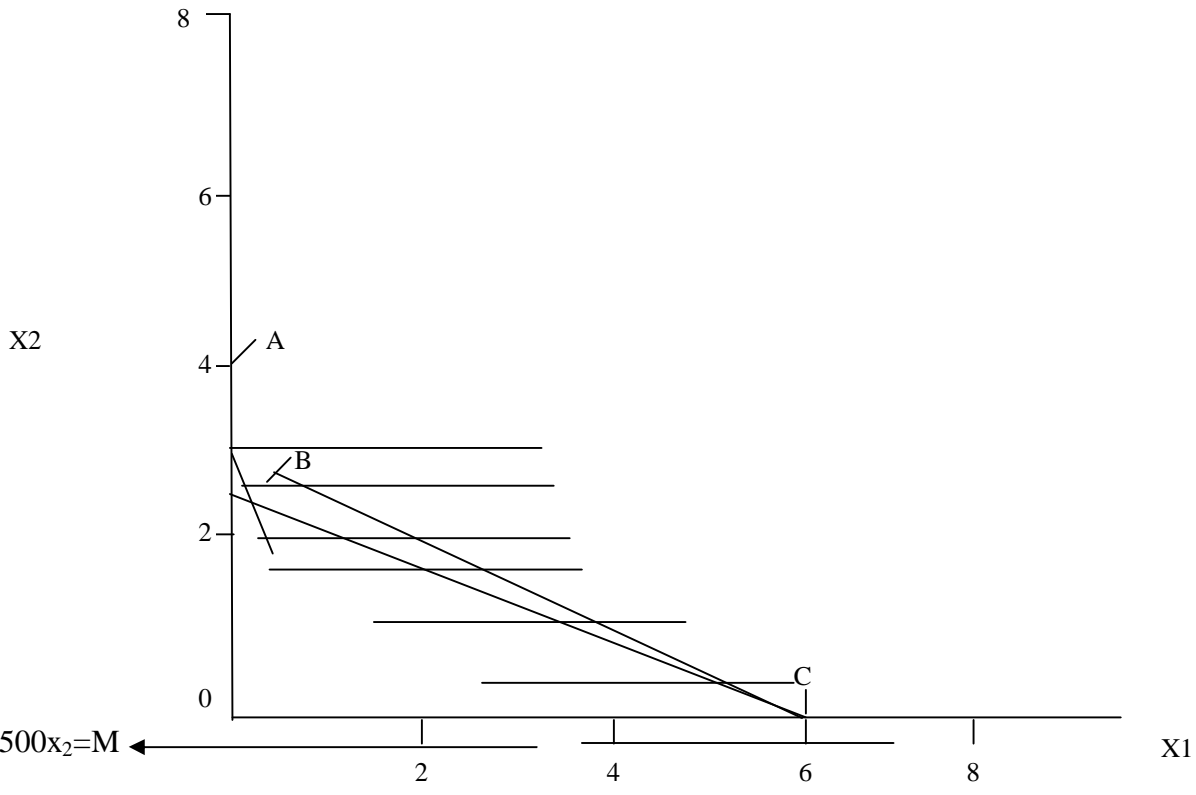
$$2x_1 + 4x_2 \geq 12$$

$$4x_1 + 12x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0$$

The feasible region for this problem is illustrated in the following Graph 4. Here each of the half planes lies above its boundary. In this case the feasible region is infinite. In this case, we are concerned with the minimization; also it is not possible to determine the maximum value. As in the previous

example let us introduce a parameter M in the objective function i.e.  $2000x_1 + 1500x_2 = M$  and draw the lines for different values of M, which is shown in the following Table 2.



**Graph 4: Graphical Linear Programming Solution**

Extreme Point	Coordinates		Objective Function $2000x_1 + 1500x_2$
	$X_1$	$X_2$	
A	$X_1 = 0$	$X_2 = 4$	6000
B	$X_1 = 0.5$	$X_2 = 2.75$	5125
C	$X_1 = 6$	$X_2 = 0$	12000

**Table 2: Shows the objective function Minimum value computation**

The minimum value is 5125 at the extreme point B, which is the value of the M (objective function). The optimum values variables are  $X_1 = 0.5$  and  $X_2 = 2.75$ .

**2.6 Multiple Optimal Solutions**

When the objective function passed through only the extreme point located at the intersection of two half planes, then the linear programming problem possess unique solutions. The previous examples i.e. Example 2.5 and Example 2.6 are of this types (which possessed unique solutions).

When the objective function coincides with one of the half planes generated by the constraints in the problem, will possess multiple optimal solutions. In this section we are going to discuss about the multiple optimal solutions of linear programming problem with the help of the following Example 2.7.

### Example 2.7:

A company purchasing scrap material has two types of scarp materials available. The first type has 30% of material X, 20% of material Y and 50% of material Z by weight. The second type has 40% of material X, 10% of material Y and 30% of material Z. The costs of the two scraps are Rs.120 and Rs.160 per kg respectively. The company requires at least 240 kg of material X, 100 kg of material Y and 290 kg of material Z. Find the optimum quantities of the two scraps to be purchased so that the company requirements of the three materials are satisfied at a minimum cost.

### Solution

First we have to formulate the linear programming model. Let us introduce the decision variables  $x_1$  and  $x_2$  denoting the amount of scrap material to be purchased. Here the objective is to minimize the purchasing cost. So, the objective function here is

Minimize

$$120x_1 + 160x_2$$

Subject to:

$$0.3x_1 + 0.4x_2 \geq 240$$

$$0.2x_1 + 0.1x_2 \geq 100$$

$$0.5x_1 + 0.3x_2 \geq 290$$

$$x_1 \geq 0; x_2 \geq 0$$

Multiply by 10 both sides of the inequalities, then the problem becomes

Minimize

$$120x_1 + 160x_2$$

Subject to:

$$3x_1 + 4x_2 \geq 2400$$

$$2x_1 + x_2 \geq 1000$$

$$5x_1 + 3x_2 \geq 2900$$

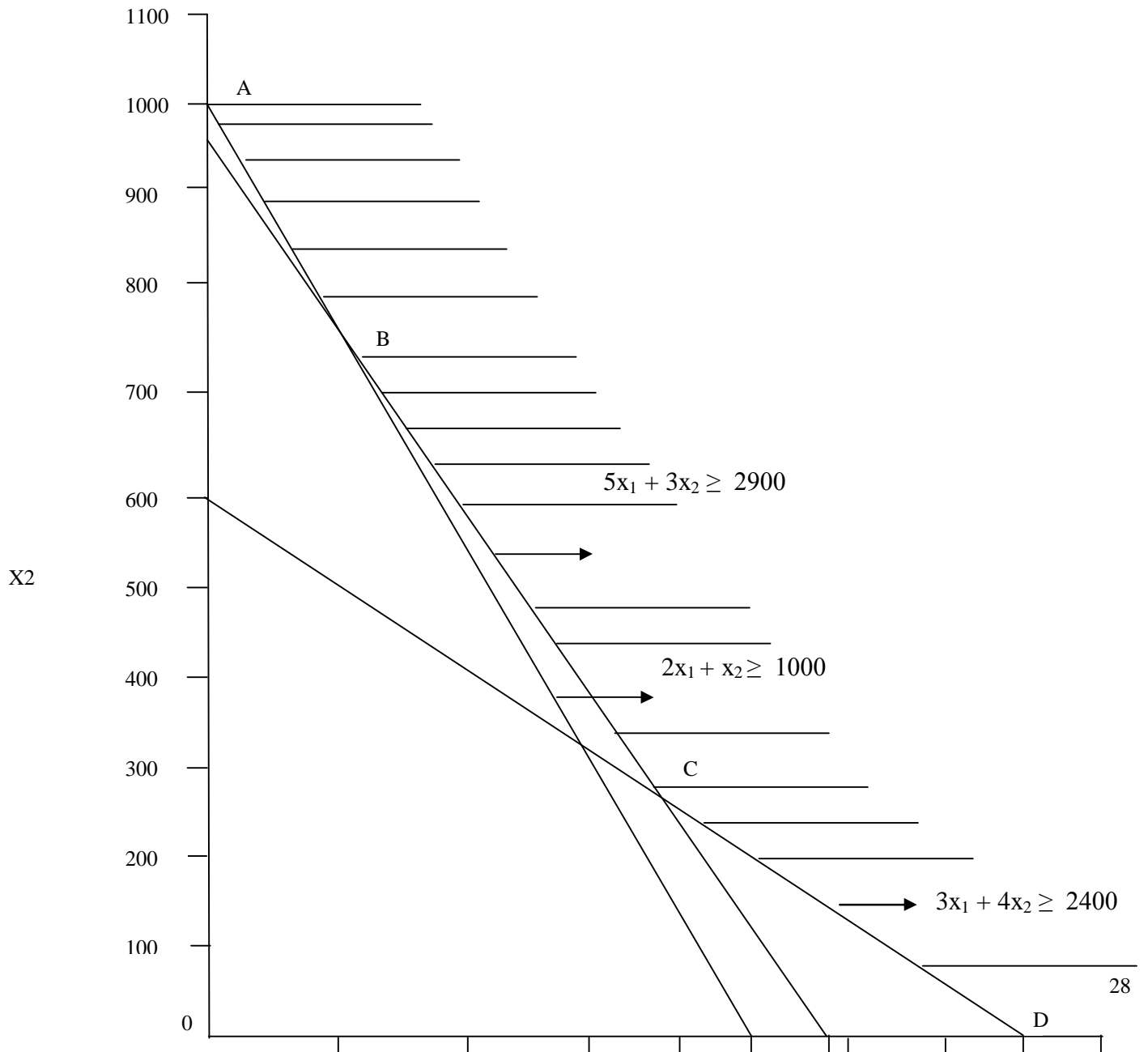
$$x_1 \geq 0; x_2 \geq 0$$

Let us introduce parameter M in the objective function i.e.  $120x_1 + 160x_2 = M$ . Then we have to determine the different values for M, which is shown in the following Table 3.

Extreme Point	Coordinates		Objective Function $120x_1 + 160x_2$
	$X_1$	$X_2$	
A	$X_1 = 0$	$X_2 = 1000$	160000
B	$X_1 = 150$	$X_2 = 740$	136400
C	$X_1 = 400$	$X_2 = 300$	96000
D	$X_1 = 800$	$X_2 = 0$	96000

**Table 3:** Shows the calculation of Minimum objective function value

Note that there are two minimum value for the objective function ( $M=96000$ ). The feasible region and the multiple solutions are indicated in the following Graph 5.



### Graph 5: Feasible Region, Multiple Optimal Solutions

The extreme points are A, B, C, and D. One of the objective functions  $120x_1 + 160x_2 = M$  family coincides with the line CD at the point C with value  $M=96000$ , and the optimum value variables are  $x_1 = 400$ , and  $x_2 = 300$ . And at the point D with value  $M=96000$ , and the optimum value variables are  $x_1 = 800$ , and  $x_2 = 0$ .

Thus, every point on the line CD minimizes objective function value and the problem contains multiple optimal solutions.

### 2.7 Unbounded Solution

When the feasible region is unbounded, a maximization problem may don't have optimal solution, since the values of the decision variables may be increased arbitrarily. This is illustrated with the help of the following problem.

Maximize

$$3x_1 + x_2$$

Subject to:

$$x_1 + x_2 \geq 6$$

$$-x_1 + x_2 \leq 6$$

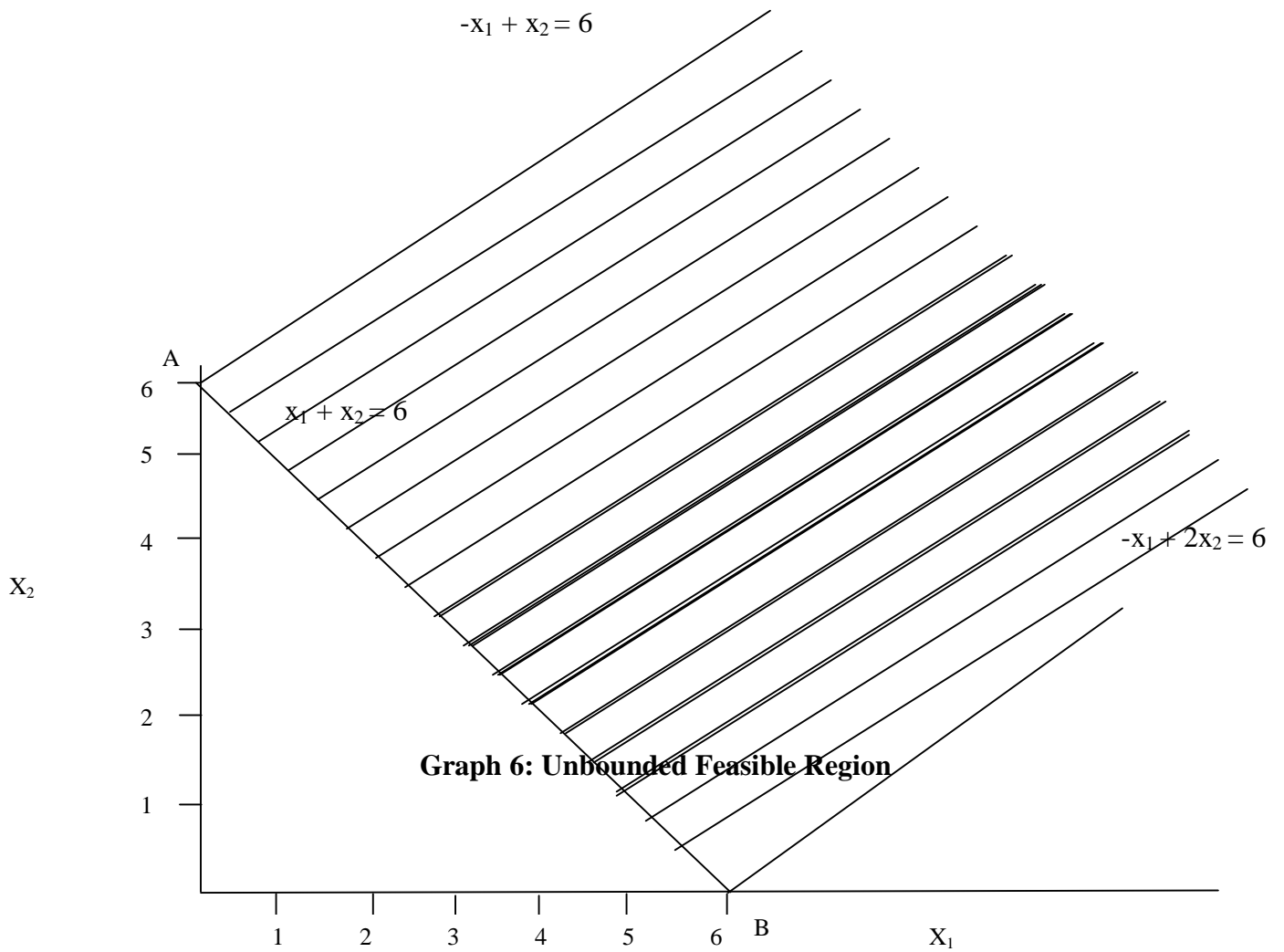
$$-x_1 + 2x_2 \geq -6$$

and

$$x_1, x_2 \geq 0$$

Graph 6 shows the unbounded feasible region and demonstrates that the objective function can be made arbitrarily large by increasing the values of  $x_1$  and  $x_2$  within the unbounded feasible region. In this case, there is no point  $(x_1, x_2)$  is optimal because there are always other feasible points for which objective function is larger. Note that it is not the unbounded feasible region alone that precludes an optimal solution. The minimization of the function subject to the constraints shown in the Graph 6 would be solved at one the extreme point (A or B).

The unbounded solutions typically arise because some real constraints, which represent a practical resource limitation, have been missed from the linear programming formulation. In such situation the problem needs to be reformulated and re-solved.



**2.8 Infeasible Solution**

A linear programming problem is said to be infeasible if no feasible solution of the problem exists. This section describes infeasible solution of the linear programming problem with the help of the following Example 2.8.

**Example 2.8:**

Minimize

$$200x_1 + 300x_2$$

Subject to:

$$0.4x_1 + 0.6x_2 \geq 240$$

$$0.2x_1 + 0.2x_2 \leq 80$$

$$0.4x_1 + 0.3x_2 \geq 180$$

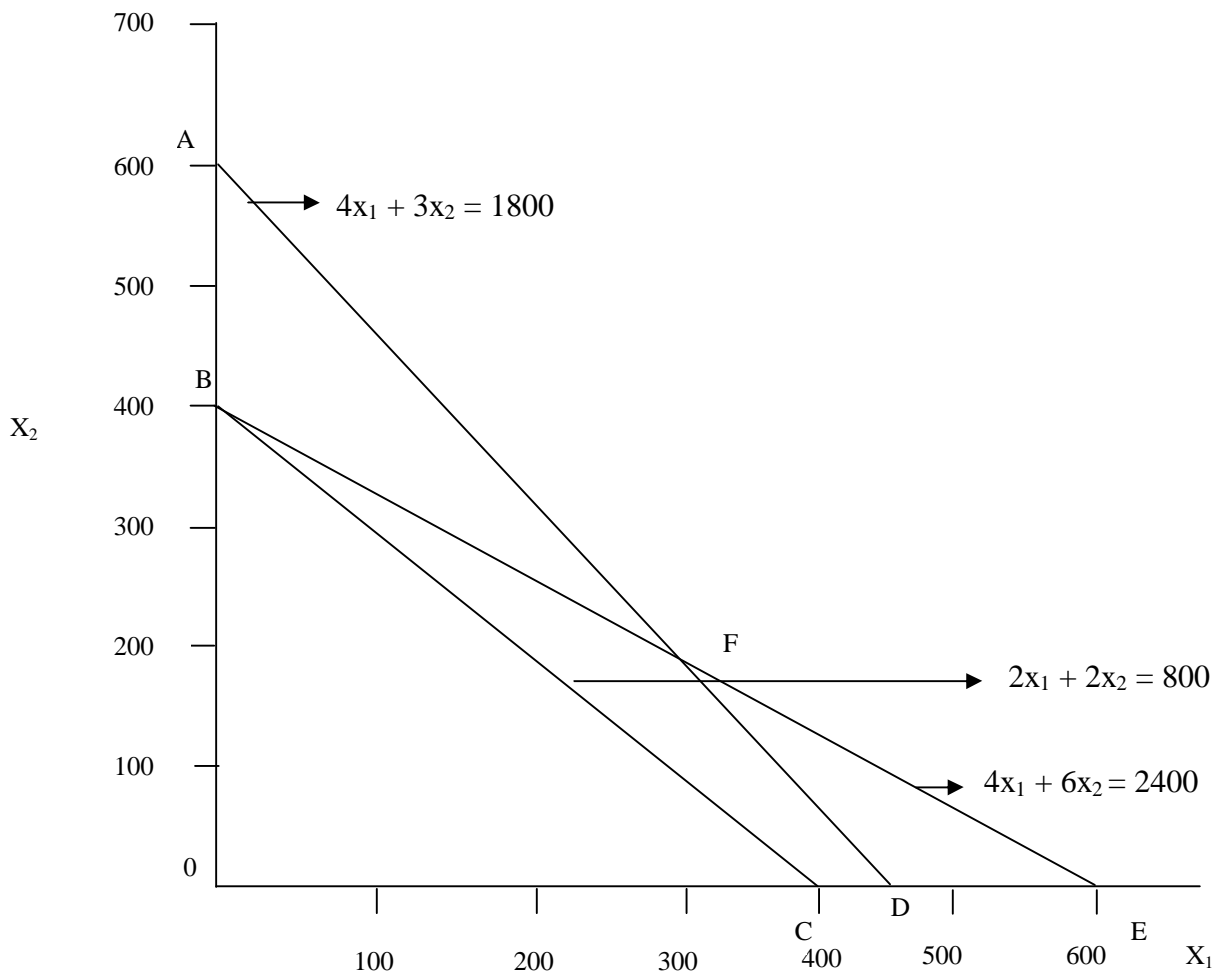
$$x_1, x_2 \geq 0$$

On multiplying both sides of the inequalities by 10, we get

$$4x_1 + 6x_2 \geq 2400$$

$$2x_1 + 2x_2 \leq 800$$

$$4x_1 + 3x_2 \geq 1800$$



**Graph 7: Infeasible Solution**

The region right of the boundary AFE includes all the solutions which satisfy the first ( $4x_1 + 6x_2 \geq 2400$ ) and the third ( $4x_1 + 3x_2 \geq 1800$ ) constraints. The region left of the BC contains all solutions which satisfy the second constraint ( $2x_1 + 2x_2 \leq 800$ ).

Hence, there is no solution satisfying all the three constraints (first, second, and third). Thus, the linear problem is infeasible. This is illustrated in the above Graph 7.

## 2.9 Summary

In Operations Research linear programming is a versatile technique with wide applications in various management problems. Linear Programming problem has a number of characteristics. That is first we have to identify the decision variable. The problem must have a well defined objective function, which are expressed in terms of the decision variables.

The objective function may have to be maximized when it indicates the profit or production or contribution. If the objective function represents cost, in this case the objective function has to be minimized.

The management problem is expressed in terms of the decision variables with the objective function and constraints. A linear programming problem is solved graphically if it contains only two variables.

## 2.10 Key Terms

**Objective Function:** is a linear function of the decision variables representing the objective of the manager/decision maker.

**Constraints:** are the linear equations or inequalities arising out of practical limitations.

**Decision Variables:** are some physical quantities whose values indicate the solution.

**Feasible Solution:** is a solution which satisfies all the constraints (including the non-negative) presents in the problem.

**Feasible Region:** is the collection of feasible solutions.

**Multiple Solutions:** are solutions each of which maximize or minimize the objective function.

**Unbounded Solution:** is a solution whose objective function is infinite.

**Infeasible Solution:** means no feasible solution.

## 2.11 Self Assessment Questions

Q1. A juice company has its products viz. canned apple and bottled juice with profit margin Rs.4 and Rs.2 respectively per unit. The following table shows the labour, equipment, and ingredients to produce each product per unit.

	Canned Apple	Bottled Juice	Total
Labour	2.0	3.0	12.0
Equipment	3.2	1.0	8.0



Ingredients	2.4	2.0	9.0
-------------	-----	-----	-----

Formulate the linear programming problem (model) specifying the product mix which will maximize the profit without exceeding the levels of resources.

Q2. An organization is interested in the analysis of two products which can be produced from the idle time of labour, machine and investment. It was notified on investigation that the labour requirement of the first and the second products was 4 and 5 units respectively and the total available man hours was 48. Only first product required machine hour utilization of one hour per unit and at present only 10 spare machine hours are available. Second product needs one unit of byproduct per unit and the daily availability of the byproduct is 12 units. According to the marketing department the sales potential of first product cannot exceed 7 units. In a competitive market, first product can be sold at a profit of Rs.6 and the second product at a profit of Rs.10 per unit.

Formulate the problem as a linear programming model. Also determine graphically the feasible region. Identify the redundant constraints if any.

Q3. Find graphically the feasible region of the linear programming problem given in Q1.

Q4. A bed mart company is in the business of manufacturing beds and pillows. The company has 40 hours for assembly and 32 hours for finishing work per day. Manufacturing of a bed requires 4 hours for assembly and 2 hours in finishing. Similarly a pillow requires 2 hours for assembly and 4 hours for finishing. Profitability analysis indicates that every bed would contribute Rs.80, while a pillow contribution is Rs.55 respectively. Find out the daily production of the company to maximize the contribution (profit).

Q5. Maximize

$$1170x_1 + 1110x_2$$

Subject to:

$$9x_1 + 5x_2 \geq 500$$

$$7x_1 + 9x_2 \geq 300$$

$$5x_1 + 3x_2 \leq 1500$$

$$7x_1 + 9x_2 \leq 1900$$

$$2x_1 + 4x_2 \leq 1000$$

$$x_1, x_2 \geq 0$$

Find graphically the feasible region and the optimal solution.

Q6. Solve the following LP problem graphically

Minimize

$$2x_1 + 1.7x_2$$

Subject to:

$$0.15x_1 + 0.10x_2 \geq 1.0$$

$$0.75x_1 + 1.70x_2 \geq 7.5$$

$$1.30x_1 + 1.10x_2 \geq 10.0$$

$$x_1, x_2 \geq 0$$

Q7. Solve the following LP problem graphically

Maximize

$$2x_1 + 3x_2$$

Subject to:

$$x_1 - x_2 \leq 1$$

$$x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Q8. Graphically solve the following problem of LP

Maximize

$$3x_1 + 2x_2$$

Subject to:

$$2x_1 - 3x_2 \geq 0$$

$$3x_1 + 4x_2 \leq -12$$

$$x_1, x_2 \geq 0$$

Q9. Solve the following problem graphically

Maximize

$$4x_1 + 4x_2$$

Subject to:

$$-2x_1 + x_2 \leq 1$$

$$x_1 \leq 2$$

$$x_1 + x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

## 2.12 Key Solutions

Q1.

Canned Apple  $x_1$

Bottled Juice  $x_2$

Maximize

$$4x_1 + 2x_2$$

Subject to:

$$2x_1 + 3x_2 \leq 12$$

$$3.2x_1 + x_2 \leq 8$$

$$2.4x_1 + 2x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

Q2.

First Product  $x_1$

Second Product  $x_2$

Maximize

$$6x_1 + 10x_2$$

Subject to:

$$4x_1 + 5x_2 \leq 48$$

$$x_1 \leq 10$$

$$\begin{array}{r} x_2 \leq 12 \\ x_1 \leq 7 \end{array}$$

$$x_1, x_2 \geq 0$$

The constraints  $x_1 \leq 10$  is redundant.

Q4.

Beds = 8

Pillows = 4

Maximum Profits is: Rs.860

Q5.

Optimum variables values are:  $x_1=271.4$ ,  $x_2=0$

The maximum value is: 317573

Q6.

Optimum variables values are:  $x_1=6.32$ ,  $x_2=1.63$

The minimum values is: 15.4

Q7.

The solution is unbounded

Q8.

The problem has no feasible solution

Q9.

The problem has multiple solutions with the following optimum variable values:

$x_1=2$ ,  $x_2=1$  or  $x_1=2/3$ ,  $x_2=7/3$

The Maximum objective function value is: 12

### 2.13 Further References

Mittal, K.V. 1976. Optimization Methods in Operations Research and Systems Analysis, Wiley Eastern Ltd, New Delhi.

Taha, H.A1999. Operations Research An Introduction, PHI Ltd., New Delhi.

Richard I.Levin, David S. Rubin, Joel P. Stinson, Everette S.Gardner, Jr.1992. Quantitative Approaches to Management, McGraw Hill, NJ.

**UNIT I****LESSON****3 LINEAR PROGRAMMING – SIMPLEX METHOD****LESSON STRUCTURE**

- 3.1 Introduction
- 3.2 Basics of Simplex Method
- 3.3 Simplex Method Computation
- 3.4 Simplex Method with More Than Two Variables
- 3.5 Two Phase and M Method
  - 3.5.1 Two Phase Method
  - 3.5.2 M Method
- 3.6 Multiple Solutions
- 3.7 Unbounded Solution
- 3.8 Infeasible Solution
- 3.9 Summary
- 3.10 Key Terms
- 3.11 Self Assessment Questions
- 3.12 Key Solutions
- 3.13 3.13 Further References

**Objectives**

After Studying this lesson, you should be able to:

- ❖ Understand the basics of simplex method
- ❖ Explain the simplex calculations
- ❖ Describe various solutions of Simplex Method
- ❖ Understand two phase and M method

### 3.1 Introduction

The Linear Programming with two variables can be solved graphically. The graphical method of solving linear programming problem is of limited application in the business problems as the number of variables is substantially large. If the linear programming problem has larger number of variables, the suitable method for solving is Simplex Method. The simplex method is an iterative process, through which it reaches ultimately to the minimum or maximum value of the objective function.

The simplex method also helps the decision maker/manager to identify the following:

- Redundant Constraints
- Multiple Solutions
- Unbounded Solution
- Infeasible Problem

### 3.2 Basics of Simplex Method

The basic of simplex method is explained with the following linear programming problem.

#### Example 3.1:

Maximize

$$60x_1 + 70x_2$$

Subject to:

$$2x_1 + x_2 \leq 300$$

$$3x_1 + 4x_2 \leq 509$$

$$4x_1 + 7x_2 \leq 812$$

$$x_1, x_2 \geq 0$$

#### Solution

First we introduce the variables

$$s_3, s_4, s_5 \geq 0$$

So that the constraints becomes equations, thus

$$2x_1 + x_2 + s_3 = 300$$

$$3x_1 + 4x_2 + s_4 = 509$$

$$4x_1 + 7x_2 + s_5 = 812$$

Corresponding to the three constraints, the variables  $s_3, s_4, s_5$  are called as slack variables. Now, the system of equation has three equations and five variables.

There are two types of solutions they are basic and basic feasible, which are discussed as follows:

#### Basic Solution

We may equate any two variables to zero in the above system of equations, and then the system will have three variables. Thus, if this system of three equations with three variables is solvable such a solution is called as basic solution.

For example suppose we take  $x_1=0$  and  $x_2=0$ , the solution of the system with remaining three variables is  $s_3=300$ ,  $s_4=509$  and  $s_5=812$ , this is a basic solution and the variables  $s_3$ ,  $s_4$ , and  $s_5$  are known as basic variables where as the variables  $x_1$ ,  $x_2$  are known as non-basic variables.

The number of basic solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is  $m$  and the number of variables including the slack variables is  $n$  then there are at most  ${}^nC_{n-m} = {}^nC_m$  basic solutions.

### Basic Feasible Solution

A basic solution of a linear programming problem is called as basic feasible solutions if it is feasible it means all the variables are non-negative. The solution  $s_3=300$ ,  $s_4=509$  and  $s_5=812$  is a basic feasible solution.

The number of basic feasible solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is  $m$  and the number of variables including the slack variables is  $n$  then there are at most  ${}^nC_{n-m} = {}^nC_m$  basic feasible solutions.

Every basic feasible solution is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of given constraints. It is impossible to identify the extreme points geometrically if the problem has several variables but the extreme points can be identified using basic feasible solutions. Since one the basic feasible solution will maximize or minimize the objective function, the searching of extreme points can be carry out starting from one basic feasible solution to another.

The Simplex Method provides a systematic search so that the objective function increases in the cases of maximization progressively until the basic feasible solution has been identified where the objective function is maximized.

### 3.3 Simplex Method Computation

This section describes the computational aspect of simplex method. Consider the following linear programming problem

$$\begin{array}{ll}
 \text{Maximize} & \\
 & 60x_1 + 70x_2 \\
 \text{Subject to:} & \\
 & 2x_1 + x_2 + s_3 = 300 \\
 & 3x_1 + 4x_2 + s_4 = 509 \\
 & 4x_1 + 7x_2 + s_5 = 812 \\
 & x_1, x_2, s_3, s_4, s_5 \geq 0
 \end{array}$$

The profit  $Z=60x_1 + 70x_2$  i.e. Maximize  $60x_1 + 70x_2$

The standard form can be summarized in a compact table form as

In this problem the slack variables  $s_3, s_4,$  and  $s_5$  provide a basic feasible solution from which the simplex computation starts. That is  $s_3=300, s_4=509$  and  $s_5=812$ . This result follows because of the special structure of the columns associated with the slacks.

If  $z$  represents profit then  $z=0$  corresponding to this basic feasible solution. We represent by  $C_B$  the coefficient of the basic variables in the objective function and by  $XB$  the numerical values of the basic variable.

So that the numerical values of the basic variables are:  $XB_1=300, XB_2=509, XB_3=812$ . The profit  $z=60x_1+70x_2$  can also expressed as  $z-60x_1-70x_2=0$ . The simplex computation starts with the first compact standard simplex table as given below:

$C_B$	Basic Variables	$C_j$ XB	60 $x_1$	70 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$
0	$s_3$	300	2	1	1	0	0
0	$s_4$	509	3	4	0	1	0
0	$s_5$	812	4	7	0	0	1
	$z$		-60	-70	0	0	0

Table 1

In the objective function the coefficients of the variables are  $CB_1=CB_2=CB_3=0$ . The topmost row of the Table 1 denotes the coefficient of the variables  $x_1, x_2, s_3, s_4, s_5$  of the objective function respectively. The column under  $x_1$  indicates the coefficient of  $x_1$  in the three equations respectively. Similarly the remaining column also formed.

On seeing the equation  $z=60x_1+70x_2$  we may observe that if either  $x_1$  or  $x_2$ , which is currently non-basic is included as a basic variable so that the profit will increase. Since the coefficient of  $x_2$  is higher we choose  $x_2$  to be included as a basic variable in the next iteration. An equivalent criterion of choosing a new basic variable can be obtained the last row of Table 1 i.e. corresponding to  $z$ .

Since the entry corresponding to  $x_2$  is smaller between the two negative values,  $x_2$  will be included as a basic variable in the next iteration. However with three constraints there can be only three basic variables.

Thus, by bringing  $x_2$  a basic variable one of the existing basic variables becomes non-basic. The question here is How to identify this variable? The following statements give the solution to this question.

Consider the first equation i.e.  $2x_1 + x_2 + s_3 = 300$



From this equation

$$2x_1 + s_3 = 300 - x_2$$

But  $x_1 = 0$ . Hence, in order that  $s_3 \geq 0$

$$300 - x_2 \geq 0$$

$$\text{i.e. } x_2 \leq 300$$

Similarly consider the second equation i.e.  $3x_1 + 4x_2 + s_4 = 509$

From this equation

$$3x_1 + s_4 = 509 - 4x_2$$

But,  $x_1 = 0$ . Hence, in order that  $s_4 \geq 0$

$$509 - 4x_2 \geq 0$$

$$\text{i.e. } x_2 \leq 509/9$$

Similarly consider the third equation i.e.  $4x_1 + 7x_2 + s_5 = 812$

From this equation

$$4x_1 + s_5 = 812 - 7x_2$$

But  $x_1 = 0$ . Hence, in order that  $s_5 \geq 0$

$$812 - 7x_2 \geq 0$$

$$\text{i.e. } x_2 \leq 812/7$$

Therefore the three equations lead to

$$x_2 \leq 300, \quad x_2 \leq 509/9, \quad x_2 \leq 812/7$$

Thus  $x_2 = \text{Min}(x_2 \leq 300, x_2 \leq 509/9, x_2 \leq 812/7)$  it means

$$x_2 = \text{Min}(x_2 \leq 300/1, x_2 \leq 509/9, x_2 \leq 812/7) = 116$$

Therefore  $x_2 = 116$

If  $x_2 = 116$ , you may note from the third equation

$$7x_2 + s_5 = 812$$

$$\text{i.e. } s_5 = 0$$

Thus, the variable  $s_5$  becomes non-basic in the next iteration.

So that the revised values of the other two basic variables are

$$s_3 = 300 - x_2 = 184$$

$$s_4 = 509 - 4 \cdot 116 = 45$$

Refer to Table 1, we obtain the elements of the next Table i.e. Table 2 using the following rules:

1. We allocate the quantities which are negative in the z-row. Suppose if all the quantities are positive, the inclusion of any non-basic variable will not increase the value of the objective function. Hence the present solution maximizes the objective function. If there are more than one negative values we choose the variable as a basic variable corresponding to which the z value is least as this is likely to increase the more profit.

2. Let  $x_j$  be the incoming basic variable and the corresponding elements of the  $j^{\text{th}}$  row column be denoted by  $Y1_j$ ,  $Y2_j$  and  $Y3_j$  respectively. If the present values of the basic variables are  $XB_1$ ,  $XB_2$  and  $XB_3$  respectively, then we can compute.

Min  $[XB_1/Y1_j, XB_2/Y2_j, XB_3/Y3_j]$  for  $Y1_j, Y2_j, Y3_j > 0$ .

Note that if any  $Y_{ij} \leq 0$ , this need not be included in the comparison. If the minimum occurs corresponding to  $XB_r/Y_{rj}$  then the  $r^{\text{th}}$  basic variable will become non-basic in the next iteration.

3. Using the following rules the Table 2 is computed from the Table 1.

- i. The revised basic variables are  $s_3$ ,  $s_4$  and  $x_2$ . Accordingly, we make  $CB_1=0$ ,  $CB_2=0$  and  $CB_3=70$ .
- ii. As  $x_2$  is the incoming basic variable we make the coefficient of  $x_2$  one by dividing each element of row-3 by 7. Thus the numerical value of the element corresponding to  $x_1$  is  $4/7$ , corresponding to  $s_3$  is  $1/7$  in Table 2.
- iii. The incoming basic variable should appear only in the third row. So we multiply the third-row of Table 2 by 1 and subtract it from the first-row of Table 1 element by element. Thus the element corresponding to  $x_2$  in the first-row of Table 2 is 0.

Therefore the element corresponding to  $x_1$  is

$$2 - 1 * 4/7 = 10/7 \text{ and the element corresponding to } s_5 \text{ is}$$

$$0 - 1 * 1/7 = -1/7$$

In this way we obtain the elements of the first and the second row in Table 2. In Table 2 the numerical values can also be calculated in a similar way.

$C_B$	Basic Variables	$C_j$ $X_B$	60 $x_1$	70 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$
0	$s_3$	184	10/7	0	1	0	-1/7
0	$s_4$	45	5/7	0	0	1	-4/7
70	$x_2$	116	4/7	1	0	0	1/7
	$z_j - c_j$		-140/7	0	0	0	70/7

Table 2

Let  $CB_1$ ,  $CB_2$ ,  $CB_3$  be the coefficients of the basic variables in the objective function. For example in Table 2  $CB_1=0$ ,  $CB_2=0$  and  $CB_3=70$ . Suppose corresponding to a variable  $J$ , the quantity  $z_j$  is defined as  $z_j = CB_1 Y_{1j} + CB_2 Y_{2j} + CB_3 Y_{3j}$ . Then the  $z$ -row can also be represented as  $Z_j - C_j$ .

For example:

$$z_1 - c_1 = 10/7 * 0 + 5/7 * 0 + 70 * 4/7 - 60 = -140/7$$

$$z_5 - c_5 = -1/7 * 0 - 4/7 * 0 + 1/7 * 70 - 0 = 70/7$$

1. Now we apply rule (1) to Table 2. Here the only negative  $z_j - c_j$  is  $z_1 - c_1 = -140/7$   
Hence  $x_1$  should become a basic variable at the next iteration.

2. We compute the minimum of the ratio

$$\text{Min} \left( \begin{array}{c} \frac{184}{7}, \frac{45}{7}, \frac{116}{7} \end{array} \right) = \text{Min} \left( \begin{array}{c} \frac{644}{5}, 63, 203 \end{array} \right) = 63$$

This minimum occurs corresponding to  $s_4$ , it becomes a non basic variable in next iteration.

3. Like Table 2, the Table 3 is computed using the rules (i), (ii), (iii) as described above.

$C_B$	Basic Variables	$C_j$ $X_B$	60 $x_1$	70 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$
0	$s_3$	94	0	0	1	-2	1
60	$x_1$	63	1	0	0	7/5	-4/5
70	$x_2$	80	0	1	0	-4/5	3/5
	$Z_j - C_j$		0	0	0	28	-6

Table 3

1.  $z_5 - c_5 < 0$  should be made a basic variable in the next iteration.

2. Now compute the minimum ratios

$$\text{Min} \left( \begin{array}{c} \frac{94}{1}, \frac{80}{\frac{3}{5}} \end{array} \right) = 94$$

Note: Since  $y_{25} = -4/5 < 0$ , the corresponding ratio is not taken for comparison. The variable  $s_3$  becomes non basic in the next iteration.

3. From the Table 3, Table 4 is calculated following the usual steps.

$C_B$	Basic Variables	$C_j$ $X_B$	60 $x_1$	70 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$
0	$s_5$	94	0	0	1	-2	1
60	$x_1$	691/5	1	0	4/5	-1/5	0
70	$x_2$	118/5	0	1	-3/5	2/5	0
	$Z_j - C_j$		0	0	6	16	0

Note that  $z_j - c_j \geq 0$  for all  $j$ , so that the objective function can't be improved any further.

Thus, the objective function is maximized for  $x_1 = 691/5$  and  $x_2 = 118/5$  and

The maximum value of the objective function is 9944.

### 3.4 Simplex Method with More Than Two Variables

In previous section we discussed the simplex method of linear programming problem with two decision variables. The simplex method computational procedure can be readily extended to linear programming problems with more than two variables. This is illustrated in this section with the help of the product mix problem given in the following Example 3.2.

#### Example 3.2

An organization has three machine shops viz. A, B and C and it produces three product viz. X, Y and Z using these three machine shops. Each product involves the operation of the machine shops. The time available at the machine shops A, B and C are 100, 72 and 80 hours respectively. The profit per unit of product X, Y and Z is \$22, \$6 and \$2 respectively. The following table shows the time unit amount of each product. Determine an appropriate product mix so as to maximize the profit.

Shops	Machine		
	A	B	C
X	10	7	2
Y	2	3	4
Z	1	2	1
<b>Available Hours</b>	100	72	80

Products	Profit/unit
X	\$22
Y	\$6
Z	\$2

#### Solution

First we have to develop linear programming formulation. The linear programming formulation of the product mix problem is:

$$\begin{aligned} &\text{Maximize} \\ &22x_1 + 6x_2 + 2x_3 \\ &\text{Subject to:} \\ &10x_1 + 2x_2 + x_3 \leq 100 \\ &7x_1 + 3x_2 + 2x_3 \leq 72 \\ &2x_1 + 4x_2 + x_3 \leq 80 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

We introduce slack variables  $s_4$ ,  $s_5$  and  $s_6$  to make the inequalities equation.

Thus, the problem can be stated as

$$\begin{aligned} &\text{Maximize} \\ &22x_1 + 6x_2 + 2x_3 \end{aligned}$$

Subject to:

$$\begin{aligned} 10x_1 + 2x_2 + x_3 + s_4 &= 100 \\ 7x_1 + 3x_2 + 2x_3 + s_5 &= 72 \\ 2x_1 + 4x_2 + x_3 + s_6 &= 80 \\ x_1, x_2, x_3, s_4, s_5, s_6 &\geq 0 \end{aligned}$$

From the above equation the simplex Table 1 can be obtained in a straight forward manner. Here the basic variables are  $s_4$ ,  $s_5$  and  $s_6$ . Therefore  $CB_1 = CB_2 = CB_3 = 0$ .

CB	Basic Variable	$C_j$ XB	22 $x_1$	6 $x_2$	2 $x_3$	0 $s_4$	0 $s_5$	0 $s_6$
0	$s_4$	100	10	2	1	1	0	0
0	$s_5$	72	7	3	2	0	1	0
0	$s_6$	80	2	4	1	0	0	1
	$z_j - c_j$		-22	-6	-2	0	0	0

Table 1

- $z_1 - c_1 = -22$  is the smallest negative value. Hence  $x_1$  should be taken as a basic variable in the next iteration.
- Calculate the minimum of the ratios

$$\text{Min} \left( \frac{100}{10}, \frac{72}{7}, \frac{80}{2} \right) = 10$$

The variable  $s_4$  corresponding to which minimum occurs is made a non basic variable.

- From the Table 1, the Table 2 is calculated using the following rules:

- The revised basic variables are  $x_1$ ,  $s_5$ ,  $s_6$ . Accordingly we make  $CB_1=22$ ,  $CB_2=0$  and  $CB_3=0$ .
- Since  $x_1$  is the incoming variable we make  $x_1$  coefficient one by dividing each element of row 1 by 10. Thus the numerical value of the element corresponding to  $x_2$  is  $2/10$ , corresponding to  $x_3$  is  $1/10$ , corresponding to  $s_4$  is  $1/10$ , corresponding to  $s_5$  is  $0/10$  and corresponding to  $s_6$  is  $0/10$  in Table 2.
- The incoming basic variable should only appear in the first row. So we multiply first row of Table 2 by 7 and subtract it from the second row of Table 1 element by element.

Thus,

The element corresponding to  $x_1$  in the second row of Table 2 is zero

$$\text{The element corresponding to } x_2 \text{ is } 3 - 7 * \frac{2}{10} = \frac{16}{10}$$

By using this way we get the elements of the second and the third row in Table 2.

Similarly, the calculation of numerical values of basic variables in Table 2 is done.

CB	Basic Variable	C <sub>j</sub> XB	22 x <sub>1</sub>	6 x <sub>2</sub>	2 x <sub>3</sub>	0 s <sub>4</sub>	0 s <sub>5</sub>	0 s <sub>6</sub>
22	x <sub>1</sub>	10	1	2/10	1/10	1/10	0	0
0	s <sub>5</sub>	2	0	16/10	13/10	-7/10	1	0
0	s <sub>6</sub>	60	0	18/5	4/5	-1/5	0	1
	z <sub>j</sub> -c <sub>j</sub>		0	-8/5	1/5	12/5	0	0

Table 2

1.  $z_2 - c_2 = -8/5$ . So  $x_2$  becomes a basic variable in the next iteration.
2. Calculate the minimum of the ratios

$$\text{Min} \left( \begin{array}{ccc} 10, & 7, & 60 \\ \frac{2}{10} & \frac{16}{10} & \frac{18}{5} \end{array} \right) = \text{Min} \left( \begin{array}{ccc} 50, & \frac{70}{16}, & \frac{300}{18} \end{array} \right) = \frac{70}{16}$$

Hence the variable  $s_5$  will be a non basic variable in the next iteration.

3. From Table 2, the Table 3 is calculated using the rules (i), (ii) and (iii) mentioned above.

CB	Basic Variable	C <sub>j</sub> XB	22 x <sub>1</sub>	6 x <sub>2</sub>	2 x <sub>3</sub>	0 s <sub>4</sub>	0 s <sub>5</sub>	0 s <sub>6</sub>
22	x <sub>1</sub>	73/8	1	0	-1/16	3/16	-1/8	0
6	x <sub>2</sub>	30/8	0	1	13/16	-7/16	5/8	0
0	s <sub>6</sub>	177/4	0	0	-17/8	11/8	-9/4	1
	z <sub>j</sub> -c <sub>j</sub>		0	0	24/16	24/16	1	0

Table 3

Note that all  $z_j - c_j \geq 0$ , so that the solution is  $x_1 = 73/8$ ,  $x_2 = 30/8$  and  $s_6 = 177/4$  maximizes the objective function.

The Maximum Profit is:  $22 * 73/8 + 6 * 30/8 = 1606/8 + 180/8$

$$= 1786/8 = \$223.25$$

### 3.5 Tow Phase and M-Method

In the last two section we discussed the simplex method was applied to linear programming problems with less than or equal to ( $\leq$ ) type constraints. Thus, there we could introduce slack variables which provide an initial basic feasible solution of the problem.

Generally, the linear programming problem can also be characterized by the presence of both 'less than or equal to' type or 'greater than or equal to ( $\geq$ )' type constraints. In such case it is not always possible to obtain an initial basic feasible solution using slack variables.

The greater than or equal to type of linear programming problem can be solved by using the following methods:

1. Two Phase Method
2. M Method

In this section we will discuss these two methods.

### 3.5.1 Two Phase Method

We discuss the Two Phase Method with the help of the following Example 3.3.

#### Example 3.3

Minimize

$$12.5x_1 + 14.5x_2$$

Subject to:

$$x_1 + x_2 \geq 2000$$

$$0.4x_1 + 0.75x_2 \geq 1000$$

$$0.075x_1 + 0.1x_2 \leq 200$$

$$x_1, x_2 \geq 0$$

#### Solution

Here the objective function is to be minimized; the values of  $x_1$  and  $x_2$  which minimized this objective function are also the values which maximize the revised objective function i.e.

Maximize

$$-12.5x_1 - 14.5x_2$$

We can multiply the second and the third constraints by 100 and 1000 respectively for the convenience of calculation.

Thus, the revised linear programming problem is:

Maximize

$$-12.5x_1 - 14.5x_2$$

Subject to:

$$x_1 + x_2 \geq 2000$$

$$40x_1 + 75x_2 \geq 100000$$

$$75x_1 + 100x_2 \leq 200000$$

$$x_1, x_2 \geq 0$$

Now we convert the two inequalities by introducing **surplus variables**  $s_3$  and  $s_4$  respectively. The third constraint is changed into an equation by introducing a **slack variable**  $s_5$ .

Thus, the linear programming problem becomes as

Maximize

$$-12.5x_1 - 14.5x_2 = -25/2x_1 - 29/2x_2$$

Subject to:

$$\begin{aligned} x_1 + x_2 - s_3 &= 2000 \\ 40x_1 + 75x_2 - s_4 &= 100000 \\ 75x_1 + 100x_2 + s_5 &= 200000 \end{aligned}$$

$$x_1, x_2, s_3, s_4, s_5 \geq 0$$

Even though the surplus variables can convert greater than or equal to type constraints into equations they are unable to provide initial basic variables to start the simplex method calculation. So we may have to introduce two more additional variables  $a_6$  and  $a_7$  called as **artificial variable** to facilitate the calculation of an initial basic feasible solution.

In this method the calculation is carried out in two phases hence two phase method.

### Phase I

In this phase we will consider the following linear programming problem

Maximize

$$-a_6 - a_7$$

Subject to:

$$\begin{aligned} x_1 + x_2 - s_3 + a_6 &= 2000 \\ 40x_1 + 75x_2 - s_4 + a_7 &= 100000 \\ 75x_1 + 100x_2 + s_5 &= 200000 \end{aligned}$$

$$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0$$

The initial basic feasible solution of the problem is

$$a_6 = 2000, a_7 = 100000 \text{ and } s_5 = 200000.$$

As the minimum value of the objective function of the Phase I is zero at the end of the Phase I calculation both  $a_6$  and  $a_7$  become zero.

CB	Basic variables	$C_j$ XB	0 $x_1$	0 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$	-1 $a_6$	-1 $a_7$
-1	$a_6$	2000	1	1	-1	0	0	1	0
-1	$a_7$	100000	40	75	0	-1	0	0	1
0	$s_5$	200000	75	100	0	0	1	0	0
		$Z_j - C_j$	-41	-76	1	1	0	0	0

Table 1

Here  $x_2$  becomes a basic variable and  $a_7$  becomes non basic variable in the next iteration. It is no longer considered for re-entry in the table.



CB	Basic variables	$C_j$ XB	0 $x_1$	0 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$	-1 $a_6$
-1	$a_6$	2000/3	7/15	0	-1	1/75	0	1
0	$x_2$	4000/3	8/15	1	0	-1/75	0	0
0	$s_5$	200000/3	65/3	0	0	4/3	1	0
		$Z_j - C_j$	-1/15	0	1	-1/75	0	0

Table 2

Then  $x_1$  becomes a basic variable and  $a_6$  becomes a non basic variable in the next iteration.

CB	Basic variables	$C_j$ XB	0 $x_1$	0 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$
0	$x_1$	10000/7	1	0	-15/7	1/35	0
0	$x_2$	4000/7	0	1	8/7	-1/35	0
0	$s_5$	250000/7	0	0	325/7	16/21	1
		$Z_j - C_j$	0	0	0	0	0

Table 3

The calculation of Phase I end at this stage. Note that, both the artificial variable have been removed and also found a basic feasible solution of the problem.

The basic feasible solution is:

$$x_1 = 10000/7, x_2 = 4000/2, s_5 = 250000/7.$$

## Phase II

The initial basic feasible solution obtained at the end of the Phase I calculation is used as the initial basic feasible solution of the problem. In this Phase II calculation the original objective function is introduced and the usual simplex procedure is applied to solve the linear programming problem.

CB	Basic variables	C <sub>j</sub> XB	-25/2 x <sub>1</sub>	-29/2 x <sub>2</sub>	0 s <sub>3</sub>	0 s <sub>4</sub>	0 s <sub>5</sub>
-25/2	x <sub>1</sub>	10000/7	1	0	-15/7	1/35	0
-29/2	x <sub>2</sub>	4000/7	0	1	8/7	-1/35	0
0	s <sub>5</sub>	250000/7	0	0	325/7	5/7	1
		z <sub>j</sub> -c <sub>j</sub>	0	0	143/14	2/35	0

Table 1

In this Table 1 all  $z_j - c_j \geq 0$  the current solution maximizes the revised objective function.

Thus, the solution of the problem is:

$$x_1 = 10000/7 = 1428 \text{ and } x_2 = 4000/7 = 571.4 \text{ and}$$

The Minimum Value of the objective function is: 26135.3

### 3.5.2 M Method

In this method also we need artificial variables for determining the initial basic feasible solution. The M method is explained in the next Example 3.4 with the help of the previous Example 3.3.

#### Example 3.4

Maximize

$$-12.5x_1 - 14.5x_2$$

Subject to:

$$\begin{aligned} x_1 + x_2 - s_3 &= 2000 \\ 40x_1 + 75x_2 - s_4 &= 100000 \\ 75x_1 + 100x_2 + s_5 &= 200000 \end{aligned}$$

$$x_1, x_2, s_3, s_4, s_5 \geq 0.$$

Introduce the artificial variables  $a_6$  and  $a_7$  in order to provide basic feasible solution in the second and third constraints. The objective function is revised using a **large positive number** say **M**.

Thus, instead of the original problem, consider the following problem i.e.

Maximize

$$-12.5x_1 - 14.5x_2 - M(a_6 + a_7)$$

Subject to:

$$\begin{aligned} x_1 + x_2 - s_3 + a_6 &= 2000 \\ 40x_1 + 75x_2 - s_4 + a_7 &= 100000 \\ 75x_1 + 100x_2 + s_5 &= 200000 \\ x_1, x_2, s_3, s_4, s_5, a_6, a_7 &\geq 0. \end{aligned}$$

The coefficient of  $a_6$  and  $a_7$  are large negative number in the objective function. Since the objective function is to be maximized in the optimum solution, the artificial variables will be zero. Therefore, the basic variable of the optimum solution are variable other than the artificial variables and hence is a basic feasible solution of the original problem.

The successive calculation of simplex tables is as follows:

CB	Basic variables	$C_j$ XB	-12.5 $x_1$	-14.5 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$	-M $a_6$	-M $a_7$
-M	$a_6$	2000	1	1	-1	0	0	1	0
-M	$a_7$	100000	40	75	0	-1	0	0	1
0	$s_5$	200000	75	100	0	0	1	0	0
		$z_j - c_j$	-41M +12.5	-76M +14.5	M	M	0	0	0

Table 1

Since M is a large positive number, the coefficient of M in the  $z_j - c_j$  row would decide the entering basic variable. As  $-76M < -41M$ ,  $x_2$  becomes a basic variable in the next iteration replacing  $a_7$ . The artificial variable  $a_7$  can't be re-entering as basic variable.

CB	Basic variables	$C_j$ XB	-12.5 $x_1$	-14.5 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$	-M $a_6$
-M	$a_6$	2000/3	7/15	0	-1	1/75	0	1
-14.5	$x_2$	4000/3	8/15	1	0	-1/75	0	0
0	$s_5$	200000/3	65/3	0	0	4/3	1	0
		$z_j - c_j$	-7/15M +143/30	0	M	-M/75 +29/150	0	0

Table 2

Now  $x_1$  becomes a basic variable replacing  $a_6$ . Like  $a_7$  the variable  $a_6$  also artificial variable so it can't be re-entering in the table.

CB	Basic variables	$C_j$ XB	-12.5 $x_1$	-14.5 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$
-12.5	$x_1$	10000/7	1	0	-15/7	1/35	0
-14.5	$x_2$	4000/7	0	1	8/7	-1/35	0
0	$s_5$	250000/7	0	0	325/7	16/21	1
		$z_j - c_j$	0	0	143/14	2/35	0

Table 3

Hence

The optimum solution of the problem is  $x_1 = 10000/7$ ,  $x_2 = 4000/7$  and

The Minimum Value of the Objective Function is: 26135.3

### 3.6 Multiple Solutions

The simplex method also helps in identifying multiple solutions of a linear programming problem. This is explained with the help of the following Example 3.5.

#### Example 3.5

Consider the following linear programming problem.

Maximize

$$2000x_1 + 3000x_2$$

Subject to:

$$6x_1 + 9x_2 \leq 100$$

$$2x_1 + x_2 \leq 20$$

$$x_1, x_2 \geq 0.$$

#### Solution

Introduce the slack variables  $s_3$  and  $s_4$ , so that the inequalities can be converted in to equation as follows:

$$6x_1 + 9x_2 + s_3 = 100$$

$$2x_1 + x_2 + s_4 = 20$$

$$x_1, x_2, s_3, s_4 \geq 0.$$

The computation of simple procedure and tables are as follows:

CB	Basic variables	C <sub>j</sub> XB	2000 x <sub>1</sub>	3000 x <sub>2</sub>	0 s <sub>3</sub>	0 s <sub>4</sub>
0	s <sub>3</sub>	100	6	9	1	0
0	s <sub>4</sub>	20	2	1	0	1
z <sub>j</sub> -c <sub>j</sub>			-2000	-3000	0	0

Table 1

CB	Basic variables	C <sub>j</sub> XB	2000 x <sub>1</sub>	3000 x <sub>2</sub>	0 s <sub>3</sub>	0 s <sub>4</sub>
0	x <sub>2</sub>	100/9	2/3	1	1/9	0
0	s <sub>4</sub>	80/9	4/3	0	-1/9	1
z <sub>j</sub> -c <sub>j</sub>			0	0	3000/9	0

Table 2

Here  $z_j - c_j \geq 0$  for all the variables so that we can't improve the simplex table any more. Hence it is optimum.

The optimum solution is  $x_1 = 0$ ,  $x_2 = 100/9$  and

The maximum value of the objective function is:  $100000/3 = 33333.33$ .

However, the  $z_j - c_j$  value corresponding to the non basic variable  $x_1$  is also zero. This indicates that there is more than one optimum solution for the problem exists.

In order to calculate the value of the alternate optimum solution we have to introduce  $x_1$  as a basic variable replacing  $s_4$ . The next Table 3 shows the computation of this.

CB	Basic variables	$C_j$ XB	2000 $x_1$	3000 $x_2$	0 $s_3$	0 $s_4$
3000	$x_2$	20/3	0	1	1/6	1/2
2000	$x_1$	20/3	1	0	-1/12	3/4
$z_j - c_j$			0	0	1000/3	3000

Table 3

Thus,

$x_1 = 20/3$ ,  $x_2 = 20/3$  also maximize the objective function and

The Maximum value of the objective function is:  $100000/3 = 33333.33$

Thus, the problem has multiple solutions.

### 3.7 Unbounded Solution

In this section we will discuss how the simplex method is used to identify the unbounded solution. This is explained with the help of the following Example 3.6.

#### Example 3.6

Consider the following linear programming problem.

Maximize

$$5x_1 + 4x_2$$

Subject to:

$$x_1 - x_2 \leq 8$$

$$x_1 \leq 7$$

$$x_1, x_2 \geq 0.$$

#### Solution :

Introduce the slack variables  $s_3$  and  $s_4$ , so that the inequalities becomes as equation as follows:

$$x_1 + s_3 = 7$$

$$x_1 - x_2 + s_4 = 8$$

$$x_1, x_2, s_3, s_4 \geq 0.$$

The calculation of simplex procedures and tables are as follows:

CB	Basic variables	$C_j$ XB	5 $x_1$	4 $x_2$	0 $s_3$	0 $s_4$
0	$s_3$	7	1	0	1	0
0	$s_4$	8	1	-1	0	1
$z_j - c_j$			-5	-4	0	0

Table 1

CB	Basic variables	$C_j$ XB	5 $x_1$	4 $x_2$	0 $s_3$	0 $s_4$
5	$x_1$	7	1	0	1	0
0	$s_4$	1	0	-1	-1	1
$z_j - c_j$			0	-4	5	0

Table 2

Note that  $z_2 - c_2 < 0$  which indicates  $x_2$  should be introduced as a basic variable in the next iteration. However, both  $y_{12} \leq 0$ ,  $y_{22} \leq 0$ .

Thus, it is not possible to proceed with the simplex method of calculation any further as we cannot decide which variable will be non basic at the next iteration. This is the criterion for unbounded solution.

**NOTE:** If in the course of simplex computation  $z_j - c_j < 0$  but  $y_{ij} \leq 0$  for all  $i$  then the problem has no finite solution.

But in this case we may observe that the variable  $x_2$  is unconstrained and can be increased arbitrarily. This is why the solution is unbounded.

### 3.8 Infeasible Solution

This section illustrates how to identify the infeasible solution using simplex method. This is explained with the help of the following Example 3.7.

#### Example 3.7

Consider the following problem.

$$\begin{aligned} &\text{Minimize} \\ &200x_1 + 300x_2 \end{aligned}$$

Subject to:

$$2x_1 + 3x_2 \geq 1200$$

$$x_1 + x_2 \leq 400$$

$$2x_1 + 3/2x_2 \geq 900$$

$$x_1, x_2 \geq 0$$

### Solution

Since it is a minimization problem we have to convert it into maximization problem and introduce the slack, surplus and artificial variables. The problem appears in the following manner after doing all these procedure.

$$\begin{aligned} &\text{Maximize} \\ &-200x_1 - 300x_2 \end{aligned}$$

Subject to:

$$2x_1 + 3x_2 - s_3 + a_6 = 1200$$

$$x_1 + x_2 + s_4 = 400$$

$$2x_1 + 3/2x_2 - s_5 + a_7 = 900$$

$$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0$$

Here the  $a_6$  and  $a_7$  are artificial variables. We use two phase method to solve this problem.

### Phase I

Maximize

$$-a_6 - a_7$$

Subject to:

$$2x_1 + 3x_2 - s_3 + a_6 = 1200$$

$$x_1 + x_2 + s_4 = 400$$

$$2x_1 + 3/2x_2 - s_5 + a_7 = 900$$

$$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0$$

The calculation of simplex procedures and tables are as follows:

CB	Basic variables	C <sub>j</sub> XB	0 x <sub>1</sub>	0 x <sub>2</sub>	0 s <sub>3</sub>	0 s <sub>4</sub>	0 s <sub>5</sub>	-1 a <sub>6</sub>	-1 a <sub>7</sub>
-1	a <sub>6</sub>	1200	2	3	-1	0	0	1	0
0	s <sub>4</sub>	400	1	1	0	1	0	0	1
-1	a <sub>7</sub>	900	2	3/2	0	0	-1	0	0
	Z <sub>j</sub> -C <sub>j</sub>		-4	-9/2	1	0	1	0	0

Table 1



CB	Basic variables	$C_j$ XB	0 $x_1$	0 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$	-1 $a_7$
0	$x_2$	400	2/3	1	-1/3	0	0	0
0	$s_4$	0	1/3	0	1/3	1	0	0
-1	$a_7$	300	1	0	1/2	0	-1	1
		$z_j - c_j$	-1	0	-1/2	0	1	0

Table 2

CB	Basic variables	$C_j$ XB	0 $x_1$	0 $x_2$	0 $s_3$	0 $s_4$	0 $s_5$	-1 $a_7$
0	$x_2$	400	0	1	-1	-2	0	0
0	$x_1$	0	1	0	1	3	0	0
-1	$a_7$	300	0	0	-1/2	-3	-1	1
		$z_j - c_j$	0	0	1/2	3	1	0

Table 3

Note that  $z_j - c_j \geq 0$  for all the variables but the artificial variable  $a_7$  is still a basic variable. This situation indicates that the problem has no feasible solution.

### 3.9 Summary

The simplex method is very useful and appropriate method for solving linear programming problem having more than two variables. The slack variables are introduced for less than or equal to type, surplus variables are introduced for greater than or equal to type of linear programming problem. The basic feasible solution is important in order to solve the problem using the simplex method.

A basic feasible solution of a system with  $m$ -equations and  $n$ -variables has  $m$  non-negative variables called as basic variables and  $n-m$  variables with value zero known as non-basic variables. The objective function is maximized or minimized at one of the basic feasible solutions.

Surplus variables can't provide the basic feasible solution instead artificial variables are used to get the basic feasible solutions and it initiates the simplex procedure. Two phase and M-Method are available to solve linear programming problem in these cases.

The simplex method is also used to identify the multiple, unbounded and infeasible solutions.

### 3.10 Key Terms

**Basic Variable:** Variable of a basic feasible solution has a non-negative value.

**Non Basic Variable:** Variable of a feasible solution has a value equal to zero.

**Artificial Variable:** A non-negative variable introduced to provide basic feasible solution and initiate the simplex procedures.

**Slack Variable:** A variable corresponding to a  $\leq$  type constraint is a non-negative variable introduced to convert the inequalities into equations.

**Surplus Variable:** A variable corresponding to a  $\geq$  type constraint is a non-negative variable introduced to convert the constraint into equations.

**Basic Solution:** System of m-equation and n-variables i.e.  $m < n$  is a solution where at least  $n-m$  variables are zero.

**Basic Feasible Solution:** System of m-equation and n-variables i.e.  $m < n$  is a solution where m variables are non-negative and  $n-m$  variables are zero.

**Optimum Solution:** A solution where the objective function is minimized or maximized.

### 3.11 Self Assessment Questions

Q1. A soft drinks company has a two products viz. Coco-cola and Pepsi with profit of \$2 an \$1 per unit. The following table illustrates the labour, equipment and materials to produce per unit of each product. Determine suitable product mix which maximizes the profit using simplex method.

	Pepsi	Coco-cola	Total Resources
Labour	3	2	12
Equipment	1	2.3	6.9
Material	1	1.4	4.9

Q2. A factory produces three using three types of ingredients viz. A, B and C in different proportions. The following table shows the requirements o various ingredients as inputs per kg of the products.

Products	Ingredients		
	A	B	C
1	4	8	8
2	4	6	4
3	8	4	0

The three profits coefficients are 20, 20 and 30 respectively. The factory has 800 kg of ingredients A, 1800 kg of ingredients B and 500 kg of ingredient C.

Determine the product mix which will maximize the profit and also find out maximum profit.

Q3. Solve the following linear programming problem using two phase and M method.

Maximize

$$12x_1 + 15x_2 + 9x_3$$

Subject to:

$$8x_1 + 16x_2 + 12x_3 \leq 250$$

$$4x_1 + 8x_2 + 10x_3 \geq 80$$

$$7x_1 + 9x_2 + 8x_3 = 105$$

$$x_1, x_2, x_3 \geq 0$$

Q4. Solve the following linear programming problem using simplex method.

Maximize

$$3x_1 + 2x_2$$

Subject to:

$$x_1 - x_2 \leq 1$$

$$x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Q5. Solve the following linear programming problem using simplex method.

Maximize

$$x_1 + x_2$$

Subject to:

$$-2x_1 + x_2 \leq 1$$

$$x_1 \leq 2$$

$$x_1 + x_2 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

Q6. Maximize

$$P = 3x_1 + 4x_2 + x_3$$

Subject to:

$$x_1 + 2x_2 + x_3 \leq 6$$

$$2x_1 + 2x_3 \leq 4$$

$$3x_1 + x_2 + x_3 \leq 9$$

$$x_1, x_2, x_3 \geq 0$$

### 3.12 Key Solutions

Q1. Coco-Cola = 20/9, Pepsi = 161/90

Maximum Profit = \$6.23

Q2.  $x_1 = 0, x_2 = 125, x_3 = 75/2$

Maximum Profit = 5375

Q3.  $x_1 = 6, x_2 = 7, x_3 = 0$

Maximum Profit = 177

Q4. Unbounded Solution

Q5.  $x_1 = 2, x_2 = 1$  or  $x_1 = 2/3, x_2 = 7/3$

Maximum Profit = 3.

Q6.  $x_1 = 2$ ,  $x_2 = 2$ ,  $x_3 = 0$

Maximum P = 14

### 3.13. Further References

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Mustafi, C.K. 1988. Operations Research Methods and Practices, Wiley Eastern Limited, New Delhi.

Levin, R and Kirkpatrick, C.A. 1978. Quantitative Approached to Management, Tata McGraw Hill, Kogakusha Ltd., International Student Edition.

Peterson R and Silver, E. A. 1979. Decision Systems for Inventory Management and Production Planning.

Handley, G and T.N. Whitin. 1983. Analysis of Inventory Systems, PHI.

Starr, M.K. and D.W. Miller. 1977. Inventory Control Theory and Practice, PHI.

## UNIT I

**4 DUAL LINEAR PROGRAMMING PROBLEMS****LESSON STRUCTURE**

- 4.1 Introduction
- 4.2 Dual Problem Formulation
- 4.3 Dual Problem Properties
- 4.4 Simple Way of Solving Dual Problem
- 4.5 Summary
- 4.6 Key Terms
- 4.7 Self Assessment Questions
- 4.8 Key Solutions
- 4.9 Further References

**Objectives**

After Studying this lesson, you should be able to:

- ❖ Understand the Dual Linear programming Problem
- ❖ Formulate a Dual Problem
- ❖ Solve the Dual Linear Programming Problem
- ❖ Understand the Properties of a Dual Problem

## 4.1 Introduction

For every linear programming problem there is a corresponding linear programming problem called the **dual**. If the original problem is a maximization problem then the dual problem is minimization problem and if the original problem is a minimization problem then the dual problem is maximization problem. In either case the final table of the dual problem will contain both the solution to the dual problem and the solution to the original problem.

The solution of the dual problem is readily obtained from the original problem solution if the simplex method is used.

The formulation of the dual problem also sometimes referred as the concept of duality is helpful for the understanding of the linear programming. The variable of the dual problem is known as the dual variables or shadow price of the various resources. The dual problem is easier to solve than the original problem. The dual problem solution leads to the solution of the original problem and thus efficient computational techniques can be developed through the concept of duality. Finally, in the competitive strategy problem solution of both the original and dual problem is necessary to understand the complete problem.

## 4.2 Dual Problem Formulation

If the original problem is in the standard form then the dual problem can be formulated using the following rules:

- ✚ The number of constraints in the original problem is equal to the number of dual variables. The number of constraints in the dual problem is equal to the number of variables in the original problem.
- ✚ The original problem profit coefficients appear on the right hand side of the dual problem constraints.
- ✚ If the original problem is a maximization problem then the dual problem is a minimization problem. Similarly, if the original problem is a minimization problem then the dual problem is a maximization problem.
- ✚ The original problem has less than or equal to ( $\leq$ ) type of constraints while the dual problem has greater than or equal to ( $\geq$ ) type constraints.
- ✚ The coefficients of the constraints of the original problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

The Dual Linear Programming Problem is explained with the help of the following Example 4.1.

### Example 4.1

Consider the following product mix problem:

Three machine shops A, B, C produces three types of products X, Y, Z respectively. Each product involves operation of each of the machine shops. The time required for each operation on various products is given as follows:

Products	Machine Shops			Profit per unit
	A	B	C	
X	10	7	2	\$12
Y	2	3	4	\$3
Z	1	2	1	\$1
Available Hours	100	77	80	

The available hours at the machine shops A, B, C are 100, 77, and 80 only. The profit per unit of products X, Y, and Z is \$12, \$3, and \$1 respectively.

**Solution:**

The formulation of Linear Programming (original problem) is as follows:

Maximize

$$12x_1 + 3x_2 + x_3$$

Subject to:

$$10x_1 + 2x_2 + x_3 \leq 100$$

$$7x_1 + 3x_2 + 2x_3 \leq 77$$

$$2x_1 + 4x_2 + x_3 \leq 80$$

$$x_1, x_2, x_3 \geq 0$$

We introduce the slack variables  $s_4, s_5$  and  $s_6$  then the equalities becomes as:

Maximize

$$12x_1 + 3x_2 + x_3$$

Subject to:

$$10x_1 + 2x_2 + x_3 + s_4 = 100$$

$$7x_1 + 3x_2 + 2x_3 + s_5 = 77$$

$$2x_1 + 4x_2 + x_3 + s_6 = 80$$

$$x_1, x_2, x_3, s_4, s_5, s_6 \geq 0$$

Form the above equations, the first simplex table is obtained is as follows:

CB	Basic Variable	C <sub>j</sub> XB	12 x <sub>1</sub>	3 x <sub>2</sub>	1 x <sub>3</sub>	0 s <sub>4</sub>	0 s <sub>5</sub>	0 s <sub>6</sub>
0	s <sub>4</sub>	100	10	2	1	1	0	0

0	$s_5$	77	7	3	2	0	1	0
0	$s_6$	80	2	4	1	0	0	1
	$z_j - c_j$		-12	-3	-1	0	0	0

Table 1

Note that the basic variables are  $s_4$ ,  $s_5$  and  $s_6$ . Therefore  $CB_1 = 0$ ,  $CB_2 = 0$ ,  $CB_3 = 0$ .

1. The smallest negative element in the above table of  $z_1 - c_1$  is -12. Hence,  $x_1$  becomes a basic variable in the next iteration.
2. Determine the minimum ratios

$$\text{Min} \left( \frac{100}{10}, \frac{72}{7}, \frac{80}{2} \right) = 10$$

Here the minimum value is  $s_4$ , which is made as a non-basic variable.

3. The next Table 2 is calculated using the following rules:

- (i) The revised basic variables are  $x_1$ ,  $s_5$ ,  $s_6$ . Accordingly we make  $CB_1=22$ ,  $CB_2=0$  and  $CB_3=0$ .
- (ii) Since  $x_1$  is the incoming variable we make  $x_1$  coefficient one by dividing each element of row 1 by 10. Thus the numerical value of the element corresponding to  $x_2$  is  $2/10$ , corresponding to  $x_3$  is  $1/10$ , corresponding to  $s_4$  is  $1/10$ , corresponding to  $s_5$  is  $0/10$  and corresponding to  $s_6$  is  $0/10$  in Table 2.
- (iii) The incoming basic variable should only appear in the first row. So we multiply first row of Table 2 by 7 and subtract it from the second row of Table 1 element by element.

Thus,

The element corresponding to  $x_1$  in the second row of Table 2 is zero

$$\text{The element corresponding to } x_2 \text{ is } 3 - 7 * \frac{2}{10} = \frac{16}{10}$$

By using this way we get the elements of the second and the third row in Table 2.

Similarly, the calculation of numerical values of basic variables in Table 2 is done.

CB	Basic Variable	$C_j$ XB	22 $x_1$	6 $x_2$	2 $x_3$	0 $s_4$	0 $s_5$	0 $s_6$
12	$x_1$	10	1	$2/10$	$1/10$	$1/10$	0	0
0	$s_5$	7	0	$16/10$	$13/10$	$-7/10$	1	0
0	$s_6$	60	0	$18/5$	$4/5$	$-1/5$	0	1
	$z_j - c_j$		0	$-3/5$	$1/5$	$6/5$	0	0

Table 2



4.  $z_2 - c_2 = -3/5$ . So  $x_2$  becomes a basic variable in the next iteration.

5. Determine the minimum of the ratios

$$\text{Min} \left( \begin{array}{ccc} 10, & 7, & 60 \\ \frac{2}{10} & \frac{16}{10} & \frac{18}{5} \end{array} \right) = \text{Min} \left( \begin{array}{ccc} 50, & 70, & 300 \\ 16 & 18 & 16 \end{array} \right) = \frac{70}{16}$$

So that the variable  $s_5$  will be a non basic variable in the next iteration.

6. From Table 2, the Table 3 is calculated using the rules (i), (ii) and (iii) mentioned above.

CB	Basic Variable	C <sub>j</sub> XB	12 x <sub>1</sub>	3 x <sub>2</sub>	1 x <sub>3</sub>	0 s <sub>4</sub>	0 s <sub>5</sub>	0 s <sub>6</sub>
12	x <sub>1</sub>	73/8	1	0	-1/16	3/16	-1/8	0
3	s <sub>5</sub>	35/8	0	1	13/16	-7/16	5/8	0
0	s <sub>6</sub>	177/4	0	0	-17/8	11/8	-9/4	1
	z <sub>j</sub> -c <sub>j</sub>		0	0	11/16	15/16	3/8	0

Table 3

Since all the  $z_i - c_j \geq 0$ , the optimum solution is as:

$$x_1 = 73/8 \text{ and } x_2 = 35/8 \text{ and}$$

The Maximum Profit is:  $\$981/8 = \$122.625$

Suppose an investor is deciding to purchase the resources A, B, C. What offers is he going to produce?

Let, assume that  $W_1, W_2$  and  $W_3$  are the offers made per hour of machine time A, B and C respectively. Then these prices  $W_1, W_2$  and  $W_3$  must satisfy the conditions given below:

- $W_1, W_2, W_3 \geq 0$
- Assume that the investor is behaving in a rational manner; he would try to bargain as much as possible so that the total annual payable to the produces would be as little as possible. This leads to the following condition:

Minimize

$$100W_1 + 77W_2 + 80W_3$$

3. The total amount offer by the investor to the three resources viz. A, B and C required to produce one unit of each product must be at least as high as the profit gained by the producer per unit.

Since, these resources enable the producer to earn the specified profit corresponding to the product he would not like to sell it for anything less assuming he is behaving rationally. This leads to the following conditions:

$$10w_1 + 7w_2 + 2w_3 \geq 12$$

$$2w_1 + 3w_2 + 4w_3 \geq 3$$

$$w_1 + 2w_2 + w_3 \geq 1$$

Thus, in this case we have a linear problem to ascertain the values of the variable  $w_1$ ,  $w_2$ ,  $w_3$ . The variables  $w_1$ ,  $w_2$  and  $w_3$  are called as **dual variables**.

**Note:**

The original (primal) problem illustrated in this example

- a. considers the objective function maximization
- b. contains  $\leq$  type constraints
- c. has non-negative constraints

This original problem is called as primal problem in the standard form.

### 4.3 Dual Problem Properties

The following are the different properties of dual programming problem:

- i. If the original problem is in the standard form, then the dual problem solution is obtained from the  $z_j - c_j$  values of slack variables.

For example: In the Example 4.1, the variables  $s_4$ ,  $s_4$  and  $s_6$  are the slack variables. Hence the dual problem solution is  $w_1 = z_4 - c_4 = 15/16$ ,  $w_2 = z_5 - c_5 = 3/8$  and  $w_3 = z_6 - c_6 = 0$ .

- ii. The original problem objective function maximum value is the minimum value of the dual problem objective function.

For example:

From the above Example 4.1 we know that the original problem maximum values is  $981/8 = 122.625$ . So that the minimum value of the dual problem objective function is

$$100 \cdot 15/16 + 77 \cdot 3/8 + 80 \cdot 0 = 981/8$$

Here the result has an important practical implication. If both producer and investor analyzed the problem then neither of the two can outmaneuver the other.

- iii. Shadow Price: A resource shadow price is its unit cost, which is equal to the increase in profit to be realized by one additional unit of the resource.

For example:

Let the minimum objective function value is expressed as:

$$100 \cdot 15/16 + 77 \cdot 3/8 + 80 \cdot 0$$

If the first resource is increased by one unit the maximum profit also increases by 15/16, which is the first dual variable of the optimum solution. Therefore, the dual variables are also referred as the resource shadow price or imputed price. Note that in the previous example the shadow price of the third resource is zero because there is already an unutilized amount, so that profit is not increased by more of it until the current supply is totally exhausted.

- iv. In the originals problem, if the number of constraints and variables is  $m$  and  $n$  then the constraint and variables in the dual problem is  $n$  and  $m$  respectively. Suppose the slack variables in the original problem is represented by  $y_1, y_1, \dots, y_n$  and the surplus variables are represented by  $z_1, z_2, \dots, z_n$  in the dual problem.
- v. Suppose, the original problem is not in a standard form, then the dual problem structure is unchanged. However, if a constraint is greater than or equal to type, the corresponding dual variable is negative or zero. Similarly, if a constraint in the original problem is equal to type, then the corresponding dual variables is unrestricted in sign.

### Example 4.2

Consider the following linear programming problem

Maximize

$$22x_1 + 25x_2 + 19x_3$$

Subject to:

$$18x_1 + 26x_2 + 22x_3 \leq 350$$

$$14x_1 + 18x_2 + 20x_3 \geq 180$$

$$17x_1 + 19x_2 + 18x_3 = 205$$

$$x_1, x_2, x_3 \geq 0$$

Note that this is a primal or original problem.

The corresponding dual problem for this problem is as follows:

Minimize

$$250w_1 + 80w_2 + 105w_3$$

Subject to:

$$18w_1 + 4w_2 + 7w_3 \geq 22$$

$$26w_1 + 18w_2 + 19w_3 \geq 25$$

$$22w_1 + 20w_2 + 18w_3 \geq 19$$

$$w_1 \geq 0, w_2 \leq, \text{ and } w_3 \text{ is unrestricted in sign (+ or -).}$$

Now, we can solve this using simplex method as usual.

### 4.4 Simple Way of Solving Dual Problem

Solving of dual problem is simple; this is illustrated with the help of the following Example 4.3.

**Example 4.3:**

Minimize

$$P = x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \geq 8$$

$$2x_1 + x_2 \geq 12$$

$$x_1 \geq 1$$

**Solution:**

Step 1: Set up the P-matrix and its transpose

$$P = \begin{pmatrix} 1 & 1 & 8 \\ 2 & 1 & 12 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$PT = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 8 & & & \end{pmatrix}$$

Step 2: Form the objective function

	$w_1$	$w_2$	$w_3$	$s_1$	$s_2$	$g$	$z$
	1	2	1	1	0	0	1
	1	1	0	0	1	0	2
	-8	-12	-1	0	0	1	0

$$12 \ 1 \ 0$$

constraints and the for the dual

$$w_1 + w_2 \leq 2$$

$$z = 8w_1 = 12w_2 + 2$$

$$+ 2w_2 + w_3 \leq 1$$

Step 3: Construct the initial simplex tableau for the dual

Since there are no negative entries in the last column above the third row, we have a standard simplex problem. The most negative number in the bottom row to the left of the last column is **-12**. This establishes the pivot column. The smallest nonnegative ratio is 1/2. The pivot element is **2** in the  $w_2$ -column.

Pivoting about the

$w_1$	$w_2$	$w_3$	$s_1$	$s_2$	$g$	$z$
1	2	1	1	0	0	1
1/2	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

**2** we get:

$w_1$	$w_2$	$w_3$	$s_1$	$s_2$	$g$	$z$
1/2	<b>1</b>	1/2	1/2	0	0	1/2
1	1	0	0	1	0	2
-8	-12	-1	0	0	1	0

$w_1$	$w_2$	$w_3$	$s_1$	$s_2$	$g$	$z$
1/2	<b>1</b>	1/2	1/2	0	0	1/2
1/2	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

The most negative entry in the bottom row to the left of the last column is **-2**. The smallest non-negative ratio is the **1/2** in the first row. This is the next pivot element.

Pivoting about the

$w_1$	$w_2$	$w_3$	$s_1$	$s_2$	$g$	$z$
<b>1/2</b>	1	1/2	1/2	0	0	1/2
1/2	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

**1/2**:

Since there are no negative entries in the bottom row and to the left of the last column, the process is complete. The solutions are at the feet of the slack variable columns.

Therefore,

The  
provided by  $x_1 = 8$

The  
is: 8

$w_1$	$w_2$	$w_3$	$s_1$	$s_2$	$g$	$z$
<b>1</b>	2	1	1	0	0	1
0	-1	-1	-1	1	0	1
0	4	7	8	0	1	8

optimum solution  
and  $x_2 = 0$

Minimum Value

#### 4.5 Summary

For every linear programming problem there is a dual problem. The variables of the dual problem are called as dual variables. The variables have economic value, which can be used for planning its resources. The dual problem solution is achieved by the simplex method calculation of the original (primal) problem. The dual problem solution has certain properties, which may be very useful for calculation purposes.

#### 4.6 Key Terms

**Original Problem:** This is the original linear programming problem, also called as primal problem.

**Dual Problem:** A dual problem is a linear programming problem is another linear programming problem formulated from the parameters of the primal problem.

**Dual Variables:** Dual programming problem variables.

**Optimum Solution:** The solution where the objective function is minimized or maximized.

**Shadow Price:** Price of a resource is the change in the optimum value of the objective function per unit increase of the resource.

#### 4.7 Self Assessment Questions

Q1. An organization manufactures three products viz. A, B and c. The required raw material per piece of product A, B and C is 2kg, 1kg, and 2kg. Assume that the total weekly availability is 50 kg. In order to produce the products the raw materials are processed on a machine by the labour force and on a weekly

the availability of machine hours is 30. Assume that the available total labour hour is 26. The following table illustrates time required per unit of the three products.

The profit per unit from the products A, B and C are #25, #30 and #40.

Formulate the dual linear programming problem and determine the optimum values of the dual variables.

Q2. Consider the following dual problem

Minimize  
 $3w_1 + 4w_2$

Subject to:  
 $3w_1 + 4w_2 \geq 24$   
 $2w_1 + w_2 \geq 10$   
 $5w_1 + 3w_2 \geq 29$   
 $w_1, w_2 \geq 0$

**4.8 Key Solutions**

Q1. Minimize

$$50w_1 + 30w_2 + 26w_3$$

Subject to:

$$2w_1 + 0.5w_2 + w_3 \geq 25$$

$$w_1 + 3w_2 + 2w_3 \geq 30$$

$$2w_1 + w_2 + w_3 \geq 40$$

$$w_1 = 50/3 = 16.6, w_2 = 0 \text{ and } w_3 = 20/3 = 6.6$$

Q2. Maximize

$$24x_1 + 10x_2 + 29x_3$$

Subject

Product	Labour Hour	Machine Hour
A	0.5	1
B	3	2
C	1	1

to:

$$3x_1 + 2x_2 + 5x_3 \leq 3$$

$$4x_1 + x_2 + 3x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0 \text{ and } w_1 = 4, w_2 = 3$$

Objective Function Maximum

Value is: 24

**4.9 Further References**

Mustafi, C.K. 1988. Operations Research Methods and Practices, Wiley Eastern Limited, New Delhi.

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Peterson R and Silver, E. A. 1979. Decision Systems for Inventory Management and Production Planning.

Levin, R and Kirkpatrick, C.A. 1978. Quantitative Approached to Management, Tata McGraw Hill, Kogakusha Ltd., International Student Edition.



**UNIT II****LESSON****1 TRANSPORTATION PROBLEM****LESSON STRUCTURE**

- 1.5 Introduction
- 1.6 Transportation Algorithm
- 1.7 Basic Feasible Solution of a  
Transportation Problem
- 1.8 Modified Distribution Method
- 1.9 Unbalanced Transportation Problem
- 1.10 Degenerate Transportation  
Problem
- 1.11 Transshipment Problem
- 1.12 Transportation Problem  
Maximization
- 1.13 Summary
- 1.14 Key Terms
- 1.15 Self Assessment Questions
- 1.16 Key Solutions
- 1.17 Further References

**Objectives**

After Studying this lesson, you should be able to:

- ❖ Formulation of a Transportation Problem
- ❖ Determine basic feasible solution using various methods
- ❖ Understand the MODI, Stepping Stone Methods for cost minimization
- ❖ Make unbalanced Transportation Problem into balanced one using appropriate method
- ❖ Solve Degenerate Problem
- ❖ Formulate and Solve Transshipment Problem
- ❖ Describe suitable method for maximizing the objective function instead of minimizing

## 1.1 Introduction

A special class of linear programming problem is **Transportation Problem**, where the objective is to minimize the cost of distributing a product from a number of **sources** (e.g. factories) to a number of **destinations** (e.g. warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given route is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

The transportation problem special feature is illustrated here with the help of following Example 1.1.

### Example 1.1:

Suppose a manufacturing company owns three factories (sources) and distribute his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

Factories	Retail Agency					Capacity
	1	2	3	4	5	
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Usually the above table is referred as Transportation Table, which provides the basic information regarding the transportation problem. The quantities inside the table are known as transportation cost per unit of product. The capacity of the factories 1, 2, 3 is 50, 100 and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100,60,50,50, and 40 respectively.

In this case, the transportation cost of one unit  
 from factory 1 to retail agency 1 is 1,  
 from factory 1 to retail agency 2 is 9,  
 from factory 1 to retail agency 3 is 13, and so on.

A transportation problem can be formulated as linear programming problem using variables with two subscripts.

Let

$x_{11}$ =Amount to be transported from factory 1 to retail agency 1  
 $x_{12}$ = Amount to be transported from factory 1 to retail agency 2  
 .....  
 .....  
 .....  
 .....  
 $x_{35}$ = Amount to be transported from factory 3 to retail agency 5.

Let the transportation cost per unit be represented by  $C_{11}, C_{12}, \dots, C_{35}$  that is  $C_{11}=1, C_{12}=9$ , and so on.

Let the capacities of the three factories be represented by  $a_1=50, a_2=100, a_3=150$ .

Let the requirement of the retail agencies are  $b_1=100, b_2=60, b_3=50, b_4=50, b_5=40$ .

Thus, the problem can be formulated as

Minimize

$$C_{11}x_{11} + C_{12}x_{12} + \dots + C_{35}x_{35}$$

Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a_1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a_2$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a_3$$

$$x_{11} + x_{21} + x_{31} = b_1$$

$$x_{12} + x_{22} + x_{32} = b_2$$

$$x_{13} + x_{23} + x_{33} = b_3$$

$$x_{14} + x_{24} + x_{34} = b_4$$

$$x_{15} + x_{25} + x_{35} = b_5$$

$$x_{11}, x_{12}, \dots, x_{35} \geq 0.$$

Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem. There are varieties of procedures, which are described in the next section.

### 1.2 Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

**Step 1:** Determine a starting basic feasible solution, using any one of the following three methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

**Step 2:** Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.

### 1.3 Basic Feasible Solution of a Transportation Problem

The computation of an initial feasible solution is illustrated in this section with the help of the example 1.1 discussed in the previous section. The problem in the example 1.1 has 8 constraints and 15 variables we can eliminate one of the constraints since  $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4 + b_5$ . Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most 7 non-zero  $X_{ij}$ . Generally, any basic feasible solution with  $m$  sources (such as factories) and  $n$  destination (such as retail agency) has at most  $m + n - 1$  non-zero  $X_{ij}$ .

The special structure of the transportation problem allows securing a non artificial basic feasible solution using one the following three methods.

4. North West Corner Method
5. Least Cost Method
6. Vogel Approximation Method

The difference among these three methods is the **quality** of the initial basic feasible solution they produce, in the sense that a better that a better initial solution yields a smaller objective value. Generally the Vogel Approximation Method produces the **best** initial basic feasible solution, and the North West Corner Method produces the **worst**, but the North West Corner Method involves least computations.

### North West Corner Method:

The method starts at the North West (upper left) corner cell of the tableau (variable  $x_{11}$ ).

**Step -1:** Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

**Step -2:** Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

**Step -3:** If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step -1.

### Example 1.2:

Consider the problem discussed in Example 1.1 to illustrate the North West Corner Method of determining basic feasible solution.

Factories	Retail Agency					Capacity
	1	2	3	4	5	
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

The allocation is shown in the following tableau:

	1	9	13	36	51	Capacity
	<b>50</b>					<del>50</del>
	24	12	16	20	1	<del>100</del> <del>50</del>
	14	33	1	23	26	<del>150</del> <del>140</del> <del>90</del> <del>40</del>
<b>Requirement</b>	<b>100</b>	<del>60</del>	<del>50</del>	<del>50</del>	<del>40</del>	
	<del>50</del>	<del>10</del>				

The arrows show the order in which the allocated (**bolded**) amounts are generated. The starting basic solution is given as

$$\begin{aligned}
 x_{11} &= 50, \\
 x_{21} &= 50, x_{22} = 50 \\
 x_{32} &= 10, x_{33} = 50, x_{34} = 50, x_{35} = 40
 \end{aligned}$$

The corresponding transportation cost is

$$50 * 1 + 50 * 24 + 50 * 12 + 10 * 33 + 50 * 1 + 50 * 23 + 40 * 26 = 4420$$

It is clear that as soon as a value of  $X_{ij}$  is determined, a row (column) is eliminated from further consideration. The last value of  $X_{ij}$  eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most  $m + n - 1$  positive  $X_{ij}$  if the transportation problem has  $m$  sources and  $n$  destinations.

### Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which  $C_{ij}$  is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly. If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

**Example 1.3:**

The least cost method of determining initial basic feasible solution is illustrated with the help of problem presented in the section 1.1.

						<b>Capacity</b>
	1	9	13	36	51	<del>50</del>
<b>50</b>						
	24	12	16	20	1	<del>100 60</del>
		<b>60</b>			<b>40</b>	
	14	33	1	23	26	
<b>50</b>			<b>50</b>	<b>50</b>		<del>150 100 50</del>
<b>Requirement</b>	<b>100</b>	<del><b>60</b></del>	<del><b>50</b></del>	<del><b>50</b></del>	<del><b>40</b></del>	
	<del><b>50</b></del>					

The Least Cost method is applied in the following manner:

We observe that  $C_{11}=1$  is the minimum unit cost in the table. Hence  $X_{11}=50$  and the first row is crossed out since the row has no more capacity. Then the minimum unit cost in the uncrossed-out row and column is  $C_{25}=1$ , hence  $X_{25}=40$  and the fifth column is crossed out. Next  $C_{33}=1$  is the minimum unit cost, hence  $X_{33}=50$  and the third column is crossed out. Next  $C_{22}=12$  is the minimum unit cost, hence  $X_{22}=60$  and the second column is crossed out. Next we look for the uncrossed-out row and column now  $C_{31}=14$  is the minimum unit cost, hence  $X_{31}=50$  and crossed out the first column since it was satisfied. Finally  $C_{34}=23$  is the minimum unit cost, hence  $X_{34}=50$  and the fourth column is crossed out.

So that the basic feasible solution developed by the Least Cost Method has transportation cost is

$$1 * 50 + 12 * 60 + 1 * 40 + 14 * 50 + 1 * 50 + 23 * 50 = 2710$$

**Note** that the minimum transportation cost obtained by the least cost method is much lower than the corresponding cost of the solution developed by using the north-west corner method.

**Vogel Approximation Method (VAM):**

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

**Step 1:** For each row (column) with strictly positive capacity (requirement), determine a **penalty** by subtracting the **smallest** unit cost element in the row (column) from the next **smallest** unit cost element in the same row (column).

**Step 2:** Identify the row or column with the **largest penalty** among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

**Step 3:** We select  $X_{ij}$  as a basic variable if  $C_{ij}$  is the **minimum cost** in the row or column with **largest penalty**. We choose the numerical value of  $X_{ij}$  as high as possible subject to the row and the column constraints. Depending upon whether  $a_i$  or  $b_j$  is the smaller of the two  $i^{\text{th}}$  row or  $j^{\text{th}}$  column is crossed out.

**Step 4:** The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

### Example 1.4:

Consider the following transportation problem

Origin	Destination				$a_i$
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
$b_j$	60	40	30	110	240

Note:  $a_i$ =capacity (supply)

$b_j$ =requirement (demand)

Now, compute the penalty for various rows and columns which is shown in the following table:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
$b_j$	60	40	30	110	240	
Row Penalty	4	15	8	3		

Look for the highest penalty in the row or column, the highest penalty occurs in the **second column** and the minimum unit cost i.e.  $c_{ij}$  in this column is  $c_{12}=22$ . Hence assign 40 to this cell i.e.  $x_{12}=40$  and cross out the second column (since second column was satisfied. This is shown in the following table:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	20	22 <b>40</b>	17	4	80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
$b_j$	60	40	30	110	240	

Row Penalty	4	15	8	3	
-------------	---	----	---	---	--

The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the **first row** and the minimum unit cost in this row is  $c_{14}=4$ , hence  $x_{14}=80$  and cross out the first row. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	22	17	4	0	13
		<b>40</b>		<b>80</b>		
2	24	37	9	7	70	2
3	32	37	20	15	50	5
$b_j$	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 8 which occurs in the **third column** and the **minimum cost** in this column is  $c_{23}=9$ , hence  $x_{23}=30$  and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	22	17	4	0	13
		<b>40</b>		<b>80</b>		
2	24	37	9	7	40	17
			<b>30</b>			
3	32	37	20	15	50	17
$b_j$	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **second row** and the **smallest cost** in this row is  $c_{24}=15$ , hence  $x_{24}=30$  and cross out the fourth column with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	22	17	4	0	13
		<b>40</b>		<b>80</b>		
2	24	37	9	7	10	17
			<b>30</b>	<b>30</b>		
3	32	37	20	15	50	17
$b_j$	60	40	30	110	240	



Row Penalty	8	15	8	8	
-------------	---	----	---	---	--

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **second row** and the **smallest cost** in this row is  $c_{21}=24$ , hence  $x_{21}=10$  and cross out the second row with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	<del>22</del>	17	4	0	13
		<b>40</b>		<b>80</b>		
2	<del>24</del>	<del>37</del>	9	7	0	17
	<b>10</b>		<b>30</b>	<b>30</b>		
3	32	37	20	15	50	17
$b_j$	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **third row** and the **smallest cost** in this row is  $c_{31}=32$ , hence  $x_{31}=50$  and cross out the third row or first column. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	<del>22</del>	17	4	0	13
		<b>40</b>		<b>80</b>		
2	<del>24</del>	<del>37</del>	9	7	0	17
	<b>10</b>		<b>30</b>	<b>30</b>		
3	<del>32</del>	37	20	15	0	17
	<b>50</b>					
$b_j$	60	40	30	110	240	
Row Penalty	8	15	8	8		

The transportation cost corresponding to this choice of basic variables is

$$22 * 40 + 4 * 80 + 9 * 30 + 7 * 30 + 24 * 10 + 32 * 50 = 3520$$

#### 1.4 Modified Distribution Method

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method.

**Step 1:** Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

**Step 2:** Introduce dual variables corresponding to the row constraints and the column constraints. If there are  $m$  origins and  $n$  destinations then there will be  $m+n$  dual variables. The dual variables corresponding to the row constraints are represented by  $u_i, i=1,2,\dots,m$  where as the dual variables corresponding to the column constraints are represented by  $v_j, j=1,2,\dots,n$ . The values of the dual variables are calculated from the equation given below

$$u_i + v_j = c_{ij} \quad \text{if } x_{ij} > 0$$

**Step 3:** Any basic feasible solution has  $m + n - 1$   $x_{ij} > 0$ . Thus, there will be  $m + n - 1$  equation to determine  $m + n$  dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

**Step 4:** If  $x_{ij}=0$ , the dual variables calculated in Step 3 are compared with the  $c_{ij}$  values of this allocation as  $c_{ij} - u_i - v_j$ . If al  $c_{ij} - u_i - v_j \geq 0$ , then by the *theorem of complementary slackness* it can be shown that the corresponding solution of the transportation problem is optimum. If one or more  $c_{ij} - u_i - v_j < 0$ , we select the cell with the least value of  $c_{ij} - u_i - v_j$  and allocate as much as possible subject to the row and column constraints. The allocations of the number of adjacent cell are adjusted so that a basic variable becomes non-basic.

**Step 5:** A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

**Example 1.5:**

For example consider the transportation problem given below:

						<b>Supply</b>
	1	9	13	36	51	<b>50</b>
	24	12	16	20	1	<b>100</b>
	14	33	1	23	26	<b>150</b>
<b>Demand</b>	<b>100</b>	<b>70</b>	<b>50</b>	<b>40</b>	<b>40</b>	<b>300</b>

**Step 1:** First we have to determine the basic feasible solution. The basic feasible solution using least cost method is

$$x_{11}=50, x_{22}=60, x_{25}=40, x_{31}=50, x_{32}=10, x_{33}=50 \text{ and } x_{34}=40$$

**Step 2:** The dual variables  $u_1, u_2, u_3$  and  $v_1, v_2, v_3, v_4, v_5$  can be calculated from the corresponding  $c_{ij}$  values, that is

$$\begin{array}{llll} u_1+v_1=1 & u_2+v_2=12 & u_2+v_5=1 & u_3+v_1=14 \\ u_3+v_2=33 & u_3+v_3=1 & u_3+v_4=23 & \end{array}$$

**Step 3:** Choose one of the dual variables arbitrarily is zero that is  $u_3=0$  as it occurs most often in the above equations. The values of the variables calculated are

$$u_1 = -13, u_2 = -21, u_3 = 0$$

$$v_1 = 14, v_2 = 33, v_3 = 1, v_4 = 23, v_5 = 22$$

**Step 4:** Now we calculate  $c_{ij} - u_i - v_j$  values for all the cells where  $x_{ij}=0$  (.e. unallocated cell by the basic feasible solution)

That is

$$\begin{aligned} \text{Cell}(1,2) &= c_{12} - u_1 - v_2 = 9 + 13 - 33 = -11 \\ \text{Cell}(1,3) &= c_{13} - u_1 - v_3 = 13 + 13 - 1 = 25 \\ \text{Cell}(1,4) &= c_{14} - u_1 - v_4 = 36 + 13 - 23 = 26 \\ \text{Cell}(1,5) &= c_{15} - u_1 - v_5 = 51 + 13 - 22 = 42 \\ \text{Cell}(2,1) &= c_{21} - u_2 - v_1 = 24 + 21 - 14 = 31 \\ \text{Cell}(2,3) &= c_{23} - u_2 - v_3 = 16 + 21 - 1 = 36 \\ \text{Cell}(2,4) &= c_{24} - u_2 - v_4 = 20 + 21 - 23 = 18 \\ \text{Cell}(3,5) &= c_{35} - u_3 - v_5 = 26 - 0 - 22 = 4 \end{aligned}$$

Note that in the above calculation all the  $c_{ij} - u_i - v_j \geq 0$  except for cell (1, 2) where  $c_{12} - u_1 - v_2 = 9 + 13 - 33 = -11$ .

Thus in the next iteration  $x_{12}$  will be a basic variable changing one of the present basic variables non-basic. We also observe that for allocating one unit in cell (1, 2) we have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net transportation cost for each unit of such reallocation is

$$-33 - 1 + 9 + 14 = -11$$

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in the cell (3, 2) will be negative. Thus, the revised basic feasible solution is

$$x_{11}=40, x_{12}=10, x_{22}=60, x_{25}=40, x_{31}=60, x_{33}=50, x_{34}=40$$

### 1.5 Unbalanced Transportation Problem

In the previous section we discussed about the balanced transportation problem i.e. the total supply (capacity) at the origins is equal to the total demand (requirement) at the destination. In this section we are going to discuss about the unbalanced transportation problems i.e. when the total supply is not equal to the total demand, which are called as **unbalanced transportation problem**.

In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost.

#### Example 1.6:

Consider the following unbalanced transportation problem

#### Warehouses

Plant	Warehouses			Supply
	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	
X	20	17	25	400
Y	10	10	20	500
<b>Demand</b>	400	400	500	

In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

Plant	Warehouses			Supply
	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	
X	20	17	25	400
Y	10	10	20	500
<b>Unsatisfied Demand</b>	0	0	0	400
<b>Demand</b>	400	400	500	1300

Now we can solve as balanced problem discussed as in the previous sections.

### 1.6. Degenerate Transportation Problem

In a transportation problem, if a basic feasible solution with  $m$  origins and  $n$  destinations has less than  $m + n - 1$  positive  $X_{ij}$  i.e. occupied cells, then the problem is said to be a **degenerate transportation problem**. The degeneracy problem does not cause any serious difficulty, but it can cause computational problem while determining the optimal minimum solution.

Therefore it is important to identify a degenerate problem as early as beginning and take the necessary action to avoid any computational difficulty. The degeneracy can be identified through the following results:

“In a transportation problem, a degenerate basic feasible solution exists if and only if some partial sum of supply (row) is equal to a partial sum of demand (column). For example the following transportation problem is degenerate. Because in this problem

$$a_1 = 400 = b_1$$

$$a_2 + a_3 = 900 = b_2 + b_3$$

Plant	Warehouses			Supply ( $a_i$ )
	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	

X	20	17	25	400
Y	10	10	20	500
<b>Unsatisfied demand</b>	0	0	0	400
<b>Demand (b<sub>j</sub>)</b>	400	400	500	1300

There is a technique called perturbation, which helps to solve the degenerate problems.

### Perturbation Technique:

The degeneracy of the transportation problem can be avoided if we ensure that no partial sum of  $a_i$  (supply) and  $b_j$  (demand) is equal. We set up a new problem where

$$\begin{aligned} a_i &= a_i + d & i &= 1, 2, \dots, m \\ b_j &= b_j & j &= 1, 2, \dots, n - 1 \\ b_n &= b_n + m_d & d &> 0 \end{aligned}$$

This modified problem is constructed in such a way that no partial sum of  $a_i$  is equal to the  $b_j$ . Once the problem is solved, we substitute  $d = 0$  leading to optimum solution of the original problem.

### Example: 1.7

Consider the above problem

Plant	Warehouses			Supply (a <sub>i</sub> )
	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	
X	20	17	25	400 + d
Y	10	10	20	500 + d
<b>Unsatisfied demand</b>	0	0	0	400 + d
<b>Demand (b<sub>j</sub>)</b>	400	400	500 + 3d	1300 + 3d

Now this modified problem can be solved by using any of the three methods viz. North-west Corner, Least Cost, or VAM.

### 1.7 Transshipment Problem

There could be a situation where it might be more economical to transport consignments in several stages that is initially within certain origins and destinations and finally to the ultimate receipt points, instead of transporting the consignments from an origin to a destination as in the transportation problem.

The movement of consignment involves two different modes of transport viz. road and railways or between stations connected by metre gauge and broad gauge lines. Similarly it is not uncommon to maintain dumps for central storage of certain bulk material. These require transshipment.

Thus for the purpose of transshipment the distinction between an origin and destination is dropped so that from a transportation problem with  $m$  origins and  $n$  destinations we obtain a transshipment problem with  $m + n$  origins and  $m + n$  destinations.

The formulation and solution of a transshipment problem is illustrated with the following Example 1.8.

**Example 1.8:**

Consider the following transportation problem where the origins are plants and destinations are depots.

**Table 1**

Plant	Depot			Supply
	X	Y	Z	
A	\$1	\$3	\$15	150
B	\$3	\$5	\$25	300
<b>Demand</b>	150	150	150	450

When each plant is also considered as a destination and each depot is also considered as an origin, there are altogether five origins and five destinations. So that some additional cost data are necessary, they are as follows:

**Table 2**

Unit transportation cost From Plant To Plant

From	To	
	Plant A	Plant B
Plant A	0	55
Plant B	2	0

**Table 3**

Unit transportation cost From Depot To Depot

From	To	
	Depot X	Depot Y
Depot X		
Depot Y		

<b>From</b>			
Depot X	0	25	2
Depot Y	2	0	3
Depot Z	55	3	0

**Table 4**

Unit transportation cost From Depot to Plant

		<b>To</b>	
		Plant A	Plant B
<b>From</b>			
Depot X		3	15
Depot Y		25	3
Depot Z		45	55

Now, from the Table 1, Table 2, Table 3, Table 4 we obtain the transportation formulation of the transshipment problem, which is shown in the Table 5.

**Table 5**  
**Transshipment Table**

	A	B	X	Y	Z	<b>Supply</b>
A	0	55	1	3	15	150+450=600
B	2	0	3	5	25	300+450=750
X	3	15	0	25	2	450
Y	25	3	2	0	3	450
Z	45	55	55	3	0	450
<b>Demand</b>	450	450	150+450= 600	150+450= 600	150+450= 600	

A buffer stock of 450 which is the total supply and total demand in the original transportation problem is added to each row and column of the transshipment problem. The resulting transportation problem has  $m + n = 5$  origins and  $m + n = 5$  destinations.

By solving the transportation problem presented in the Table 5, we obtain

$$\begin{array}{llllll}
 x_{11}=150 & x_{13}=300 & x_{14}=150 & x_{21}=300 & x_{22}=450 & x_{33}=300 \\
 x_{35}=150 & x_{44}=450 & x_{55}=450 & & & 
 \end{array}$$

The transshipment problem explanation is as follows:

1. Transport  $x_{21}=300$  from plant B to plant A. This increase the availability at plant A to 450 units including the 150 originally available from A.
2. From plant A transport  $x_{13}=300$  to depot X and  $x_{14}=150$  to depot Y.
3. From depot X transport  $x_{35}=150$  to depot Z.

Thus, the total cost of transshipment is:

$$2*300 + 3 * 150 + 1*300 + 2*150 = \$1650$$

Note: The consignments are transported from plants A, B to depots X, Y, Z only according to the transportation Table 1, the minimum transportation cost schedule is  $x_{13}=150$   $x_{21}=150$   $x_{22}=150$  with a minimum cost of 3450.

Thus, transshipment **reduces** the cost of consignment movement.

### 1.8 Transportation Problem Maximization

There are certain types of transportation problem where the objective function is to be maximized instead of minimized. These kinds of problems can be solved by converting the maximization problem into minimization problem. The conversion of maximization into minimization is done by subtracting the unit costs from the highest unit cost of the table.

The maximization of transportation problem is illustrated with the following Example 1.9.

#### Example 1.9:

A company has three factories located in three cities viz. X, Y, Z. These factories supplies consignments to four dealers viz. A, B, C and D. The dealers are spread all over the country. The production capacity of these factories is 1000, 700 and 900 units per month respectively. The net return per unit product is given in the following table.

Factory	Dealers				capacity
	A	B	C	D	
X	6	6	6	4	1000
Y	4	2	4	5	700
Z	5	6	7	8	900
<b>Requirement</b>	900	800	500	400	2600

Determine a suitable allocation to maximize the total return.



This is a maximization problem. Hence first we have to convert this in to minimization problem. The conversion of maximization into minimization is done by subtracting the unit cost of the table from the highest unit cost.

Look the table, here 8 is the highest unit cost. So, subtract all the unit cost from the 8, and then we get the revised minimization transportation table, which is given below.

Factory	Dealers				capacity
	A	B	C	D	
X	2	2	2	4	1000 = a1
Y	4	6	4	3	700 = a2
Z	3	2	1	0	900 = a3
<b>Requirement</b>	900=b1	800=b2	500=b3	400=b4	2600

Now we can solve the problem as a minimization problem.

The problem here is degenerate, since the partial sum of  $a_1=b_2+b_3$  or  $a_3=b_3$ . So consider the corresponding perturbed problem, which is shown below.

Factory	Dealers				capacity
	A	B	C	D	
X	2	2	2	4	1000+d
Y	4	6	4	3	700+d
Z	3	2	1	0	900+d
<b>Requirement</b>	900	800	500	400+3d	2600+3d

First we have to find out the basic feasible solution. The basic feasible solution by lest cost method is  $x_{11}=100+d$ ,  $x_{22}=700-d$ ,  $x_{23}=2d$ ,  $x_{33}=500-2d$  and  $x_{34}=400+3d$ .

Once if the basic feasible solution is found, next we have to determine the optimum solution using MODI (Modified Distribution Method) method. By using this method we obtain

$$\begin{array}{lll} u_1+v_1=2 & u_1+v_2=2 & u_2+v_2=6 \\ u_2+v_3=4 & u_3+v_3=1 & u_3+v_4=0 \end{array}$$

Taking  $u_1=0$  arbitrarily we obtain

$$u_1=0, u_2=4, u_3=1 \text{ and } v_1=2, v_2=3, v_3=0$$

On verifying the condition of optimality, we know that

$$C_{12}-u_1-v_2 < 0 \quad \text{and} \quad C_{32}-u_3-v_2 < 0$$

So, we allocate  $x_{12}=700-d$  and make readjustment in some of the other basic variables.

The revised values are:

$$x_{11}=200+d, \quad x_{12}=800, \quad x_{21}=700-d, \quad x_{23}=2d, \quad x_{33}=500-3d, \quad \text{and} \quad x_{34}=400+3d$$

$$\begin{array}{lll} u_1+v_1=2 & u_1+v_2=2 & u_2+v_1=4 \\ u_2+v_3=4 & u_3+v_3=1 & u_3+v_4=0 \end{array}$$

Taking  $u_1=0$  arbitrarily we obtain

$$\begin{array}{l} u_1=0, \quad u_2=2, \quad u_3=-1 \\ v_1=2, \quad v_2=2, \quad v_3=2, \quad v_4=1 \end{array}$$

Now, the optimality condition is satisfied.

Finally, taking  $d=0$  the optimum solution of the transportation problem is

$$X_{11}=200, \quad x_{12}=800, \quad x_{21}=700, \quad x_{33}=500 \quad \text{and} \quad x_{34}=400$$

Thus, the maximum return is:

$$6*200 + 6*800 + 4*700 + 7*500 + 8*400 = 15500$$

## 1.9 Summary

Transportation Problem is a special kind of linear programming problem. Because of the transportation problem special structure the simplex method is not suitable. But which may be utilized to make efficient computational techniques for its solution.

Generally transportation problem has a number of origins and destination. A certain amount of consignment is available in each origin. Similarly, each destination has a certain demand/requirements. The transportation problem represents amount of consignment to be transported from different origins to destinations so that the transportation cost is minimized with out violating the supply and demand constraints.

There are two phases in the transportation problem. First is the determination of basic feasible solution and second is the determination of optimum solution.

There are three methods available to determine the basic feasible solution, they are

1. North West Corner Method
2. Least Cost Method or Matrix Minimum Method
3. Vogel's Approximation Method (VAM)

In order to determine optimum solution we can use either one of the following method

1. Modified Distribution (MODI) Method
- Or
2. Stepping Stone Method

Transportation problem can be generalized into a Transshipment Problem where transportation of consignment is possible from origin to origin or destination as well as destination to origin or

destination. The transshipment problem may be result in an economy way of shipping in some situations.

**1.10 Key Terms**

**Origin:** is the location from which the shipments are dispatched.

**Destination:** is the location to which the shipments are transported.

**Unit Transportation Cost:** is the transportation cost per unit from an origin to destination.

**Perturbation Technique:** is a method of modifying a degenerate transportation problem in order to solve the degeneracy.

**1.11 Self Assessment Questions**

Q1. Four companies viz. W, X, Y and Z supply the requirements of three warehouses viz. A, B and C respectively. The companies' availability, warehouses requirements and the unit cost of transportation are given in the following table. Find an initial basic feasible solution using

- a. North West Corner Method
- b. Least Cost Method
- c. Vogel Approximation Method (VAM)

Company	Warehouses			Supply
	A	B	C	
W	10	8	9	15
X	5	2	3	20
Y	6	7	4	30
Z	7	6	9	35
<b>Requirement</b>	25	26	49	100

Q2. Find the optimum Solution of the following Problem using MODI method.

Source	Destination			Capacity
	1	2	3	
A 42	8	9	10	30
B	9	11	11	
C	10	12	9	28
<b>Demand</b>	35	40	25	100

Q3. The ABT transport company ships truckloads of food grains from three sources viz. X, Y, Z to four mills viz. A, B, C, D respectively. The supply and the demand together with the unit transportation cost per truckload on the different routes are described in the following transportation table. Assume that the unit transportation costs are in hundreds of dollars. Determine the optimum minimum shipment cost of transportation using MODI method.

Source	Mill				Supply
	A	B	C	D	
X	10	2	20	11	15
Y	12	7	9	20	25
Z	4	14	16	18	10
<b>Demand</b>	5	15	15	15	

Q4. An organization has three plants at X, Y, Z which supply to warehouses located at A, B, C, D, and E respectively. The capacity of the plants is 800, 500 and 900 per month and the requirement of the warehouses is 400, 400, 500, 400 and 800 units respectively. The following table shows the unit transportation cost.

	A	B	C	D	E
X	\$5	\$8	\$6	\$6	\$3
Y	\$4	\$7	\$7	\$6	\$6
Z	\$8	\$4	\$6	\$6	\$3

Determine an optimum distribution for the organization in order to minimize the total cost of transportation.

Q5. Solve the following transshipment problem

Consider a transportation problem has two sources and three depots. The availability, requirements and unit cost are as follows:

Source	Depot			Availability
	D1	D2	D3	
S1	9	8	1	30
S2	1	7	8	30
<b>Requirement</b>	20	20	20	<b>60</b>

In addition to the above, suppose that the unit cost of transportation from source to source and from depot to depot are as:

**Source**

		<b>Source</b>	
		S1	S2
<b>Source</b>	S1	0	1
	S2	2	0

		<b>Depot</b>		
		D1	D2	D3
<b>Depot</b>	D1	0	2	1
	D2	2	0	9
	D3	1	9	0

Find out minimum transshipment cost of the problem and also compare this cost with the corresponding minimum transportation cost.

Q6. Saravana Store, T.Nagar, Chennai interested to purchase the following type and quantities of dresses

Dress Type	V	W	X	Y	Z
Quantity	150	100	75	250	200

Four different dress makers are submitted the tenders, who undertake to supply not more than the quantities indicated below:

Dress Maker	A	B	C	D
Dress Quantity	300	250	150	200

Saravana Store estimates that its profit per dress will vary according to the dress maker as indicates in the following table:

	V	W	X	Y	Z
A	2.75	3.5	4.25	2.25	1.5
B	3	3.25	4.5	1.75	1
C	2.5	3.5	4.75	2	1.25
D	3.25	2.75	4	2.5	1.75

Determine how should the orders to be places for the dresses so as to maximize the profit.

### 1.12 Key Solutions

Q1. a.  $x_{11} = 15$ ,  $x_{21} = 10$ ,  $x_{22} = 10$ ,  $x_{32} = 16$ ,  $x_{33} = 14$ ,  $x_{43} = 35$   
Minimum Cost is: 753

b.  $x_{13} = 15$ ,  $x_{22} = 20$ ,  $x_{33} = 30$ ,  $x_{41} = 25$ ,  $x_{42} = 6$ ,  $x_{43} = 4$   
Minimum Cost is: 542

c.  $x_{13}=15$ ,  $x_{22}=20$ ,  $x_{33}=30$ ,  $x_{41}=25$ ,  $x_{42}=6$ ,  $x_{43}=4$   
Minimum Cost is: 542

Q2.  $x_{11}=2$ ,  $x_{12}=40$ ,  $x_{21}=30$ ,  $x_{31}=3$ ,  $x_{33}=25$   
Minimum Transportation Optimal cost is: 901.

Q3.  $x_{12}=5$ ,  $x_{14}=10$ ,  $x_{22}=10$ ,  $x_{23}=15$ ,  $x_{31}=5$ ,  $x_{34}=5$   
Minimum Optimal Cost is: \$435

Q4.  $x_{15}=800$ ,  $x_{21}=400$ ,  $x_{24}=100$ ,  $x_{32}=400$ ,  $x_{33}=200$ ,  $x_{34}=300$ ,  $x_{43}=300$  (supply shortage)  
Minimum Cost of Transportation is: \$9200

Q5. Transportation Problem  
 $S1-D2=10$ ,  $S1-D3=20$ ,  $S2-D1=20$ ,  $S2-D2=10$  and  
Minimum Transportation Cost is: 100

Transshipment Problem

$x_{11}=60$ ,  $x_{12}=10$ ,  $x_{15}=20$ ,  $x_{22}=50$ ,  $x_{23}=40$ ,  $x_{33}=40$ ,  $x_{34}=20$ ,  $x_{44}=60$ ,  $x_{55}=60$  and  
Minimum Transshipment Cost is: 100

Q6. 150 dresses of V and 50 dresses of Z by Dress Maker A  
250 dresses of Y by Dress Maker B  
150 dress of Z by Dress Maker C  
100 dress of W and 75 dresses of X by Dress Maker D

Maximum Profit is: 1687.50

### 1.13 Further References

Hamdy A Taha. 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Mustafi, C.K. 1988. Operations Research Methods and Practice, Wiley Eastern Ltd., New Delhi.

Mittal, K.V. 1976. Optimization Methods in Operations Research and Systems Analysis, Wiley Eastern Ltd., New Delhi.

**UNIT II****LESSON****2 ASSIGNMENT PROBLEM****LESSON STRUCTURE**

- 2.1 Introduction
- 2.2 Assignment Problem Structure and Solution
- 2.3 Unbalanced Assignment Problem
- 2.4 Infeasible Assignment Problem
- 2.5 Maximization in an Assignment Problem
- 2.6 Crew Assignment Problem
- 2.7 Summary
- 2.8 Key Solutions
- 2.9 Self Assessment Questions
- 2.10 Key Answers
- 2.11 Further References

**Objectives**

After Studying this lesson, you should be able to:

- ❖ Assignment Problem Formulation
- ❖ How to solve the Assignment Problem
- ❖ How to solve the unbalanced problem using appropriate method
- ❖ Make appropriate modification when some problems are infeasible
- ❖ Modify the problem when the objective is to maximize the objective function
- ❖ Formulate and solve the crew assignment problems



### 2.1 Introduction

The Assignment Problem can define as follows:

Given  $n$  facilities,  $n$  jobs and the effectiveness of each facility to each job, here the problem is to assign each facility to one and only one job so that the measure of effectiveness is optimized. Here the optimization means Maximized or Minimized. There are many management problems has a assignment problem structure. For example, the head of the department may have 6 people available for assignment and 6 jobs to fill. Here the head may like to know which job should be assigned to which person so that all tasks can be accomplished in the shortest time possible. Another example a container company may have an empty container in each of the location 1, 2,3,4,5 and requires an empty container in each of the locations 6, 7, 8,9,10. It would like to ascertain the assignments of containers to various locations so as to minimize the total distance. The third example here is, a marketing set up by making an estimate of sales performance for different salesmen as well as for different cities one could assign a particular salesman to a particular city with a view to maximize the overall sales.

Note that with  $n$  facilities and  $n$  jobs there are  $n!$  possible assignments. The simplest way of finding an optimum assignment is to write all the  $n!$  possible arrangements, evaluate their total cost and select the assignment with minimum cost. Bust this method leads to a calculation problem of formidable size even when the value of  $n$  is moderate. For  $n=10$  the possible number of arrangements is 3268800.

### 2.2 Assignment Problem Structure and Solution

The structure of the Assignment problem is similar to a transportation problem, is as follows:

		Jobs				
		1	2	...	n	
Workers	1	$c_{11}$	$c_{12}$	...	$c_{1n}$	1
	2	$c_{21}$	$c_{21}$	...	$c_{2n}$	1
	.	.	.	.	.	.
	.	.	.	.	.	.
	.	.	.	.	.	.
n	$c_{n1}$	$c_{n2}$	...	$c_{nn}$	1	
		1	1	...	1	

The element  $c_{ij}$  represents the measure of effectiveness when  $i^{\text{th}}$  person is assigned  $j^{\text{th}}$  job. Assume that the overall measure of effectiveness is to be minimized. The element  $x_{ij}$  represents the number of  $i^{\text{th}}$  individuals assigned to the  $j^{\text{th}}$  job. Since  $i^{\text{th}}$  person can be assigned only one job and  $j^{\text{th}}$  job can be assigned to only one person we have the following

$$x_{i1} + x_{i2} + \dots + x_{in} = 1, \text{ where } i = 1, 2, \dots, n$$

$$x_{1j} + x_{2j} + \dots + x_{nj} = 1, \text{ where } j = 1, 2, \dots, n$$

and the objective function is formulated as

$$\text{Minimize } c_{11}x_{11} + c_{12}x_{12} + \dots + c_{nn}x_{nn}$$

$$x_{ij} \geq 0$$

The assignment problem is actually a special case of the transportation problem where  $m = n$  and  $a_i = b_j = 1$ . However, it may be easily noted that any basic feasible solution of an assignment problem contains  $(2n - 1)$  variables of which  $(n - 1)$  variables are zero. Because of this high degree of degeneracy the usual computation techniques of a transportation problem become very inefficient. So, hat a separate computation technique is necessary for the assignment problem.

The solution of the assignment problem is based on the following results:

“If a constant is added to every element of a row/column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original assignment problem and vice versa”. – This result may be used in two different methods to solve the assignment problem. If in an assignment problem some cost elements  $c_{ij}$  are negative, we may have to convert them into an equivalent assignment problem where all the cost elements are non-negative by adding a suitable large constant to the cost elements of the relevant row or column, and then we look for a feasible solution which has zero assignment cost after adding suitable constants to the cost elements of the various rows and columns. Since it has been assumed that all the cost elements are non-negative, this assignment must be optimum. On the basis of this principle a computational technique known as Hungarian Method is developed. The Hungarian Method is discussed as follows.

### **Hungarian Method:**

The Hungarian Method is discussed in the form of a series of computational steps as follows, when the objective function is that of minimization type.

#### **Step 1:**

From the given problem, find out the cost table. Note that if the number of origins is not equal to the number of destinations then a dummy origin or destination must be added.

**Step 2:**

In each row of the table find out the smallest cost element, subtract this smallest cost element from each element in that row. So, that there will be at least one zero in each row of the new table. This new table is known as First Reduced Cost Table.

**Step 3:**

In each column of the table find out the smallest cost element, subtract this smallest cost element from each element in that column. As a result of this, each row and column has at least one zero element. This new table is known as Second Reduced Cost Table.

**Step 4:**

Now determine an assignment as follows:

1. For each row or column with a single zero element cell that has not be assigned or eliminated, box that zero element as an assigned cell.
2. For every zero that becomes assigned, cross out all other zeros in the same row and for column.
3. If for a row and for a column there are two or more zero and one can't be chosen by inspection, choose the assigned zero cell arbitrarily.
4. The above procedures may be repeated until every zero element cell is either assigned (boxed) or crossed out.

**Step 5:**

An optimum assignment is found, if the number of assigned cells is equal to the number of rows (and columns). In case we had chosen a zero cell arbitrarily, there may be an alternate optimum. If no optimum solution is found i.e. some rows or columns without an assignment then go to **Step 6**.

**Step 6:**

Draw a set of lines equal to the number of assignments which has been made in **Step 4**, covering all the zeros in the following manner

1. Mark check ( $\checkmark$ ) to those rows where no assignment has been made.
2. Examine the checked ( $\checkmark$ ) rows. If any zero element cell occurs in those rows, check ( $\checkmark$ ) the respective columns that contains those zeros.
3. Examine the checked ( $\checkmark$ ) columns. If any assigned zero element occurs in those columns, check ( $\checkmark$ ) the respective rows that contain those assigned zeros.
4. The process may be repeated until now more rows or column can be checked.
5. Draw lines through all unchecked rows and through all checked columns.

**Step 7:**

Examine those elements that are not covered by a line. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them.

Add this smallest element to every element that lies at the intersection of two lines. Then the resulting matrix is a new revised cost table.

### Example 2.1:

#### Problem

A work shop contains four persons available for work on the four jobs. Only one person can work on any one job. The following table shows the cost of assigning each person to each job. The objective is to assign person to jobs such that the total assignment cost is a minimum.

		Jobs			
		1	2	3	4
Persons	A	20	25	22	28
	B	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

#### Solution

As per the Hungarian Method

#### Step 1: The cost Table

		Jobs			
		1	2	3	4
Persons	A	20	25	22	28
	B	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

**Step 2:** Find the First Reduced Cost Table

		<b>Jobs</b>			
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>Persons</b>	<b>A</b>	0	5	2	8
	<b>B</b>	0	3	8	2
	<b>C</b>	2	0	4	7
	<b>D</b>	2	0	1	1

**Step 3:** Find the Second Reduced Cost Table

		<b>Jobs</b>			
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>Persons</b>	<b>A</b>	0	5	1	7
	<b>B</b>	0	3	7	1
	<b>C</b>	2	0	3	6
	<b>D</b>	2	0	0	0

**Step 4:** Determine an Assignment

By examine row A of the table in Step 3, we find that it has only one zero (cell A1) box this zero and cross out all other zeros in the boxed column. In this way we can eliminate cell B1.

Now examine row C, we find that it has one zero (cell C2) box this zero and cross out (eliminate) the zeros in the boxed column. This is how cell D2 gets eliminated.

There is one zero in the column 3. Therefore, cell D3 gets boxed and this enables us to eliminate cell D4.

Therefore, we can box (assign) or cross out (eliminate) all zeros.

The resultant table is shown below:

		<b>Jobs</b>			
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>Persons</b>	<b>A</b>	<span style="border: 1px solid black; padding: 2px;">0</span>	5	1	7
	<b>B</b>	0	3	7	1
	<b>C</b>	<del>2</del>		3	6
	<b>D</b>	2	0		0

**Step 5:** 0

The solution obtained in Step 4 is not optimal. Because we were able to make three assignments when four were required.

**Step 6:** ~~×~~ 0 ~~×~~

Cover all the zeros of the table shown in the Step 4 with three lines (since already we made three assignments).

Check row B since it has no assignment. Note that row B has a zero in column 1, therefore check column 1. Then we check row A since it has a zero in column 1. Note that no other rows and columns are checked. Now we may draw three lines through unchecked rows (row C and D) and the checked column (column 1). This is shown in the table given below:

		<b>Jobs</b>			
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>Persons</b>	<b>A</b>	<span style="border: 1px solid black; padding: 2px;">0</span>	5	1	7
	<b>B</b>	0	3	7	1
	<b>C</b>	<del>2</del>		3	6
	<b>D</b>	2	0		0

0

**Step 7:**

Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (C1 and D1) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

		<b>Jobs</b>			
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>Persons</b>	<b>A</b>	0	4	0	6
	<b>B</b>	0	2	6	0
	<b>C</b>	3	0	3	6
	<b>D</b>	3	0	0	0

**Step 8:**

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

**Step 9:**

Determine an assignment

Examine each of the four rows in the table given in Step 7, we may find that it is only row C which has only one zero box this cell C2 and cross out D2.

Note that all the remaining rows and columns have two zeros. Choose a zero arbitrarily, say A1 and box this cell so that the cells A3 and B1 get eliminated.

Now row B (cell B4) and column 3 (cell D4) has one zero box these cells so that cell D4 is eliminated.

Thus, all the zeros are either boxed or eliminated. This is shown in the following table

		<b>Jobs</b>			
		1	2	3	4
<b>Persons</b>	<b>A</b>	<span style="border: 1px solid black; padding: 2px;">0</span>	4	<del>8</del>	6
	<b>B</b>	0	2	6	
	<b>C</b>	<del>8</del>		3	<span style="border: 1px solid black; padding: 2px;">0</span>
	<b>D</b>	3	0		0

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above table is optimal.

0

The total cost of assignment is: 78 that is  $A1 + B4 + C2 + D3$

$$20 + 17 + 17 + 24 = 78$$

**2.3 Unbalanced Assignment Problem**     ~~×~~     0     ~~×~~

In the previous section we assumed that the number of persons to be assigned and the number of jobs were same. Such kind of assignment problem is called as **balanced assignment problem**. Suppose if the number of person is different from the number of jobs then the assignment problem is called as **unbalanced**.

If the number of jobs is less than the number of persons, some of them can't be assigned any job. So that we have to introduce one or more dummy jobs of zero duration to make the unbalanced assignment problem into balanced assignment problem. This balanced assignment problem can be solved by using the Hungarian Method as discussed in the previous section. The persons to whom the dummy jobs are assigned are left out of assignment.

Similarly, if the number of persons is less than number of jobs then we have introduce one or more dummy persons with zero duration to modify the unbalanced into balanced and then the problem is solved using the Hungarian Method. Here the jobs assigned to the dummy persons are left out.



**Example 2.2:**

Solve the following unbalanced assignment problem of minimizing the total time for performing all the jobs.

		<b>Jobs</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>
<b>Workers</b>	<b>A</b>	5	2	4	2	5
	<b>B</b>	2	4	7	6	6
	<b>C</b>	6	7	5	8	7
	<b>D</b>	5	2	3	3	4
	<b>E</b>	8	3	7	8	6
	<b>F</b>	3	6	3	5	7

**Solution**

In this problem the number of jobs is less than the number of workers so we have to introduce a dummy job with zero duration.

The revised assignment problem is as follows:

		<b>Jobs</b>					
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>Workers</b>	<b>A</b>	5	2	4	2	5	0
	<b>B</b>	2	4	7	6	6	0
	<b>C</b>	6	7	5	8	7	0
	<b>D</b>	5	2	3	3	4	0
	<b>E</b>	8	3	7	8	6	0
	<b>F</b>	3	6	3	5	7	0

Now the problem becomes balanced one since the number of workers is equal to the number jobs. So that the problem can be solved using Hungarian Method.

**Step 1:** The cost Table

		<b>Jobs</b>					
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>Workers</b>	<b>A</b>	5	2	4	2	5	0
	<b>B</b>	2	4	7	6	6	0
	<b>C</b>	6	7	5	8	7	0
	<b>D</b>	5	2	3	3	4	0
	<b>E</b>	8	3	7	8	6	0
	<b>F</b>	3	6	3	5	7	0

**Step 2:** Find the First Reduced Cost Table

		<b>Jobs</b>					
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>Workers</b>	<b>A</b>	5	2	4	2	5	0
	<b>B</b>	2	4	7	6	6	0
	<b>C</b>	6	7	5	8	7	0
	<b>D</b>	5	2	3	3	4	0
	<b>E</b>	8	3	7	8	6	0
	<b>F</b>	3	6	3	5	7	0

**Step 3:** Find the Second Reduced Cost Table

		<b>Jobs</b>					
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>Workers</b>	<b>A</b>	3	0	1	0	1	0
	<b>B</b>	0	2	4	4	2	0
	<b>C</b>	4	5	2	6	3	0
	<b>D</b>	3	0	0	1	0	0
	<b>E</b>	6	1	4	6	2	0
	<b>F</b>	1	4	0	3	3	0

**Step 4:** Determine an Assignment

By using the Hungarian Method the assignment is made as follows:

		<b>Jobs</b>					
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>Workers</b>	<b>A</b>	3	<del>0</del>	1	<span style="border: 1px solid black; padding: 2px;">0</span>	1	<del>0</del>
	<b>B</b>		2	4	4	2	0
	<b>C</b>	<span style="border: 1px solid black; padding: 2px;">0</span>	5	2	6	3	<del>0</del>
	<b>D</b>	3	0	0	1		0
	<b>E</b>	6	1	4	6	2	<span style="border: 1px solid black; padding: 2px;">0</span>
	<b>F</b>	1	4		3	3	0
			<del>0</del>	<del>0</del>		<span style="border: 1px solid black; padding: 2px;">0</span>	<del>0</del>

**Step 5:**

The solution obtained in Step 4 is not optimal. Because we were able to make five assignments when six were required.



**Step 6:**

Cover all the zeros of the table shown in the Step 4 with five lines (since already we made five assignments).

Check row E since it has no assignment 0 te that row B has a zero in ~~column 6~~, therefore check column 6. Then we check row C since it has a zero in column 6. Note that no other rows and columns are checked. Now we may draw five lines through unchecked rows (row A, B, D and F) and the checked column (column 6). This is shown in the table given below:

		<b>Jobs</b>					
		1	2	3	4	5	6
<b>Workers</b>	<b>A</b>	3	<del>8</del>	1	<span style="border: 1px solid black; padding: 2px;">0</span>	1	<del>8</del>
	<b>B</b>		2	4	4	2	0
	<b>C</b>	<span style="border: 1px solid black; padding: 2px;">0</span>	5	2	6	3	<del>6</del>
	<b>D</b>	3	0	0	1		0
	<b>E</b>	6	1	4	6	2	<span style="border: 1px solid black; padding: 2px;">0</span>
	<b>F</b>	1	4		3	3	0

—	<del>X</del>	<del>X</del>	—	<span style="border: 1px solid black; padding: 2px;">0</span>	<del>X</del>
---	--------------	--------------	---	---	--------------

**Step 7:**  
Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (A6, B6, D6 and F6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

		<b>Jobs</b>					
		1	2	3	4	5	6
		—					
				<span style="border: 1px solid black; padding: 2px;">0</span>			<del>X</del>

<b>Workers</b>	<b>A</b>	3	0	1	0	1	1
	<b>B</b>	0	2	4	4	2	1
	<b>C</b>	3	4	1	5	2	0
	<b>D</b>	3	0	0	1	0	1
	<b>E</b>	5	0	3	5	1	0
	<b>F</b>	1	4	0	3	3	1

**Step 8:**

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

**Step 9:**

Determine an assignment

		<b>Jobs</b>					
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>Workers</b>	<b>A</b>	3	<del>0</del>	1	0	1	1
	<b>B</b>		2	4	4	2	1
	<b>C</b>	0	4	1	5	2	
	<b>D</b>	3	0	0	1		1
	<b>E</b>	5		3	5	1	0
	<b>F</b>	1	4		3	3	1

~~×~~      ~~×~~      0

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above table is optimal.

Thus, the worker A is assigned to Job4, worker B is assigned to job 1, worker C is assigned to job 6, worker D is assigned to job 5, worker E is assigned to job 2, and worker F is assigned to job 3. Since the Job 6 is dummy so that worker C can't be assigned.

The total minimum time is: 14 that is  $A4 + B1 + D5 + E2 + F3$

$$2 + 2 + 4 + 3 + 3 = 14$$

### Example 2.3:

A marketing company wants to assign three employees viz. A, B, and C to four offices located at W, X, Y and Z respectively. The assignment cost for this purpose is given in following table.

		Offices			
		W	X	Y	Z
Employees	A	160	220	240	200
	B	100	320	260	160
	C	100	200	460	250

### Solution

Since the problem has fewer employees than offices so that we have introduced a dummy employee with zero cost of assignment.

The revised problem is as follows:

		Offices			
		W	X	Y	Z

<b>Employees</b>	<b>A</b>	160	220	240	200
	<b>B</b>	100	320	260	160
	<b>C</b>	100	200	460	250
	<b>D</b>	0	0	0	0

Now the problem becomes balanced. This can be solved by using Hungarian Method as in the case of Example 2.2. Thus as per the Hungarian Method the assignment made as follows:

Employee A is assigned to Office X, Employee B is assigned to Office Z, Employee C is assigned to Office W and Employee D is assigned to Office Y. Note that D is empty so that no one is assigned to Office Y.

The minimum cost of assignment is:  $220 + 160 + 100 = 480$

## 2.4 Infeasible Assignment Problem

Sometimes it is possible a particular person is incapable of performing certain job or a specific job can't be performed on a particular machine. In this case the solution of the problem takes into account of these restrictions so that the infeasible assignment can be avoided.

The infeasible assignment can be avoided by assigning a very high cost to the cells where assignments are restricted or prohibited. This is explained in the following Example 2.4.

### Example 2.4:

A computer centre has five jobs to be done and has five computer machines to perform them. The cost of processing of each job on any machine is shown in the table below.

		<b>Jobs</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>



	<b>1</b>	70	30	X	60	30
<b>Computer Machines</b>	<b>2</b>	X	70	50	30	30
	<b>3</b>	60	X	50	70	60
	<b>4</b>	60	70	20	40	X
	<b>5</b>	30	30	40	X	70

Because of specific job requirement and machine configurations certain jobs can't be done on certain machines. These have been shown by X in the cost table. The assignment of jobs to the machines must be done on a one to one basis. The objective here is to assign the jobs to the available machines so as to minimize the total cost without violating the restrictions as mentioned above.

### Solution

#### Step 1: The cost Table

Because certain jobs cannot be done on certain machines we assign a high cost say for example 500 to these cells i.e. cells with X and modify the cost table. The revised assignment problem is as follows:

		<b>Jobs</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>
	<b>1</b>	70	30	500	60	30
<b>Computer Machines</b>	<b>2</b>	500	70	50	30	30
	<b>3</b>	60	500	50	70	60
	<b>4</b>	60	70	20	40	500
	<b>5</b>	30	30	40	500	70

Now we can solve this problem using Hungarian Method as discussed in the previous sections.

**Step 2:** Find the First Reduced Cost Table

		<b>Jobs</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>
<b>Computer Machines</b>	<b>1</b>	40	0	470	30	0
	<b>2</b>	470	40	20	0	0
	<b>3</b>	10	450	0	20	10
	<b>4</b>	40	50	0	20	480
	<b>5</b>	0	0	10	470	40

**Step 3:** Find the Second Reduced Cost Table

		<b>Jobs</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>

<b>Computer Machines</b>	<b>1</b>	40	0	470	30	0
	<b>2</b>	470	40	20	0	0
	<b>3</b>	10	450	0	20	10
	<b>4</b>	40	50	0	20	480
	<b>5</b>	0	0	10	470	40

**Step 4:** Determine an Assignment

		<b>Jobs</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>
<b>Computer Machines</b>	<b>1</b>	40	0	470	30	<del>0</del>
	<b>2</b>	470	40	20		0
	<b>3</b>	10	450		0	<del>10</del>
	<b>4</b>	40	50	0	20	480
	<b>5</b>		0	0	470	40

**Step 5:**

The solution obtained in Step 4 is not optimal. Because we were able to make four assignments when five were required.

**Step 6:**

Cover all the zeros of the table shown in the Step 4 with four lines (since already we made four assignments).

0	<del>0</del>
---	--------------

Check row 4 since it has no assignment. Note that row 4 has a zero in column 3, therefore check column 3. Then we check row 3 since it has a zero in column 3. Note that no other rows and columns are checked. Now we may draw four lines through unchecked rows (row 1, 2, 3 and 5) and the checked column (column 3). This is shown in the table given below:

		Jobs				
		1	2	3	4	5
Computer Machines	1	<del>40</del>	0	470	<del>30</del>	<del>0</del>
	2	470	40	20		0
	3	<del>10</del>	450		0	<del>10</del>
	4	40	50	0	20	480
	5		0	0	470	40

**Step 7:**

Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 10. Subtract this smallest element from the uncovered cells and add 1 to elements (A6, B6, D6 and F6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

		Jobs				
		1	2	3	4	5
Computer Machines	1	0	<del>0</del>	471	30	0
	2	470	40	21	0	0
	3	0	440	0	10	0
	4	30	40	0	10	470
	5	0	0	11	470	40

**Step 8:**

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

**Step 9:**

Determine an assignment

		<b>Jobs</b>				
		1	2	3	4	5
<b>Computer Machines</b>	<b>1</b>	40	<del>80</del>	471	30	<span style="border: 1px solid black; padding: 2px;">0</span>
	<b>2</b>	470	40	21		0
	<b>3</b>		440	0	<span style="border: 1px solid black; display: inline-block; width: 20px; height: 20px;"></span>	<del>80</del>
	<b>4</b>	30	40		10	470
	<b>5</b>	<span style="border: 1px solid black; padding: 2px;">0</span>		<del>80</del>	470	<del>80</del>

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above table is optimal.

0

Thus, the Machine1 is assigned to Job5, Machine 2 is assigned to job4, Machine3 is assigned to job1, Machine4 is assigned to job3 and Machine5 is assigned to job2.

The minimum assignment cost is: 170

~~80~~      0

**2.5 Maximization in an Assignment Problem**

There are situations where certain facilities have to be assigned to a number of jobs so as to maximize the overall performance of the assignment. In such cases the problem can be converted into a minimization problem and can be solved by using Hungarian Method. Here the conversion of maximization problem into a minimization can be done by subtracting all the elements of the cost table from the highest value of that table.

**Example 2.5:**

Consider the problem of five different machines can do any of the required five jobs with different profits resulting from each assignment as illustrated below:

		<b>Machines</b>				
		1	2	3	4	5

<b>Jobs</b>	<b>1</b>	40	47	50	38	50
	<b>2</b>	50	34	37	31	46
	<b>3</b>	50	42	43	40	45
	<b>4</b>	35	48	50	46	46
	<b>5</b>	38	72	51	51	49

Find out the maximum profit through optimal assignment.

### Solution

This is a maximization problem, so that first we have to find out the highest value in the table and subtract all the values from the highest value. In this case the highest value is 72.

The new revised table is given below:

		<b>Machines</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>
<b>Jobs</b>	<b>1</b>	32	35	22	34	22
	<b>2</b>	22	38	35	41	26
	<b>3</b>	22	30	29	32	27
	<b>4</b>	37	24	22	26	26
	<b>5</b>	34	0	21	21	23

This can be solved by using the Hungarian Method.

By solving this, we obtain the solution is as follows:

Jobs	Machines
1	3
2	5
3	1
4	4
5	2

The maximum profit through this assignment is: 264

## 2.6 Crew Assignment Problem

The crew assignment problem is explained with the help of the following problem

### Problem:

A trip from Chennai to Coimbatore takes six hours by bus. A typical time table of the bus service in both the direction is given in the Table 1. The cost of providing this service by the company based on the time spent by the bus crew i.e. driver and conductor away from their places in addition to service times. The company has five crews. The condition here is that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours for the return trip. Also the company has guest house facilities for the crew of Chennai as well as at Coimbatore.

Find which line of service is connected with which other line so as to reduce the waiting time to the minimum.

Table 1

Departure from Chennai	Route Number	Arrival at Coimbatore	Arrival at Chennai	Route Number	Departure from Coimbatore
06.00	1	12.00	11.30	a	05.30
07.30	2	13.30	15.00	b	09.00
11.30	3	17.30	21.00	c	15.00
19.00	4	01.00	00.30	d	18.30
00.30	5	06.30	06.00	e	00.00

### Solution

For each line the service time is constant so that it does not include directly in the computation. Suppose if the entire crew resides at Chennai then the waiting times in hours at Coimbatore for different route connections are given below in Table 2.

If route 1 is combined with route a, the crew after arriving at Coimbatore at 12 Noon start at 5.30 next morning. Thus the waiting time is 17.5 hours. Some of the assignments are infeasible. Route c

leaves Coimbatore at 15.00 hours. Thus the crew of route 1 reaching Coimbatore at 12 Noon are unable to take the minimum stipulated rest of four hours if they are asked to leave by route c. Hence 1-c is an infeasible assignment. Thus its cost is M (a large positive number).

**Table 2**

<b>Route</b>	<b>a</b>	<b>b</b>	<b>c</b>	<b>d</b>	<b>e</b>
<b>1</b>	17.5	21	M	6.5	12
<b>2</b>	16	19.5	M	5	10.5
<b>3</b>	12	15.5	21.5	M	6.5
<b>4</b>	4.5	8	4	17.5	23
<b>5</b>	23	M	8.5	12	17.5

Similarly, if the crews are assumed to reside at Coimbatore then the waiting times of the crew in hours at Chennai for different route combinations are given below in Table 3.

**Table 3**

<b>Route</b>	<b>a</b>	<b>b</b>	<b>c</b>	<b>d</b>	<b>e</b>
<b>1</b>	18.5	15	9	5.5	M
<b>2</b>	20	16.5	10.5	7	M
<b>3</b>	M	20.5	14.5	11	5.5
<b>4</b>	7.5	M	22	18.5	13
<b>5</b>	13	9.5	M	M	18.5



Suppose, if the crew can be instructed to reside either at Chennai or at Coimbatore, minimum waiting time from the above operation can be computed for different route combination by choosing the minimum of the two waiting times (shown in the Table 2 and Table 3). This is given in the following Table 4.

**Table 4**

Route	a	b	c	d	e
1	17.5*	15	9	5.5	12*
2	16*	16.5	10.5	5*	10.5*
3	12*	15.5*	14.5	11	5.5
4	4.5*	8*	14*	17.5*	13
5	13	9.5	8.5*	12*	17.5*

**Note:** The asterisk marked waiting times denotes that the crew are based at Chennai; otherwise they are based at Coimbatore.

Now we can solve the assignment problem (presented in Table 4) using Hungarian Method.

**Step 1:** Cost Table (Table 5)

**Table 5**

Route	a	b	c	d	e
1	17.5*	15	9	5.5	12*
2	16*	16.5	10.5	5*	10.5*
3	12*	15.5*	14.5	11	5.5
4	4.5*	8*	14*	17.5*	13
5	13	9.5	8.5*	12*	17.5*

**Step 2:** Find the First Reduced cost table (Table 6)

**Table 6**

<b>Route</b>	<b>a</b>	<b>b</b>	<b>c</b>	<b>d</b>	<b>e</b>
<b>1</b>	12	9.5	3.5	0	6.5
<b>2</b>	11	11.5	5.5	0	5.5
<b>3</b>	6.5	10	9	5.5	0
<b>4</b>	0	3.5	9.5	13	8.5
<b>5</b>	4.5	1	0	3.5	9

**Step 3:** Find the Second Reduced cost table (Table 7)

**Table 7**

<b>Route</b>	<b>a</b>	<b>b</b>	<b>c</b>	<b>d</b>	<b>e</b>
<b>1</b>	12	8.5	3.5	0	6.5
<b>2</b>	11	10.5	5.5	0	5.5
<b>3</b>	6.5	9	9	5.5	0
<b>4</b>	0	2.5	9.5	13	8.5
<b>5</b>	4.5	0	0	3.5	9

**Step 4:** Determine an Assignment (Table 8)

**Table 8**

Route	a	b	c	d	e
1	12	8.5	3.5	0	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	<del>5.5</del>	
4		2.5	9.5	13	8.5
5	4.5		0	3.5	0

**Step 5:** The solution obtained in Step 4 is not optimal since the number of assignments are less than the number of rows (columns).

**Step 6:** Check (✓) row 2 since 0 has no assignment and note that row 2 has a zero in column d, therefore check (✓) column d also. Then check row 1 since it has zero in column d. Draw the lines through the unchecked rows and checked column using 4 lines (only 4 assignments are made). This is shown in Table 9.

**Table 9**

Route	a	0	<del>c</del>	d	e
1	12	8.5	3.5	0	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	<del>5.5</del>	
4		2.5	9.5	13	8.5
5	4.5		0	3.5	0

0

**Step 7:** Develop a new revised table (Table 10)

Take the smallest element from the elements not covered by the lines in this case 3.5 is the smallest element. Subtract all the uncovered elements from 3.5 and add 3.5 to the elements lie at the intersection of two lines (cells 3d, 4d and 5d). The new revised table is presented in Table 10.

**Table 10**

<b>Route</b>	<b>a</b>	<b>b</b>	<b>c</b>	<b>d</b>	<b>e</b>
<b>1</b>	8.5	5	0	0	3
<b>2</b>	7.5	7	2	0	2
<b>3</b>	6.5	9	9	9	0
<b>4</b>	0	2.5	9.5	16.5	8.5
<b>5</b>	4.5	0	0	7	9

**Step 8:** Go to Step 4 and repeat the procedure until an optimal solution if arrived.

**Step 9:** Determine an Assignment (Table 11)

**Table 11**

<b>Route</b>	<b>a</b>	<b>b</b>	<b>c</b>	<b>d</b>	<b>e</b>
--------------	----------	----------	----------	----------	----------

<b>1</b>	8.5	5	<span style="border: 1px solid black; padding: 2px;">0</span>	<del>∞</del>	3
<b>2</b>	7.5	7	2		2
<b>3</b>	6.5	9	9	<span style="border: 1px solid black; padding: 2px;">0</span>	
<b>4</b>		2.5	9.5	16.5	8.5
<b>5</b>	4.5		0	7	<span style="border: 1px solid black; padding: 2px;">0</span>

The assignment illustrated in the above Table 11 is **optimal** since the **number of assignments is equal to the number of rows (columns)**.

Thus, the routes to be prepared to achieve the minimum waiting time are as follows:

$$\boxed{0} - c, 2 - d, 3 - e, 4 - a \text{ and } 5 - b$$

By referring Table 5, we can obtain the waiting times of these assignments as well as the residence (guest house) of the crews. This is presented in the following Table 12.

**Table 12**

Routes	Residence of the Crew	Waiting Time
1 - c	<span style="border: 1px solid black; padding: 2px;">0</span> Coimbatore	9
2 - d	Chennai	5
3 - e	Coimbatore	5.5
4 - a	Chennai	4.5
5 - b	Coimbatore	9.5

**2.7 Summary**

The assignment problem is used for the allocation of a number of persons to a number of jobs so that the total time of completion is minimized. The assignment problem is said to be **balanced** if it has equal number of person and jobs to be assigned. If the number of persons (jobs) is different from the number of jobs (persons) then the problem is said to be **unbalanced**. An unbalanced assignment problem can be solved by converting into a balanced assignment problem. The conversion is done by introducing dummy person or a dummy job with zero cost.

Because of the special structure of the assignment problem, it is solved by using a special method known as **Hungarian Method**.

**2.8 Key Terms**

**Cost Table:** The completion time or cost corresponding to every assignment is written down in a table form if referred as a cost table.

**Hungarian Method:** is a technique of solving assignment problems.

**Assignment Problem:** is a special kind of linear programming problem where the objective is to minimize the assignment cost or time.

**Balanced Assignment Problem:** is an assignment problem where the number of persons equal to the number of jobs.

**Unbalanced Assignment Problem:** is an assignment problem where the number of jobs is not equal to the number of persons.

**Infeasible Assignment Problem:** is an assignment problem where a particular person is unable to perform a particular job or certain job cannot be done by certain machines.

## 2.9 Self Assessment Questions

Q1. A tourist company owns a one car in each of the five locations viz. L1, L2, L3, L4, L5 and a passengers in each of the five cities C1, C2, C3, C4, C5 respectively. The following table shows the distant between the locations and cities in kilometer. How should be cars be assigned to the passengers so as to minimize the total distance covered.

		Cities				
		C1	C 2	C3	C4	C5
Locations	L1	120	110	115	30	36
	L2	125	100	95	30	16
	L3	155	90	135	60	50
	L4	160	140	150	60	60
	L5	190	155	165	90	85

Q2. Solve the following assignment problem

1            2            3            4            5

1	Rs.3	Rs.8	Rs.2	Rs.10	Rs.3
2	Rs.8	Rs.7	Rs.2	Rs.9	Rs.7
3	Rs.6	Rs.4	Rs.2	Rs.7	Rs.5
4	Rs.8	Rs.4	Rs.2	Rs.3	Rs.5
5	Rs.9	Rs.10	Rs.6	Rs.9	Rs.10

Q3. Work out the various steps of the solution of the Example 2.3.

Q4. A steel company has five jobs to be done and has five softening machines to do them. The cost of softening each job on any machine is given in the following cost matrix. The assignment of jobs to machines must be done on a one to one basis. Here is the objective is to assign the jobs to the machines so as to minimize the total assignment cost without violating the restrictions.

		<b>Jobs</b>				
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>
<b>Softening Machines</b>	<b>1</b>	80	30	X	70	30
	<b>2</b>	70	X	60	40	30
	<b>3</b>	X	80	60	80	70
	<b>4</b>	70	80	30	50	X
	<b>5</b>	30	30	50	X	80

Q5. Work out the various steps of the solution of the problem presented in Example 2.5.

Q6. A marketing manager wants to assign salesman to four cities. He has four salesmen of varying experience. The possible profit for each salesman in each city is given in the following table. Find out an assignment which maximizes the profit.

		<b>Cities</b>			
		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>Salesmen</b>	<b>1</b>	25	27	28	38
	<b>2</b>	28	34	29	40
	<b>3</b>	35	24	32	33
	<b>4</b>	24	32	25	28

Q7. Shiva's three wife, Rani, Brinda, and Fathima want to earn some money to take care of personal expenses during a school trip to the local beach. Mr. Shiva has chosen three chores for his wife: washing, cooking, sweeping the cars. Mr. Shiva asked them to submit bids for what they feel was a fair pay for each of the three chores. The three wife of Shiva accept his decision. The following table summarizes the bid received.

		<b>Chores</b>		
		<b>Washing</b> <b>1</b>	<b>Cooking</b> <b>2</b>	<b>Sweeping</b> <b>3</b>
<b>Wife's</b>	<b>Rani</b>	25	18	17
	<b>Brinda</b>	17	25	15
	<b>Fathima</b>	18	22	32

Q8. Solve the following problem

		<b>Office</b>			
		<b>O1</b>	<b>O2</b>	<b>O3</b>	<b>O4</b>



<b>Employees</b>	<b>E1</b>	2600	3200	3400	3000
	<b>E2</b>	2000	4200	3600	2600
	<b>E3</b>	2000	3000	5600	4000

Q9. The railway operates seven days a week has a time table shown in the following table. Crews (Driver and Guard) must have minimum rest of six hours between trans. Prepare the combination of trains that minimizes waiting time away from the city. Note that for any given combination the crew will be based at the city that results in the smaller waiting time and also find out for each combination the city where the crew should be based at.

Train No.	Departure at Bangalore	Arrival at Chennai	Train No.	Departure at Chennai	Arrival at Bangalore
101	7 AM	9 AM	201	9 AM	11 AM
102	9 AM	11 AM	202	10 AM	12 Noon
103	1.30 PM	3.30 PM	203	3.30 PM	5.30 PM
104	7.30 PM	9.30 PM	204	8 PM	10 PM

## 2.10 Key Solutions

Q1. L1 – C1, L2 – C3, L3 – C2, L4 – C4, L5 – C5 and  
Minimum Distance is: 450

Q2. 1 – 5, 2 – 3, 3 – 2, 4 – 4, 5 – 1 and  
Minimum Cost is: Rs.21

Q4. 1 – 2, 2 – 4, 3 – 3, 4 – 4, 5 – 1 and  
Minimum Assignment Cost is:

Q6. 1 - 1, 2 – 4, 3 – 3, 4 – 2 and  
Maximum Profit is: 139

Q7. Rani – Cooking, Brinda – Sweeping, Fathima – Washing and  
Minimum Bids Rate is: 51

Q8. E1 – O2, E2 – O4, E3 – O1  
Since E4 is empty, Office O3 cannot be assigned to any one.  
Minimum Cost is: 7800

Q9.

<b>Trains</b>	<b>Cities</b>
201 – 103	Bangalore
202 – 104	Chennai

### **2.11 Further References**

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Cooper, L and D. Steinberg, 1974. Methods and Applications of Linear Programmings, Saunders, Philadelphia, USA.

Mustafi, C.K. 1988. Operations Research Methods and Practices, Wiley Eastern Limited, New Delhi.

**UNIT II****LESSON****3 INTRODUCTION TO INVENTORY MANAGEMENT****LESSON STRUCTURE**

- 3.1 Introduction
- 3.2 Objectives of Inventory
- 3.3 Inventory is an Essential Requirement
- 3.4 Basic Functions of Inventory
- 3.5 Types of Inventory
- 3.6 Factors Affecting Inventory
- 3.7 Summary
- 3.8 Key Terms
- 3.9 Self Assessment Questions
- 3.10 Further References

**Objectives**

After Studying this lesson, you should be able to:

- ❖ Understand what is inventory
- ❖ Describe various inventory concepts
- ❖ Describe the objectives of inventory
- ❖ Explain the functions of inventory
- ❖ Describe requirements of inventory
- ❖ Explain different types of inventory
- ❖ Describe different factors affecting inventory

### 3.1 Introduction

Simply inventory is a stock of physical assets. The physical assets have some economic value, which can be either in the form of material, men or money. Inventory is also called as an idle resource as long as it is not utilized. Inventory may be regarded as those goods which are procured, stored and used for day to day functioning of the organization.

Inventory can be in the form of physical resource such as raw materials, semi-finished goods used in the process of production, finished goods which are ready for delivery to the consumers, human resources, or financial resources such as working capital etc.

Inventories means measures of power and wealth of a nation or of an individual during centuries ago. That is a business man or a nation's wealth and power were assessed in terms of grammes of gold, heads of cattle, quintals of rice etc.

In recent past, inventories mean measure of business failure. Therefore, businessmen have started to put more emphasis on the liquidity of assets as inventories, until fast turnover has become a goal to be pursued for its own sake.

Today inventories are viewed as a large potential risk rather than as a measure of wealth due to the fast developments and changes in product life. The concept of inventories at present has necessitated the use of scientific techniques in the inventory management called as inventory control.

Thus, inventory control is the technique of maintaining stock items at desired levels. In other words, inventory control is the means by which material of the correct quality and quantity is made available as and when it is needed with due regard to economy in the holding cost, ordering costs, setup costs, production costs, purchase costs and working capital.

Inventory Management answers two questions viz. How much to order? and when to order? Management scientist insisting that the inventory is an very essential requirement. Why? This is illustrated in the next section with the help of materials conversion process diagram.

### 3.2 Objectives of Inventory

Inventory has the following main objectives:

- To supply the raw material, sub-assemblies, semi-finished goods, finished goods, etc. to its users as per their requirements at right time and at right price.
- To maintain the minimum level of waste, surplus, inactive, scrap and obsolete items.
- To minimize the inventory costs such as holding cost, replacement cost, breakdown cost and shortage cost.
- To maximize the efficiency in production and distribution.
- To maintain the overall inventory investment at the lowest level.
- To treat inventory as investment which is risky? For some items, investment may lead to higher profits and for others less profit.

### 3.3 Inventory is an Essential Requirement

Inventory is a part and parcel of every facet of business life. Without inventory no business activity can be performed, whether it being a manufacturing organization or service organization such as libraries, banks, hospitals etc. Irrespective of the specific organization, inventories are reflected by way of a conversion process of inputs to outputs. The conversion process is illustrated in the figure 3.1 as given below.

From the figure 3.1 we can see that there may be stock pints at three stages viz. Input, Conversion Process and Output. The stocks at input are called raw materials whereas the stocks at the output are called products. The stocks at the conversion process may be called finished or semi-finished goods or sometimes may be raw material depending on the requirement of the product at conversion process, where the input and output are based on the market situations of uncertainty, it becomes physically impossible and economically impractical for each stock item to arrive exactly where it is required and when it is required.

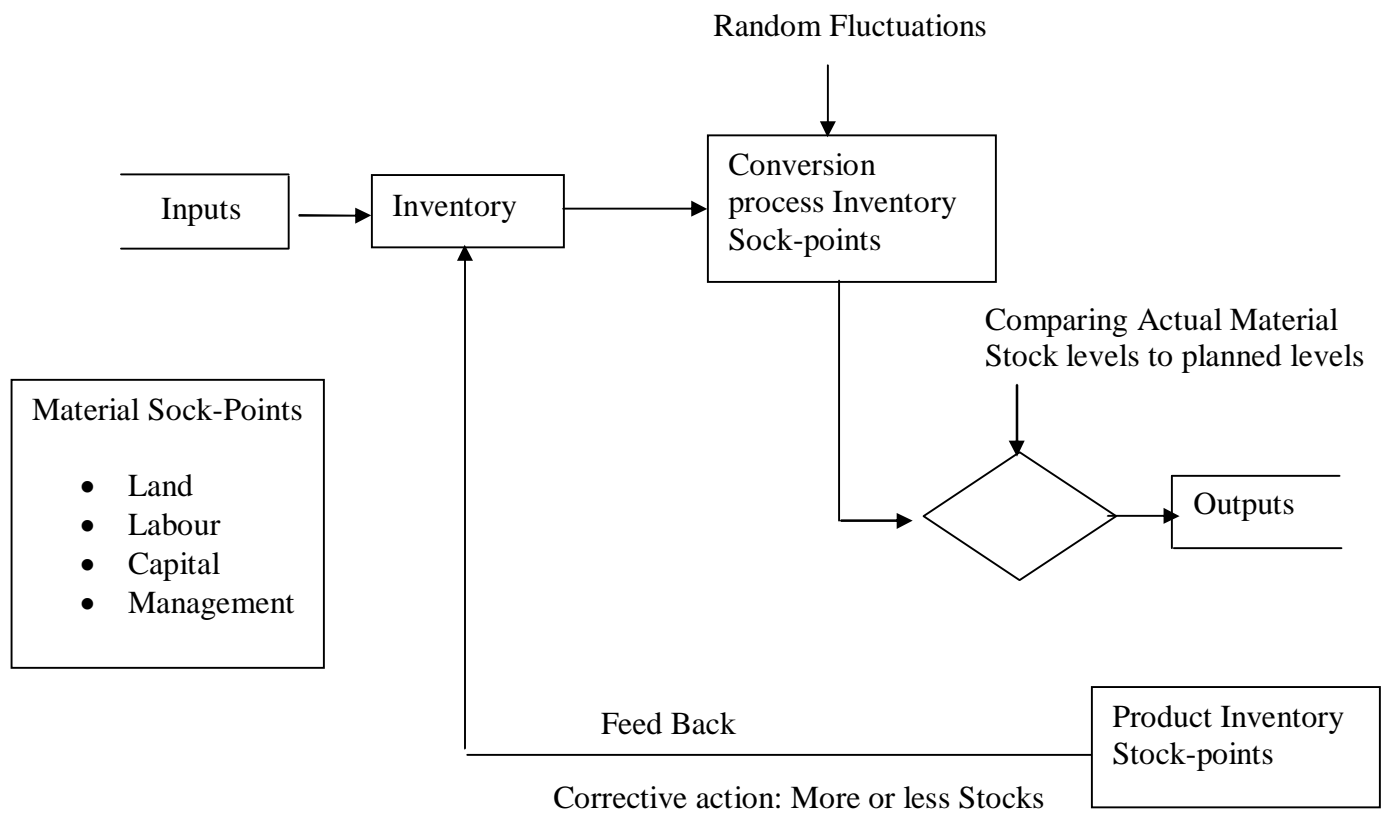


Fig: 3.1 Materials Conversion Process

Even it is physically possible to deliver the stock when it is required, it costs more expensive. This is the basic reason for carrying the inventory. Thus, inventories play an essential and pervasive role in any organization because they make it possible:

- ✓ To meet unexpected demand
- ✓ To achieve return on investment
- ✓ To order largest quantities of goods, components or materials from the suppliers at advantageous prices
- ✓ To provide reasonable customer service through supplying most of the requirements from stock without delay
- ✓ To avoid economically impractical and physically impossible delivering/getting right amount of stock at right time of required
- ✓ To maintain more work force levels
- ✓ To facilitate economic production runs
- ✓ To advantage of shipping economies
- ✓ To smooth seasonal or critical demand
- ✓ To facilitate the intermittent production of several products on the same facility
- ✓ To make effective utilization of space and capital
- ✓ To meet variations in customer demand
- ✓ To take the advantage of price discount
- ✓ To hedge against price increases
- ✓ To discount quantity

### 3.4 Basic Functions of Inventory

The important basic function of inventory is

- Increase the profitability- through manufacturing and marketing support. But zero inventory manufacturing- distribution system is not practically possible, so it is important to remember that each rupee invested in inventory should achieve a specific goal. The other inventory basic functions are
  - Geographical Specialization
  - Decoupling
  - Balancing supply and demand and
  - Safety stock

### Inventory Investment Alternative

Investment is most important and major part of asset, which should be required to produce a minimum investment return. The MEC (Marginal Efficiency of Capital) concept holds that an organization should invest in those alternatives that produce a higher investment return than capital to borrow. The following figure 3.2 shows that investment alternative A on the MEC curve is acceptable.

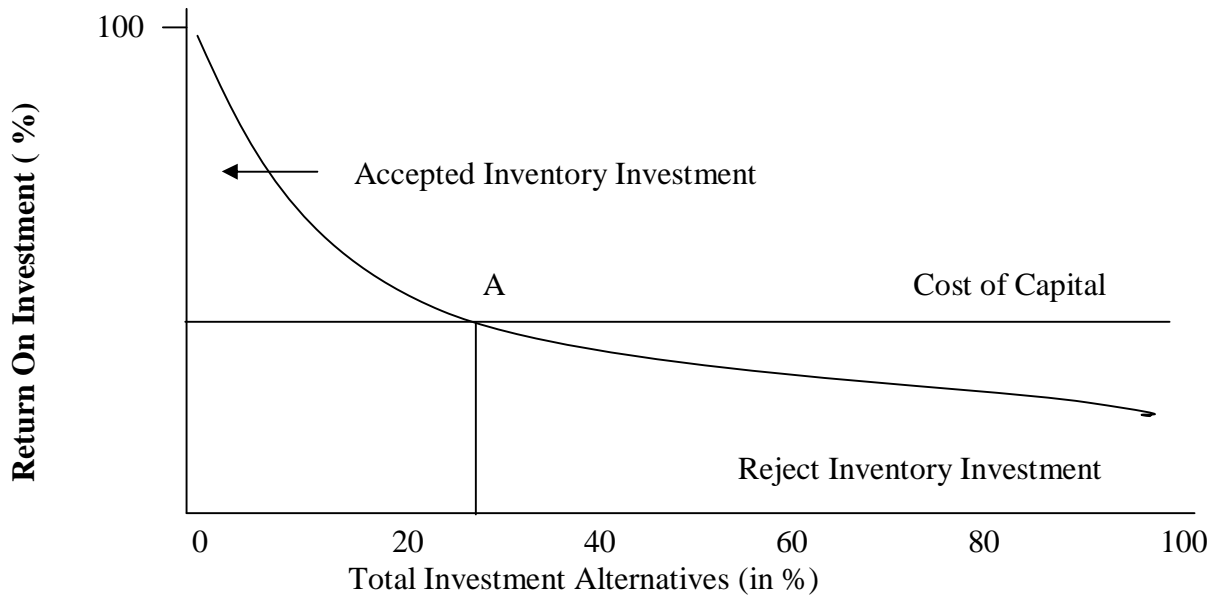


Fig 3.2 Typical MEC curve

The curve shows that about 20% of the inventory investment alternatives will produce a return on investment above the capital cost.

**Geographical Specialization**

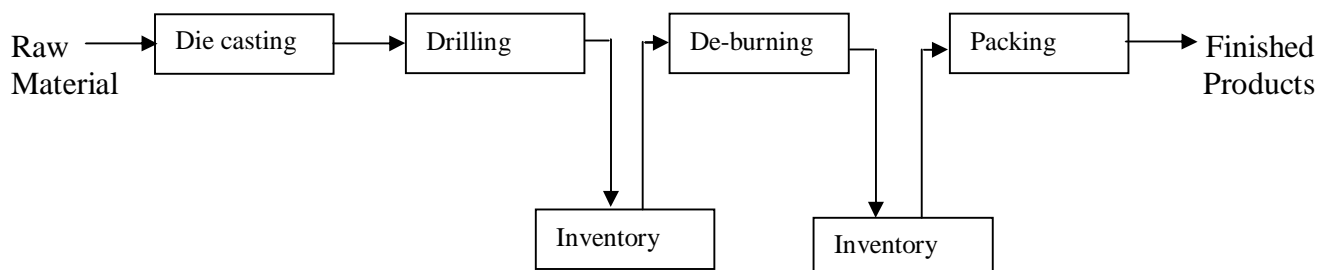
Another basic inventory function is to allow the geographical specialization individual operating units. There is a considerable distance between the economical manufacturing location and demand areas due to factors of production such as raw material, labour, water, power. So that the goods from various manufacturing locations are collected at a simple warehouse or plant to assemble in final product or to offer consumers a single mixed product shipment. This also provides economic specialization between manufacturing and distribution units/locations of an organization.

**Decoupling**

The provision of maximum efficiency of operations within a single facility is also one of the important basic functions of the inventory. This is achieved by decoupling, which is done by breaking operations apart so that one operation(s) supply is independent of another(s) supply.

The decoupling function serves in two ways of purposes, they are

1. Inventories are needed to reduce the dependencies among successive stages of operations so that shortage of materials, breakdowns or other production fluctuations at one stage do not cause later stage to shut down. This is illustrated in the following figure 3.3 in an engineering unit.



**Fig: 3.3 Decoupling of operation using inventory**

The figure shows that the de-burning, packing could continue to operate from inventories should die-casting and drilling be shut down or they can be decoupled from the production processes that precede them.

2. One organizational unit schedules its operations independently of another organizational unit. For example: Consider an automobile organization, here assembly process can be schedule separately from engine built up operation, and each can be decoupled from final automobile assembly operations through in process inventories.

**Supply and Demand Balancing**

The function of Balancing concerns elapsed time between manufacturing and using the product. Balancing inventories exist to reconcile supply with demand. The most noticeable example of balancing is seasonal production and year round usage like sugar, rice, woolen textiles, etc. Thus the investment of balancing inventories links the economies of manufacturing with variations of usage.

**Safety Stock**

The safety stock also called as buffer stock. The function of safety stock concerns short range variations in either replacement or demand. Determination of the safety stock size requires a great deal of inventory planning. Safety stock provides protection against two types of uncertainty, they are

1. Sales in excess of forecast during the replenishment period
2. Delays in replenishment

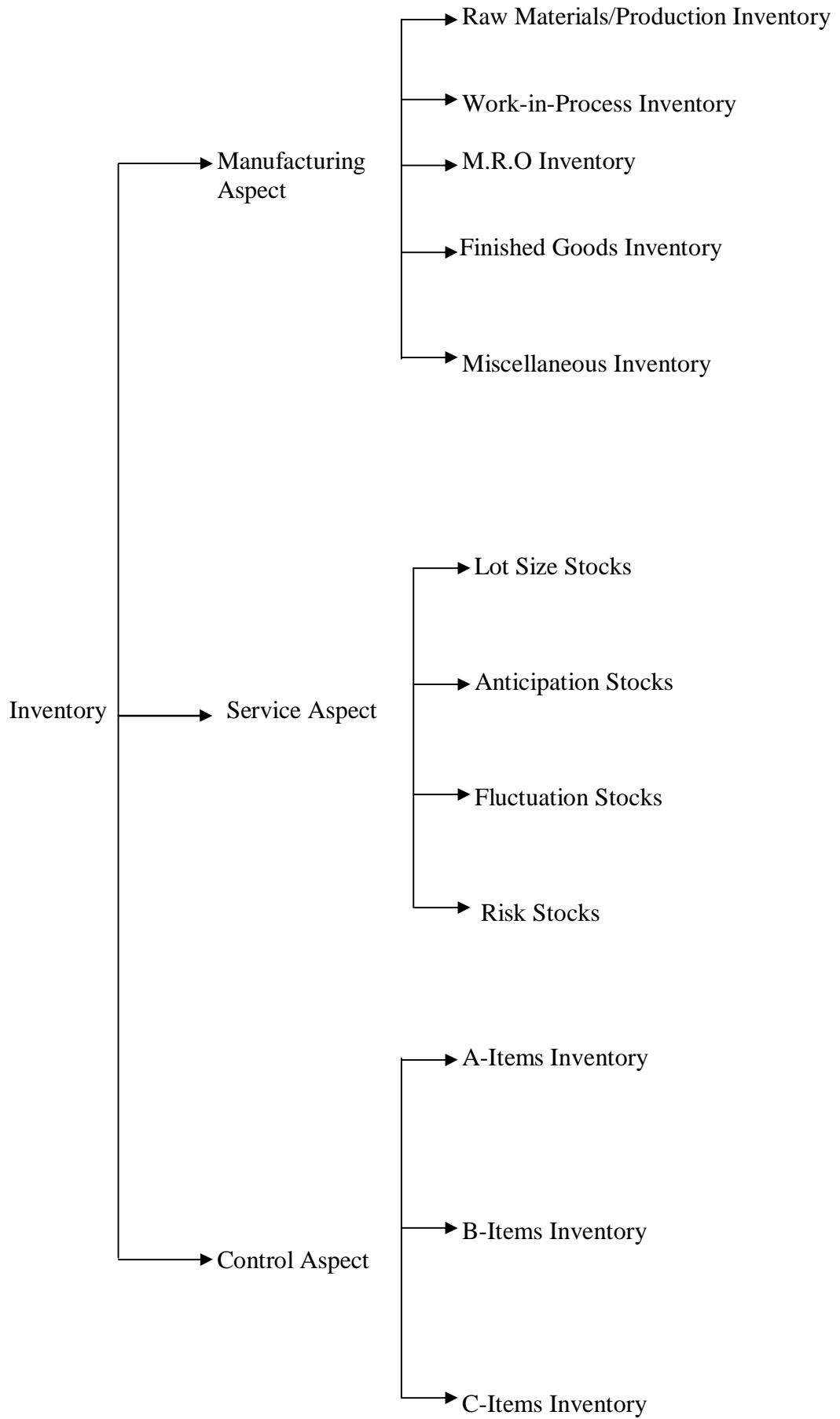
Thus, the inventories committed to safety stocks denote the greatest potential for improved performances. There are different techniques are available to develop safety stocks.

**3.5 Types of Inventory**

Inventory may be classified into manufacturing, service and control aspects, which is illustrated in the figure 3.4 as given below:



**Fig: 3.4 Types of Inventory.**



Each inventory type is discussed in detail are as follows:

### **Raw Material/Manufacturing Inventory**

There are five types of Manufacturing Inventory, they are

#### **Production Inventory**

Items going to final product such a raw materials, sub-assemblies purchased from outside are called production inventory.

#### **Work-in-Process Inventory**

The items in the form of semi-finished or products at different stage of production process are known as work-in-process inventory.

#### **M.R.O. Inventory**

Maintenance, Repair and Operating supplies such as spare parts and consumable stores, which do not go into final product but are consumed during the production process.

#### **Finished Goods Inventory**

Finished Goods Inventory includes the products ready for dispatch to the consumers or distributors/retailers.

#### **Miscellaneous Inventory**

Items excluding those mentioned above, such as waste, scrap, obsolete, and un-saleable items arising from the main production process, stationery used in the office and other items required by office, factory and other departments, etc. are called miscellaneous inventory.

#### **Service Inventory**

The service inventory can be classified into four types, they are:

#### **Lot Size Stocks**

Lot size means purchasing in lots. The reasons for this is to

- Obtain quantity discounts
- Minimize receiving and handling costs
- Reduce purchase and transport costs

For example: It would be uneconomical for a textile factory to buy cotton everyday rather than in bulk during the cotton season.

#### **Anticipation Stocks**

Anticipation stocks are kept to meet predictable changes in demand or in availability of raw materials.

For example: The purchase of potatoes in the potato season for sale of roots preservation products throughout the year.

**Fluctuation Stocks**

Fluctuation stocks are carried to ensure ready supplies to consumers in the face of irregular fluctuations in their demands

**Risk Stocks**

Risk stocks are the items required to ensure that there is no risk of complete production breakdown. Risk stocks are critical and important for production.

**Control Inventory**

A good way of examining an inventory control is: to make ABC classification, which is also known as ABC analysis. ABC analysis means the “control” will be “Always Better” if we start with the ABC classification of inventory.

The ABC concepts classifies inventories into three groups in terms of percentage of total value and percentage of number of inventory items, this is illustrated in the figure 3.5 and 3.6 as given below.

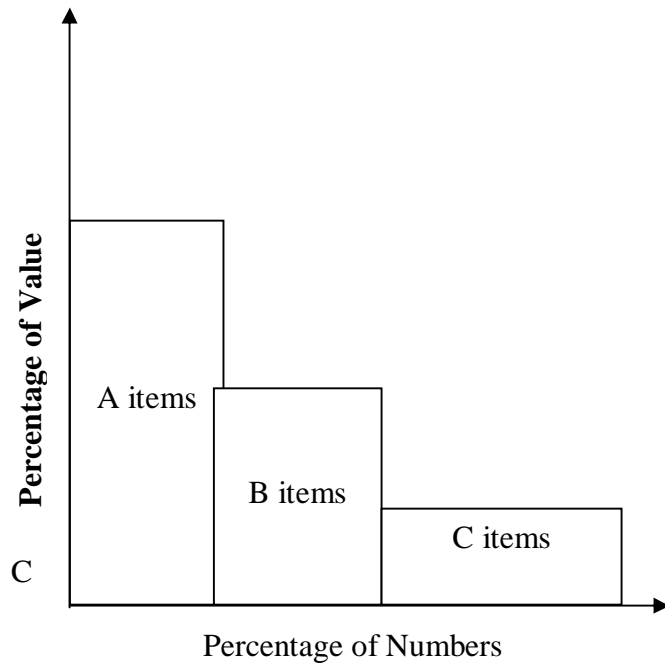


Fig: 3.5 ABC Classification (Frequency Form)

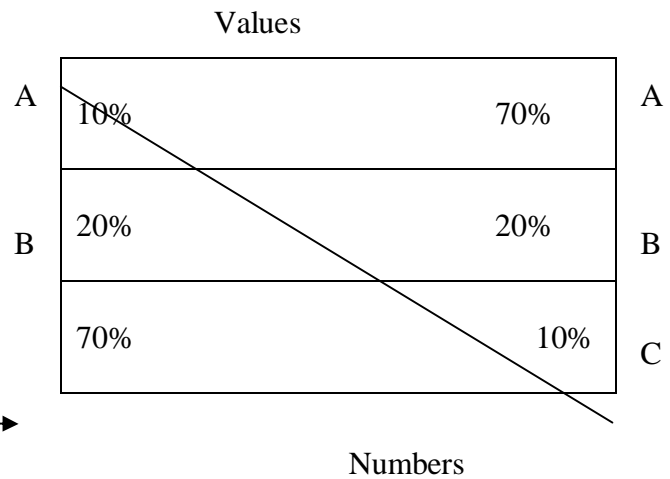


Fig: 3.6 ABC Classification (Tabular Form)

The three groups of inventory items are called A-items group, B-items group, C-items group, which are explained as follows:

A-items Group: This constitutes 10% of the total number of inventory items and 70% of total money value for all the items.

B-items Group: This constitutes 20% of the total number of inventory items and 20 % of total money value for all the items.

C-items Group: This constitutes 70% of the total number of inventory items and 10 % of total money value for all the items. This is just opposite of A-items group.

The ABC classification provides us clear indication for setting properties of control to the items, and A-class item receive the importance first in every respect such as tight control, more security, and high operating doctrine of the inventory control.

The coupling of ABC classification with VED classification enhances the inventory control efficiency. VED classification means Vital, Essential and Desirable Classification. From the above description, it may be noted that ABC classification is based on the logic of proportionate value while VED classification based on experience, judgment, etc. The ABC /VED classification is presented in the following figure 3.7.

	V	E	D	Total
A	3	5	2	10
B	5	7	8	20
C	10	40	20	70
Total	18	52	30	100

Fig: 3.7 ABC/VED Classifications

- This is an example of a particular case
- The values are expressed in percentage

Note that the total number of categories becomes nine.

### 3.6 Factors Affecting Inventory

The main problem of inventory control is to answer tow questions viz.

1. How much to order? and
2. When to order?

These questions are answered by developing a inventory model, which is based on the consideration of the main aspects of inventory viz. demand and cost. There are many factors related with these tow main factors (Demand and Cost). In this section we will discuss these different factors.

The different factors are:

- Economic Parameters
- Demand
- Ordering Cycle
- Delivery Lag
- Time Horizon

- Stages of Inventory
- Number of Supply Echelons
- Number and Availability of Items
- Government's and Organization's Policy

### **Economic Parameters**

There are different types of economic parameters, they are:

- Purchase Price
- Procurement Costs
- Selling Price
- Holding Costs
- Shortage costs
- Information Processing System Operating Costs

### **Purchase Cost**

The cost of the item is the direct manufacturing cost if it is produced in in-house or the cost paid to the supplier for the item received. This cost usually equal to the purchase price. When the marketing price goes on fluctuating, inventory planning is based on the average price mostly it is called as a fixed price. When price discounts can be secured or when large production runs may result in a decrease in the production cost, the price factor is of special interest.

### **Procurement Costs**

The costs of placing a purchase order is known as ordering costs and the costs of initial preparation of a production system (if in-house manufacturing) is called as set up cost. These costs are called as procurement cost, but these costs vary directly with each purchase order placed or with set up made and are normally assumed independent of the quantity ordered or produced.

Procurement costs include costs of transportation of items ordered, expediting and follow up, goods receiving and inspection, administration (includes telephone bills, computer cost, postage, salaries of the persons working for tendering, purchasing, paper work, etc.), payment processing etc. This cost is expressed as the cost per order/setup.

### **Holding Costs**

The holding costs also called as carrying costs. The cost associated with holding/carrying of stocks is called holding cost or carrying cost or possession cost. Holding costs includes handling/carrying cost, maintenance cost, insurance, safety measures, warehouse rent, depreciation, theft, obsolescence, salaries, interest on the locked money, etc. Thus, by considering all these elements the storage cost is expressed either as per unit of item held per unit of time or as a percentage of average money value of investment held. Therefore the size of all these holding costs usually increases or decreases in proportion to the amount of inventory that is carried.

### **Shortage Costs**

These costs are penalty costs as a result of running out of stock at the time of item is required. There are different forms of shortage cost, which is illustrated in the following figure 3.8. One form of the shortage costs is called as back order on the selling side or backlogging cost on the production side when the unsatisfied demand can be satisfied at later stage that is consumers has to wait till they gets the supply.

The second form of shortage costs is called as lost sales costs on the selling side or no backlogging costs on the production side, when the unsatisfied demand is lost or the consumers goes some where else instead of waiting for the supply.

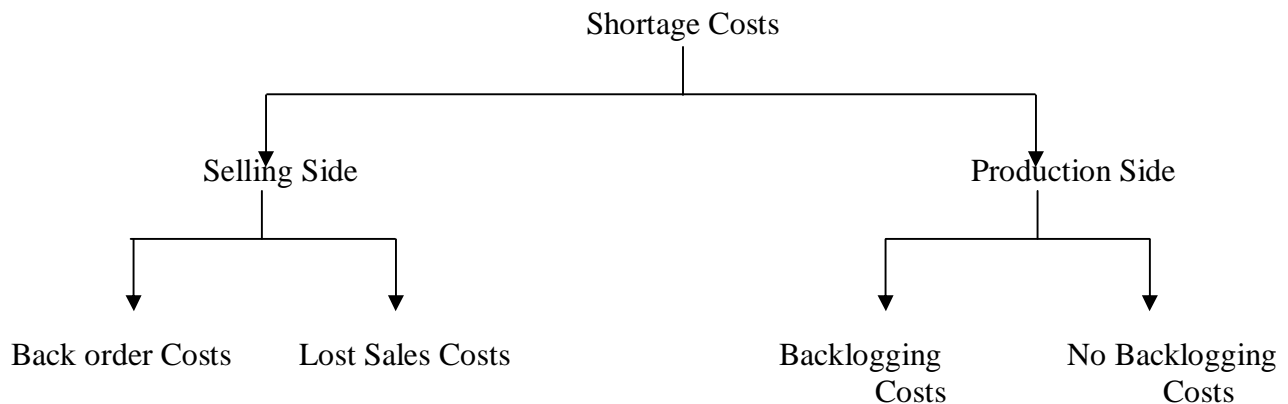


Fig: 3.8 Nature of Shortage Costs

These includes the costs of production stoppage, overtime payments, idle machine, loss of goodwill, loss of sales opportunity, special order at higher price, loss of profits etc.

### **Information System Operation Costs**

Today there are more inventory records should be maintain in the organization, so that some person must update the records either by hand or by using computer. If the inventory levels are not recorded daily, this operating cost is incurred in obtaining accurate physical inventory record counts. The operating costs are fixed.

### **Demand**

A commodity demand pattern may be deterministic or probabilistic.

#### **Deterministic Demand**

In this case, the demand is assumed that the quantities of commodity needed over subsequent periods of time are known with certainty. This is expressed over equal time periods in terms of known constant demands or in terms of variable demands. The two cases are called as static and dynamic demands.

#### **Probabilistic Demand**

This occurs when requirements over a certain time period are not known with certainty but their pattern can be denoted by a known probability distribution. In this case, the probability distribution is said to be

stationary or non-stationary over time periods. The terms stationary and non-stationary are equivalent to the terms static and dynamic in the deterministic demand.

For a given time period the demand may be instantaneously satisfied at the beginning of the time period or uniformly during that time period. The effect of uniform and instantaneous demand directly reflects on the total cost of carrying inventory.

### **Ordering Cycle**

The ordering cost is related with the inventory situation time measurement. An ordering cycle can be identified by the time period between two successive placements of orders. The later may be initiated in one of two ways as:

- Periodic Review
- Continuous Review

#### **Periodic Review**

In this case, the orders are placed at equally intervals of time.

#### **Continuous Review**

In this case, an inventory record is updated continuously until a certain lower limit is reached at this point a new order is placed. Some times this is referred as the two-bin system.

### **Delivery Lag**

The requirement of the inventory is felt and an order is placed, it may be delivered instantaneously or some times it may be needed before delivery if affected. The time period between the placement of the requisition for an item and its receipt for actual use is called as delivery lag. The delivery lags also known as lead time.

There are four different types of lead time, they are

- Administrative Lead Time
- Transportation Lead Time
- Suppliers Lead Time
- Inspection Lead Time

The Inspection Lead Time and Administrative Lead Time can be fixed in nature, where as the Transportation Lead Time and Suppliers Lead Time can never be fixed. It means generally the lead time may be deterministic or probabilistic.

### **Time Horizon**

Time horizon is, the planning period over which inventory is to be controlled. The planning period may be finite or infinite in nature. Generally, inventory planning is done on annual basis in most of the organizations.

**Stages of Inventory**

In the sequential production process, if the items/parts are stocked at more than one point they are called multi-stage inventories. This is illustrated in the following figure 3.9.

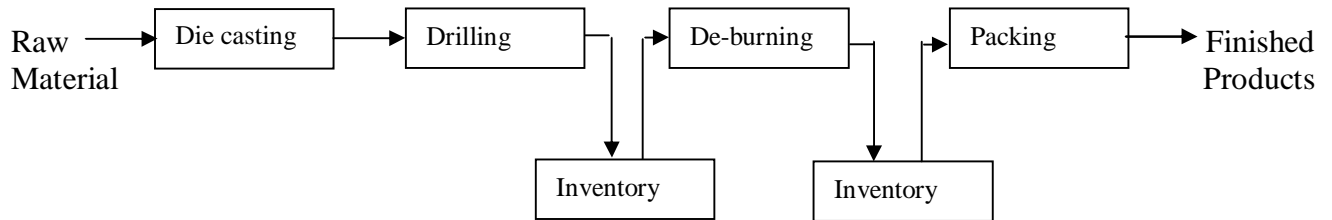


Fig: 3.9 Multistage Inventories

**Number of Supply Echelons**

Already we saw that there are several stocking points in the inventory system. These stocking points are organized in such that one points act as a supply source for some other points. For example, the production factories supplies the products to warehouse and the warehouse supplies to the retailer and then to the consumers. In this the each level of movement of the product is called on echelon. This is illustrated in the figure 3.10 given below.

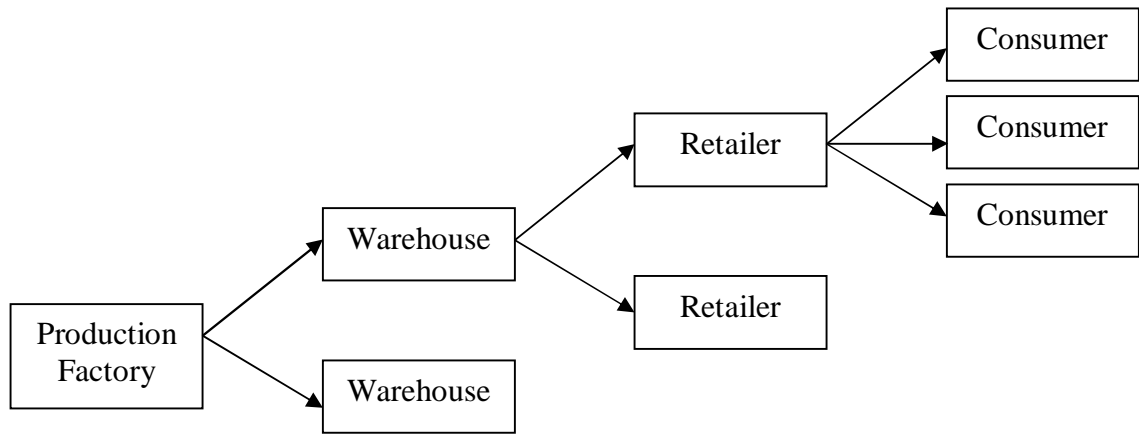


Fig: 3.10 Multi Echelon Supply Systems

**Number and Availability of Items**

Due to different marker situations some times supply position is poorly affected, which in turn affects the poison of inventory in the organizations.

Generally inventory includes more than one item. Therefore the number of items in inventory affects the situation when these items complete for limited total capital or limited space.

**Government’s and Organization’s Policy**



There are different governments and as well as organization policies such as import and export, availability of capital, land, labour, pollutions systems, etc. The government has laid down some policy norms for items to be imported as well as for other items like highly inflammable, explosive and other important materials. Similarly, an organization also has certain policies based on the availability of capital, labour, etc. All these policies affect organization inventories level.

We have discussed different factors that affect the inventory in an organization (above). These factors are responsible for the development of proper inventory system is called as characteristics of inventory.

### 3.7 Summary

This lesson illustrated the introduction of inventory and inventory management/control. This lesson also illustrated the objectives of inventory, inventory functions, inventory types and the various factors affecting the inventory.

### 3.8 Key Terms

**Carrying Cost:** Cost of maintaining one unit of an item in the stock per unit of time (normally one year). The carrying cost also called as Holding Cost.

**Decoupling:** Use of inventories to break apart operations so that one operations supply is independent of another.

**Backlog:** Accumulation of unsatisfied demands

**Delivery Lag:** Time between the placing of an order for the item and receipt of the items for use.

**ABC Classification:** Classifications of inventories in terms of annual usage value in different categories of high value (A), medium value (B) and low value (C).

**VED Classification:** Vital Essential Desirable Classification. This is based on experience/judgment. VED classification when coupled with ABC classification enhances the inventory control efficiency.

### 3.9 Self Assessment Questions

Q1. What is inventory and their Objectives?

Q2. Discuss different types of inventory.

Q3. What are the major functions of inventory in an organization?

Q4. Explain different factors affect the inventory?

Q5. Some business peoples think that inventory as necessary evil while others think inventory as an asset. What is your view?

### 3.10 Further References

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Mustafi, C.K. 1988. Operations Research Methods and Practices, Wiley Eastern Limited, New Delhi.

Levin, R and Kirkpatrick, C.A. 1978. Quantitative Approached to Management, Tata McGraw Hill, Kogakusha Ltd., International Student Editon.

Levin and Rubin. Quantitative Techniques for Managers, PHI, New Delhi.

## UNIT II

### LESSON

## 4 INVENTORY MODELS

### LESSON STRUCTURE

- 4.1 Introduction
- 4.2 Deterministic Inventory Model
- 4.3 Deterministic Single Item Inventory Model
  - 4.3.1 Economic Order Quantity Model I
  - 4.3.2 Economic Order Quantity Model II
  - 4.3.3 Economic Production Quantity Model
  - 4.3.4 Price Discounts Model
  - 4.3.5 Dynamic Demand Models
- 4.4 Deterministic Multi Items Inventory Model
  - 4.4.1 Unknown Cost Structure Model
  - 4.4.2 Known Cost Structure Model
- 4.5 Probabilistic Inventory Models
  - 4.5.1 Single Period Probabilistic Model
  - 4.5.2 Single Period Discrete Probabilistic Demand Model
- 4.6 Summary
- 4.7 Key Terms
- 4.8 Self Assessment Questions
- 4.9 Further References

### Objectives

After Studying this lesson, you should be able to:

- ❖ Understand Simple Deterministic Inventory Models
- ❖ Understand Simple Probabilistic Inventory Models
- ❖ Develop Simple Deterministic Inventory Models
- ❖ Illustrate the use of Simple Inventory Models in Practical Situations
- ❖ Briefly explain Single Item Inventory Models
- ❖ Briefly explain Multi Item Inventory Models
- ❖ Describe static and dynamic Inventory Models
- ❖ Develop Single Period Probabilistic Models

## 4.1 Introduction

The methodology for inventory situation modeling is based on four concepts, they are:

1. Examine the inventory situation, list characteristics and assumption related to the inventory situation.
2. Develop the total annual relevant cost equation in narrative form as:

$$\boxed{\text{Total annual relevant cost}} = \boxed{\text{Item cost}} + \boxed{\text{Procurement cost}} + \boxed{\text{Carrying cost (cycle stocks + safety stocks)}} + \boxed{\text{Stock out costs (cost/sales back order)}}$$

3. Convert the total annual relevant cost equation from narrative form into the shorthand logic of mathematics.
4. Optimize the cost equation by finding the optimum for how much to order (also called order quantity), when to re-order (also called re-order point) and the total annual relevant cost.

In general, the situation of inventory can be classified into two types viz. deterministic and stochastic.

Deterministic- in this variables are known with certainty

Stochastic – in this variables are probabilistic

This lesson briefly outlines Deterministic Inventory Models and Probabilistic (Discrete Demand Distribution Model) Inventory Models.

In this section we will discuss deterministic inventory models and later we will discuss probabilistic inventory models (section 4.5).

## 4.2 Deterministic Inventory Models

There are different deterministic inventory models, they are:

- a. Deterministic single item Inventory Models
  - i. EOQ – Economic Order Quantity Model – I
  - ii. EOQ – Economic Order Quantity Model – II (instantaneous supply when shortages are allowed)
  - iii. EPQ – Economic Production Model (Gradual supply case and shortage not allowed)
  - iv. Price Discounts Model (instantaneous supply with no shortages)
  - v. Dynamic Demand Models
- b. Deterministic multi-item Inventory Models
  - i. Unknown cost structure Model
  - ii. Known cost structure Model

### 4.3 Deterministic Single Item inventory Models

Inventory models with all the known parameters with certainty are known as deterministic inventory model. In this section we will discuss the deterministic inventory models for single item.

#### 4.3.1 Economic Order Quantity (EOQ) Model I

The EOQ concept applies to the items which are replenished periodically into inventory in lots covering several periods' needs, subject to the following conditions:

- Consumption of item or sales or usage is uniform and continuous
- The item is replenished in lots or batches, either by manufacturing or by purchasing

#### Description

The EOQ model is described under the following situations:

- a. Demand is deterministic and it is denoted by  $D$  units per year.
- b. Price per Unit or cost of purchase is  $C$ .
- c. Planning period is one year.
- d. Ordering Cost or Procurement Cost or Replenishment Cost is  $C_o$ . Suppose if the items are manufactured it is known as set up cost.
- e. Holding Cost (or carrying cost) is  $C_h$  per unit of item per one year time period. The  $C_h$  is expressed either in terms of cost per unit per period or in terms of percentage charge of the purchase price.
- f. Shortage Cost (mostly it is back order cost) is  $C_s$  per unit per year.
- g. Order Size is  $Q$ .
- h. Cycle period of replenishment is  $t$ .
- i. Delivery lag/Lead Time is  $L$  (expressed in units of time)

In this section we will discuss about instantaneous supply when shortages are not allowed. That is whatever is demanded, is supplied immediately after the lead time. If we assume these constraints, a graph of inventory against time will be look like a regular saw-tooth pattern as given below (Fig: 4.1).

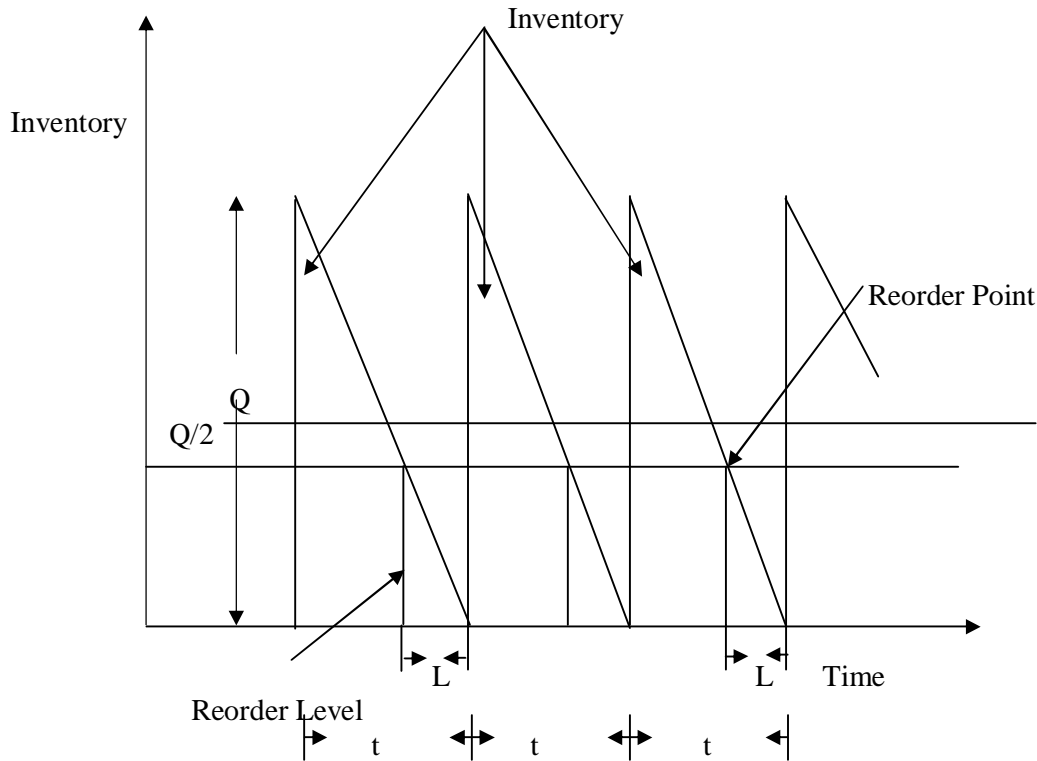


Fig: 4.1. Saw-Tooth Inventory Model

- In this model we assumed that the shortages are not allowed, it means that shortage cost is prohibitive or  $C_s$  is too much large or infinite.
- Everything is so known and regular, there is no need of safety stock.
- Inventory will run out altogether just as the next lot is received.

The different levels of inventories for this model are fixed as follows:

- Minimum level = Safety Stock (Buffer Stock)
- Maximum level = Minimum level + EOQ
- Reorder level = Minimum level + Lead Time Consumption

-- In this case safety stock is not needed, so that safety stock is zero i.e. Minimum level = 0.

-- Maximum Inventory is the ORDER SIZE (lot size).

-- Maximum Inventory is the ORDER SIZE (lot size).

Therefore, the average inventory per cycle =  $\frac{1}{2}(\text{Maximum level} + \text{Minimum level})$ .

Here cycle is the intermittent pattern, in which inventory vary from maximum to minimum and then back to maximum.

-- Maximum inventory is Q

Therefore, the average inventory per cycle =  $\frac{1}{2}(Q+0) = Q/2$ , and the average inventory is time independent.

In this case, the Total Annual Relevant Cost is as follows:

$$\begin{aligned}
 & \boxed{\text{Total annual relevant cost (TC)}} = \boxed{\text{Annual Purchase Cost (PC) + Annual Carrying Cost (CC)}} + \boxed{\text{Annual Ordering Cost (OC)}} \\
 & = (\text{Price/Unit}) \left( \begin{array}{c} \text{Quantity} \\ \text{Purchased} \\ \text{Per Year} \end{array} \right) + \left( \begin{array}{c} \text{Ordering} \\ \text{Cost/Order} \end{array} \right) \left( \begin{array}{c} \text{Number of} \\ \text{Orders placed /} \\ \text{year} \end{array} \right) + \left( \begin{array}{c} \text{Carrying Cost} \\ \text{per unit} \end{array} \right) \left( \begin{array}{c} \\ \\ \end{array} \right) \\
 & \text{Average number of units carried}
 \end{aligned}$$

-----→ eq.1

Note that,

$$\text{Number of Orders/Year} = \text{Annual Demand/Order Size}$$

$$= D/Q$$

Thus, the eq.1 is written as:

$$TC = CD + C_o D/Q + C_h Q/2 \quad \text{-----→ eq.2}$$

The EOQ or Order Size is that quantity, which minimizes the Total Cost. Total Cost is the sum of Fixed Cost and Variable Cost. The Fixed Cost (CD) is independent of Order Size while the variable Cost is dependent on the Order Size (Q). Since, the fixed cost does not play any in minimization or maximization process, only variable cost will be minimized here.

For the total cost to be minimum, the first order derivative of TC is zero, that is,

$$dTC/dQ = -C_o D/Q^2 + C_h/2 = 0 \quad \text{-----→ eq.3} \quad \text{or} \quad C_o D/Q = C_h Q/2 \quad \text{-----→ eq.4}$$

or

$$\text{Annual Ordering Cost} = \text{Annual Carrying Cost} \quad \text{-----→ eq.5}$$

The eq.5 may also be obtained from the following fig: 4.2

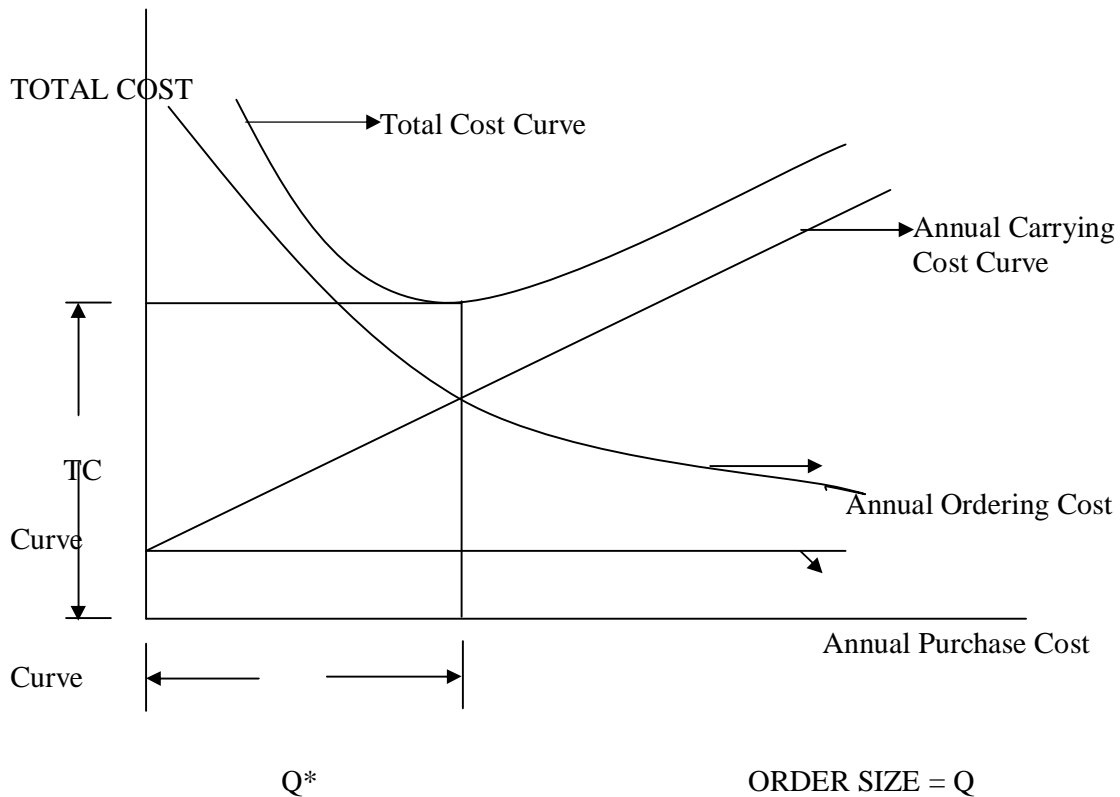


Fig: 4.2. Inventory Control Cost Trade-offs

Now, if we examine the eq.2 that is the total cost equation, we obtain the relationship between the fixed cost and variable cost. This relationship is shown in the above fig.4.2. Note that the total cost curve has the lowest value just above the intersection of the ordering cost curve and carrying cost curve, and also at the intersection annual cost is equal to the annual carrying cost.

From the eq.4, now we will get

$$EOQ = Q^* = \sqrt{\frac{2C_oD}{C_h}} \text{ -----} \rightarrow \text{eq.6}$$

The cycle period  $t = \frac{\text{Optimal Order Quantity}}{\text{Annual Demand}}$  or  $t^* = \frac{Q^*}{D} = \sqrt{\frac{2C_o}{C_h D}} \text{ -----} \rightarrow \text{eq.7}$

$N =$  Total number of orders per year , which is the reciprocal of cycle period ( $1/t^*$ )

That is

$$N = \frac{D}{Q^*} = \sqrt{\frac{C_h D}{2C_o}} \text{ -----} \rightarrow \text{eq.8}$$

$$Q = \sqrt{\frac{2C_o D}{C_h}}$$

The annual cost = TC = CD +  $\sqrt{2C_o C_h D}$  -----> eq.9

Lead Time Consumption = (Lead time in years) \* (Demand Rate per year)

Minimum Level = 0  
Maximum Level = Q\*  
Reorder Level = LD

Let us see few example of this case.

### Example 4.1

A manufacturer uses Rs.20, 000 worth of an item during the year. Manufacturer estimated the ordering cost as Rs.50 per order and holding costs as 12.5% of average inventory value. Find the optimal order size, number of orders per year, time period per order and total cost.

### Solution

Given that:

D = Rs.20, 000  
Co = Rs.50  
Ch = 12.5% of average inventory value / unit

Total Cost = TC =  $\frac{25D}{Q} + (0.125) \frac{Q}{D}$  , where Q is order size in Rs.

By applying the equations (eq.6) to (eq.9), we will get Q\*, t\*, N

$$Q^* = \sqrt{\frac{2C_o D}{C_h}}$$

$$= \sqrt{\frac{2 * 50 * 20000}{0.125}} = \text{Rs.4000}$$

$$t^* = \sqrt{\frac{2C_o}{C_h D}}$$

$$= \sqrt{\frac{2 * 50}{(0.125) * (20000)}} = \frac{1}{5} \text{ years} = 73 \text{ days}$$

$$N = \frac{1}{t^*} = 5$$

Note: TC means in this case variable cost only



$$TC^* = \sqrt{2 * 50 * 0.125 * 20000} = \text{Rs.500}$$

Therefore

**Order Size = Q = Rs.4000**  
**Number of order / year = N = 5**  
**Time period / order = t\* = 73 days**  
**Total Cost = TC\* = Rs.500**

**Example 4.2**

A manufacturer uses an item at a uniform rate of 25,000 units per year. Assume that no shortage is allowed and delivery is at an infinite rate. The ordering, receiving and hauling cost is Rs.23 per order, while inspection cost is Rs.22 per order. Interest costs is Rs.0.056 and deterioration and obsolescence cost is Rs.0.004 respectively per year for each item actually held in inventory plus Rs.0.02 per year per unit based on the maximum number of units in inventory.

Determine the EOQ. If lead time is 40 days, find reorder level.

**Solution**

Given that

Demand = D = 25000 units/year  
 Ordering Cost = Co = 23 + 22 = Rs.45 per order  
 Storage cost Ch = 0.056 + 0.004 = Rs0.060 (based on actual inventory (=average inventory))  
 Storage cost Ch = Rs.0.02 per unit/year (based on maximum inventory)

$$\begin{aligned} \text{Total Variable Cost} = TC &= \frac{25 * 25000}{Q} + \frac{0.060 * Q}{2} + 0.02 * Q \\ &= \frac{625000}{Q} + \frac{0.1 * Q}{2} \end{aligned}$$

Thus,

$$Q^* = \sqrt{\frac{2C_o D}{C_h}} = Q^* = \sqrt{\frac{2 * 25 * 25000}{0.1}} = 3535.5 \text{ units (3535 units approximately)}$$

$$\text{Reorder level} = L * D = \frac{40 * 25000}{365} = 2739.7 \text{ units}$$

That is = 2740 units

Therefore

**EOQ = Q\* = 3535 units**  
**Reorder level = 2740 units**

Look in to the fig.4.2; in this the total cost curve is almost flat near the minimum cost point. This indicates that small variations in optimal order size will not change the total cost appreciably. For this purpose we will examine the model of sensitivity in the next section.

### **Model Sensitivity**

In order to examine sensitivity of the model, we compare the sensitivity of the total costs (TC) for any operating system with the total variable costs for an optimal inventory system (TC\*) by using the ratio  $TC/TC^*$ . To do this, we have to calculate  $TC/TC^*$  as a function of  $Q/Q^*$ .

Therefore

$$TC = \left( \frac{C_o D}{Q} + \frac{C_h Q}{2} \right) \Bigg/ \left( \frac{C_o D}{Q^*} + \frac{C_h Q^*}{2} \right) \text{ -----} \rightarrow \text{eq.10}$$

Now by substituting  $Q^* = \sqrt{2C_o D/C_h}$  into equation eq.10 and solving algebraically we get the relationship as

$$\frac{TC}{TC^*} = \frac{1}{2} \left[ \frac{Q^* + Q}{Q^*} \right] \text{ -----} \rightarrow \text{eq.11}$$

, which is shown in the following figure fig.4.3

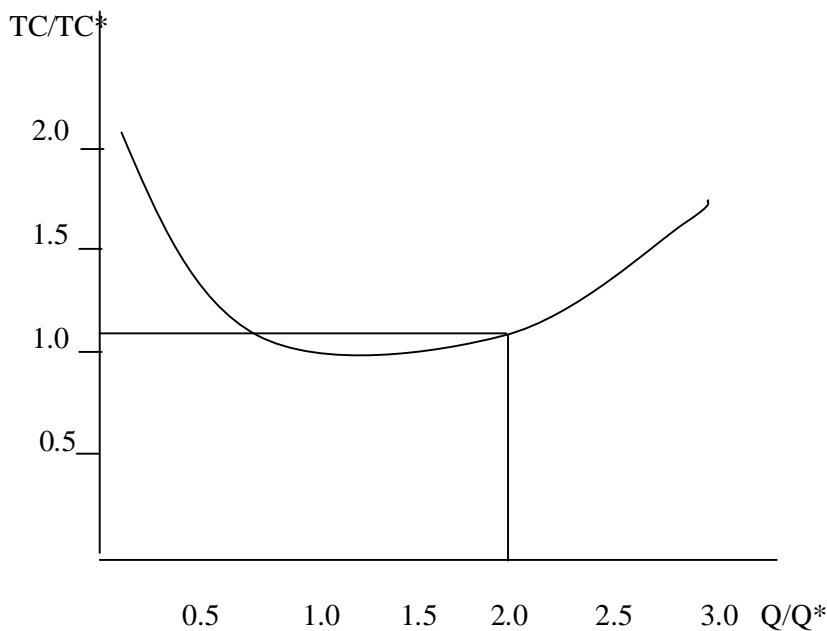


Fig: 4.3 Inventory Sensitivity (in case of simple lot size)

According to this if Q is off from optimal either direction by a factor of 2; costs are increased by only 25%. This has an important practical implication.

The model sensitivity is explained with the help of the following example, so that we will understand more.

**Example 4.3**

Consider the Example 4.1. The sensitivity of total cost if order size is Rs.4000, then we will get that

$$TC/TC^* = \frac{1}{2} \left[ \frac{4000 + 4000}{4000} \right] = 1$$



From this we will get

$$EOQ = Q^* = \sqrt{\frac{(2C_o D)}{C_h} \left( \frac{C_o + C_h}{C_s} \right)} \text{-----} \rightarrow \text{eq.13}$$

$$\text{Inventory Level} = I^* = \sqrt{\frac{(2C_o D)}{C_h} \left( \frac{C_s}{C_o + C_h} \right)} \text{-----} \rightarrow \text{eq.14}$$

$$\text{Shortage Level} = Q^* - I^* \text{-----} \rightarrow \text{eq.15}$$

$$\text{Cycle Period} = t^* = \frac{Q^*}{D} \text{-----} \rightarrow \text{eq.16}$$

$$\text{Number of Orders/Year} = \frac{1}{t^*}$$

Therefore

$$\text{Total Variable Cost} = TC^* = \sqrt{\frac{2C_o C_h C_s D}{(C_h + C_s)}} \text{-----} \rightarrow \text{eq.18}$$

Thus, if we compare the total variable cost of Model I and Model II we will see that

$$\sqrt{2C_o C_h D} > \sqrt{\frac{2C_o C_h C_s D}{(C_h + C_s)}} \text{---- This implies that the annual}$$

Cost when shortage is allowed is less than the annual inventory management cost when shortages are not allowed. That is shortage should be allowed whenever the shortage cost is not very large for reducing the total cost.

#### Example 4.4

Consider the following problem.

##### Problem

The demand for an inventory item each costing Re5, is 20000 units per year. The ordering cost is Rs.10. The inventory carrying cost is 30% based on the average inventory per year. Stock out cost is Rs.5 per unit of shortage incurred. Find out various parameters.

##### Solution

Given that

$$\text{Demand} = D = 20000$$

$$\text{Ordering Cost} = C_o = \text{Rs.10}$$

$$\text{Carrying Cost} = C_h = 30\% \text{ of Re } 5 = \frac{30}{100} * 5 = 1.5$$

$$\text{Stock out Cost} = C_s = \text{Rs.}5$$

Now we have to determine the various parameter of EOQ Model II such as EOQ, Inventory Level, Shortage Level, Cycle Period, number of orders/year and Total Cost.

$$\begin{aligned} \text{EOQ} = Q^* &= \sqrt{\frac{(2C_oD)}{C_h} \left( \frac{C_o+C_h}{C_s} \right)} \\ &= \sqrt{\left( \frac{2*10*20000}{0.30} \right) \left( \frac{0.30+10}{5} \right)} = 1657 \text{ units} \end{aligned}$$

$$\begin{aligned} \text{Inventory Level} = I^* &= \sqrt{\frac{(2C_oD)}{C_h} \left( \frac{C_s}{C_o+C_h} \right)} \\ \text{Inventory Level} = I^* &= \sqrt{\left( \frac{2*10*20000}{0.30} \right) \left( \frac{5}{10+0.30} \right)} = 804 \text{ units} \end{aligned}$$

$$\text{Shortage Level} = Q^* - I^* = 1657 - 804 = 853 \text{ units}$$

$$\text{Cycle Period} = t^* = \frac{Q^*}{D} = \frac{1657}{20000} = 30.24 \text{ days} = 30 \text{ days}$$

$$\text{Number of Orders/Year} = \frac{1}{t^*} = \frac{1}{Q^*/D} = \frac{D}{Q^*} = \frac{20000}{1657} = 12 \text{ Orders/year}$$

$$\begin{aligned} \text{Total Cost} &= \sqrt{\frac{2C_oC_hC_sD}{(C_h + C_s)}} \\ &= \sqrt{\left( \frac{2*10*0.30*5*20000}{0.30+5} \right)} = \text{RS.}336.4 \end{aligned}$$

#### 4.3.3 Economic Production Quantity (EPQ) Model

Here we will discuss about Graduate Supply case when Shortages are not allowed. EOQ model is more common in retail situation, while economic production quantity EPQ is basically associated with manufacturing environment. EPQ shows that over a period of time inventory gradually built and the consumption go side by side where production rate is higher than that of consumption rate.

Assumption (a) to (i) of EOQ Model I also hold good for this model. In this model the Order Size (Q) is taken as Production Size, the annual production rate is taken as P such that  $P > D$ , otherwise, if  $P \leq D$ , the item will be used as fast as it is produced. This situation is illustrated in the following figure fig: 4.5.

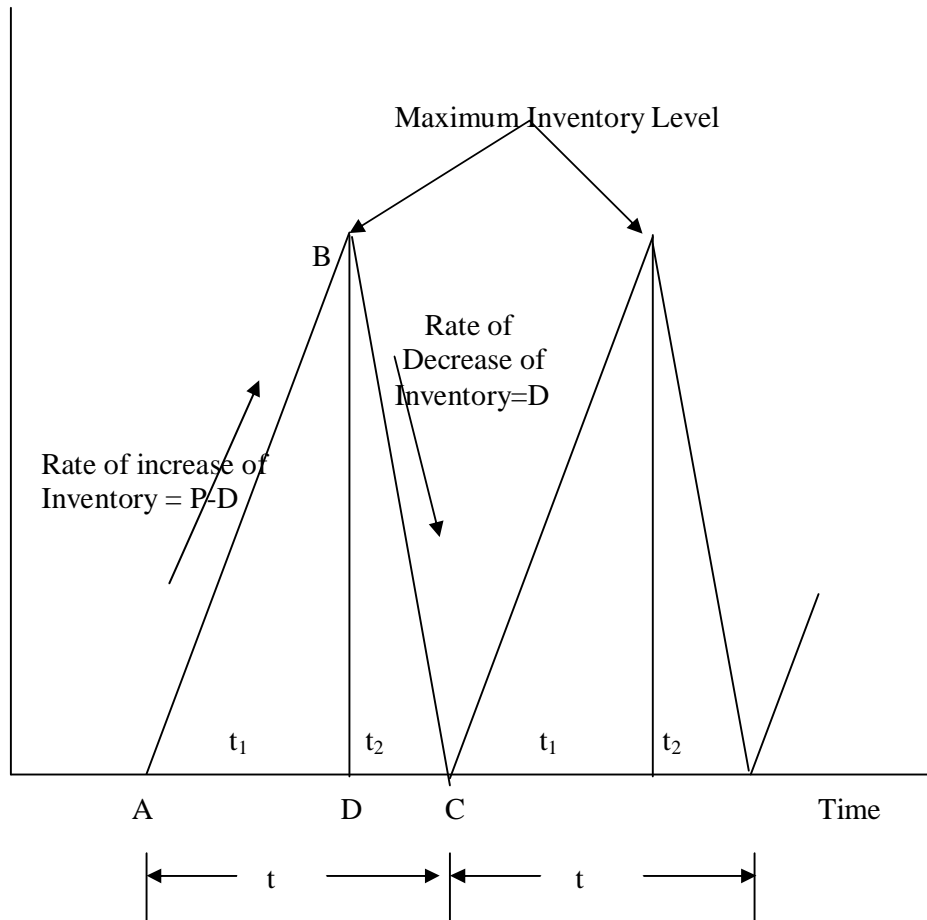


Fig: 4.5 Gradual Replacement Inventory Situation

In the above figure,  $t$  = Cycle Time  
 $t_1$  = Production Time  
 $t_2$  = Depletion Time and

$t = t_1 + t_2$  of maximum inventory level BD.

$$\text{Production Time} = t_1 = \frac{Q}{P}$$

$$\text{Cycle Time} = t = \frac{Q}{D}$$

$$\begin{aligned} \text{Maximum Inventory Level} = BD &= (P - D) * t_1 \\ &= (P - D) * \frac{Q}{P} \end{aligned}$$

Minimum Inventory Level = 0

$$\begin{aligned} \text{Average Inventory Carried} &= \frac{(P - D) \frac{Q}{P} + 0}{2} \\ &= \frac{(P - D) Q}{2P} \end{aligned}$$



Total Variable Cost/Year = Annual Setup Cost + Annual Carrying Cost

$$= \frac{C_o D}{Q} + C_h \frac{(P - D)}{P} \frac{Q}{2} \quad \text{-----} \rightarrow \text{eq.19}$$

Thus,

$$EPQ = Q^* = \sqrt{\frac{2C_o D}{\frac{C_h(P-D)}{P}}} \quad \text{-----} \rightarrow \text{eq.20}$$

$$\text{Total Variable Cost} = TC^* = \sqrt{2C_o C_h (P-D) \frac{D}{P}} \quad \text{-----} \rightarrow \text{eq.21}$$

The economic production quantity model (gradual supply case and shortage not allowed) is explained with the help of following Example 4.5.

### Example 4.5

#### Problem

An inventory item unit is used at the rate of 200/day, and can be manufactured at a rate of 700/day. It costs Rs.3000 to set up the manufacturing process and Rs.0.2 per unit per day held in inventory based on the actual inventory any time. Assume that shortage is not allowed.

Find out the minimum cost and the optimum number of units per manufacturing run.

#### Solution

Given that

Demand,  $D = 200$  units

Production,  $P = 700$  units

Set up Cost,  $C_o = \text{Rs.}3000$

Holding Cost,  $C_h = \text{Rs.}0.2$

$$\begin{aligned} \tilde{N} \text{ Minimum Cost} = TC^* &= \sqrt{\frac{2C_o C_h (P-D) D}{P}} \\ &= \sqrt{\frac{2 * 3000 * 0.2 (700-200) (200)}{700}} = \text{Rs.}414 \end{aligned}$$

$$EPQ = Q^* = \sqrt{\frac{2C_o D}{\frac{C_h(P-D)}{P}}} = \sqrt{\frac{2C_o D}{C_h} * P}$$

$$\frac{C_h(P-D)}{P} \quad C_h(P-D)$$

$$= \sqrt{\frac{2 \times 3000 \times 200 \times 700}{0.2(700-200)}} = 2898.2 = 2898 \text{ units}$$

Therefore

**Minimum Cost = Rs.414**

**Number of units per Manufacturing Run = 2898 units**

#### 4.3.4 Price Discounts Model

In this section we will discuss, instantaneous supply with no shortages. We know very well that whenever we make bulk purchasing of items there may be some discount in price is usually offered by the suppliers. As far as discount concern, there are two types:

1. Incremental Discount – discount allowed only for items which are in excess of the specified amount. In this case, all the prices offered in different slabs are applicable in finding the total cost.
2. All units Discount – discount allowed for all the items purchased. In this, only one price at any one slab is applicable for finding the total cost.

Here we are going to discuss only all units' discount type.

##### **Advantages of Bulk Purchase**

Buying in bulk may result in the following advantages:

- less unit price
- less ordering cost
- cheaper transportation
- fewer stock outs
- sellers preferential treatment

##### **Disadvantages of Bulk Purchase**

Bulk purchase also has the following disadvantages in addition to the above advantages:

- high carrying cost
- lower stock turnover
- huge capital required
- less flexibility
- older stocks
- heavy deterioration
- heavy depreciation

In case of purchased items, if the discounts are allowed, the price C may vary according to the following pattern:

$$C = P_0 \text{ if purchase quantity } Q = Q_0 < q_1$$

$$= P_1 \text{ if purchase quantity } q_1 \leq Q_1 < q_2$$

.....

.....

$$= P_n \text{ if purchase quantity } q_n < Q_n$$

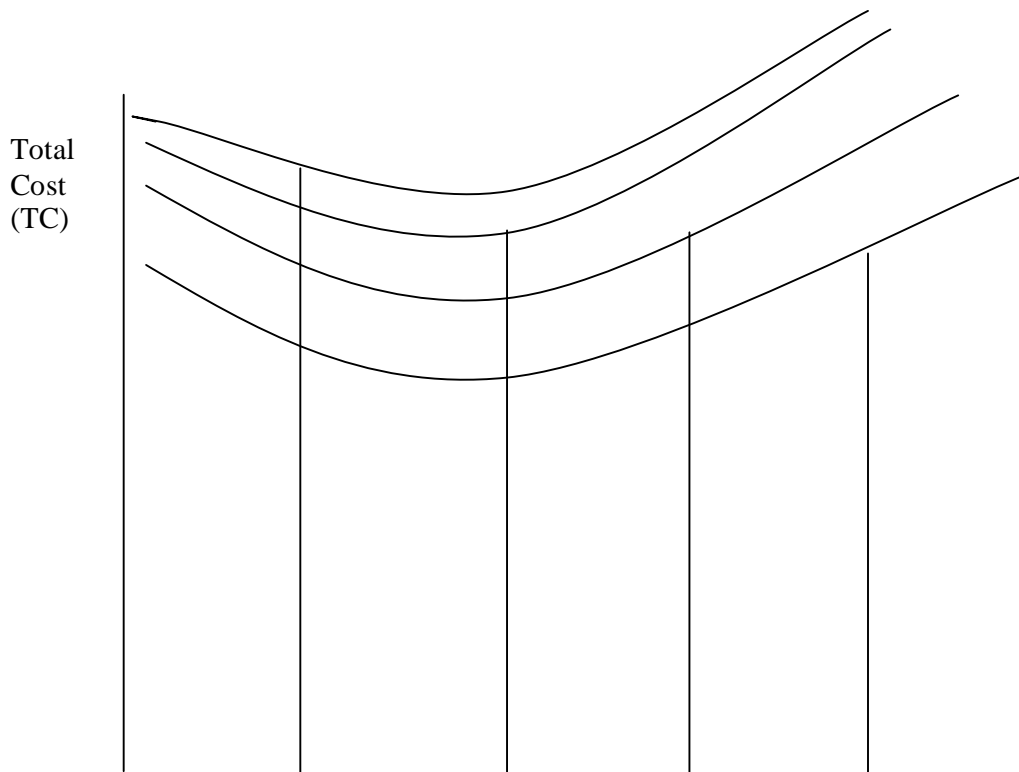
where  $P_{j-1}$  is  $> P_j$ , for  $j = 1, 2, \dots, n$ . -----> eq.22  
 $P_j$  = Price per unit for the  $j^{\text{th}}$  lot size

Suppose, shortages are not permitted, the total cost per year is obtained by the following set of relations:

$$TC(Q_j) = DP_j + C_oD + 1 C_h Q_j \quad , \text{ where } q_j \leq Q_j < q_{j+1} \quad \text{-----> eq.23}$$

and  $C_h = ip_j$ ,  $i$  being percentage change for  $j=0$  to  $n$ . Since price (or unit cost) varies with purchase size ( $Q$ ), the fixed cost term  $CD$  in equation eq.23 can't be omitted for minimizing the total cost (TC).

Equation eq.23 for quantity discounts are represented in the figure fig: 4.6.



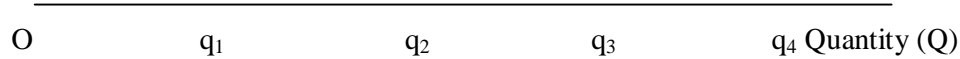


Fig: 4.6 Quantity Discount

The heavy curve on the various price discounts shows feasible portion of the total cost which is a step function. Therefore, for determining the overall optimum, the following steps of procedure is adopted:

**Step 1:** Find EOQ for the lowest price

That is compute  $Q_n^* = \sqrt{\frac{2CoD}{iP_n}}$

If  $Q_n^* \geq q_n$ , the optimum order quantity is  $Q_n^*$ .

If  $Q_n^* < q_n$ , then go to Step 2.

**Step 2:** Compute  $Q_{n-1}^* = \sqrt{\frac{2CoD}{iP_{n-1}}}$  for the next lowest price.

If  $Q_{n-1}^* \geq q_{n-1}$ , then compare the total cost  $TC_{n-1}$  for purchasing  $Q_{n-1}^*$  with the total cost  $TC_n$  for purchasing quantity  $q_n$  and select least cost purchase quantity.

If  $Q_{n-1}^* < q_{n-1}$  then go to Step 3.

**Step 3:** Compute  $Q_{n-2}^* = \sqrt{\frac{2CoD}{iP_{n-2}}}$  for the next lowest price.

If  $Q_{n-2}^* \geq q_{n-2}$ , then compare the total cost  $TC_{n-2}$ ,  $TC_{n-1}$  and  $TC_n$  for purchase quantities  $Q_{n-2}^*$ ,  $q_{n-1}$  and  $q_n$  respectively and select the optimum purchase quantity.

If  $Q_{n-2}^* < q_{n-2}$  then go to Step 4.

**Step 4:** Continue the procedure until  $Q_{n-j}^* \geq q_{n-j}$ . Then compare the total costs  $TC_{n-j}$  with  $TC_{n-j+1}$ , ...,  $TC_{n-1}$ ,  $TC_n$  for purchase quantities  $Q_{n-j}^*$ ,  $q_{n-j+1}$ , ...,  $q_{n-1}$ ,  $q_n$  respectively, and select the optimum purchase quantity.

The price discount model is explained with the help of the following examples.

**Example 4.6**

Consider the following problem, which explains the price discount model.

**Problem**

Suppose, annual demand for an item is 1500 units, ordering cost is Rs.250, inventory carrying charge is 12% of the purchase price per year and the purchase prices are:

$$\begin{aligned}
 P1 &= \text{Rs.}5 && \text{for purchasing } Q1 < 250 \\
 P2 &= \text{Rs.}4.25 && \text{for purchasing } 250 \leq Q2 < 500 \\
 P3 &= \text{Rs.}3.75 && \text{for purchasing } 500 \leq Q3
 \end{aligned}$$

Find out the optimum purchase quantity.

**Solution**

Given that

Demand,  $D = 1500$  units

Ordering Cost,  $C_o = \text{Rs.}250$

Carrying Cost,  $Ch = 12\% = i$

This problem belongs to the Step 1. So that as per Step 1

$$Q_3^* = \sqrt{\frac{2C_oD}{iP_n}} = \sqrt{\frac{2*250*1500}{(0.12)(3.75)}} = 1290 \text{ units}$$

Note that  $1290 > 500$ , Optimum Purchase Quantity = 1290 units.

**Example 4.7**

**Problem**

Consider the problem in Example 4.6 with ordering cost of Rs.10 only. Find out the optimum purchase quantity.

**Solution**

Given that

Demand,  $D = 1500$  units

Ordering Cost,  $C_o = \text{Rs.}10$

Carrying Cost,  $Ch = 12\% = i$

In this case,

$$Q_3^* = \sqrt{\frac{2C_oD}{iP_n}} = \sqrt{\frac{2*10*1500}{(0.12)(3.75)}} = 258 \text{ units}$$

Since  $258 < 500 = q_3$ , so that we may have to compute

$$Q_2^* = \sqrt{\left( \frac{2C_oD}{iP_n} \right)} = \sqrt{\left( \frac{2*10*1500}{(0.12)(4.25)} \right)} = 242 \text{ units}$$

Since  $242 < 250 = q_2$ , next we may have to compute

$$Q_1^* = \sqrt{\frac{2CoD}{iP_n}} = \sqrt{\frac{2*10*1500}{(0.12)(5)}} = 224 \text{ units}$$

Since  $224 > 0$ , now we have to compare the total cost for purchasing  $Q_1^* = 224$ ,  $q_2 = 250$  and  $q_3 = 500$  units respectively.

From equation eq.23

$$TC(Q_j) = DP_j + C_oD + 1 C_hQ_j, \text{ where } q_j \leq Q_j < q_{j+1}$$

We will get

$$TC_1 \text{ (for purchasing 224)} = 5*1500 + \frac{10*1500}{224} + \frac{1}{2} (0.12) (5)224 = 7634.2$$

$$TC_2 \text{ (for } Q_2=250) = 4.25*1500 + \frac{10*1500}{250} + \frac{1}{2} (0.12) (4.25)250 = 6498.75$$

$$TC_3 \text{ (for } Q_3=500) = 3.75*1500 + \frac{10*1500}{500} + \frac{1}{2} (0.12) (3.75)500 = 5767.5$$

The EPQ for his problem is  $Q_3^* = 500$  units

#### 4.3.5 Dynamic Demand Models

In this model, assume that demand is known with certainty, and although may vary from one period to the next period. There are five types of dynamic demand inventory models, they are:

- i) Production Inventory Model (Incremental Cost Method)
- ii) Dynamic Inventory Model (Prescribed Rule Method)
- iii) Dynamic Inventory Model (Fixed EOQ Method)

The above five dynamic demand models of inventory are discussed in details in the following subsequent sections.

##### i) Production Inventory Model

This is also called as **Incremental Cost Method**. This situation is explained with the help of the following example 4.8.

##### Example 4.8

Consider the following problem.

##### Problem

A production factory has a fixed weekly cyclic demand as follows:

Days	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Demand in units	9	17	2	0	19	9	14

Here the policy is to maintain constant daily production seven days a week. Shortage cost is Rs.4 per unit per day and storage cost depends upon the size of Q, the quantity carried, as follows:

Cost per unit for one day(Rs.)	=1	=4	=10
Q	$1 \leq Q \leq 3$	$4 \leq Q \leq 20$	$20 \leq Q$

And the charges are based on the situation at the end of the day.

Determine the optimal starting stock level.

**Solution**

Let the production rate be the average of the total sales which is

$$= 9 + 17 + 2 + 0 + 19 + 9 + 14 = 10 \text{ units/days}$$

Day	On hand stock start of day	Demand	Inventory	Shortage cost (Rs.)	Carrying cost (Rs.)
7 Mon	8	9	-1	4	0
Tue	-1+10=9	17	-8	32	0
Next Wed	-8+10+=2	2	0	0	0
we will Thu	0+10=10	0	10	0	44
find the Fri	10+10=20	19	1	0	1
total Sat	1+10=11	9	2	0	2
weekly Sun	2+10=12	14	-2	8	0
cost for Mon	-2+10=8	Total Cost		44	+ 43 = 87

different starting stocks, which is illustrated in the following tables.

Table 4.1 shows for starting stock 8, Table 4.2 shows for starting stock 9 and Table 4.3 shows for starting stock 10.

Table  
4.1  
Cost  
Analysis  
s with  
Initial

Day	On hand stock start of day	Demand	Inventory	Shortage cost (Rs.)	Carrying cost (Rs.)
Mon	9	9	0	0	0
Tue	0+10=10	17	-7	28	0
Wed	-7+10+=3	2	1	0	1
Thu	1+10=11	0	11	0	44
Fri	11+10=21	19	2	0	2
Sat	2+10=12	9	3	0	3
Sun	3+10=13	14	-1	4	0
Mon	-1+10=9	Total Cost		32	+ 50 = 82

Stock 8

Day	On hand stock start of day	Demand	Inventory	Shortage cost (Rs.)	Carrying cost (Rs.)
Mon	10	9	1	0	1
Tue	1+10=11	17	-6	24	0
Wed	-6+10+=4	2	2	0	2
Thu	2+10=12	0	12	0	48
Fri	12+10=22	19	3	0	3
Sat	3+10=13	9	4	0	16
Sun	4+10=14	14	0	0	0
Mon	0+10=10	Total Cost		24	+ 70 = 94

Table 4.2 Cost Analysis with Initial Stock 9

Table 4.3 Cost Analysis with Initial Stock 10

Therefore

The optimal solution for starting stock of 9 units is on MONDAY.

Minimum Total Cost for this is Rs.82.



Procedure for solving such incremental problem is cost analysis, which is self explanatory.

**ii) Dynamic Inventory Model (Prescribed Rule Method)**

Some times, the organization dealing with inventory may prescribe some rule of procurement of items of inventory. For example

- Procuring every month
- Procuring every three months

The prescribed rule method is explained with the help of the following Example 4.9.

**Example 4.9**

Consider the following problem.

**Problem**

An organization estimates the demand of an item as follows:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Demand	140	98	62	134	20	72	22	164	139	170	248	51	1320

The organization has decided that their ordering cost is Rs.54 and carrying charge per unit per month is 2% at the end of each month. The cost of item per unit is Rs.20. Assume that the supply is instantaneous, there in no stock outs and no lead time. Only full month requirement is ordered.

**Solution**

The following table is used for determining the total cost for the policy of quarterly ordering.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Starting Inventory	0	160	62	0	92	72	0	303	139	0	299	51	--
Replenishment	300	--	--	226	--	--	325	--	--	469	--	--	1320
Requirements	140	98	62	134	20	72	22	164	139	170	248	51	1320
Ending Inventory	160	62	0	92	72	0	303	139	0	299	51	0	1178

Table 4.4 Total Cost Analysis for Quarterly Policy

Therefore

$$\text{Total Carrying Cost} = 1178 \times (0.02) \times (20) = \text{Rs.}471.2$$

$$\text{Total Replenishment Cost} = 4 \times 54 = \text{Rs.}216$$

$$\text{Total Annual Cost} = 216 + 471.2 = \text{Rs.}687.2$$

**iii) Dynamic Inventory Model (Fixed EOQ Method)**

The Fixed EOQ Method is explained with the help of the following Example 4.10.

**Example 4.10**

Consider the problem of Example 4.9.

**Problem**

An organization estimates the demand of an item as follows:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Demand	140	98	62	134	20	72	22	164	139	170	248	51	1320

The organization has decided that their ordering cost is Rs.54 and carrying charge per unit per month is 2% at the end of each month. The cost of item per unit is Rs.20. Assume that the supply is instantaneous, there in no stock outs and no lead time. Only full month requirement is ordered.

**Solution**

$$\text{Average Monthly Demand} = \frac{\Sigma \text{Demand}}{\Sigma \text{Month}} = \frac{1320}{12} = 110 \text{ units/month}$$

$$\text{EOQ} = \sqrt{\frac{2 * 54 * 110}{(0.02) * (20)}} = 172 \text{ (approximately)}$$

In this case the organization has to ordered full month requirement, therefore 172 lies between 140 and 248 units, but 172 is more closer to 140 than 248, so that the organization order one month requirement at the beginning of the January.

Similarly, at the beginning of the February, the organization order two month requirement.

The detailed result is illustrated in the following Table 4.5

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Starting Inventory	0	0	62	0	92	72	0	164	0	0	0	0	--
Replenishment Requirements	140	160	--	226	--	--	186	--	139	170	248	51	1320
Ending Inventory	0	62	0	92	72	0	164	0	0	0	0	0	390

Table 4.5 EOQ Total Cost Analysis

Therefore

$$\begin{aligned} \text{Total Ordering Cost} &= 8 * 54 = \text{Rs.}432 \\ \text{Total Carrying Cost} &= 390 * 0.4 = \text{Rs.}156 \end{aligned}$$

Thus, the Total Cost is reduced if EOQ policy is used instead of three month (quarterly) rule.

**4.4 Deterministic Multi Item Inventory Models**

When there is more than one item in the inventory is called as multi item inventory. Since this contains more items, the inventory control requires special type of care. This type of

inventory problems may have different types of constraints like capital, cost structure, storage space, purchasing load etc. As the number of constraints increase the problem becomes more complex. In this section we will discuss some examples of this type of inventory.

There are two types of multi item inventory model which is based on the structure of the cost, they are:

1. Model with Unknown Cost Structure
2. Model with Known Cost Structure

#### 4.4.1 Unknown Cost Structure Model

In India most of the organizations do not maintain the proper inventory records which may provide sufficient cost information to generate the two basic parameters viz. procurement cost and carrying cost of inventory control. Some organizations in some situations have not developed cost structure related to inventory control, but still they wish to minimize total cost of inventory. There may be critical situations, in which an organization may need to take immediate actions to improve the situation without considering the structure of cost.

The use of inventory models without cost information is impossible, but we will show here that it is possible to get many of benefits of inventory techniques even when carrying cost and ordering cost are not known. In such a problem, there are two different approaches, they are:

1. Minimize the Total Carrying Cost while keeping the number of orders/year fixed  
or
2. Minimize the Total Number of orders/year while keeping the same level of inventory.

#### Minimize the Total Carrying Cost while keeping the number of orders/year fixed Model

As per the EOQ model

$$Q^* = \sqrt{\frac{2CoD}{C_h}} \quad \text{or} \quad Q^* = \sqrt{\frac{2CoD}{C_h}} * \sqrt{D}$$

Suppose  $\alpha = \sqrt{\frac{2Co}{C_h}}$  = Constant, because ordering cost and carrying cost are deterministic values.

$$C_h$$

Therefore  $Q^* = \alpha * \sqrt{D}$  -----> eq.24

The equation eq.24 tells that EOQ is proportional to the square root of demand for any inventory item of control.

For this equation, we get

$$\alpha = \frac{Q^*}{\sqrt{D}} = \frac{Q^* \sqrt{D}}{D} = \frac{\sqrt{D}}{(D/Q^*)} = \frac{\text{Square root of Demand}}{\text{Number of orders/year}} \text{ -----} \rightarrow \text{eq.25}$$

Since  $\alpha$  is constant for any single item of inventory, we take  $\alpha$  as the constant for all the inventory items. Thus, we take

$$\alpha = \frac{\sum \sqrt{D}}{\Sigma(D/Q^*)} \text{ or } \alpha = \frac{\text{Sum of square roots of Demand of each inventory item}}{\text{Sum of the number of orders/year for each item}} \text{ -} \rightarrow \text{eq.26}$$

This model is explained with the help of the following Example 4.11.

**Example 4.11**

Consider the following problem.

**Problem**

An organization has the following procurement pattern of six items irrespective of their demand level. Reduce the inventory levels while keeping total number of orders/year fixed.

Item No.	Number of Orders/Year	Demand (Rs.)	Order Size (Rs.)	Average Inventory
1	5	2000,000	500000	250000
2	5	700,000	260000	130000
3	5	100,000	23500	11750
4	5	9,000	10000	5000
5	5	5,000	700	350
6	5	2,700	500	250

**Solution**

First we find the value of  $\alpha = \frac{\sum \sqrt{D}}{\Sigma(D/Q^*)}$  -----  $\rightarrow$  eq.27

Therefore  $\sum \sqrt{D} = 1414 + 836 + 316 + 95 + 70 + 52 = \text{Rs.}2783$

$\Sigma(D/Q^*) = 5 + 5 + 5 + 5 + 5 + 5 = 30$

Thus,  $\alpha = \frac{\sum \sqrt{D}}{\Sigma(D/Q^*)} = \frac{2783}{30} = 92.7$

Now we will analyze the ordering quantity, which is illustrated in the following Table 4.6.

Item No.	Demand (D) (Rs.)	$\sqrt{D}$	$\alpha$	EOQ= $\alpha \sqrt{D}$	Number of orders/year = $D/\text{EOQ}$	Average Inventory = EOQ/ $\Sigma$ Number of orders/year
1	2000,000	1414	92.7	131077.8	15.258	4369
2	700,000	836	92.7	77497.2	9.0	2583
3	100,000	316	92.7	29293.2	3.41	976
4	9,000	95	92.7	8806.5	1.0	293

5	5,000	70	92.7	6489	0.77	216
6	2,700	52	92.7	4820.4	0.56	160
					Σ 29.998	Σ 8597

Table 4.6 Analysis of Ordering Quantity

Thus, according to the policy of the organization ordering for six items a year each item, Total Average Inventory becomes **Rs.397350**.

But, as per the new schedule as obtained in the above Table , the Average Inventory is **Rs.8597** which is much less and at the same time total Number of Orders remains same.

Therefore, substantial savings can still be achieved when cost information is known.

**Minimize the Total Number of orders/year (or purchasing workload) while keeping the same level of inventory Model**

We will know that

$$Q^* = \alpha \sqrt{D} \quad \text{or} \quad \alpha = \frac{Q^*}{\sqrt{D}}$$

Now onwards, we will take  $\alpha = \frac{\sum Q^*}{\sum \sqrt{D}}$  -----→ eq.28 for all inventory items.

This model is explained with the help of the previous Example 4.11.

Here we will discuss about how to minimize the number of orders or purchasing workload.

**Problem**

An organization has the following procurement pattern of six items irrespective of their demand level. Reduce the inventory levels while keeping total number of orders/year fixed.

Item No.	Number of Orders/Year	Demand (Rs.)	Order Size (Rs.)	Average Inventory
1	5	2000,000	500000	250000
2	5	700,000	260000	130000
3	5	100,000	23500	11750
4	5	9,000	10000	5000
5	5	5,000	700	350
6	5	2,700	500	250

**Solution**

Here  $K = \frac{\text{Sum of the order size}}{\text{Sum of square root of Demand}}$

$$K = \frac{50000+260000+23500+10000+700+500}{1414+836+316+95+70+52} = \frac{794700}{2783} = 285.555$$

Item No.	Demand (D) (Rs.)	$\sqrt{D}$	$\alpha$	$EOQ = \alpha\sqrt{D}$	Number of orders/year = $D/EOQ$	Average Inventory = $EOQ/2$
1	2000,000	1414	285.555	403774.77	4.95	201887.385
2	700,000	836	285.555	238723.98	2.93	119361.99
3	100,000	316	285.555	90235.38	1.10	45117.69
4	9,000	95	285.555	27127.725	0.33	13563.8625
5	5,000	70	285.555	19988.85	0.25	9994.425
6	2,700	52	285.555	14848.86	0.18	7424.43
$\Sigma EOQ = 794700$					$\Sigma 9.74$	$\Sigma 397349.8$

Table 4.7 Analysis for Reducing Number of Orders

Thus, from the Table 4.7, it is obvious that the total purchasing workload has reduced by 35%. Therefore, there is a definite saving of cost by applying these methods for multiple items even when cost information is not known.

#### 4.4.2 Known Cost Structure Model

We may classify the models with known cost structures into two main types, they are:

- Model without Limitations
- Model with Limitations

Here we will discuss these models, if we know the complete cost structure. In this model let us consider the following symbolic notation:

- $D_j$  - Demand
- $C_{oj}$  - Ordering Cost
- $C_{hj}$  - Holding Cost or Carrying Cost for  $j^{\text{th}}$  item respectively.

#### Model without Limitations

In any situation the items may be purchased according to their individual economic order quantities, if there are no restrictions for the items being storing.

In this case,

Total Variable Cost per annum for n-items can be represented as

$$TC = \sum_{J=1}^n \left( \frac{C_{oj}D_j}{Q_j} + \frac{C_{hj}Q_j}{2} \right) \text{ and } \dots \dots \dots \rightarrow \text{eq.29}$$



The optimum Order Size for each item is

$$Q_j^* = \sqrt{\frac{2C_{oj}D_j}{C_{hj}}} \quad \text{for } j = 1, \dots, n \text{ -----} \rightarrow \text{eq.30}$$

**Model with Limitations**

We know that the different constraints on inventories are available capital, order size per year, storage space, etc. In this case we will consider a single constraint for the discussion of this model. Note that the constraints capital and storage space are interchangeable.

Let

$Q_j$  is the order quantity for item  $j$   
 $F_j$  is the storage space (or floor space) for one unit of item  $j$   
 (or)  $F_j$  – the capital requirement for one unit of item  $j$ , since the storage

space and

the capital are interchangeable.

$F$  is the total available storage space (floor space) or available capital

Then the constraints are as follows:

$$\sum_{j=1}^n F_j Q_j + F_2 Q_2 + \dots \dots \dots F_n Q_n \leq F \text{ -----} \rightarrow \text{eq.31}$$

Here the objective is to minimize the total inventory cost expressed by the equation eq.29. So that in order to solve this problem first we have to convert the constraint problem into unconstrained minimization problem and then the optimum result is obtained.

The problem is to minimize the function known as Lagrange Function:

$$L(\rho, Q_1, Q_2, \dots, Q_n) = \sum_{j=1}^n \left[ \left( \frac{C_{oj}D_j}{Q_j} + \frac{C_{hj}Q_j}{2} \right) + \rho \left( \sum_{j=1}^n F_j Q_j - F \right) \right] \text{ -----} \rightarrow \text{eq.32}$$

Then, the values of optimal sizes are

$$Q_j^* = \sqrt{\frac{2C_{oj}D_j}{(C_{hj} + 2\rho F_j)}} \quad \text{for } j = 1, \dots, n \text{ -----} \rightarrow \text{eq.33}$$

The known cost structures model is explained with the following Example 4.12.

**Example 4.12**

Consider the following problem.

**Problem**

A retail shop purchases three items of inventory viz. A, B and C respectively. The shop works on the limitations that the shop is not able to invest more than Rs.40000 at any time. The other shop relevant information is given in the following table:

Item	A	B	C
Demand rage (units/year) – Dj	20000	1000	2000
Purchase Cost/unit – Cj	40	200	100
Ordering Cost/Order – Coj	100	150	200
Holding or Carrying Cost - J	40%	40%	40%

**Solution**

In the absence of constraints, the Optimal Order Sizes are:

$$Q1 = \sqrt{\frac{2 * 100 * 20000}{(.40) (40)}} = 500$$

$$Q2 = \sqrt{\frac{2 * 150 * 1000}{(.40)(200)}} = 61$$

$$Q3 = \sqrt{\frac{2 * 200 * 2000}{(.40) (100)}} = 141$$

Now we know the optimal sizes, so that with these optimal sizes we may determine the maximum investment, that is

$$\begin{aligned} \text{Maximum Investment} &= 500 * 40 + 61 * 200 + 141 * 100 \\ &= 20000 + 12200 + 14100 \\ &= \text{Rs. } 46300 \end{aligned}$$

Note that the Maximum Investment Rs.46300 is greater than the allowable investment capacity i.e. Rs.40000 in inventory.

Therefore, equation eq.33 is used with the following alteration

$$Q_j^* = \sqrt{\frac{2 \cdot C_{oj} \cdot D_j}{C_j(J+2\rho)}} \text{ for } j = 1, 2, 3. \text{ Since, there are only three items. ---} \rightarrow \text{eq.34}$$

Note that here  $\rho = \text{is the solution of the equation.}$

$$Q_j^* = \sum_{j=1}^3 \left[ \sqrt{\frac{2 \cdot C_{oj} \cdot D_j}{C_j(J+2\rho)}} \cdot C_j \right] = 40000$$

or

$$\sqrt{\frac{40 \cdot 20000 \cdot 100}{0.1 + \rho}} + \sqrt{\frac{150 \cdot 1000 \cdot 200}{0.1 + \rho}} + \sqrt{\frac{200 \cdot 2000 \cdot 100}{0.1 + \rho}} = 40000$$

or

$$\rho = 0.16899919$$

Now we will substitute the value of  $\rho$  in equation eq.34

Therefore, now the order size becomes

$$Q^*_1 = 368$$

$$Q^*_2 = 45$$

$$Q^*_3 = 104$$

That is,

If limitations are not imposed on the purchase of quantities,

$$\text{The optimal Total Cost} = \frac{2 \cdot 100 \cdot 20000}{500} + \frac{2 \cdot 150 \cdot 1000}{61} + \frac{2 \cdot 200 \cdot 2000}{141}$$

$$= 8000 + 4918 + 5674$$

$$= \text{Rs.18592}$$

But, under the limitations,

The optimal Total Cost =  $\frac{2*100*20000}{368+2*0.17*40} + \frac{2*150*1000}{45+2*0.17*200} + \frac{2*200*2000}{104+2*0.17*100}$   
 = 18934.19, which is higher than the total cost without limitations.

**Note:**

Many times the application of the equation

$$Q^*_j = \sqrt{\frac{2C_{oj}D_j}{(C_{hj}+2\rho F_j)}} \quad \text{for } j = 1, \dots, n \text{ -----} \rightarrow \text{eq.33}$$

to find optimal order quantities under limitations does not help to obtain the result. Therefore, we have to apply trial and error procedure in the following manner:

- i) First determine the EOQ's for all type of inventory items without considering the limitation that is taking  $\rho=0$  and then find  $Q_j$ . If these values satisfy the constraint (eq.31), then this solution becomes optimal because the constraint is not active.
- ii) If the constraint is not satisfied by the values obtained under (i) above, we give some value to  $\rho$  (arbitrarily but institutively) for example say  $\rho=h$  and solve for  $Q_{js}$ .  $Q_{js}$  satisfy the constraint, these are optimal quantities. Otherwise, we interpolate or extrapolate the value  $\rho$  in between 0 and h or beyond h. With this value of  $\rho$ , the order sizes obtained will be approximately optimal.

**4.5 Probabilistic Inventory Models**

In previous sections, we have discussed simple deterministic inventory models where each and every influencing factor is completely known. Generally in actual business environment complete certainty never occurs. Therefore, here we will discuss some practical situations of inventory problems by relaxing the condition of certainty for some of the factors.

The major influencing factors for the inventory problems are Demand, Price and Lead Time. There are also other factors like Ordering Cost, Carrying Cost or Holding Cost and Stock out Costs, but their nature is not so much disturbing. Because of this their estimation provides almost, on the average, as known as values. Even Price can also be averaged out to reflect the condition of certainty. But there are situations where Price fluctuations are too much in the market and hence they influence the inventory decisions. Similarly, the demand variations or consumption variation of an item as well as the lead time variation influence the overall inventory policy. In this section we will discuss single period probabilistic models.

**4.5.1 Single Period Probabilistic Models**

Single Period Discrete Probabilistic Model deals with the inventory situation of the items like perishable goods, seasonal goods and spare parts requiring one time purchase only. These items demand may be discrete or continuous. In these models the lead time is very much important because purchases are made only once.

In single period model, the problem is analyzed using incremental (or marginal) analysis and the decision procedure consists of a sequence of steps. In such cases, there are two types of cost involved. There are Under Stocking Cost and Over Stocking Cost. These two costs describe opportunity losses incurred when the number of units stocked is not exactly equal to the number of units actually demanded.

In this section we will use the following symbols:

D = Demand for each unit of item (or a random variable)

Q = Number of units stocked or to be purchased

C1 = Under Stocking Cost some times also known as over ordering cost. This is an opportunity loss associated with each unit left unsold i.e.

$$C1 = S - Ch/2 - Cs$$

C2 = Over Stocking Cost some times also known as under ordering cost. This is an opportunity loss due to not meeting the demand, i.e

$$C2 = C + Ch - V$$

Where

C = cost/unit

Ch = carrying cost/unit for the entire period

Cs = shortage cost

V = salvage value

S = selling price

In this section we are going to discuss only discrete demand distribution.

#### **4.5.2 Single Period Discrete Probabilistic Demand Model (Discrete Demand Distribution)**

Here we will discuss the following methods of solving the single period discrete probabilistic demand.

- a. Incremental Analysis Method
- b. Payoff Matrix Method

##### **a. Incremental Analysis Method**

The Cost equation is developed as follows:

For any quantity in stock Q, only D units are consumed (or demanded). Then for specific period of time, the cost associated with Q units in stock is either:

- i)  $(Q-D)C_2$ , where  $D$  is the number of units demanded or consumed is less than or equal to the number of units  $Q$  in stock. That is  $D \leq Q$ .
- ii)  $(D-Q)C_1$ , where the number of units required is greater than the number of units  $Q$  in stock. That is  $D > Q$ .

We know that  $D$  is a random variable, so its probability distribution of demand is known.  $\rho(D)$  represents the probability that the demand is  $D$  units, such that total probability is one.

$$\text{That is } \rho(0) + \rho(1) + \dots + \rho(D) + \dots = \sum_{D=0}^{\infty} \rho(D) = 1 \text{ -----} \rightarrow \text{eq. p1}$$

The total expected cost is the sum of expected cost of under-stocking and over-stocking. Therefore

The Total Expected Cost, say  $f(Q)$ , is given by

$$f(Q) = C_2 \sum_{D=0}^Q (Q-D) \rho(D) + C_1 \sum_{D=Q+1}^{\infty} (D-Q) \rho(D) \text{ -----} \rightarrow \text{eq.p2}$$

Suppose,  $Q^*$  is the optimal quantity stocked, then the total expected cost  $f(Q^*)$  will be minimum. Thus, if we stock one unit more or less than the optimal quantity, the total expected cost will be higher than the optimal.

Thus,

$$\begin{aligned} f(Q^{*+1}) &= C_2 \sum_{D=0}^{Q^{*+1}} (Q^{*+1}-D) \rho(D) + C_1 \sum_{D=Q^{*+2}}^{\infty} (D-Q^{*+1}) \rho(D) \\ &= C_2 \sum_{D=0}^{Q^*} (Q^*-D) \rho(D) + C_2 \sum_{D=Q^{*+1}}^{\infty} (D-Q^*) \rho(D) + C_2 \sum_{D=0}^{Q^*} \rho(D) - C_1 \sum_{D=Q^{*+1}}^{\infty} \rho(D) \\ &= f(Q^*) + (C_2+C_1) \sum_{D=0}^{Q^*} \rho(D) - C_1 \text{ -----} \rightarrow \text{eq.p3} \end{aligned}$$

Thus,

$$f(Q^{*+1}) - f(Q^*) = (C_2+C_1) \rho(D \leq Q^*) - C_1 \geq 0 \text{ -----} \rightarrow \text{eq.p4}$$

$$\text{where } \rho(D \leq Q^*) = \sum_{D=0}^{Q^*} \rho(D) \text{ - is a cumulative probability.}$$

Similarly, we obtain

$$f(Q^*-1) - f(Q^*) = C_1 - (C_2+C_1) \rho(D \leq Q^*-1) \geq 0 \text{ -----} \rightarrow \text{eq.p5}$$

We will obtain the following from the above equations eq.p4 and eq.p5

$$\rho(D \leq Q^* - 1) \leq \frac{C_1}{C_2 + C_1} \leq \rho(D \leq Q^*) \quad \text{-----} \rightarrow \text{eq.p6}$$

Therefore, the optimal stock level  $Q^*$  satisfies the relationship (eq.p6).

Note that for practical application of (eq.p6), the three step procedure is as follows:

**Step 1:**

From the data, prepare a table showing the probability  $\rho(D)$ , and cumulative probability  $\rho(D \leq Q)$  for each reasonable value of  $D$ .

**Step 2:**

Calculate the ratio  $\frac{C_1}{C_2 + C_1}$  which is called as service level.

**Step 3:**

Determine the value of  $Q$  which satisfies the inequality eq.p6.

This situation is explained with the following Example 4.13.

**Example 4.13**

An organization stocks seasonal products at the start of the season and cannot reorder. The inventory item costs him Rs.35 each and he sells at Rs.50 each. For any item that cannot be met on demand, the organization has estimated a goodwill cost of Rs.25. Any unsold item will have a salvage value of Rs.20. Holding cost during the period is estimated to be 10% of the price. The probability distribution of demand is as follows:

Units Stocked	2	3	4	5	6
Probability of Demand $\rho(D=Q)$	0.35	0.25	0.20	0.15	0.05

Determine the Optimum Number of Items to be stocked.

**Solution**

Now we have to follow the above sequence of steps.

**Step 1:**

We will prepare the Table 4.8 containing the data regarding demand distribution as follows:

Units Stocked	Probability of Demand $\rho(D=Q)$	Cumulative Probability $\rho(D \leq Q)$
2	0.35	0.35
3	0.25	0.60
4	0.20	0.80
5	0.15	0.95
6	0.05	1.00

Table 4.8 Probability Distribution of Demand

Step 2:

Calculate the ratio  $\frac{C1}{C2+C1}$  which is called as service level.

We see that

$$S=50, C=35, Ch=0.1*35= 3.5, V=20, Cs=25$$

Therefore

$$C2 = C+Ch-V = 35+3.5-20 = 18.5$$

$$C1 = S-C-\frac{Ch}{2} + Cs = 50-35-\frac{3.5}{2} + 25 = 38.25$$

$$\text{Thus, } \frac{C1}{C2+C1} = \frac{38.25}{18.5+38.25} = 0.6740$$

Step 3:

Look into the Table 4.8, the ratio 0.6740 lies between cumulative probabilities of 0.60 and 0.80 which in turn reflect the values of Q as 3 and 4 (units stocked).

$$\text{That is } \rho(D \leq 3) = 0.60 < 0.6740 < 0.80 = \rho(D \leq 4)$$

**Therefore, the optimal number of units to stock is 4 units.**

### Cost of Under Stocking Estimation

Suppose, in the previous Example 4.13, the under stocking cost is not known, but the decision maker policy is to maintain a stock level of say 5 units. We can determine for what values of C1 (under estimating cost) does  $Q^*=5$ ?

In this case, we have the following inequality:

$$\rho(D \leq 4) \leq \frac{C1}{C2+C1} \leq \rho(D \leq 5)$$

$$\text{That is } \rho(D \leq 4) \leq \frac{C1}{18.5+C1} \leq \rho(D \leq 5) \text{ or}$$

Or



$$0.80 \leq \frac{C1}{18.5+C1} \leq 0.95$$

So that the Minimum Value of C1 is determine by letting

$$\frac{C1}{18.5+C1} = 0.80 \quad \text{or} \quad C1 = \frac{(0.80)(18.5)}{(1-0.80)} = \text{Rs.}74$$

Similarly, the Maximum Value of C1 is determine by letting

$$\frac{C1}{18.5+C1} = 0.95 \quad \text{or} \quad C1 = \frac{(0.95)(18.5)}{(1-0.95)} = \text{Rs.}351.5$$

Therefore  $74 \leq C1 \leq 351.5$

### Perishable Products Inventory

Many of the organization manage merchandise which contains negligible utility if it is not sold almost immediately. The examples of such kind of products are newspaper, fresh produce, printed programmes for special events and other perishable products. Generally such inventory items have high mark-up. The major difference between the wholesale cost and the retail price is due to the risk vendor faces in stocking the inventory. Vendor faces obsolescence costs on the one hand and opportunity costs on the other.

All this kind of problems can be very easily solved with the help of the above discussed model. This is explained in the following Example 4.14.

#### Example 4.14

A boy selling newspaper, he buys papers for Rs.0.45 each and sells them for Rs.0.70 each. The condition here is the boy cannot return unsold newspapers. The following table shows the daily demand distribution. If each days demand is independent of the pervious days demand, how many news papers should he order each day?

Number of Customers	240	250	260	270	280	290	300	310	320	330
Probability	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05

#### Solution

Step 1:

Prepare a following Table 4.9 showing the probability  $\rho(D)$ , and cumulative probability  $\rho(D \leq Q)$  for each reasonable value of D.

Number of Customers	240	250	260	270	280	290	300	310	320	330
Probability	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05
Cumulative Probability	0.01	0.04	0.10	0.20	0.40	0.65	0.80	0.90	0.95	1.00

Table 4.9 Probability Discrete Distribution of Demand

Step 2:

$$\frac{C1}{C2+C1} = \frac{0.25}{0.45+0.25} = \frac{0.25}{0.70} = 0.357$$

Step 3:

Thus, the Value of Q such that

$$\rho(D \leq Q^*-1) \leq 0.357 \leq \rho(D \leq Q^*) \quad \text{this gives}$$

$$Q^* = 280$$

Therefore **The newspaper boy should buy 280 papers each day.**

### b. Payoff Matrix Method

The Payoff Matrix Method of single period Discrete Probabilistic Demand Model is explained with the help of the following Example 4.15.

#### Example 4.15

Consider the example 4.13 i.e.

An organization stocks seasonal products at the start of the season and cannot reorder. The inventory item costs him Rs.35 each and he sells at Rs.50 each. For any item that cannot be met on demand, the organization has estimated a goodwill cost of Rs.25. Any unsold item will have a salvage value of Rs.20. Holding cost during the period is estimated to be 10% of the price. The probability distribution of demand is as follows:

Units Stocked	2	3	4	5	6
Probability of Demand $\rho(D=Q)$	0.35	0.25	0.20	0.15	0.05

Determine the Optimum Number of Items to be stocked.

In this case the organization has five reasonable courses of action. The organization can stock the items from 2 to 6 units. There is no possible reason to stock more than 6 items

since the organization can never sell more than 6 items and there is no possible reason for ordering less than 2 items. Since there are five courses of action for stocking and five levels of demand, it follows that there are 25 combinations of one course of action and one level of demand. For these 25 combinations, we can determine the organization **payoffs** in the form of **payoff matrix**.

As per the information of cost given in the problem, the payoffs are obtained for the following two situations:

- When demand is not more than the stock level
- When demand is more than the stock quantity.

That is

Payoffs For	$Q \geq D$	$Q < D$
Item Cost	$-35Q$	$-35Q$
Sale of items	$50D$	$50Q$
Goodwill Cost	-	$-25(D-Q)$
Salvage Cost	$20(Q-D)$	-
Holding Cost	$-3.5(Q-D)-3.5D/2$	$-3.5Q/2$
Total payoff	$-18.50Q+31.75D$	$38.25Q-25D$

The payoff matrix will be 5X5. Each element of the matrix can be computed by above total payoffs for demand less than, equal to, or greater than the order size (Q). When demand is less than or equal to the order size, we have the following contributions to the payoff.

Here the organization buys the items for Rs.35Q and the organization sells D of them for Rs.50D, the organization earns salvage of Rs.20 (Q-D) for unsold items, and the organization incurs holding cost of Rs.(.10)(35)(Q-D) on unsold items an average holding cost of (.10)(35)D/2 on the sold items during the period. Thus the total payoff becomes  $-128.5Q+31.75D$  for demand less or equal to order size.

If demand is more than the order size, the contributory payoff will consist of the following:

- Purchase Cost Rs.35Q
- Selling Profit of Rs.50Q
- Goodwill Cost Rs.25(D-Q) and
- Holding Cost of Rs. (0.10)(35)Q/2

Thus, the total payoff for demand more than order size is  $38.25Q-25D$ .

The payoff matrix is as follows (Table 4.10):

	Units Demanded D					
	2	3	4	5		
6						
Units stocked or	2	26.50	1.50	-23.50	-48.5	-

73.5						
Order Size (Q)	3	8.0	39.75	14.75	-10.25	-
35.25						
	4	-10.5	21.25	53.0	28.0	
3.0						
	5	-29.0	2.75	34.5	66.25	
41.25						
	6	-47.5	-15.75	16.0	47.75	
79.5						
Probability of Demand		0.35	0.25	0.20	0.15	
0.05						

Table 4.10 Payoff Matrix

Now we will determine the expected payoff for each order size or courses of action. The procedure for computing the expected values is simple, as follows:

Procedure: For any given course of action multiply each possible payoff for that course of action by the corresponding probability of the given level of demand and add all of these products up.

Thus, for first course of action of order size 2 units, the expected value of payoff is:

$$(26.5)(0.35) + (1.5)(0.25) + (-23.5)(0.2) + (-48.5)(0.15) + (-73.5)(0.05) = \text{Rs.}-6$$

For order size 3 units

$$(8.0)(0.35) + (39.75)(0.25) + (14.75)(0.2) + (-10.25)(0.15) + (-35.25)(0.05) = \text{Rs.}12.3875$$

For order size 4 units

$$(-10.5)(0.35) + (21.25)(0.25) + (53)(0.2) + (28)(0.15) + (3)(0.05) = \text{Rs.}20.2625$$

For order size 5 units

$$(-29)(0.35) + (2.75)(0.25) + (34.5)(0.2) + (66.25)(0.15) + (41.25)(0.05) = \text{Rs.}9.4375$$

For order size 6 units

$$(-47.5)(0.35) + (-15.75)(0.25) + (16)(0.2) + (47.75)(0.15) + (79.5)(0.05) = \text{Rs.}-6.225$$

Therefore, we compute all the expected values:

Order Size Q	2	3	4	5	6
Expected value	Rs.-6	Rs.12.3875	<b>Rs.20.2625</b>	Rs.9.4375	Rs.-6.225

**Here the objective is to select course of action which provides the highest payoff. Thus, the organization should order for 4 units for the highest expected payoff value of Rs.20.2625.**

Note: If we compare the two methods i.e. incremental analysis and payoff matrix, if we see the solution that incremental analysis provides only the optimum level of purchase quantity and does not indicate about the level of expected profit. But, the payoff matrix method provides both the answers i.e. optimum purchase quantity as well as the optimum expected profit.

The interesting is here, we may also convert the payoff matrix to opportunity cost matrix, where the opportunity cost is, in short, a cost sustained because the decision taken is not the best in terms of the level of demand which actually occurs.

The computation of opportunity cost matrix from the payoff matrix is very easy. Take any column of the payoff matrix corresponding to a specific level of demand and select the largest payoff if the payoffs are profits, the smallest payoff if the payoffs are costs. Then subtract each payoff in the same column from the largest payoff to get the corresponding opportunity costs in the case of profits. If it is cost, subtract the smallest payoff from each payoff in the same column to get the opportunity costs.

In this Example 4.15, we may obtain the opportunity cost matrix as follows (Table 4.11):

	Units Demanded D				
	2	3	4	5	6
Order size					
2	0	38.25	76.5	114.75	153.0
3	18.5	0	38.25	76.5	114.75
4	37.0	18.5	0	38.25	76.5
5	55.5	37.0	18.5	0	38.0
6	74.0	55.5	37.0	18.5	0
Probability of Demand	0.35	0.25	0.20	0.15	0.05

Table 4.11 Opportunity Cost Matrix

Now, we have to determine the expected opportunity costs for each alternative courses of action. The objective is to select the course of action which provides minimum expected opportunity costs.

Therefore,

The expected opportunity cost for the first alternative course of action of order size 2 is:

$$(0)(.35) + (38.25)(.25) + (76.5)(.2) + (114.75)(0.15) + (153)(.05) = \text{Rs.}49.725$$

For order size 3 units

$$(18.5)(.35) + (0)(.25) + (38.25)(.2) + (76.5)(.15) + (114.75)(.05) = \text{Rs.}31.3375$$

For order size 4 units

$$(37)(.35) + (18.5)(.25) + (0)(.2) + (38.25)(.15) + (76.5)(.05) = \text{Rs.}27.1375$$

For order size 5 units

$$(55.5)(.35) + (37)(.25) + (18.5)(.2) + (0)(.15) + (38)(.05) = \text{Rs.}34.275$$

For order size 6 units

$$(74)(.35) + (55.5)(.25) + (37)(.2) + (18.5)(.15) + (0)(.05) = \text{Rs.}49.95$$

That is, the following are the obtained expected opportunity costs;

Order Size Q	2	3	4	5	6
Expected Cost	Rs. 49.725	Rs.31.3375	<b>Rs.27.1375</b>	Rs.34.275	Rs.49.95

Thus, the decision is to select the minimum expected cost is that, the organization should store **4 units** for the lowest cost of **Rs.27.1375**

### Relationship between the Payoff Matrix and Opportunity Cost Matrix

Here, we may find a relationship between the payoff matrix and the opportunity cost matrix, as follows:

Let

$$\text{EOC} = K - \text{EP}$$

Where,

EOC = Expected Opportunity Cost

EP = Expected Payoff or profit

K = Constant or

$$K = (26.5)(.35) + (39.75)(.25) + (53)(.2) + (66.25)(.15) + (79.5)(.05) \\ = 43.735$$

That is K = sum of the expected value of the largest elements in each column of the payoff matrix.

= the expected value of the payoffs for all the best courses of action.

Or

The expected opportunity cost for a given courses of action = K(43.735) - Expected pay of for each

courses of action -

→(eq1)

Thus, it is obvious from the above equation eq1 that the maximum value of EP will simultaneously produce, the minimum of EOC. The two analyses namely, the payoff matrix method and the opportunity cost matrix method produce the same result.

Suppose, if the original matrix is in terms of costs it can by similar reasoning, be shown that the above relationship (eq1) will be of the following form:

$$EOC = EP - K$$

In this case EP is in terms of costs.

#### 4.6 Summary

In this lesson various deterministic inventory models have been developed for various operating conditions. Here we discussed single item inventory and as well as multi item inventory models. In this lesson we also discussed probabilistic discrete demand models for single period inventory items.

#### 4.7 Key Terms

- Inventory** – stores of goods or stocks.
- Ordering Cost** - Cost involved in placing an order.
- Procurement Cost** – Same as ordering cost.
- Replenishment Cost** – Same as ordering cost.
- Set up Cost** – Cost associated with the setting of machine for production.
- Shortage Cost** – Costs associated with the demand when stocks have been depleted, it is generally called as back order costs.
- Safety Stock** – Extra Stocks.
- Perishable Product** – The inventories that deteriorate with time.
- Deterministic Model** – An inventory model where all the factors are completely known.
- Discrete Probability Distribution** – A probability distribution in which the variable is allowed to take only limited number of values.
- Expected Opportunity Cost** – Expected value of the variable indicating opportunity costs.
- Expected payoff** – Expected value of the variable indicating payoffs.
- Expected Value** – The average value or mean.
- Minimum Value** – This is also known as safety stock or the buffer stock.
- Maximum Value** – Level of inventory beyond which inventory is not allowed.
- Payoff** – The benefit which accrues form a given combination of decision alternative courses of action and state of nature.
- Reorder Level** – The stock level which is sufficient for the lead time consumption, and an order is initiated when inventory dips to this level.
- Under Stocking Cost** – Cost relating to the out of stock situation under the probabilistic

situation.

**Over Stocking Cost** – This is the cost of keeping more units than demanded.

**Opportunity Cost Matrix** – Matrix of opportunity Costs.

#### 4.8 Self Assessment Questions

Q1. A Production unit uses Rs.10,000 worth of an item during the year. The production units estimated the ordering cost as Rs.25 per order and holding cost as 12.5 percent of the average inventory value. Determine the optimal order size, number of orders per year, time period per order and total cost.

**Answer**

Order Size =  $Q^* = \text{Rs.}2000$

No. of orders per year =  $N = 5$

Time period per order =  $t^* = 73$  days

Total Cost =  $TC^* = \text{Rs.}250$

Q2. The usage of an inventory item each costing Re 1, is 10000 units/year and the ordering cost is Rs.10, carrying charge is 20% based on the average inventory per year, stock out cost is Rs.5 per unit of shortage incurred. Determine EOQ, inventory level, shortage level, cycle period, number of order per year and the total cost.

**Answer**

EOQ =  $Q^* = 1020$  units

Inventory Level =  $I^* = 980$  units

Shortage Level = 40 units

Cycle Period =  $t^* = 37$  days

No. of orders/year = 10

Total Cost = Rs.400

Q3. The demand for a unit of item is at the rate of 200 per day and can be produced at a rate of 800 per day. It costs Rs.5000 to set up the production process and Rs.0.2 per unit per day held in inventory based on the actual inventory any time. Assume that the shortage is not allowed. Find out the minimum cost and the optimum number of units per production run.

**Answer**

Hint:  $D=200$   $P=800$   $Co=\text{Rs.}5000$   $Ch=\text{Rs.}0.2$

$Q^* = 3651$  units

$TC^* = \text{Rs.}547.7$

Q4. The demand for an item is 2400 units per year. The ordering cost is Rs.100, inventory holding cost is 24 percent of the purchase price per year. Determine the optimum purchase quantity if the purchase prices are as follows:



$P1 = \text{Rs.}10$  for purchasing  $Q1 < 500$   
 $P2 = \text{Rs.}9.25$  for purchasing  $500 \leq Q2 < 750$   
 $P3 = \text{Rs.}8.75$  for purchasing  $750 \leq Q3$

**Answer**

Economic Purchase Quantity = EPQ =  $Q^* = 750$  units

Q5. A company follows the following procurement pattern of five items irrespective of their level of demand. Reduce the inventory levels while keeping the same total number of orders per annum.

Item	Demand/Year (\$)	Number of orders/year	Order size (\$)	Average Inventory
1	1000,000	5	250,000	125,000
2	640,000	5	160,000	80,000
3	90,000	5	22,500	11,250
4	2,500	5	625	350
5	1,600	5	400	200

**Answer**

According to the company policy, ordering five times a year each item, total average inventory becomes \$.216800.

But after the analysis of ordering quantity the average inventory becomes \$.7698.32, which is much less, at the same time the number of orders almost remain same. Thus, substantial savings can still be achieved when cost information is not known.

Q6. Suppose the carrying cost is 30% per unit/year, unit price is Rs.4, and the ordering cost is Rs.30 per order for an item used in an organization in the following pattern:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Demand	200	220	150	170	210	200	180	220	170	200	160	140

Determine the ordering schedule and the total inventory cost using the following method.

1. Prescribed Rule Method
2. Fixed EOQ Method

Q7. The following numbers indicate the annual values in dollars of some thirty inventory items of materials selected at random. Carry out an ABC analysis and list out the values of three items viz. A-items, B-items and C-items.

1	2	3	9	75	3
4	6	13	2	3	12

100	2	7	40	15	55
1	12	25	15	8	10
1	20	30	1	4	5

Q8. An organization stocking two products. The organization has limited storage space and can't store more than 40 units. The following are the demand distribution for the products:

Product 1		Product 2	
Demand	Probability of Demand	Demand	Probability of Demand
10	0.10	10	0.05
20	0.20	20	0.20
30	0.35	30	0.30
40	0.25	40	0.13
50	0.10	50	0.10

If the inventory holding cost is Rs.10 (product 1) and Rs15 (product 2) per unit of the ending inventories, the shortage costs are Rs.20 and Rs.50 per unit at the ending shortage for the first and second products respectively. Determine the economic order quantities for both the products.

#### 4.9 Further References

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Mustafi, C.K. 1988. Operations Research Methods and Practices, Wiley Eastern Limited, New Delhi.

Levin, R and Kirkpatrick, C.A. 1978. Quantitative Approached to Management, Tata McGraw Hill, Kogakusha Ltd., International Student Edition.

Peterson R and Silver, E. A. 1979. Decision Systems for Inventory Management and Production Planning.

Handley, G and T.N. Whitin. 1983. Analysis of Inventory Systems, PHI.

Starr, M.K. and D.W. Miller. 1977. Inventory Control Theory and Practice, PHI.

## UNIT III

### NETWORK PROBLEMS

#### Introduction

A network consists of several destinations or jobs which are linked with one another. A manager will have occasions to deal with some network or other. Certain problems pertaining to networks are taken up for consideration in this unit.

#### LESSON 1

#### SHORTEST PATH PROBLEM

##### LESSON OUTLINE

- The description of a shortest path problem.
- The determination of the shortest path.

##### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand a shortest path problem
- understand the algorithm for a shortest path problem
- work out numerical problems

##### THE PROBLEM

Imagine a salesman or a milk vendor or a post man who has to cover certain previously earmarked places to perform his daily routines. It is assumed that all the places to be visited by him are connected well for a suitable mode of transport. He has to cover all the locations. While doing so, if he visits the same place again and again on the same day, it will be a loss of several resources such as time, money, etc. Therefore he shall place a constraint upon himself not to visit the same place again and again on the same day. He shall be in a position to determine a route which would enable him to cover all the locations, fulfilling the constraint.

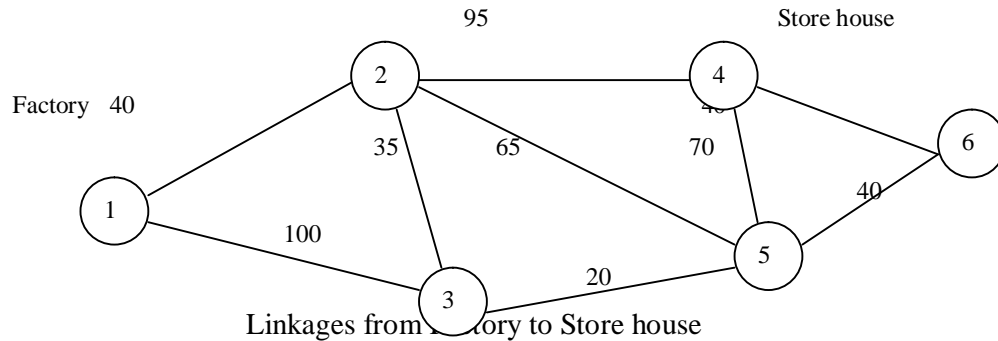
The shortest route method aims to find how a person can travel from one location to another, keeping the total distance traveled to the minimum. In other words, it seeks to identify the shortest route to a series of destinations.

##### EXAMPLE

Let us consider a real life situation involving a shortest route problem.

A leather manufacturing company has to transport the finished goods from the factory to the store house. The path from the factory to the store house is through certain

intermediate stations as indicated in the following diagram. The company executive wants to identify the path with the shortest distance so as to minimize the transportation cost. The problem is to achieve this objective.



The shortest route technique can be used to minimize the total distance from a node designated as the **starting node or origin** to another node designated as the **final node**.

In the example under consideration, the origin is the factory and the final node is the store house.

### STEPS IN THE SHORTEST ROUTE TECHNIQUE

The procedure consists of starting with a set containing a node and enlarging the set by choosing a node in each subsequent step.

#### Step 1:

First, locate the origin. Then, find the node nearest to the origin. Mark the distance between the origin and the nearest node in a box by the side of that node.

In some cases, it may be necessary to check several paths to find the nearest node.

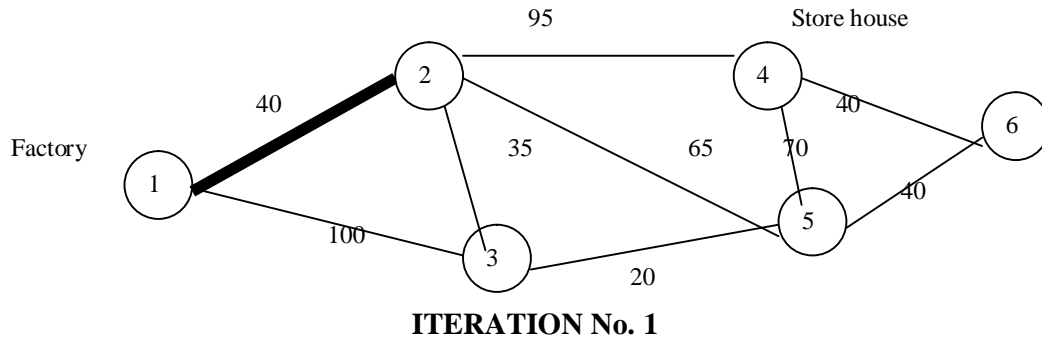
#### Step 2:

Repeat the above process until the nodes in the entire network have been accounted for. The last distance placed in a box by the side of the ending node will be the distance of the shortest route. We note that the distances indicated in the boxes by each node constitute the shortest route to that node. These distances are used as intermediate results in determining the next nearest node.

### SOLUTION FOR THE EXAMPLE PROBLEM

Looking at the diagram, we see that node 1 is the origin and the nodes 2 and 3 are neighbours to the origin. Among the two nodes, we see that node 2 is at a distance of 40 units from node 1 whereas node 3 is at a distance of 100 units from node 1. The minimum of {40, 100} is 40. Thus, the node nearest to the origin is node 2, with a distance of 40 units. So, out of the two nodes 2 and 3, we select node 2. We form a set of nodes {1, 2} and construct a path connecting the node 2 with node 1 by a thick line and mark the distance of 40 in a box by the side of node 2. This first iteration is shown in the following diagram.

40
----



Now we search for the next node nearest to the set of nodes  $\{1, 2\}$ . For this purpose, consider those nodes which are neighbours of either node 1 or node 2. The nodes 3, 4 and 5 fulfill this condition. We calculate the following distances.

The distance between nodes 1 and 3 = 100.

The distance between nodes 2 and 3 = 35.

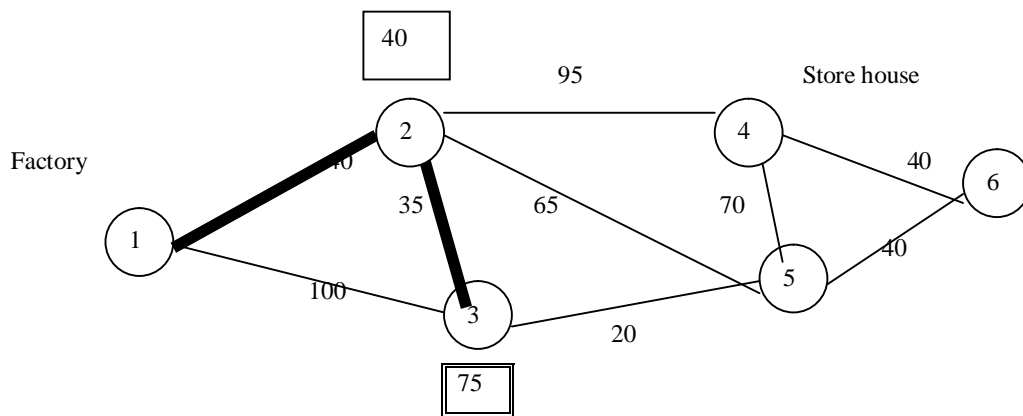
The distance between nodes 2 and 4 = 95.

The distance between nodes 2 and 5 = 65.

Minimum of  $\{100, 35, 95, 65\} = 35$ .

Therefore, node 3 is the nearest one to the set  $\{1, 2\}$ . In view of this observation, the set of nodes is enlarged from  $\{1, 2\}$  to  $\{1, 2, 3\}$ . For the set  $\{1, 2, 3\}$ , there are two possible paths, viz. Path  $1 \rightarrow 2 \rightarrow 3$  and Path  $1 \rightarrow 3 \rightarrow 2$ . The Path  $1 \rightarrow 2 \rightarrow 3$  has a distance of  $40 + 35 = 75$  units while the Path  $1 \rightarrow 3 \rightarrow 2$  has a distance of  $100 + 35 = 135$  units.

Minimum of  $\{75, 135\} = 75$ . Hence we select the path  $1 \rightarrow 2 \rightarrow 3$  and display this path by thick edges. The distance 75 is marked in a box by the side of node 3. We obtain the following diagram at the end of Iteration No. 2.

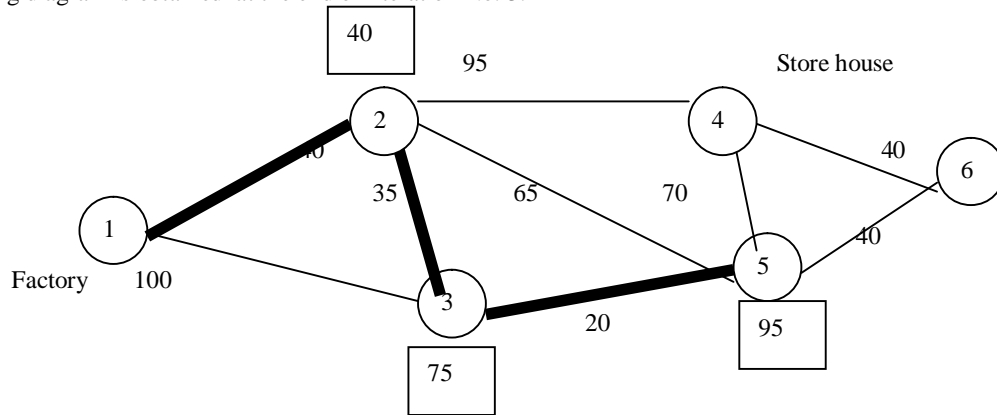


### REPEATING THE PROCESS

We repeat the process. The next node nearest to the set  $\{1, 2, 3\}$  is either node 4 or node 5.

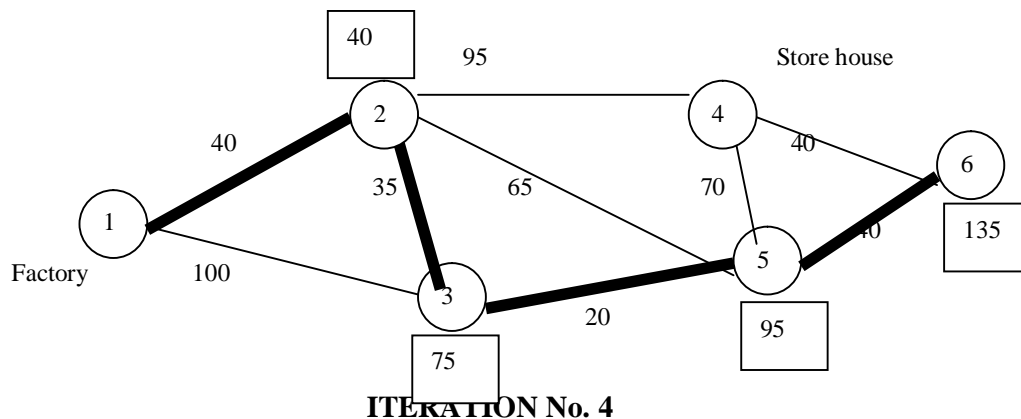
Node 4 is at a distance of 95 units from node 2 while node 2 is at a distance of 40 units from node 1. Thus, node 4 is at a distance of  $95 + 40 = 135$  units from the origin.

As regards node 5, there are two paths viz.  $2 \rightarrow 5$  and  $3 \rightarrow 5$ , providing a link to the origin. We already know the shortest routes from nodes 2 and 3 to the origin. The minimum distances have been indicated in boxes near these nodes. The path  $3 \rightarrow 5$  involves the shortest distance. Thus, the distance between nodes 1 and 5 is 95 units (20 units between nodes 5 and 3 + 75 units between node 3 and the origin). Therefore, we select node 5 and enlarge the set from  $\{1, 2, 3\}$  to  $\{1, 2, 3, 5\}$ . The distance 95 is marked in a box by the side of node 5. The following diagram is obtained at the end of Iteration No. 3.



### ITERATION No. 3

Now 2 nodes remain, viz., nodes 4 and 6. Among them, node 4 is at a distance of 135 units from the origin (95 units from node 4 to node 2 + 40 units from node 2 to the origin). Node 6 is at a distance of 135 units from the origin (40 + 95 units). Therefore, nodes 4 and 6 are at equal distances from the origin. If we choose node 4, then travelling from node 4 to node 6 will involve an additional distance of 40 units. However, node 6 is the ending node. Therefore, we select node 6 instead of node 4. Thus the set is enlarged from  $\{1, 2, 3, 5\}$  to  $\{1, 2, 3, 5, 6\}$ . The distance 135 is marked in a box by the side of node 6. Since we have got a path beginning from the start node and terminating with the stop node, we see that the solution to the given problem has been obtained. We have the following diagram at the end of Iteration No. 4.



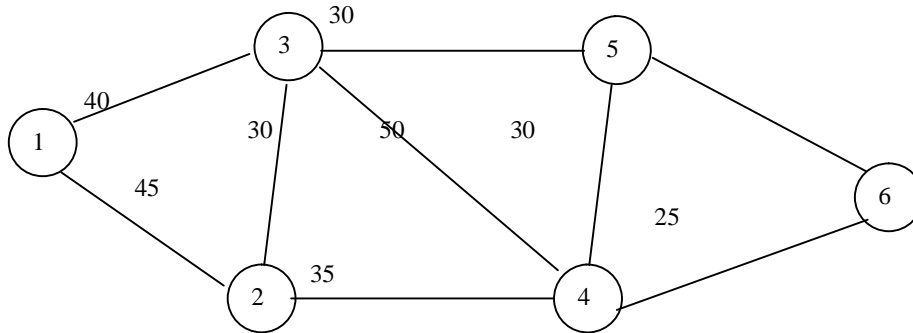
### ITERATION No. 4

## MINIMUM DISTANCE

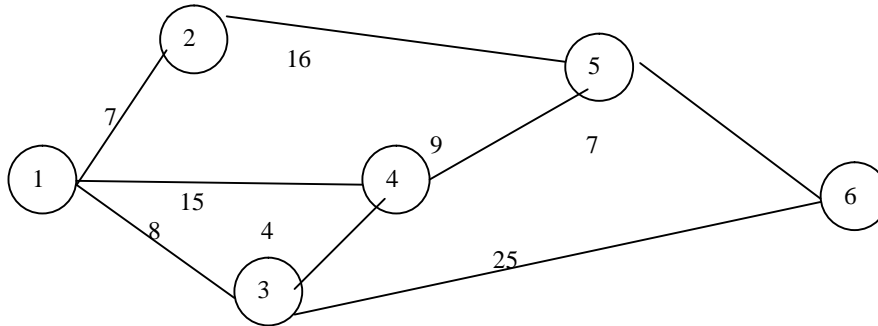
Referring to the above diagram, we see that **the shortest route** is provided by the path  $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6$  with a minimum distance of 135 units.

## QUESTIONS

1. Explain the shortest path problem.
2. Explain the algorithm for a shortest path problem
3. Find the shortest path of the following network:



4. Determine the shortest path of the following network:



## LESSON 2

### MINIMUM SPANNING TREE PROBLEM

#### LESSON OUTLINE

- The description of a minimum spanning tree problem.
- The identification of the minimum spanning tree.

#### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand a minimum spanning tree problem
- understand the algorithm for minimum spanning tree problem
- locate the minimum spanning tree



- carry out numerical problems

**Tree:** A minimally connected network is called a tree. If there are  $n$  nodes in a network, it will be a tree if the number of edges =  $n-1$ .

### Minimum spanning tree algorithm

**Problem :** Given a connected network with weights assigned to the edges, it is required to find out a tree whose nodes are the same as those of the network.

The weight assigned to an edge may be regarded as the distance between the two nodes with which the edge is incident.

#### Algorithm:

The problem can be solved with the help of the following algorithm.

The procedure consists of selection of a node at each step.

**Step 1:** First select any node in the network. This can be done arbitrarily. We will start with this node.

**Step 2:** Connect the selected node to the nearest node.

**Step 3:** Consider the nodes that are now connected. Consider the remaining nodes. If there is no node remaining, then stop. On the other hand, if some nodes remain, among them find out which one is nearest to the nodes that are already connected. Select this node and go to Step 2.

Thus the method involves the repeated application of Steps 2 and 3. Since the number of nodes in the given network is finite, the process will end after a finite number of steps. The algorithm will terminate with step 3.

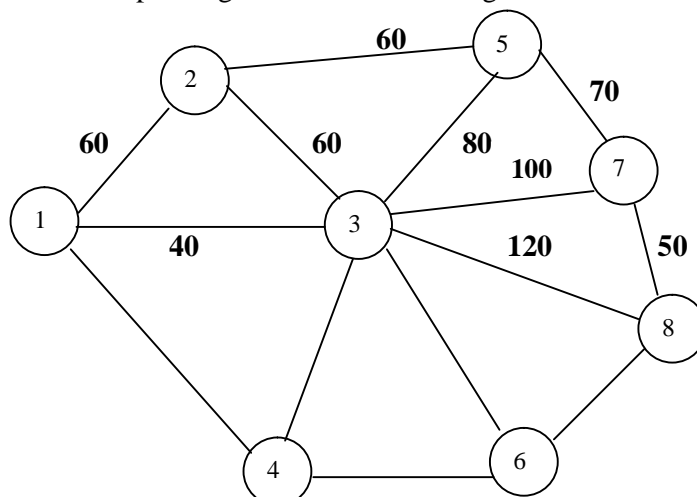
#### How to break ties:

While applying the above algorithm, if some nodes remain in step 3 and if there is a tie in the nearest node, then the tie can be broken arbitrarily.

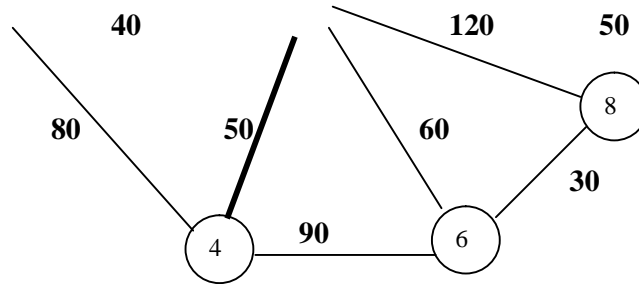
As a consequence of tie, we may end up with more than one optimal solution.

#### Problem 1:

Determine the minimum spanning tree for the following network.





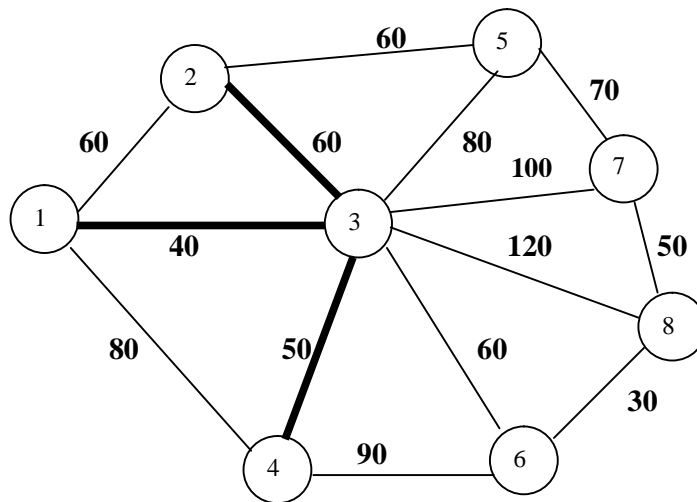


**Iteration No. 2**

Next go to step 3.

Now the connected nodes are 1, 3 and 4. The remaining nodes are 2, 5, 6, 7 and 8. Node 2 is at a distance of 60 from node 1. Nodes 5, 6, 7 and 8 are not adjacent to node 1. All of the nodes 2, 5, 6, 7 and 8 are adjacent to node 3. Among them, nodes 2 and 6 are nearer to node 3, with equal distance of 60.

Node 6 is adjacent to node 4, at a distance of 90. Now there is a tie between nodes 2 and 6. The tie can be broken arbitrarily. So we select node 2. Connect node 3 to Node 2 by a thick line. This is Iteration No. 3.

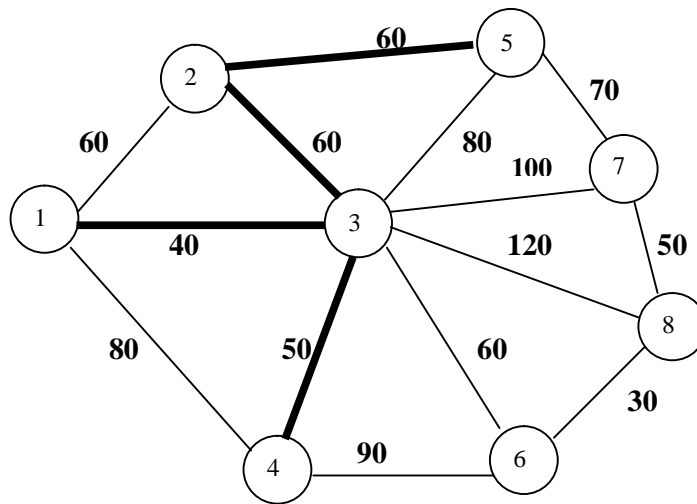


**Iteration No. 3**

We continue the above process.

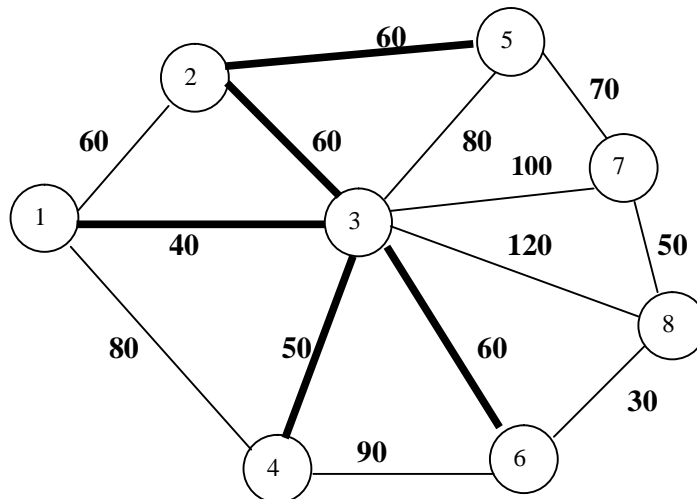
Now nodes 1, 2, 3 and 4 are connected. The remaining nodes are 5, 6, 7 and 8. None of them is adjacent to node 1. Node 5 is adjacent to node 2 at a distance of 60. Node 6 is at a distance of 60 from node 3. Node 6 is at a distance of 90 from node 4. There is a tie between

nodes 5 and 6. We select node 5. Connect node 2 to node 5 by a thick line. This is Iteration No. 4.



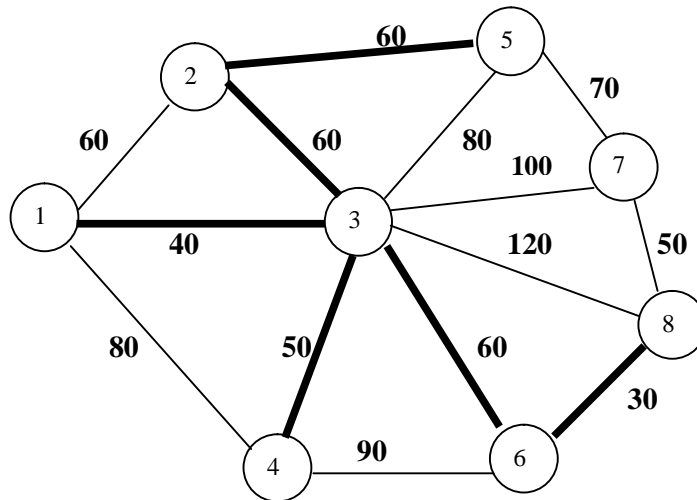
**Iteration No. 4**

Now nodes 1, 2, 3, 4 and 5 are connected. The remaining nodes are 6, 7 and 8. Among them, node 6 is at the shortest distance of 60 from node 3. So, connect node 3 to node 6 by a thick line. This is Iteration No. 5.



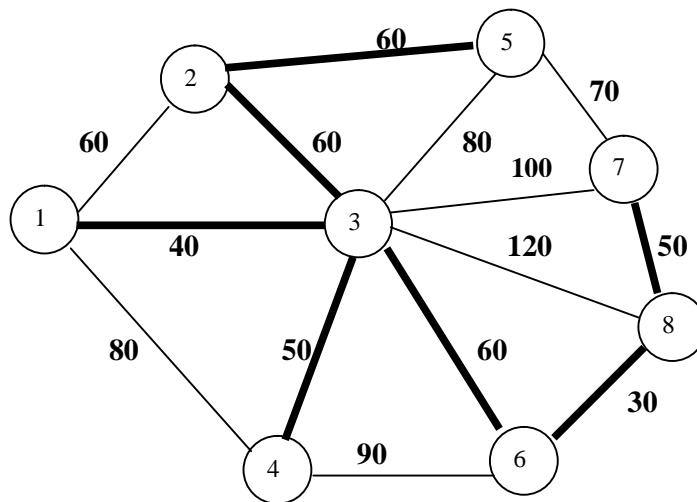
**Iteration No. 5**

Now nodes 1, 2, 3, 4, 5 and 6 are connected. The remaining nodes are 7 and 8. Among them, node 8 is at the shortest distance of 30 from node 6. Consequently we connect node 6 to node 8 by a thick line. This is Iteration No. 6.



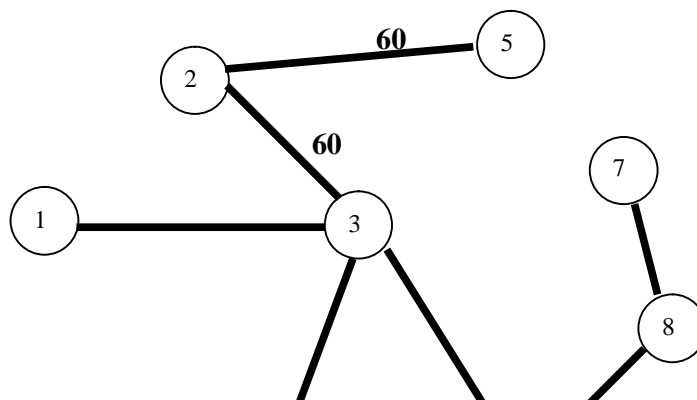
**Iteration No. 6**

Now nodes 1, 2, 3, 4, 5, 6 and 8 are connected. The remaining node is 7. It is at the shortest distance of 50 from node 8. So, connect node 8 to node 7 by a thick line. This is Iteration No.7.



**Iteration No. 7**

Now all the nodes 1, 2, 3, 4, 5, 6, 7 and 8 are connected by seven thick lines. Since no node is remaining, we have reached the stopping condition. Thus we obtain the following minimum spanning tree for the given network.



40

50

50

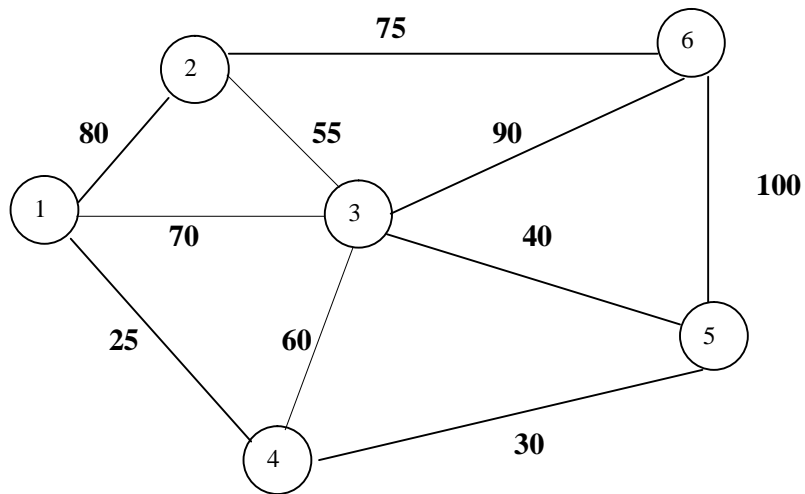
60

30

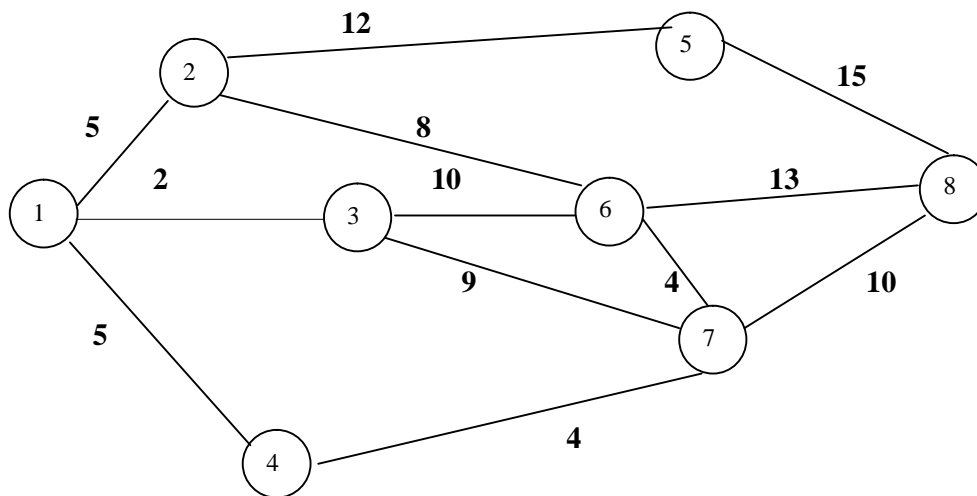
### Minimum Spanning Tree

#### QUESTIONS

1. Explain the minimum spanning tree algorithm.
2. From the following network, find the minimum spanning tree.



3. Find the minimum spanning tree of the following network:





## LESSON 3 PROJECT NETWORK

### LESSON OUTLINE

- The key concepts
- Construction of project network diagram

### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

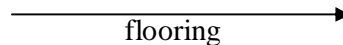
- understand the definitions of important terms
- understand the development of project network diagram
- work out numerical problems

### KEY CONCEPTS

Certain key concepts pertaining to a project network are described below:

#### 1. Activity

An activity means a work. A project consists of several activities. An activity takes time. It is represented by an arrow in a diagram of the network. For example, an activity in house construction can be flooring. This is represented as follows:



Construction of a house involves various activities. Flooring is an activity in this project. We can say that a project is completed only when all the activities in the project are completed.

#### 2. Event

It is the beginning or the end of an activity. Events are represented by circles in a project network diagram. The events in a network are called the **nodes**. **Example:**



Starting a punching machine is an activity. Stopping the punching machine is another activity.

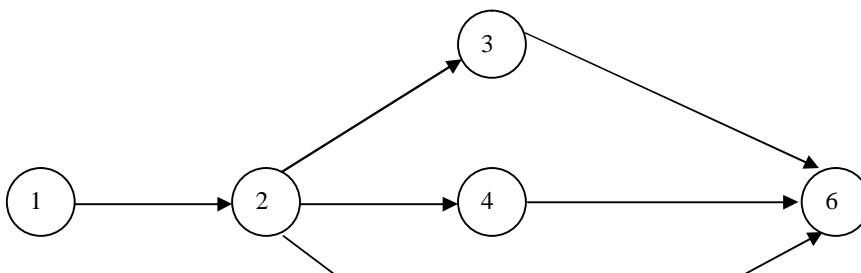
#### 3. Predecessor Event

The event just before another event is called the predecessor event.

#### 4. Successor Event

The event just following another event is called the successor event.

**Example:** Consider the following.





In this diagram, event 1 is predecessor for the event 2.

Event 2 is successor to event 1.

Event 2 is predecessor for the events 3, 4 and 5.

Event 4 is predecessor for the event 6.

Event 6 is successor to events 3, 4 and 5.

### 5. Network

A network is a series of related activities and events which result in an end product or service. The activities shall follow a prescribed sequence. For example, while constructing a house, laying the foundation should take place before the construction of walls. Fitting water tapes will be done towards the completion of the construction. Such a sequence cannot be altered.

### 6. Dummy Activity

A dummy activity is an activity which does not consume any time. Sometimes, it may be necessary to introduce a dummy activity in order to provide connectivity to a network or for the preservation of the logical sequence of the nodes and edges.

### 7. Construction of a Project Network

A project network consists of a finite number of events and activities, by adhering to a certain specified sequence. There shall be a **start event** and an **end event (or stop event)**. All the other events shall be between the start and the end events. The activities shall be marked by directed arrows. An activity takes the project from one event to another event.

An event takes place at a point of time whereas an activity takes place from one point of time to another point of time.

## CONSTRUCTION OF PROJECT NETWORK DIAGRAMS

### Problem 1:

Construct the network diagram for a project with the following activities:

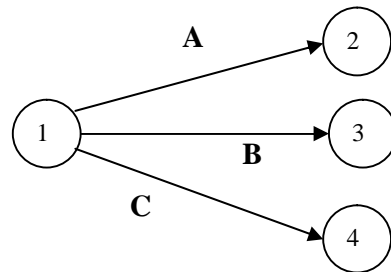
Activity	Name of	Immediate
----------	---------	-----------

Event→Event	Activity	Predecessor Activity
1→2	A	-
1→3	B	-
1→4	C	-
2→5	D	A
3→6	E	B
4→6	F	C
5→6	G	D

**Solution:**

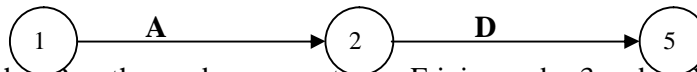
The start event is node 1.

The activities A, B, C start from node 1 and none of them has a predecessor activity. A joins nodes 1 and 2; B joins nodes 1 and 3; C joins nodes 1 and 4. So we get the following:

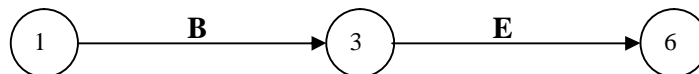


This is a part of the network diagram that is being constructed.

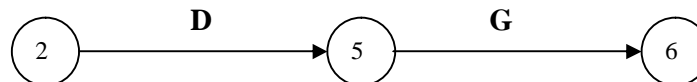
Next, activity D has A as the predecessor activity. D joins nodes 2 and 5. So we get



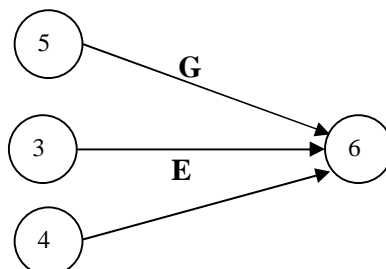
Next, activity E has B as the predecessor activity. E joins nodes 3 and 6. So we get



Next, activity G has D as the predecessor activity. G joins nodes 5 and 6. Thus we obtain



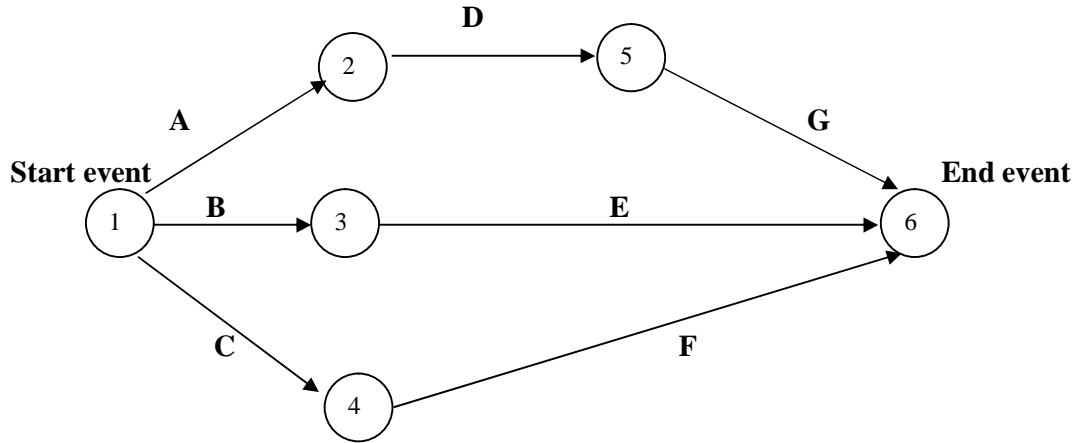
Since activities E, F, G terminate in node 6, we get



**F**

6 is the end event.

Combining all the pieces together, the following network diagram is obtained for the given project:



We validate the diagram by checking with the given data.

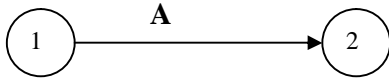
**Problem 2:**

Develop a network diagram for the project specified below:

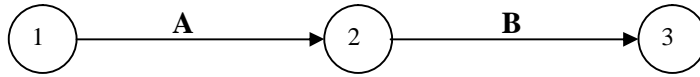
Activity	Immediate Predecessor Activity
A	-
B	A
C, D	B
E	C
F	D
G	E, F

**Solution:**

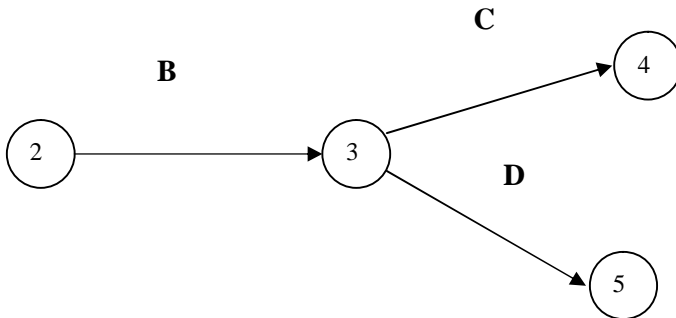
Activity A has no predecessor activity. i.e., It is the first activity. Let us suppose that activity A takes the project from event 1 to event 2. Then we have the following representation for A:



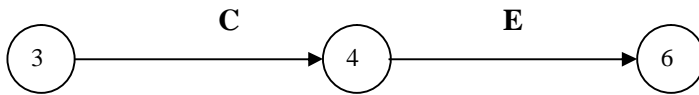
For activity B, the predecessor activity is A. Let us suppose that B joins nodes 2 and 3. Thus we get



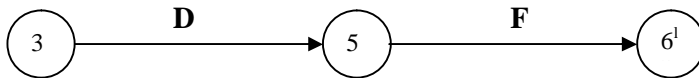
Activities C and D have B as the predecessor activity. Therefore we obtain the following:



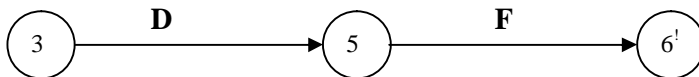
Activity E has D as the predecessor activity. So we get



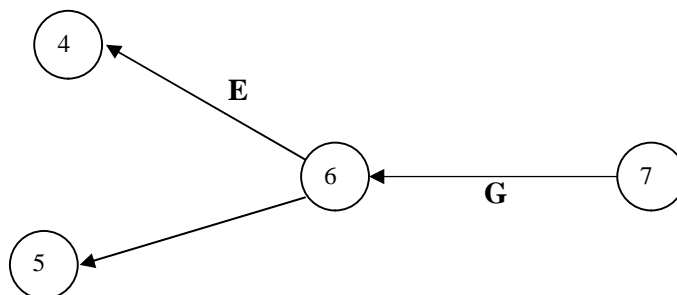
Activity F has D as the predecessor activity. So we get



Activity G has E and F as predecessor activities. This is possible only if nodes 6 and 6<sup>l</sup> are one and the same. So, rename node 6<sup>l</sup> as node 6. Then we get



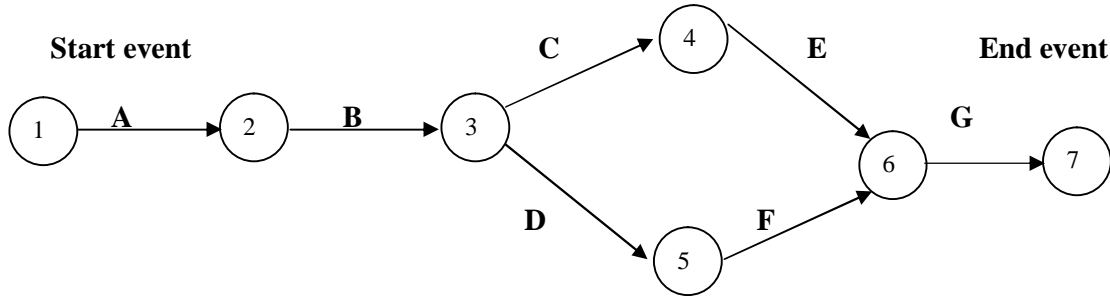
and



**F**

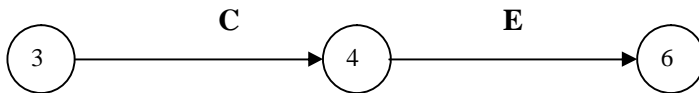
G is the last activity.

Putting all the pieces together, we obtain the following diagram the project network:

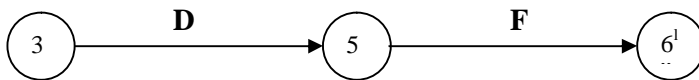


The diagram is validated by referring to the given data.

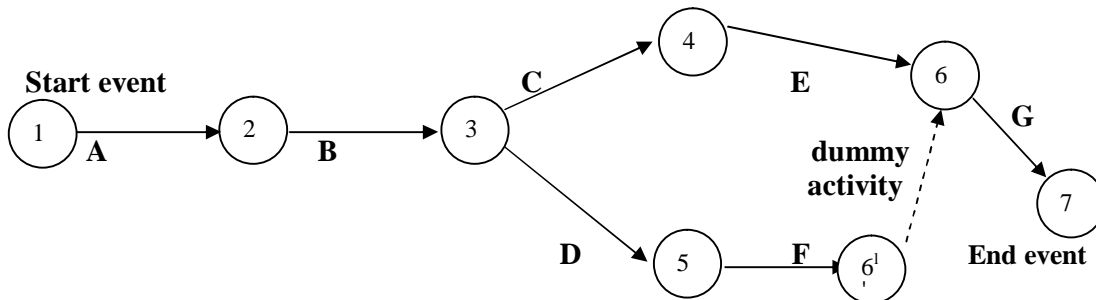
**Note:** An important point may be observed for the above diagram. Consider the following parts in the diagram



and



We took nodes 6 and 6<sup>1</sup> as one and the same. Instead, we can retain them as different nodes. Then, in order to provide connectivity to the network, we join nodes 6<sup>1</sup> and 6 by a dummy activity. Then we arrive at the following diagram for the project network:



**QUESTIONS:**

1. Explain the terms: event, predecessor event, successor event, activity, dummy activity, network.
2. Construct the network diagram for the following project:

Activity	Immediate Predecessor Activity
A	-
B	-
C	A
D	B
E	A
F	C, D
G	E
H	E
I	F, G
J	H, I

## LESSON 4

### CRITICAL PATH METHOD (CPM)

#### LESSON OUTLINE

- The concepts of critical path and critical activities
- Location of the critical path
- Evaluation of the project completion time

#### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the definitions of critical path and critical activities
- identify critical path and critical activities
- determine the project completion time

#### INTRODUCTION

The critical path method (CPM) aims at the determination of the time to complete a project and the important activities on which a manager shall focus attention.

#### ASSUMPTION FOR CPM

In CPM, it is assumed that precise time estimate is available for each activity.

#### PROJECT COMPLETION TIME

From the start event to the end event, the time required to complete all the activities of the project in the specified sequence is known as the project completion time.

#### PATH IN A PROJECT

A continuous sequence, consisting of nodes and activities alternatively, beginning with the start event and stopping at the end event of a network is called a path in the network.

#### CRITICAL PATH AND CRITICAL ACTIVITIES

Consider all the paths in a project, beginning with the start event and stopping at the end event. For each path, calculate the time of execution, by adding the time for the individual activities in that path.

The path with the largest time is called the **critical path** and the activities along this path are called the **critical activities** or **bottleneck activities**. The activities are called critical

because they cannot be delayed. However, a non-critical activity may be delayed to a certain extent. Any delay in a critical activity will delay the completion of the whole project. However, a certain permissible delay in a non –critical activity will not delay the completion of the whole project. It shall be noted that delay in a non-critical activity beyond a limit would certainly delay the completion the whole project. Sometimes, there may be several critical paths for a project. A project manager shall pay special attention to critical activities.

**Problem 1:**

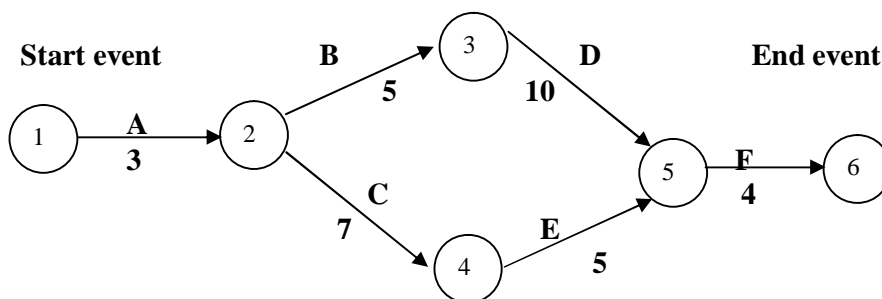
The following details are available regarding a project:

Activity	Predecessor Activity	Duration (Weeks)
A	-	3
B	A	5
C	A	7
D	B	10
E	C	5
F	D,E	4

Determine the critical path, the critical activities and the project completion time.

**Solution:**

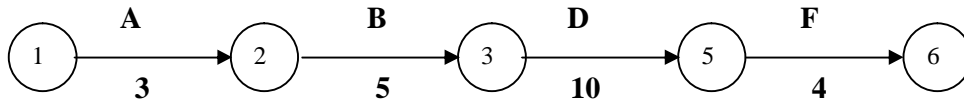
First let us construct the network diagram for the given project. We mark the time estimates along the arrows representing the activities. We obtain the following diagram:



Consider the paths, beginning with the start node and stopping with the end node. There are two such paths for the given project. They are as follows:

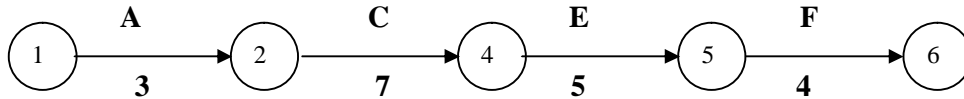
**Path I**





with a time of  $3 + 5 + 10 + 4 = 22$  weeks.

**Path II**



with a time of  $3 + 7 + 5 + 4 = 19$  weeks.

Compare the times for the two paths. Maximum of  $\{22, 19\} = 22$ . We see that path I has the maximum time of 22 weeks. Therefore, path I is the critical path. The critical activities are A, B, D and F. The project completion time is 22 weeks.

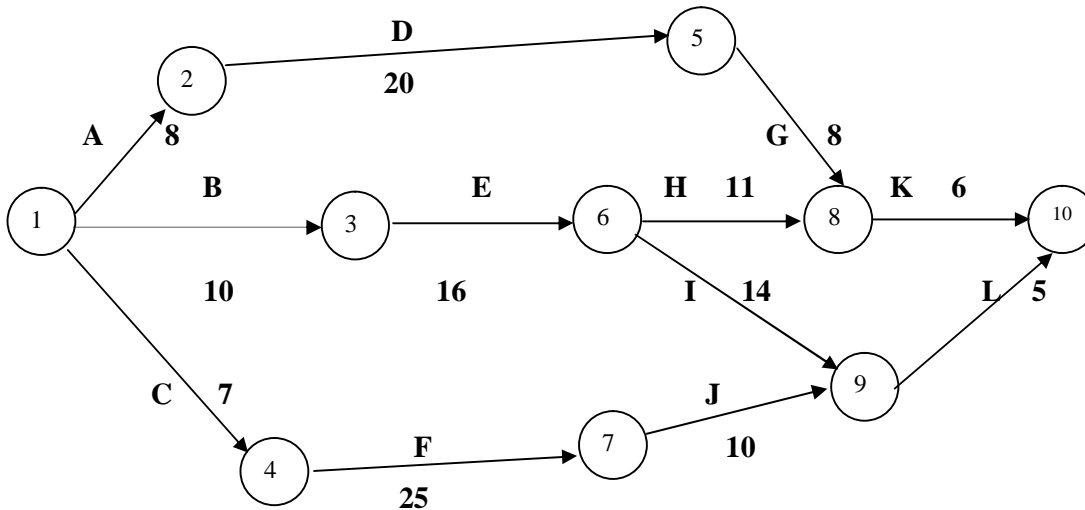
We notice that C and E are non-critical activities.

Time for path I - Time for path II =  $22 - 19 = 3$  weeks.

Therefore, together the non-critical activities can be delayed upto a maximum of 3 weeks, without delaying the completion of the whole project.

**Problem 2:**

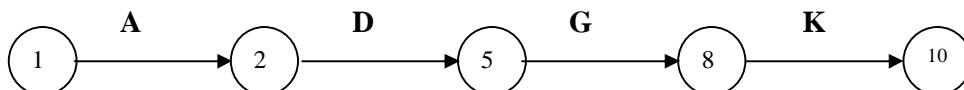
Find out the completion time and the critical activities for the following project:



**Solution:**

In all, we identify 4 paths, beginning with the start node of 1 and terminating at the end node of 10. They are as follows:

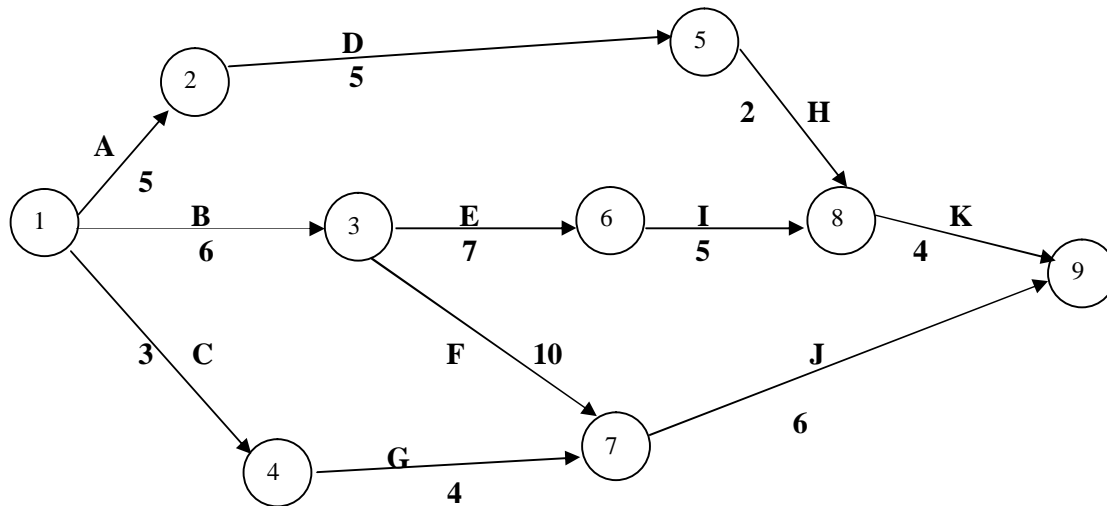
**Path I**





1-4	3
2-5	5
3-6	7
3-7	10
4-7	4
5-8	2
6-8	5
7-9	6
8-9	4

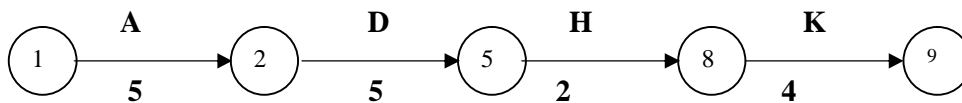
**Solution:** We have the following network diagram for the project:



**Solution:**

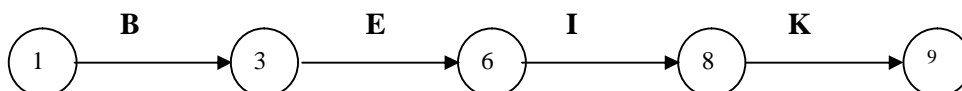
We assert that there are 4 paths, beginning with the start node of 1 and terminating at the end node of 9. They are as follows:

**Path I**



Time for the path =  $5 + 5 + 2 + 4 = 16$  weeks.

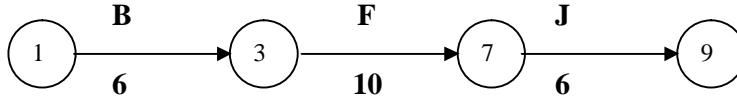
**Path II**



6                      7                      5                      4

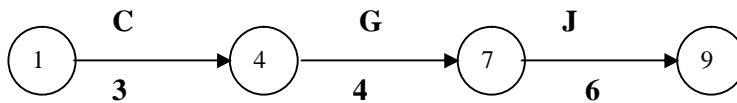
Time for the path =  $6 + 7 + 5 + 4 = 22$  weeks.

**Path III**



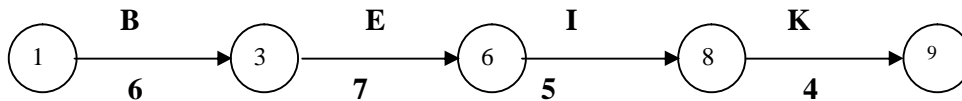
Time for the path =  $6 + 10 + 6 = 16$  weeks.

**Path IV**



Time for the path =  $3 + 4 + 6 = 13$  weeks.

Compare the times for the four paths. Maximum of  $\{16, 22, 16, 13\} = 22$ . We see that the following path has the maximum time and so it is the critical path:



The critical activities are B, E, I and K. The non-critical activities are A, C, D, F, G, H and J.

The project completion time is 22 weeks.

**QUESTIONS:**

1. Explain the terms: critical path, critical activities.
2. The following are the time estimates and the precedence relationships of the activities in a project network:

Activity	IMMEDIATE Predecessor Activity	time estimate (weeks)
A	-	4
B	-	7
C	-	3
D	A	6
E	B	4
F	B	7
G	C	6
H	E	10
I	D	3
J	F, G	4
K	H, I	2

Draw the project network diagram. Determine the critical path and the project completion time.

## LESSON 5

### PERT

#### LESSON OUTLINE

- The concept of PERT
- Estimates of the time of an activity
- Determination of critical path
- Probability estimates

#### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the importance of PERT
- locate the critical path
- determine the project completion time
- find out the probability of completion of a project before a stipulated time

#### INTRODUCTION

Programme Evaluation and Review Technique (PERT) is a tool that would help a project manager in project planning and control. It would enable him in continuously monitoring a project and taking corrective measures wherever necessary. This technique involves statistical methods.

#### ASSUMPTIONS FOR PERT

Note that in CPM, the assumption is that precise time estimate is available for each activity in a project. However, one finds most of the times that this is not practically possible.

In PERT, we assume that it is not possible to have precise time estimate for each activity and instead, probabilistic estimates of time alone are possible. A multiple time estimate approach is followed here. In probabilistic time estimate, the following 3 types of estimate are possible:

1. Pessimistic time estimate ( $t_p$ )
2. Optimistic time estimate ( $t_o$ )
3. Most likely time estimate ( $t_m$ )

The optimistic estimate of time is based on the assumption that an activity will not involve any difficulty during execution and it can be completed within a short period. On the other hand, a pessimistic estimate is made on the assumption that there would be unexpected

problems during the execution of an activity and hence it would consume more time. The most likely time estimate is made in between the optimistic and the pessimistic estimates of time. Thus the three estimates of time have the relationship

$$t_o \leq t_m \leq t_p.$$

Practically speaking, neither the pessimistic nor the optimistic estimate may hold in reality and it is the most likely time estimate that is expected to prevail in almost all cases. Therefore, it is preferable to give more weight to the most likely time estimate.

We give a weight of 4 to most likely time estimate and a weight of 1 each to the pessimistic and optimistic time estimates. We arrive at a time estimate ( $t_e$ ) as the weighted average of these estimates as follows:

$$t_e = \frac{t_o + 4 t_m + t_p}{6}$$

Since we have taken 6 units ( 1 for  $t_p$ , 4 for  $t_m$  and 1 for  $t_o$  ), we divide the sum by 6. With this time estimate, we can determine the project completion time as applicable for CPM.

Since PERT involves the average of three estimates of time for each activity, this method is very practical and the results from PERT will be have a reasonable amount of reliability.

### MEASURE OF CERTAINTY

The 3 estimates of time are such that

$$t_o \leq t_m \leq t_p.$$

Therefore the range for the time estimate is  $t_p - t_o$ .

The time taken by an activity in a project network follows a distribution with a standard deviation of one sixth of the range, approximately.

i.e., The standard deviation =  $\sigma = \frac{t_p - t_o}{6}$

and the variance =  $\sigma^2 = \left( \frac{t_p - t_o}{6} \right)^2$

The certainty of the time estimate of an activity can be analysed with the help of the variance. The greater the variance, the more uncertainty in the time estimate of an activity.

#### Problem 1:

Two experts A and B examined an activity and arrived at the following time estimates.

Expert	Time Estimate
--------	---------------

	$t_o$	$t_m$	$t_p$
A	4	6	8
B	4	7	10

Determine which expert is more certain about his estimates of time:

**Solution:**

$$\text{Variance } (\sigma^2) \text{ in time estimates} = \left( \frac{t_p - t_o}{6} \right)^2$$

$$\text{In the case of expert A, the variance} = \left( \frac{8-4}{6} \right)^2 = \frac{4}{9}$$

$$\text{As regards expert B, the variance} = \left( \frac{10-4}{6} \right)^2 = 1$$

So, the variance is less in the case of A. Hence, it is concluded that the expert A is more certain about his estimates of time.

### Determination of Project Completion Time in PERT

**Problem 2:**

Find out the time required to complete the following project and the critical activities:

Activity	Predecessor Activity	Optimistic time estimate ( $t_o$ , days)	Most likely time estimate ( $t_m$ , days)	Pessimistic time estimate ( $t_p$ , days)
A	-	2	4	6
B	A	3	6	9
C	A	8	10	12
D	B	9	12	15
E	C	8	9	10
F	D, E	16	21	26
G	D, E	19	22	25
H	F	2	5	8
I	G	1	3	5

**Solution:**

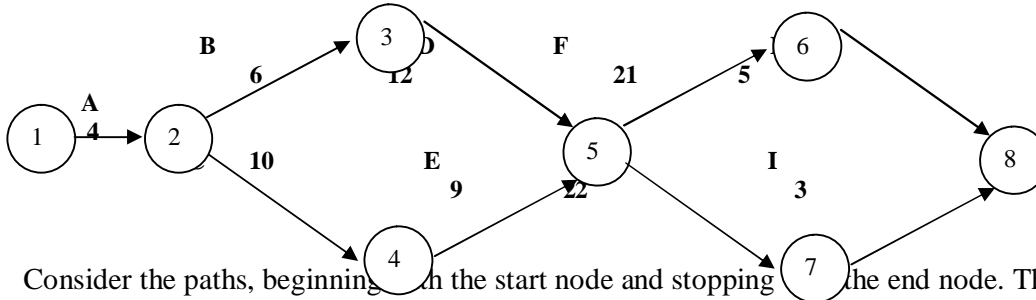
From the three time estimates  $t_p$ ,  $t_m$  and  $t_o$ , calculate  $t_e$  for each activity. We obtain the following table:

Activity	Optimistic time estimate ( $t_o$ )	4 x Most likely time estimate	Pessimistic time estimate ( $t_p$ )	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4t_m + t_p}{6}$
A	2	16	6	24	4
B	3	24	9	36	6
C	8	40	12	60	10
D	9	48	15	72	12
E	8	36	10	54	9
F	16	84	26	126	21



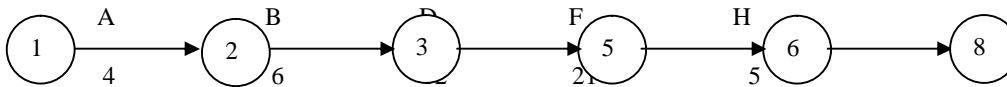
G	19	88	25	132	22
H	2	20	8	30	5
I	1	12	5	18	3

Using the single time estimates of the activities, we get the following network diagram for the project.



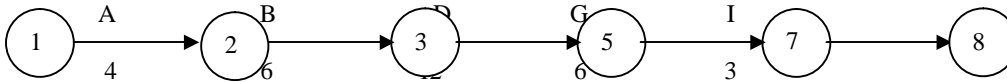
Consider the paths, beginning from the start node and stopping at the end node. There are four such paths for the given project. They are as follows:

**Path I**



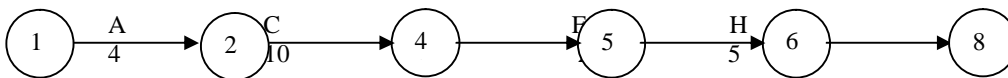
Time for the path:  $4+6+12+21+5 = 48$  days.

**Path II**



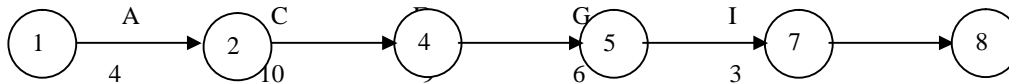
Time for the path:  $4+6+12+ 6+3 = 31$  days.

**Path III**



Time for the path:  $4+10+9+ 21+5 = 49$  days.

**Path IV**



Time for the path:  $4+10+9+ 6+3 = 32$  days.

Compare the times for the four paths.

Maximum of  $\{48, 31, 49, 32\} = 49$ .

We see that Path III has the maximum time.

Therefore the critical path is Path III. i.e.,  $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 8$ .

The critical activities are A, C, E, F and H.

The non-critical activities are B, D, G and I.

Project time (Also called project length) = 49 days.

**Problem 3:**

Find out the time, variance and standard deviation of the project with the following time estimates in weeks:

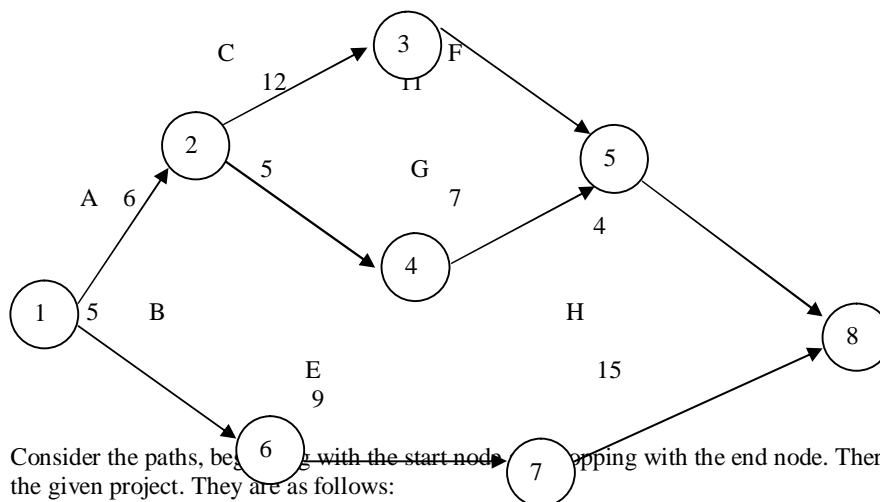
Activity	Optimistic time estimate ( $t_o$ )	Most likely time estimate ( $t_m$ )	Pessimistic time estimate ( $t_p$ )
1-2	3	6	9
1-6	2	5	8
2-3	6	12	18
2-4	4	5	6
3-5	8	11	14
4-5	3	7	11
6-7	3	9	15
5-8	2	4	6
7-8	8	16	18

**Solution:**

From the three time estimates  $t_p$ ,  $t_m$  and  $t_o$ , calculate  $t_e$  for each activity. We obtain the following table:

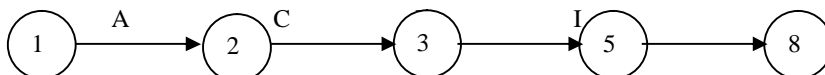
Activity	Optimistic time estimate ( $t_o$ )	4 x Most likely time estimate	Pessimistic time estimate ( $t_p$ )	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4t_m + t_p}{6}$
1-2	3	24	9	36	6
1-6	2	20	8	30	5
2-3	6	48	18	72	12
2-4	4	20	6	30	5
3-5	8	44	14	66	11
4-5	3	28	11	42	7
6-7	3	36	15	54	9
5-8	2	16	6	24	4
7-8	8	64	18	90	15

With the single time estimates of the activities, we get the following network diagram for the project.



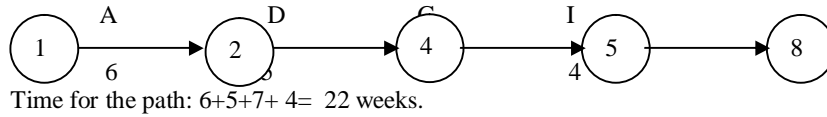
Consider the paths, beginning with the start node and ending with the end node. There are three such paths for the given project. They are as follows:

**Path I**

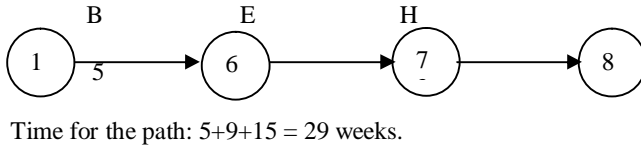


6                      12                      11                      4  
Time for the path:  $6+12+11+4 = 33$  weeks.

**Path II**



**Path III**



Compare the times for the three paths.  
Maximum of {33, 22, 29} = 33.  
It is noticed that Path I has the maximum time.  
Therefore the critical path is Path I. i.e.,  $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 8$   
The critical activities are A, C, F and I.  
The non-critical activities are B, D, G and H.  
Project time = 33 weeks.

**Calculation of Standard Deviation and Variance for the Critical Activities:**

Critical Activity	Optimistic time estimate ( $t_o$ )	Most likely time estimate ( $t_m$ )	Pessimistic time estimate ( $t_p$ )	Range ( $t_p - t_o$ )	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: 1→2	3	6	9	6	1	1
C: 2→3	6	12	18	12	2	4
F: 3→5	8	11	14	6	1	1
I: 5→8	2	4	6	4	2/3	4/9

Variance of project time (Also called Variance of project length) =  
Sum of the variances for the critical activities =  $1+4+1+4/9 = 58/9$  Weeks.  
Standard deviation of project time =  $\sqrt{\text{Variance}} = \sqrt{58/9} = 2.54$  weeks.

**Problem 4**

A project consists of seven activities with the following time estimates. Find the probability that the project will be completed in 30 weeks or less.

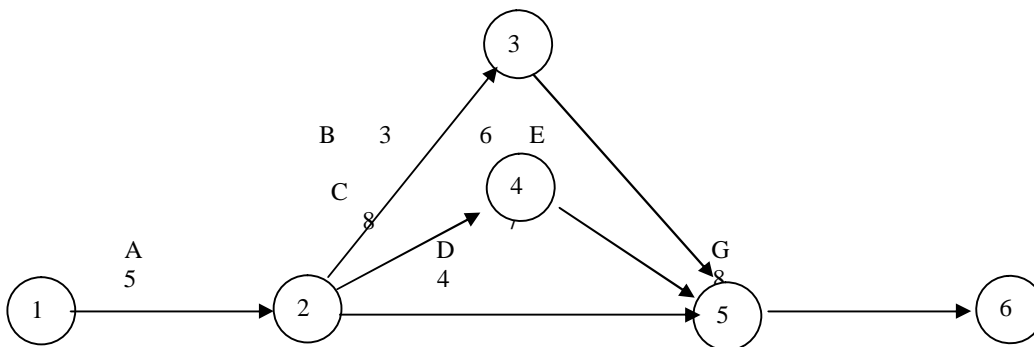
Activity	Predecessor Activity	Optimistic time estimate ( $t_o$ , days)	Most likely time estimate ( $t_m$ , days)	Pessimistic time estimate ( $t_p$ , days)
A	-	2	5	8
B	A	2	3	4
C	A	6	8	10
D	A	2	4	6
E	B	2	6	10
F	C	6	7	8
G	D, E, F	6	8	10

**Solution:**

From the three time estimates  $t_p$ ,  $t_m$  and  $t_o$ , calculate  $t_e$  for each activity. The results are furnished in the following table:

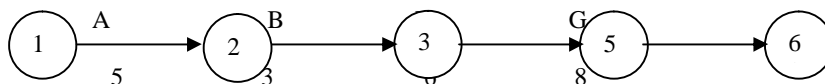
Activity	Optimistic time estimate ( $t_o$ )	4 x Most likely time estimate	Pessimistic time estimate ( $t_p$ )	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4t_m + t_p}{6}$
A	2	20	8	30	5
B	2	12	4	18	3
C	6	32	10	48	8
D	2	16	6	24	4
E	2	24	10	36	6
F	6	28	8	42	7
G	6	32	10	48	8

With the single time estimates of the activities, the following network diagram is constructed for the project.



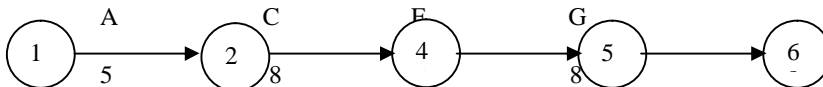
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

**Path I**



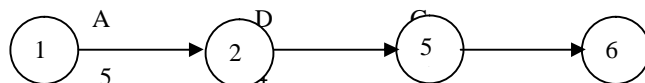
Time for the path:  $5+3+6+8 = 22$  weeks.

**Path II**



Time for the path:  $5+8+7+ 8 = 28$  weeks.

**Path III**



Time for the path:  $5+4+8 = 17$  weeks.

Compare the times for the three paths.

Maximum of  $\{22, 28, 17\} = 28$ .

It is noticed that Path II has the maximum time.

Therefore the critical path is Path II. i.e., 1 → 2 → 4 → 5 → 6.  
The critical activities are A, C, F and G.  
The non-critical activities are B, D and E.  
Project time = 28 weeks.

**Calculation of Standard Deviation and Variance for the Critical Activities:**

Critical Activity	Optimistic time estimate ( $t_o$ )	Most likely time estimate ( $t_m$ )	Pessimistic time estimate ( $t_p$ )	Range ( $t_p - t_o$ )	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: 1→2	2	5	8	6	1	1
C: 2→4	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$
F: 4→5	6	7	8	2	$\frac{1}{3}$	$\frac{1}{9}$
G: 5→6	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$

Standard deviation of the critical path =  $\sqrt{2} = 1.414$

The standard normal variate is given by the formula

$$Z = \frac{\text{Given value of } t - \text{Expected value of } t \text{ in the critical path}}{\text{SD for the critical path}}$$

So we get  $Z = \frac{30 - 28}{1.414} = 1.414$

We refer to the Normal Probability Distribution Table.  
Corresponding to  $Z = 1.414$ , we obtain the value of 0.4207  
We get  $0.5 + 0.4207 = 0.9207$   
Therefore the required probability is 0.92  
i.e., There is 92% chance that the project will be completed before 30 weeks. In other words, the chance that it will be delayed beyond 30 weeks is 8%

**QUESTIONS:**

1. Explain how time of an activity is estimated in PERT.
2. Explain the measure of certainty in PERT.
3. The estimates of time in weeks of the activities of a project are as follows:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	A	8	11	20
C	A	10	15	20
D	B	12	18	24

E	C	8	13	24
F	C	4	7	16
G	D,F	14	18	28
H	E	10	12	14
I	G,H	7	10	19

Determine the critical activities and the project completion time.

4. Draw the network diagram for the following project. Determine the time, variance and standard deviation of the project.:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	12	14	22
B	-	16	17	24
C	A	14	15	16
D	A	13	18	23
E	B	16	18	20
F	D,E	13	14	21
G	C,F	6	8	10

5. Consider the following project with the estimates of time in weeks:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	-	3	5	7
C	A	5	6	13
D	A	4	8	12
E	B,C	5	6	13
F	D,E	6	8	14

Find the probability that the project will be completed in 27 weeks.

**NORMAL DISTRIBUTION TABLE**

Area Under Standard Normal Distribution

	<b>0.00</b>	<b>0.01</b>	<b>0.02</b>	<b>0.03</b>	<b>0.04</b>	<b>0.05</b>	<b>0.06</b>	<b>0.07</b>	<b>0.08</b>	<b>0.09</b>
<b>0.0</b>	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
<b>0.1</b>	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
<b>0.2</b>	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
<b>0.3</b>	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
<b>0.4</b>	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
<b>0.5</b>	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
<b>0.6</b>	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
<b>0.7</b>	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
<b>0.8</b>	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
<b>0.9</b>	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
<b>1.0</b>	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
<b>1.1</b>	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
<b>1.2</b>	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
<b>1.3</b>	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
<b>1.4</b>	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
<b>1.5</b>	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
<b>1.6</b>	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
<b>1.7</b>	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
<b>1.8</b>	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
<b>1.9</b>	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
<b>2.0</b>	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
<b>2.1</b>	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
<b>2.2</b>	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
<b>2.3</b>	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
<b>2.4</b>	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
<b>2.5</b>	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
<b>2.6</b>	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
<b>2.7</b>	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
<b>2.8</b>	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
<b>2.9</b>	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
<b>3.0</b>	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

**LESSON 6**

**EARLIEST AND LATEST TIMES**



## LESSON OUTLINE

- The concepts of earliest and latest times
- The concept of slack
- Numerical problems

## LEARNING OBJECTIVES

### *After reading this lesson you should be able to*

- understand the concepts of earliest and latest times
- understand the concept of slack
- calculate the earliest and latest times
- find out the slacks
- identify the critical activities
- carry out numerical problems

## INTRODUCTION

A project manager has the responsibility to see that a project is completed by the stipulated date, without delay. Attention is focused on this aspect in what follows.

### **Key concepts**

Certain key concepts are introduced below.

### **EARLIEST TIMES OF AN ACTIVITY**

We can consider (i) Earliest Start Time of an activity and (ii) Earliest Finish Time of an activity.

Earliest Start Time of an activity is the earliest possible time of starting that activity on the condition that all the other activities preceding to it were began at the earliest possible times.

Earliest Finish Time of an activity is the earliest possible time of completing that activity. It is given by the formula.

The Earliest Finish Time of an activity = The Earliest Start Time of the activity + The estimated duration to carry out that activity.

### **LATEST TIMES OF AN ACTIVITY**

We can consider (i) Latest Finish Time of an activity and (ii) Latest Start Time of an activity.

Latest Finish Time of an activity is the latest possible time of completing that activity on the condition that all the other activities succeeding it are carried out as per the plan of the management and without delaying the project beyond the stipulated time.

Latest Start Time of an activity is the latest possible time of beginning that activity. It is given by the formula

Latest Start Time of an activity = The Latest Finish Time of the activity - The estimated duration to carry out that activity.

### **TOTAL FLOAT OF AN ACTIVITY**

Float seeks to measure how much delay is acceptable. It sets up a control limit for delay.

The total float of an activity is the time by which that activity can be delayed without delaying the whole project. It is given by the formula

$$\text{Total Float of an Activity} = \text{Latest Finish Time of the activity} - \text{Earliest Finish Time of that activity.}$$

It is also given by the formula

$$\text{Total Float of an Activity} = \text{Latest Start Time of the activity} - \text{Earliest Start Time of that activity.}$$

Since a delay in a critical activity will delay the execution of the whole project, the total float of a critical activity must be zero.

### **EXPECTED TIMES OF AN EVENT**

An event occurs at a point of time. We can consider (i) Earliest Expected Time of Occurrence of an event and (ii) Latest Allowable Time of Occurrence an event.

The Earliest Expected Time of Occurrence of an event is the earliest possible time of expecting that event to happen on the condition that all the preceding activities have been completed.

The Latest Allowable Time of Occurrence of an event is the latest possible time of expecting that event to happen without delaying the project beyond the stipulated time.

### **PROCEDURE TO FIND THE EARLIEST EXPECTED TIME OF AN EVENT**

**Step 1.** Take the Earliest Expected Time of Occurrence of the Start Event as zero.

**Step 2.** For an event other than the Start Event, find out all paths in the network which connect the Start node with the node representing the event under consideration.

**Step 3.** In the “**Forward Pass**” (i.e., movement in the network from left to right), find out the sum of the time durations of the activities in each path identified in Step 2.

**Step 4.** The path with the longest time in Step 3 gives the Earliest Expected Time of Occurrence of the event

#### **Working Rule for finding the earliest expected time of an event:**

For an event under consideration, locate all the predecessor events and identify their earliest expected times. With the earliest expected time of each event, add the time duration of the activity connecting that event to the event under consideration. The maximum among all these values gives the Earliest Expected Time of Occurrence of the event.

## PROCEDURE TO FIND THE LATEST ALLOWABLE TIME OF AN EVENT

We consider the “**Backward Pass**” (i.e., movement in the network from right to left).

The latest allowable time of occurrence of the End Node must be the time of completion of the project. Therefore it shall be equal to the time of the critical path of the project.

**Step 1.** Identify the latest allowable time of occurrence of the End Node.

**Step 2.** For an event other than the End Event, find out all paths in the network which connect the End node with the node representing the event under consideration.

**Step 3.** In the “**Backward Pass**” (i.e., movement in the network from right to left), subtract the time durations of the activities along each such path.

**Step 4.** The Latest Allowable Time of Occurrence of the event is determined by the path with the longest time in Step 3. In other words, the smallest value of time obtained in Step 3 gives the Latest Allowable Time of Occurrence of the event.

### Working Rule for finding the latest allowable time of an event:

For an event under consideration, locate all the successor events and identify their latest allowable times. From the latest allowable time of each successor event, subtract the time duration of the activity that begins with the event under consideration. The minimum among all these values gives the Latest Allowable Time of Occurrence of the event.

### SLACK OF AN EVENT

The allowable time gap for the occurrence of an event is known as the slack of that event. It is given by the formula

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

### SLACK OF AN ACTIVITY

The slack of an activity is the float of the activity.

#### Problem 1:

The following details are available regarding a project:

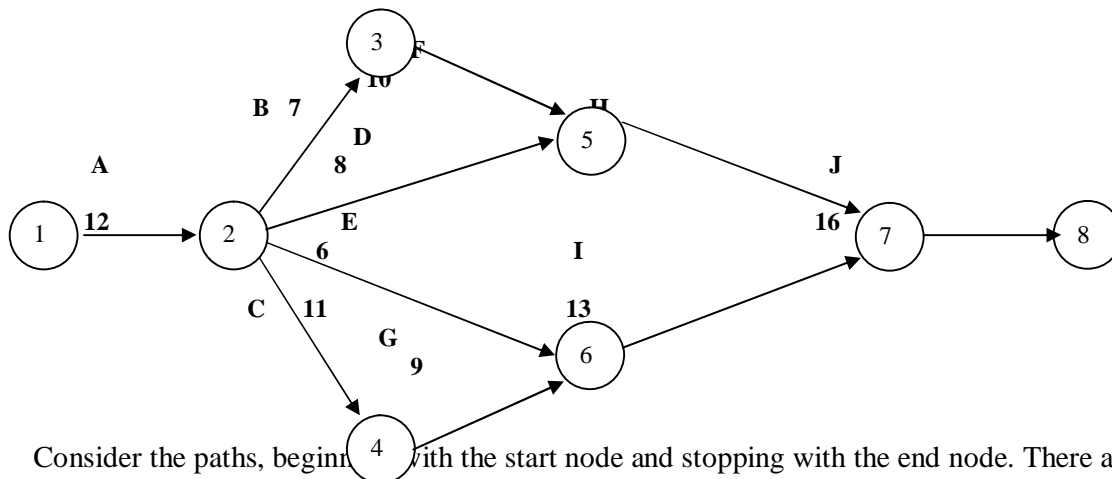
Activity	Predecessor Activity	Duration (Weeks)
A	-	12

B	A	7
C	A	11
D	A	8
E	A	6
F	B	10
G	C	9
H	D, F	14
I	E, G	13
J	H, I	16

Determine the earliest and latest times, the total float for each activity, the critical activities and the project completion time.

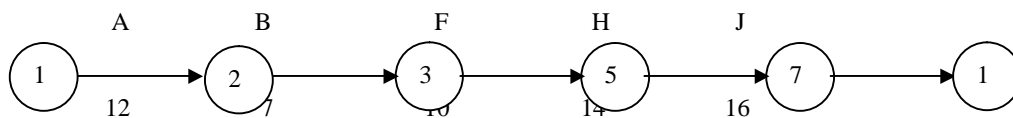
**Solution:**

With the given data, we construct the following network diagram for the project.



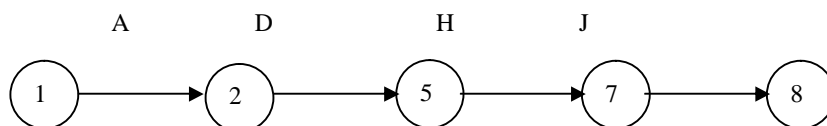
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

**Path I**



Time of the path =  $12 + 7 + 10 + 14 + 16 = 59$  weeks.

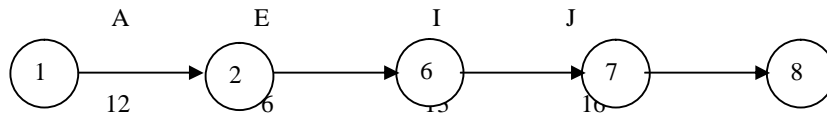
**Path II**



12                  8                  14                  16

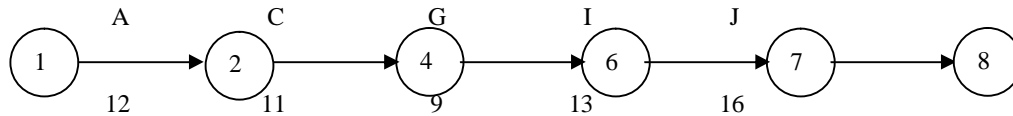
Time of the path = 12 + 8 + 14 + 16 = 50 weeks.

**Path III**



Time of the path = 12 + 6 + 13 + 16 = 47 weeks.

**Path IV**



Time of the path = 12 + 11 + 9 + 13 + 16 = 61 weeks.

Compare the times for the four paths. Maximum of {51, 50, 47, 61} = 61. We see that the maximum time of a path is 61 weeks.

**Forward pass:**

Calculation of Earliest Expected Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = 0 + 12 = 12
3	Time for Node 2 + Time for Activity B = 12 + 7 = 19
4	Time for Node 2 + Time for Activity C = 12 + 11 = 23
5	Max {Time for Node 2 + Time for Activity D, Time for Node 3 + Time for Activity F} = Max {12 + 8, 19 + 10} = Max {20, 29} = 29
6	Max {Time for Node 2 + Time for Activity E, Time for Node 4 + Time for Activity G} = Max {12 + 6, 23 + 9} = Max {18, 32} = 32
7	Max {Time for Node 5 + Time for Activity H, Time for Node 6 + Time for Activity I} = Max {29 + 14, 32 + 13} = Max {43, 45} = 45
8	Time for Node 7 + Time for Activity J = 45 + 16 = 61

Using the above values, we obtain the Earliest Start Times of the activities as follows:

Activity	Earliest Start Time (Weeks)
----------	-----------------------------

A	0
B	12
C	12
D	12
E	12
F	19
G	23
H	29
I	32
J	45

**Backward pass:**

Calculation of Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
8	Maximum time of a path in the network = 61
7	Time for Node 8 - Time for Activity J = $61 - 16 = 45$
6	Time for Node 7 - Time for Activity I = $45 - 13 = 32$
5	Time for Node 7 - Time for Activity H = $45 - 14 = 31$
4	Time for Node 6 - Time for Activity G = $32 - 9 = 23$
3	Time for Node 5 - Time for Activity F = $31 - 10 = 21$
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C, Time for Node 5 - Time for Activity D, Time for Node 6 - Time for Activity E} = Min {21 - 7, 23 - 11, 31 - 8, 32 - 6} = Min {14, 12, 23, 26} = 12
1	Time for Node 2 - Time for Activity A = $12 - 12 = 0$

Using the above values, we obtain the Latest Finish Times of the activities as follows:

Activity	Latest Finish Time (Weeks)
J	61
I	45
H	45
G	32
F	31

E	32
D	31
C	23
B	21
A	12

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	12	0	12	0	12	0
B	7	12	19	14	21	2
C	11	12	23	12	23	0
D	8	12	20	23	31	11
E	6	12	18	26	32	14
F	10	19	29	21	31	2
G	9	23	32	23	32	0
H	14	29	43	31	45	2
I	13	32	45	32	45	0
J	16	45	61	45	61	0

The activities with total float = 0 are A, C, G, I and J. They are the critical activities.  
Project completion time = 61 weeks.

**Problem 2:**

The following are the details of the activities in a project:

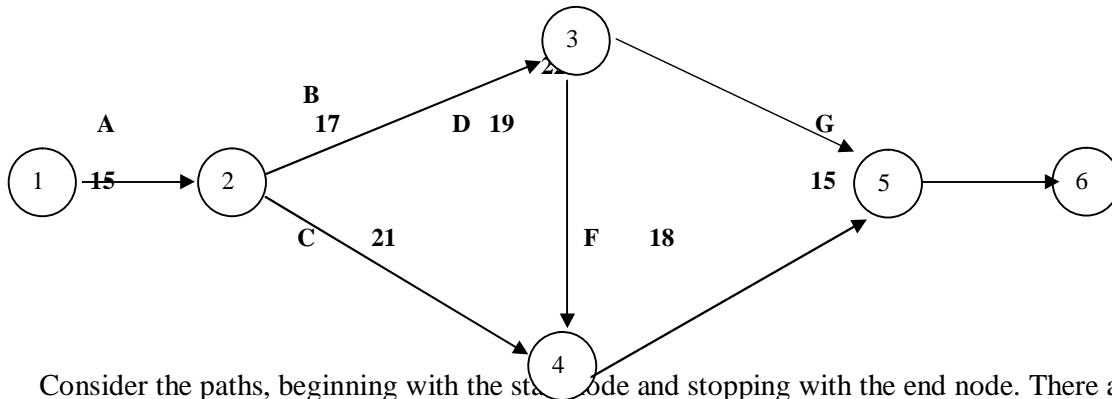
Activity	Predecessor Activity	Duration (Weeks)
A	-	15
B	A	17
C	A	21
D	B	19
E	B	22

F	C, D	18
G	E, F	15

Calculate the earliest and latest times, the total float for each activity and the project completion time.

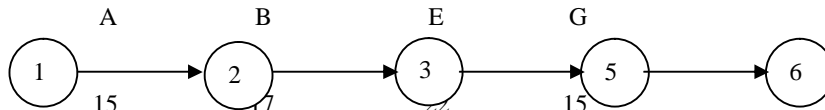
**Solution:**

The following network diagram is obtained for the given project.



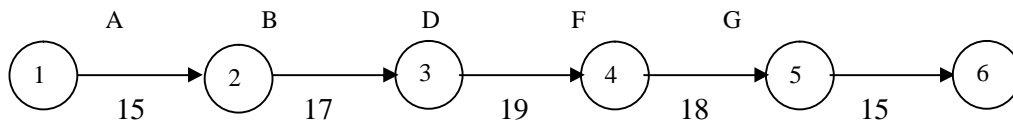
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

**Path I**



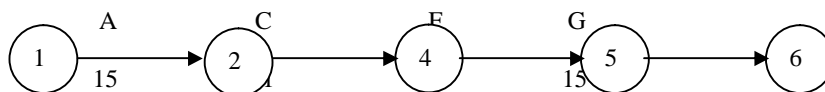
Time of the path = 15 + 17 + 22 + 15 = 69 weeks.

**Path II**



Time of the path = 15 + 17 + 19 + 18 + 15 = 84 weeks.

**Path III**



Time of the path = 15 + 21 + 18 + 15 = 69 weeks.

Compare the times for the three paths. Maximum of {69, 84, 69} = 84. We see that the maximum time of a path is 84 weeks.



**Forward pass:**

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = $0 + 15 = 15$
3	Time for Node 2 + Time for Activity B = $15 + 17 = 32$
4	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity D} = Max { $15 + 21, 32 + 19$ } = Max {36, 51} = 51
5	Max {Time for Node 3 + Time for Activity E, Time for Node 4 + Time for Activity F} = Max { $32 + 22, 51 + 18$ } = Max {54, 69} = 69
6	Time for Node 5 + Time for Activity G = $69 + 15 = 84$

Calculation of Earliest Time for Activities

Activity	Earliest Start Time (Weeks)
A	0
B	15
C	15
D	32
E	32
F	51
G	69

**Backward pass:**

Calculation of the Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
6	Maximum time of a path in the network = 84
5	Time for Node 6 - Time for Activity G = $84 - 15 = 69$
4	Time for Node 5 - Time for Activity F = $69 - 18 = 51$
3	Min {Time for Node 4 - Time for Activity D, Time for Node 5 - Time for Activity E} = Min { $51 - 19, 69 - 22$ } = Min {32, 47} = 32
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C} = Min { $32 - 17, 51 - 21$ } = Min {15, 30} = 15
1	Time for Node 2 - Time for Activity A = $15 - 15 = 0$

Calculation of the Latest Finish Times of the activities

Activity	Latest Finish Time (Weeks)
G	84

F	69
E	69
D	51
C	51
B	32
A	15

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	15	0	15	0	15	0
B	17	15	32	15	32	0
C	21	15	36	30	51	15
D	19	32	51	32	51	0
E	22	32	54	47	69	15
F	18	51	69	51	69	0
G	15	69	84	69	84	0

The activities with total float = 0 are A, B, D, F and G. They are the critical activities.  
Project completion time = 84 weeks.

### Problem 3:

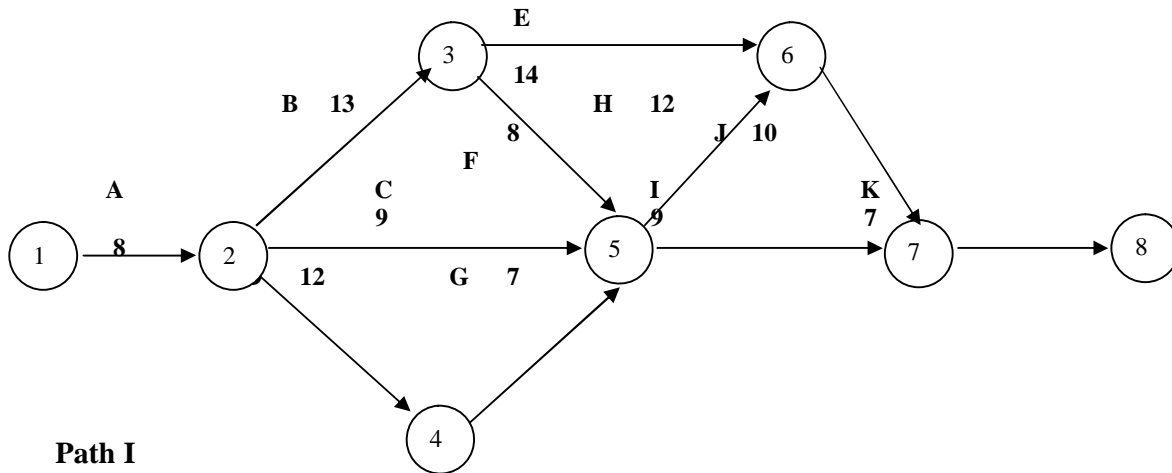
Consider a project with the following details:

Name of Activity	Predecessor Activity	Duration (Weeks)
A	-	8
B	A	13
C	A	9
D	A	12
E	B	14
F	B	8
G	D	7
H	C, F, G	12
I	C, F, G	9
J	E, H	10
K	I, J	7

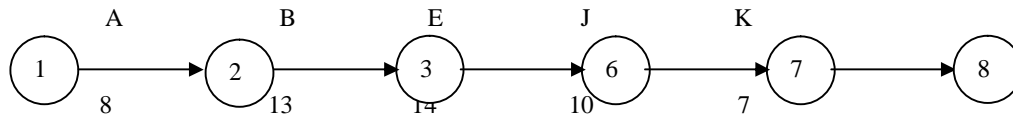
Determine the earliest and latest times, the total float for each activity, the critical activities, the slacks of the events and the project completion time.

**Solution:**

The following network diagram is got for the given project:

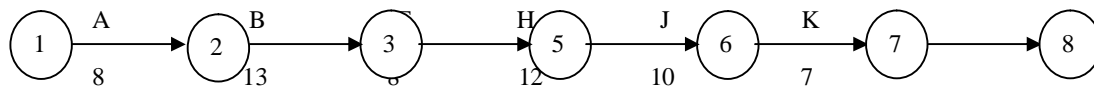


**Path I**



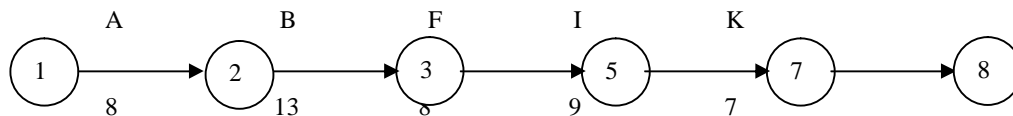
Time of the path = 8 + 13 + 14 + 10 + 7 = 52 weeks.

**Path II**



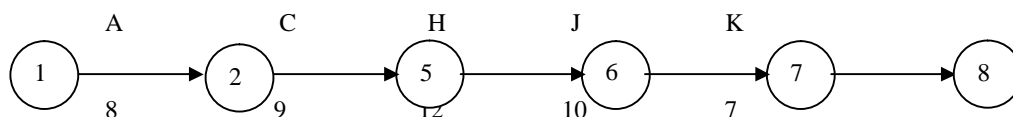
Time of the path = 8 + 13 + 12 + 10 + 7 = 50 weeks.

**Path III**



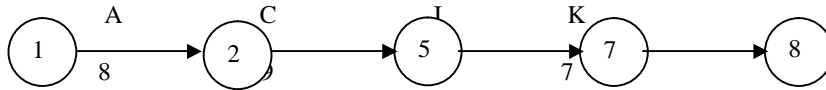
Time of the path = 8 + 13 + 8 + 9 + 7 = 45 weeks.

**Path IV**



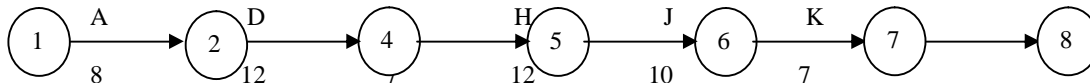
Time of the path = 8 + 9 + 12 + 10 + 7 = 46 weeks.

**Path V**



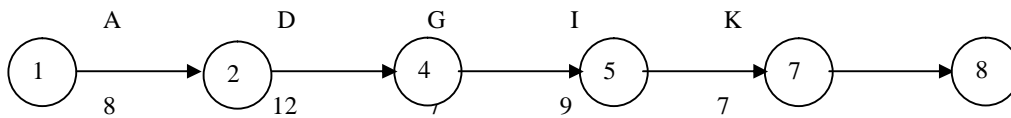
Time of the path =  $8 + 9 + 9 + 7 = 33$  weeks.

**Path VI**



Time of the path =  $8 + 12 + 12 + 10 + 7 = 56$  weeks.

**Path VII**



Time of the path =  $8 + 12 + 7 + 9 + 7 = 43$  weeks.

Compare the times for the three paths. Maximum of  $\{52, 58, 45, 46, 33, 56, 43\} = 58$ .  
We see that the maximum time of a path is 58 weeks.

**Forward pass:**

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = $0 + 8 = 8$
3	Time for Node 2 + Time for Activity B = $8 + 13 = 21$
4	Time for Node 2 + Time for Activity D = $8 + 12 = 20$
5	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity F, Time for Node 4 + Time for Activity G} = Max { $8 + 9$ , $21 + 8$ , $20 + 7$ } = Max {17, 29, 27} = 29
6	Max {Time for Node 3 + Time for Activity E, Time for Node 5 + Time for Activity H} = Max { $21 + 14$ , $29 + 12$ } = Max {35, 41} = 41
7	Max {Time for Node 5 + Time for Activity I, Time for Node 6 + Time for Activity J} = Max { $29 + 9$ , $41 + 10$ } = Max {38, 51} = 51
8	Time for Node 7 + Time for Activity J = $51 + 7 = 58$

Earliest Start Times of the activities

Activity	Earliest Start Time (Weeks)
A	0
B	8
C	8
D	8
E	21
F	21
G	20
H	29
I	29
J	41
K	51

**Backward pass:**

Calculation of Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
8	Maximum time of a path in the network = 58
7	Time for Node 8 - Time for Activity K = $58 - 7 = 51$
6	Time for Node 7 - Time for Activity J = $51 - 10 = 41$

5	Min {Time for Node 6 - Time for Activity H, Time for Node 7 - Time for Activity I} = Min {41 - 12, 51 - 9} = Min {29, 42} = 29
4	Time for Node 5 - Time for Activity G = 29 - 7 = 22
3	Min {Time for Node 5 - Time for Activity F, Time for Node 6 - Time for Activity E} = Min {29 - 8, 41 - 14} = Min {21, 27} = 21
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity D, Time for Node 5 - Time for Activity C} = Min {21 - 13, 22 - 12, 29 - 9} = Min {8, 10, 20} = 8
1	Time for Node 2 - Time for Activity A = 8 - 8 = 0

Latest Finish Times of the activities

Activity	Latest Finish Time (Weeks)
K	58
J	51
I	51
H	41
G	29
F	29
E	41
D	22
C	29
B	21
A	8

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	8	0	8	0	8	0
B	13	8	21	8	21	0
C	9	8	17	20	29	12
D	12	8	20	10	22	2
E	14	21	35	27	41	6
F	8	21	29	21	29	0

G	7	20	27	22	29	2
H	12	29	41	29	41	0
I	9	29	38	42	51	13
J	10	41	51	41	51	0
K	7	51	58	51	58	0

The activities with total float = 0 are A, B, F, H, J and K. They are the critical activities.  
Project completion time = 58 weeks.

### Calculation of slacks of the events

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

Event (Node)	Earliest Expected Time of Occurrence of Event	Latest Allowable Time of Occurrence of Event	Slack of the Event
1	0	0	0
2	8	8	0
3	21	21	0
4	20	22	2
5	29	29	0
6	41	41	0
7	51	51	0
8	58	58	0

### Interpretation:

On the basis of the slacks of the events, it is concluded that the occurrence of event 4 may be delayed upto a maximum period of 2 weeks while no other event cannot be delayed.

### QUESTIONS

1. Explain the terms: The earliest and latest times of the activities of a project.
2. Explain the procedure to find the earliest expected time of an event.
3. Explain the procedure to find the latest allowable time of an event.
4. What is meant by the slack of an activity? How will you determine it?
5. Consider the project with the following details:

activity	Duration (weeks)
1→2	1
2→3	3

2→4	7
3→4	5
3→5	8
4→5	4
5→6	1

Determine the earliest and the latest times of the activities. Calculate the total float for each activity and the slacks of the events.



## LESSON 7

### CRASHING OF A PROJECT

#### LESSON OUTLINE

- The idea of crashing of a project
- The criterion of selection of an activity for crashing
- Numerical problems

#### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the concept of crashing of a project
- choose an activity for crashing
- work out numerical problems

#### THE MEANING OF CRASHING:

The process of shortening the time to complete a project is called crashing and is usually achieved by putting into service additional labour or machines to one activity or more activities. Crashing involves more costs. A project manager would like to speed up a project by spending as minimum extra cost as possible. Project crashing seeks to minimize the extra cost for completion of a project before the stipulated time.

#### STEPS IN PROJECT CRASHING:

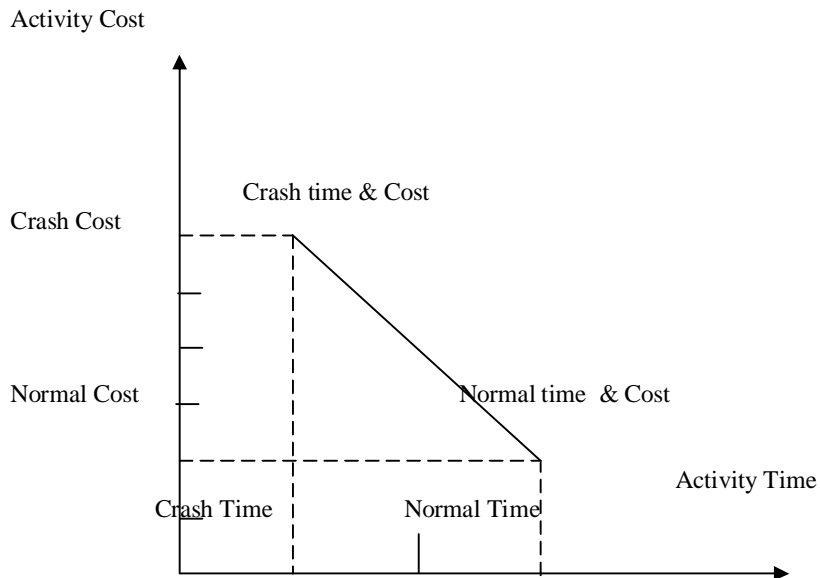
**Assumption:** It is assumed that there is a linear relationship between time and cost.

Let us consider project crashing by the critical path method. The following four-step procedure is adopted.

**Step 1:** Find the critical path with the normal times and normal costs for the activities and identify the critical activities.

**Step 2:** Find out the crash cost per unit time for each activity in the network. This is calculated by means of the following formula.

$$\text{Crash cost} / \text{Time period} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$



**Step 3:** Select an activity for crashing. The **criteria for the selection** is as follows:

Select the activity on the critical path with the smallest crash cost per unit time. Crash this activity to the maximum units of time as may be permissible by the given data.

Crashing an activity requires extra amount to be spent. However, even if the company is prepared to spend extra money, the activity time cannot be reduced beyond a certain limit in view of several other factors.

In step 1, we have to note that reducing the time of an activity along the critical path alone will reduce the completion time of a project. Because of this reason, we select an activity along the critical path for crashing.

In step 3, we have to consider the following question:

If we want to reduce the project completion time by one unit, which critical activity will involve the least additional cost?

On the basis of the least additional cost, a critical activity is chosen for crashing. If there is a tie between two critical activities, the tie can be resolved arbitrarily.

**Step 4:** After crashing an activity, find out which is the critical path with the changed conditions. Sometimes, a reduction in the time of an activity in the critical path may cause a non-critical path to become critical. If the critical path with which we started is still the longest path, then go to Step 3. Otherwise, determine the new critical path and then go to Step 3.

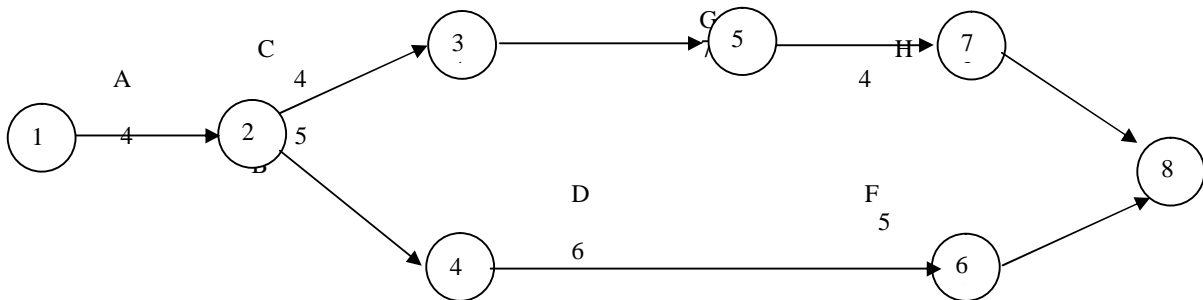
**Problem 1:** A project has activities with the following normal and crash times and cost:

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	4	3	8,000	9,000
B	A	5	3	16,000	20,000
C	A	4	3	12,000	13,000
D	B	6	5	34,000	35,000
E	C	6	4	42,000	44,000
F	D	5	4	16,000	16,500
G	E	7	4	66,000	72,000
H	G	4	3	2,000	5,000

Determine a crashing scheme for the above project so that the total project time is reduced by 3 weeks.

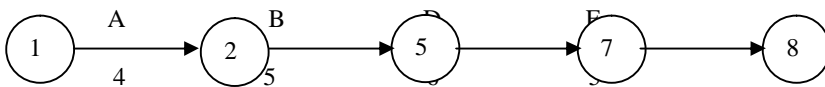
**Solution:**

We have the following network diagram for the given project with **normal costs**:



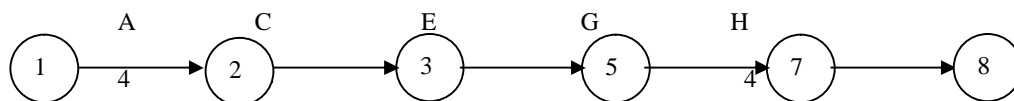
Beginning from the Start Node and terminating with the End Node, there are two paths for the network as detailed below:

**Path I:**



The time for the path = 4 + 5 + 6 + 5 = 20 weeks.

**Path II:**



The time for the path = 4 + 4 + 6 + 7 + 4 = 25 weeks.

Maximum of {20, 25} = 25.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. The non-critical activities are B, D and F.

Given that the normal time of activity A is 4 weeks while its crash time is 3 weeks. Hence the time of this activity can be reduced by one week if the management is prepared to spend an additional amount. However, the time cannot be reduced by more than one week even if the management may be prepared to spend more money. The normal cost of this activity is Rs. 8,000 whereas the crash cost is Rs. 9,000. From this, we see that crashing of activity A by one week will cost the management an extra amount of Rs. 1,000. In a similar fashion, we can work out the crash cost per unit time for the other activities also. The results are provided in the following table.

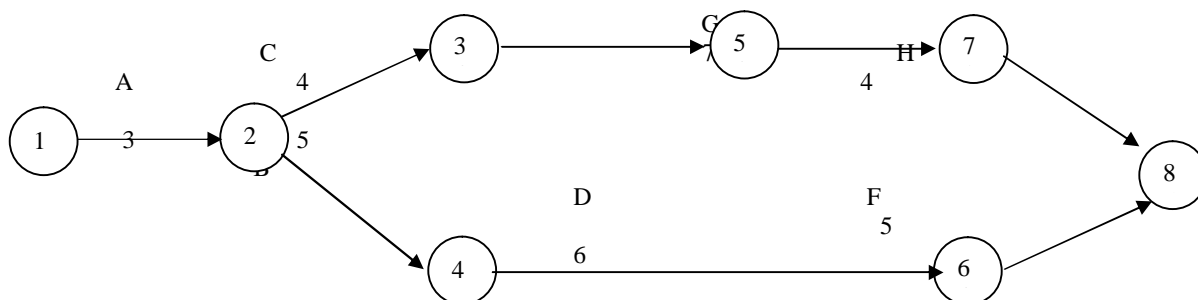
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	4	3	8,000	9,000	1,000	1	1,000
B	5	3	16,000	20,000	4,000	2	2,000
C	4	3	12,000	13,000	1,000	1	1,000
D	6	5	34,000	35,000	1,000	1	1,000
E	6	4	42,000	44,000	2,000	2	1,000
F	5	4	16,000	16,500	500	1	500
G	7	4	66,000	72,000	6,000	1	6,000
H	4	3	2,000	5,000	3,000	1	3,000

A non-critical activity can be delayed without delaying the execution of the whole project. But, if a critical activity is delayed, it will delay the whole project. Because of this reason, we have to select a critical activity for crashing. Here we have to choose one of the activities A, C, E, G and H. The crash cost per unit time works out as follows:

Rs. 1,000 for A; Rs. 1,000 for C; Rs. 1,000 for E; Rs. 6,000 for G; Rs. 3,000 for H.

The maximum among them is Rs. 1,000. So we have to choose an activity with Rs. 1,000 as the crash cost per unit time. However, there is a tie among A, C and E. The tie can be resolved arbitrarily. Let us select A for crashing. We reduce the time of A by one week by spending an extra amount of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



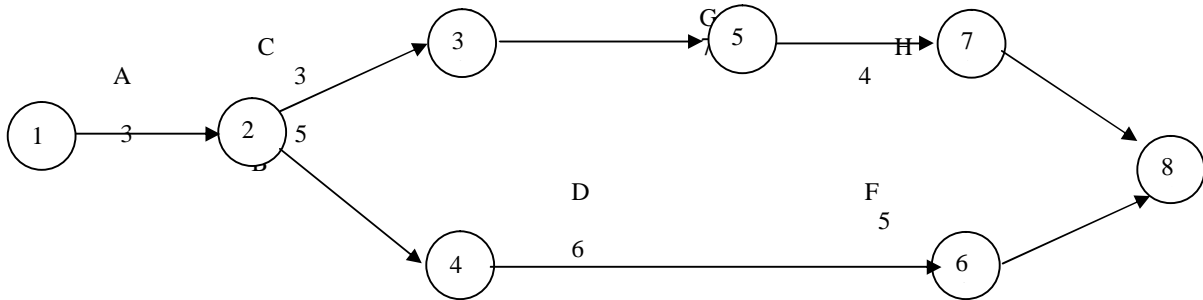
The revised time for Path I = 3 + 5 + 6 + 5 = 19 weeks.

The time for Path II = 3 + 4 + 6 + 7 + 4 = 24 weeks.

Maximum of {19, 24} = 24.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. However, the time for A cannot be reduced further. Therefore, we have to consider C, E, G and H for crashing. Among them, C and E have the least crash cost per unit time. The tie between C and E can be resolved arbitrarily. Suppose we reduce the time of C by one week with an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The time for Path I = 3 + 5 + 6 + 5 = 19 weeks.  
 The time for Path II = 3 + 3 + 6 + 7 + 4 = 23 weeks.  
 Maximum of {19, 23} = 23.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. Now the time for A or C cannot be reduced further. Therefore, we have to consider E, G and H for crashing. Among them, E has the least crash cost per unit time. Hence we reduce the time of E by one week with an extra cost of Rs. 1,000.

By the given condition, we have to reduce the project time by 3 weeks. Since this has been accomplished, we stop with this step.

**Result:** We have arrived at the following crashing scheme for the given project:

Reduce the time of A, C and E by one week each.

Project time after crashing is 22 weeks.

Extra amount required = 1,000 + 1,000 + 1,000 = Rs. 3,000.

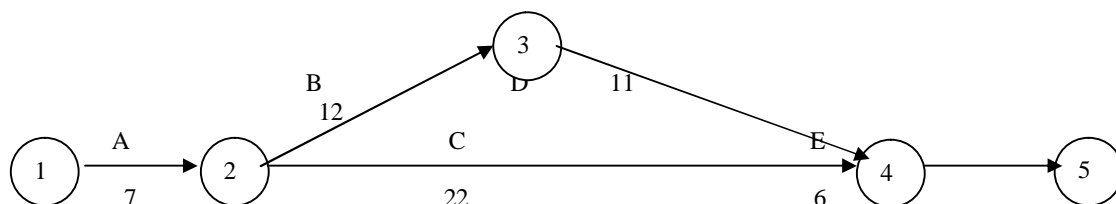
**Problem 2:**

The management of a company is interested in crashing of the following project by spending an additional amount not exceeding Rs. 2,000. Suggest how this can be accomplished.

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	7	6	15,000	18,000
B	A	12	9	11,000	14,000
C	A	22	21	18,500	19,000
D	B	11	10	8,000	9,000
E	C, D	6	5	4,000	4,500

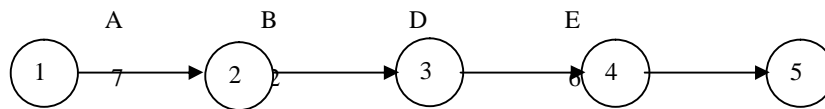
**Solution:**

We have the following network diagram for the given project with **normal costs**:



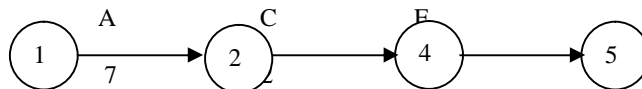
There are two paths for this project as detailed below:

**Path I:**



The time for the path =  $7 + 12 + 11 + 6 = 36$  weeks.

**Path II:**



The time for the path =  $7 + 22 + 6 = 35$  weeks.

Maximum of  $\{36, 35\} = 36$ .

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C.  
The crash cost per unit time for the activities in the project are provided in the following table.

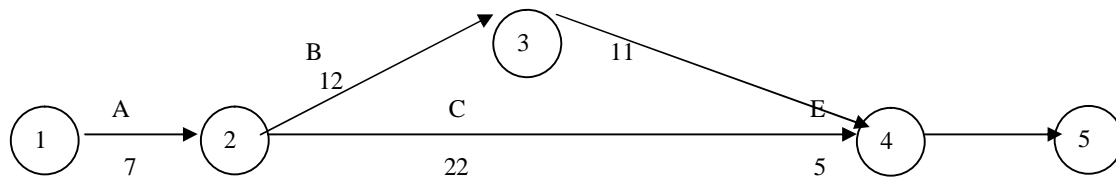
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	7	6	15,000	18,000	3,000	1	3,000
B	12	9	11,000	14,000	3,000	3	1,000
C	22	21	18,500	19,000	500	1	500
D	11	10	8,000	9,000	1,000	1	1,000
E	6	5	4,000	4,500	500	1	500

We have to choose one of the activities A, B, D and E for crashing. The crash cost per unit time is as follows:

Rs. 3,000 for A; Rs. 1,000 for B; Rs. 1,000 for D; Rs. 500 for E.

The least among them is Rs. 500. So we have to choose the activity E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I =  $7 + 12 + 11 + 5 = 35$  weeks.

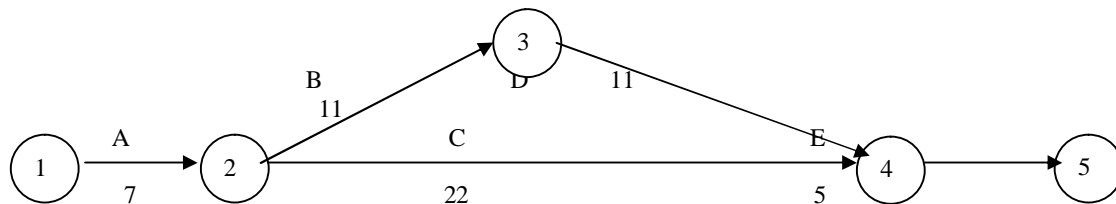
The time for Path II =  $7 + 22 + 5 = 34$  weeks.

Maximum of {35, 34} = 35.

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C.

The time of E cannot be reduced further. So we cannot select it for crashing. Next B and have the smallest crash cost per unit time. Let us select B for crashing. Let us reduce the time of E by one week at an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I =  $7 + 11 + 11 + 5 = 34$  weeks.



The time for Path II =  $7 + 22 + 5 = 34$  weeks.

Maximum of  $\{34, 34\} = 34$ .

Since both paths have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time. In path I, the activities are A, B, D and E. In path II, the activities are A, C and E.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this state has been met, we stop with this step.

**Result:** The following crashing scheme is suggested for the given project:  
Reduce the time of E, B and C by one week each.

Project time after crashing is 33 weeks.

Extra amount required =  $500 + 1,000 + 500 = \text{Rs. } 2,000$ .

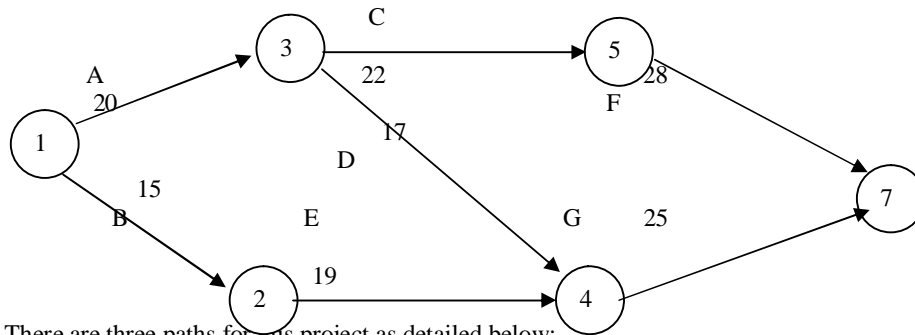
**Problem 3:**

The manager of a company wants to apply crashing for the following project by spending an additional amount not exceeding Rs. 2,000. Offer your suggestion to the manager.

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	20	19	8,000	10,000
B	-	15	14	16,000	19,000
C	A	22	20	13,000	14,000
D	A	17	15	7,500	9,000
E	B	19	18	4,000	5,000
F	C	28	27	3,000	4,000
G	D, E	25	24	12,000	13,000

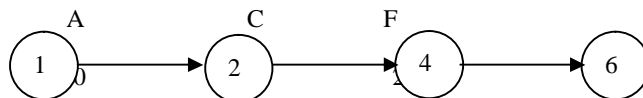
**Solution:**

We have the following network diagram for the given project with **normal costs**:



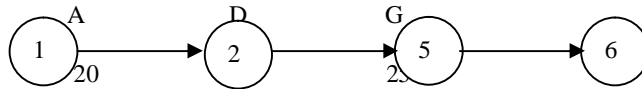
There are three paths for this project as detailed below:

**Path I:**



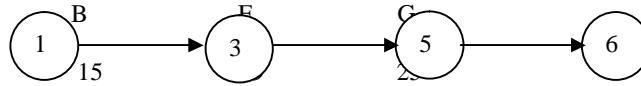
The time for the path = 20 + 22 + 28 = 70 weeks.

**Path II:**



The time for the path = 20 + 17 + 25 = 62 weeks.

**Path III:**



The time for the path = 15 + 19 + 25 = 59 weeks.

Maximum of {70, 62, 59} = 70.

Therefore Path I is the critical path and the critical activities are A, C and F. The non-critical activities are B, D, E and G.

The crash cost per unit time for the activities in the project are provided in the following table

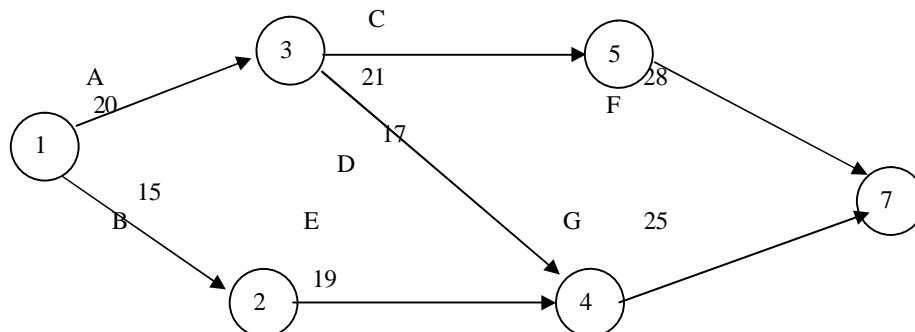
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	20	19	8,000	10,000	2,000	1	2,000
B	15	14	16,000	19,000	3,000	1	3,000
C	22	20	13,000	14,000	1,000	2	500
D	17	15	7,500	9,000	1,500	2	750
E	19	18	4,000	5,000	1,000	1	1,000
F	28	27	3,000	4,000	1,000	1	1,000
G	25	24	12,000	13,000	1,000	1	1,000

We have to choose one of the activities A, C and F for crashing. The crash cost per unit time is as follows:

Rs. 2,000 for A; Rs. 500 for C; Rs. 1,000 for F.

The least among them is Rs. 500. So we have to choose the activity C for crashing. We reduce the time of C by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I =  $20 + 21 + 28 = 69$  weeks.

The time for Path II =  $20 + 17 + 25 = 62$  weeks.

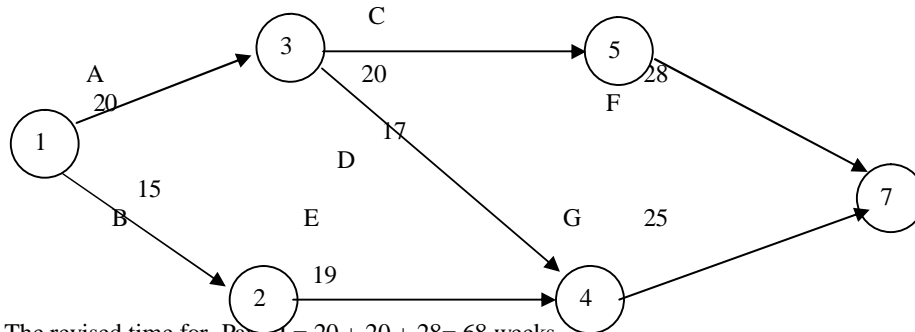
The time for Path III =  $15 + 19 + 25 = 69$  weeks.  
Maximum of  $\{69, 62, 69\} = 69$ .

Since paths I and III have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time.

In path I, the activities are A, C and F. In path III, the activities are B, E and G.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I =  $20 + 20 + 28 = 68$  weeks.  
The time for Path II =  $20 + 17 + 25 = 62$  weeks.

The time for Path III =  $15 + 19 + 25 = 69$  weeks.  
Maximum of  $\{68, 62, 69\} = 69$ .

Therefore path III is the critical activities. Hence we have to select an activity from Path III for crashing. We see that the crash cost per unit time is as follows:

Rs. 3,000 for B; Rs. 1,000 for E; Rs. 1,000 for G.

The least among them is Rs. 1,000. So we can select either E or G for crashing. Let us select E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 1,000.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this condition has been reached, we stop with this step.

**Result:** The following crashing scheme is suggested for the given project:

Reduce the time of C by 2 weeks and that of E by one week.

Project time after crashing is 67 weeks.

Extra amount required =  $2 \times 500 + 1,000 = \text{Rs. } 2,000$ .

## QUESTIONS

1. Explain the concept of crashing of a project.
2. Explain the criterion for selection of an activity for crashing.



UNIT IV  
**GAME THEORY & GOAL PROGRAMMING**

LESSON 1 BASIC CONCEPTS IN GAME THEORY

LESSON OUTLINE

- Introduction to the theory of games
- The definition of a game
- Competitive game
- Managerial applications of the theory of games
- Key concepts in the theory of games
- Types of games

LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the concept of a game
- grasp the assumptions in the theory of games
- appreciate the managerial applications of the theory of games
- understand the key concepts in the theory of games
- distinguish between different types of games

Introduction to game theory

Game theory seeks to analyse competing situations which arise out of conflicts of interest. Abraham Maslow's hierarchical model of human needs lays emphasis on fulfilling the basic needs such as food, water, clothes, shelter, air, safety and security. There is conflict of interest between animals and plants in the consumption of natural resources. Animals compete among themselves for securing food. Man competes with animals to earn his food. A man also competes with another man. In the past, nations waged wars to expand the territory of their rule. In the present day world, business organizations compete with each other in getting the market share. The conflicts of interests of human beings are not confined to the basic needs alone. Again considering Abraham Maslow's model of human needs, one can realize that conflicts also arise due to the higher levels of human needs such as love, affection, affiliation, recognition, status, dominance, power, esteem, ego, self-respect, etc. Sometimes one witnesses clashes of ideas of intellectuals also. Every intelligent and rational participant in a conflict wants to be a winner but not all participants can be the winners at a time. The situations of conflict gave birth to Darwin's theory of the 'survival of the fittest'. Nowadays

the concepts of conciliation, co-existence, co-operation, coalition and consensus are gaining ground. Game theory is another tool to examine situations of conflict so as to identify the courses of action to be followed and to take appropriate decisions in the long run. Thus this theory assumes importance from managerial perspectives. The pioneering work on the theory of games was done by von Neumann and Morgenstern through their publication entitled 'The Theory of Games and Economic Behaviour' and subsequently the subject was developed by several experts. This theory can offer valuable guidelines to a manager in 'strategic management' which can be used in the decision making process for merger, take-over, joint venture, etc. The results obtained by the application of this theory can serve as an early warning to the top level management in meeting the threats from the competing business organizations and for the conversion of the internal weaknesses and external threats into opportunities and strengths, thereby achieving the goal of maximization of profits. While this theory does not describe any procedure to play a game, it will enable a participant to select the appropriate strategies to be followed in the pursuit of his goals. The situation of failure in a game would activate a participant in the analysis of the relevance of the existing strategies and lead him to identify better, novel strategies for the future occasions.

#### Definitions of game theory

There are several definitions of game theory. A few standard definitions are presented below.

In the perception of Robert Mockler, "Game theory is a mathematical technique helpful in making decisions in situations of conflicts, where the success of one part depends at the expense of others, and where the individual decision maker is not in complete control of the factors influencing the outcome".

The definition given by William G. Nelson runs as follows: "Game theory, more properly **the theory of games of strategy**, is a mathematical method of analyzing a conflict. The alternative is not between this decision or that decision, but between this strategy or that strategy to be used against the conflicting interest".

In the opinion of Matrin Shubik, "Game theory is a method of the study of decision making in situation of conflict. It deals with human processes in which the individual decision-unit is not in complete control of other decision-units entering into the environment".

According to von Neumann and Morgenstern, "The 'Game' is simply the totality of the rules which describe it. Every particular instance at which the game is played – in a particular way – from beginning to end is a 'play'. The game consists of a sequence of moves, and the play of a sequence of choices".

J.C.C McKinsey points out a valid distinction between two words, namely 'game' and 'play'. According to him, "game refers to a particular realization of the rules".

In the words of O.T. Bartos, "The theory of games can be used for 'prescribing' how an intelligent person should go about resolving social conflicts, ranging all the way from open warfare between nations to disagreements between husband and wife".

Martin K Starr gave the following definition: "Management models in the competitive sphere are usually termed game models. By studying game theory, we can obtain substantial information into management's role under competitive conditions, even though much of the game theory is neither directly operational nor implementable".

According to Edwin Mansfield, "A game is a competitive situation where two or more persons pursue their own interests and no person can dictate the outcome. Each player, an entity with the same interests, make his own decisions. A player can be an individual or a group".

### **Assumptions for a Competitive Game**

Game theory helps in finding out the best course of action for a firm in view of the anticipated countermoves from the competing organizations. A competitive situation is a competitive game if the following properties hold:

1. The number of competitors is finite, say N.
2. A finite set of possible courses of action is available to each of the N competitors.
3. A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that all the players select their courses of action simultaneously. As a result, no competitor will be in a position to know the choices of his competitors.
4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

### **Managerial Applications of the Theory of Games**

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

- 1) Analysis of the market strategies of a business organization in the long run.
- 2) Evaluation of the responses of the consumers to a new product.
- 3) Resolving the conflict between two groups in a business organization.
- 4) Decision making on the techniques to increase market share.



- 5) Material procurement process.
- 6) Decision making for transportation problem.
- 7) Evaluation of the distribution system.
- 8) Evaluation of the location of the facilities.
- 9) Examination of new business ventures and
- 10) Competitive economic environment.

### **Key concepts in the Theory of Games**

Several of the key concepts used in the theory of games are described below:

**Players:**

The competitors or decision makers in a game are called the players of the game.

**Strategies:**

The alternative courses of action available to a player are referred to as his strategies.

**Pay off:**

The outcome of playing a game is called the pay off to the concerned player.

**Optimal Strategy:**

A strategy by which a player can achieve the best pay off is called the optimal strategy for him.

**Zero-sum game:**

A game in which the total payoffs to all the players at the end of the game is zero is referred to as a zero-sum game.

**Non-zero sum game:**

Games with “less than complete conflict of interest” are called non-zero sum games. The problems faced by a large number of business organizations come under this category. In such games, the gain of one player in terms of his success need not be completely at the expense of the other player.

**Payoff matrix:**

The tabular display of the payoffs to players under various alternatives is called the payoff matrix of the game.

**Pure strategy:**

If the game is such that each player can identify one and only one strategy as the optimal strategy in each play of the game, then that strategy is referred to as the best strategy for that player and the game is referred to as a game of pure strategy or a pure game.

**Mixed strategy:**

If there is no one specific strategy as the 'best strategy' for any player in a game, then the game is referred to as a game of mixed strategy or a mixed game. In such a game, each player has to choose different alternative courses of action from time to time.

**N-person game:**

A game in which N-players take part is called an N-person game.

**Maximin-Minimax Principle :**

The maximum of the minimum gains is called the maximin value of the game and the corresponding strategy is called the maximin strategy. Similarly the minimum of the maximum losses is called the minimax value of the game and the corresponding strategy is called the minimax strategy. If both the values are equal, then that would guarantee the best of the worst results.

**Negotiable or cooperative game:**

If the game is such that the players are taken to cooperate on any or every action which may increase the payoff of either player, then we call it a negotiable or cooperative game.

**Non-negotiable or non-cooperative game:**

If the players are not permitted for coalition then we refer to the game as a non-negotiable or non-cooperative game.

**Saddle point:**

A saddle point of a game is that place in the payoff matrix where the maximum of the row minima is equal to the minimum of the column maxima. The payoff at the saddle point is called **the value of the game** and the corresponding strategies are called the **pure strategies**.

**Dominance:**

One of the strategies of either player may be inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones.

**Types of Games:**

There are several classifications of a game. The classification may be based on various factors such as the number of participants, the gain or loss to each participant, the number of strategies available to each participant, etc. Some of the important types of games are enumerated below.

**Two person games and n-person games:**

In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

**Zero sum game and non-zero sum game:**

If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is either positive or negative but not zero, then the game is called a non-zero sum game

**Games of perfect information and games of imperfect information:**

A game of perfect information is the one in which each player can find out the strategy that would be followed by his opponent. On the other hand, a game of imperfect information is the one in which no player can know in advance what strategy would be adopted by the competitor and a player has to proceed in his game with his guess works only.

**Games with finite number of moves / players and games with unlimited number of moves:**

A game with a finite number of moves is the one in which the number of moves for each player is limited before the start of the play. On the other hand, if the game can be continued over an extended period of time and the number of moves for any player has no restriction, then we call it a game with unlimited number of moves.

**Constant-sum games:**

If the sum of the game is not zero but the sum of the payoffs to both players in each case is constant, then we call it a constant sum game. It is possible to reduce such a game to a zero-sum game.

**2x2 two person game and 2xn and mx2 games:**

When the number of players in a game is two and each player has exactly two strategies, the game is referred to as 2x2 two person game.

A game in which the first player has precisely two strategies and the second player has three or more strategies is called an 2xn game.

A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

**3x3 and large games:**

When the number of players in a game is two and each player has exactly three strategies, we call it a 3x3 two person game.

Two-person zero sum games are said to be larger if each of the two players has 3 or more choices.

The examination of 3x3 and larger games involves difficulties. For such games, the technique of linear programming can be used as a method of solution to identify the optimum strategies for the two players.

**Non-constant games :**

Consider a game with two players. If the sum of the payoffs to the two players is not constant in all the plays of the game, then we call it a non-constant game.

Such games are divided into negotiable or cooperative games and non-negotiable or non-cooperative games.

**QUESTIONS**

1. Explain the concept of a game.
2. Define a game.
3. State the assumptions for a competitive game.
4. State the managerial applications of the theory of games.
5. Explain the following terms: strategy, pay-off matrix, saddle point, pure strategy and mixed strategy.
6. Explain the following terms: two person game, two person zero sum game, value of a game, 2xn game and mx2 game.

## LESSON 2

### TWO-PERSON ZERO SUM GAMES

#### LESSON OUTLINE

- The concept of a two-person zero sum game
- The assumptions for a two-person zero sum game
- Minimax and Maximin principles

#### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the concept of a two-person zero sum game
- have an idea of the assumptions for a two-person zero sum game
- understand Minimax and Maximin principles
- solve a two-person zero sum game
- interpret the results from the payoff matrix of a two-person zero sum game

#### Definition of two-person zero sum game

A game with only two players, say player A and player B, is called a two-person zero sum game if the gain of the player A is equal to the loss of the player B, so that the total sum is zero.

#### **Payoff matrix:**

When players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a payoff matrix.

Since the game is zero sum, the gain of one player is equal to the loss of other and vice-versa. Suppose A has m strategies and B has n strategies. Consider the following payoff matrix.

$$\begin{array}{c}
 \text{Player A's strategies} \\
 A_1 \\
 \vdots \\
 A_m
 \end{array}
 \begin{array}{c}
 \text{Player B's strategies} \\
 B_1 \quad B_2 \quad \cdots \quad B_n \\
 \left[ \begin{array}{cccc}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
 \end{array} \right]
 \end{array}$$

Player A wishes to gain as large a payoff  $a_{ij}$  as possible while player B will do his best to reach as small a value  $a_{ij}$  as possible where the gain to player B and loss to player A be  $(-a_{ij})$ .

**Assumptions for two-person zero sum game:**

For building any model, certain reasonable assumptions are quite necessary. Some assumptions for building a model of two-person zero sum game are listed below.

- a) Each player has available to him a finite number of possible courses of action. Sometimes the set of courses of action may be the same for each player. Or, certain courses of action may be available to both players while each player may have certain specific courses of action which are not available to the other player.
- b) Player A attempts to maximize gains to himself. Player B tries to minimize losses to himself.
- c) The decisions of both players are made individually prior to the play with no communication between them.
- d) The decisions are made and announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
- e) Both players know the possible payoffs of themselves and their opponents.

**Minimax and Maximin Principles**

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games.

The objective of game theory is to know how these players must select their respective strategies, so that they may optimize their payoffs. Such a criterion of decision making is referred to as minimax-maximin principle. This principle in games of pure strategies leads to the best possible selection of a strategy for both players.

For example, if player A chooses his  $i^{\text{th}}$  strategy, then he gains at least the payoff  $\min a_{ij}$ , which is minimum of the  $i^{\text{th}}$  row elements in the payoff matrix. Since his objective is to maximize his payoff, he can choose strategy  $i$  so as to make his payoff as large as possible. i.e., a payoff which is not less than  $\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij}$ .

Similarly player B can choose  $j^{\text{th}}$  column elements so as to make his loss not greater than  $\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{ij}$ .

If the maximin value for a player is equal to the minimax value for another player, i.e.

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij} = V = \min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{ij}$$

then the game is said to have a saddle point (equilibrium point) and the corresponding strategies are called optimal strategies. If there are two or more saddle points, they must be equal.

The amount of payoff, i.e.,  $V$  at an equilibrium point is known as the **value of the game**.

The optimal strategies can be identified by the players in the long run.

**Fair game:**

The game is said to be fair if the value of the game  $V = 0$ .

**Problem 1:**

Solve the game with the following pay-off matrix.

		<b>Player B</b>				
		Strategies				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player A Strategies	1	-2	5	-3	6	7
	2	4	6	8	-1	6
	3	8	2	3	5	4
	4	15	14	18	12	20

**Solution:**

First consider the minimum of each row.

Row	Minimum Value
1	-3
2	-1
3	2
4	12

Maximum of  $\{-3, -1, 2, 12\} = 12$

Next consider the maximum of each column.

Column	Maximum Value
1	15
2	14
3	18
4	12
5	20

Minimum of  $\{15, 14, 18, 12, 20\} = 12$

We see that the maximum of row minima = the minimum of the column maxima. So the game has a saddle point. The common value is 12. Therefore the value  $V$  of the game = 12.

**Interpretation:**

In the long run, the following best strategies will be identified by the two players:

The best strategy for player A is strategy 4.

The best strategy for player B is strategy IV.

The game is favourable to player A.

**Problem 2:**

Solve the game with the following pay-off matrix

		<b>Player Y</b>				
		Strategies				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player X Strategies	1	9	12	7	14	26
	2	25	35	20	28	30
	3	7	6	-8	3	2
	4	8	11	13	-2	1

**Solution:**

First consider the minimum of each row.

Row	Minimum Value
1	7
2	20
3	-8
4	-2

$$\text{Maximum of } \{7, 20, -8, -2\} = 20$$

**Next consider the maximum of each column.**

Column	Maximum Value
1	25
2	35
3	20
4	28
5	30

$$\text{Minimum of } \{25, 35, 20, 28, 30\} = 20$$

It is observed that the maximum of row minima and the minimum of the column maxima are equal. Hence the given the game has a saddle point. The common value is 20.

This indicates that the value V of the game is 20.

**Interpretation.**



The best strategy for player X is strategy 2.

The best strategy for player Y is strategy III.

The game is favourable to player A.

**Problem 3:**

Solve the following game:

		<b>Player B</b>			
		Strategies			
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>
Player A Strategies	1	1	-6	8	4
	2	3	-7	2	-8
	3	5	-5	-1	0
	4	3	-4	5	7

Solution

First consider the minimum of each row.

Row	Minimum Value
1	-6
2	-8
3	-5
4	-4

$$\text{Maximum of } \{-6, -8, -5, -4\} = -4$$

**Next consider the maximum of each column.**

Column	Maximum Value
1	5
2	-4
3	8
4	7

$$\text{Minimum of } \{5, -4, 8, 7\} = -4$$

Since the  $\max \{\text{row minima}\} = \min \{\text{column maxima}\}$ , the game under consideration has a saddle point. The common value is  $-4$ . Hence the value of the game is  $-4$ .

**Interpretation.**

The best strategy for player A is strategy 4.

The best strategy for player B is strategy II.

Since the value of the game is negative, it is concluded that the game is favourable to player B.

**QUESTIONS**

1. What is meant by a two-person zero sum game? Explain.
2. State the assumptions for a two-person zero sum game.
3. Explain Minimax and Maximin principles.
4. How will you interpret the results from the payoff matrix of a two-person zero sum game? Explain.
5. What is a fair game? Explain.
6. Solve the game with the following pay-off matrix.

		<b>Player B</b>				
		Strategies				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player A Strategies	1	7	5	2	3	9
	2	10	8	7	4	5
	3	9	12	0	2	1
	4	11	-2	-1	3	4

Answer: Best strategy for A: 2

Best strategy for B: IV

V = 4

The game is favourable to player A

7. Solve the game with the following pay-off matrix.

		<b>Player B</b>				
		Strategies				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player A Strategies	1	-2	-3	8	7	0
	2	1	-7	-5	-2	3
	3	4	-2	3	5	-1
	4	6	-4	5	4	7

Answer: Best strategy for A: 3

Best strategy for B: II

V = -2

The game is favourable to player B

8. Solve the game with the following pay-off matrix.

	<b>Player B</b>	
	I	II
<b>Player A</b>	6	4
	7	5

Answer: Best strategy for A: 2

Best strategy for B: II

$$V = 5$$

The game is favourable to player A

9. Solve the following game and interpret the result.

	<b>Player B</b>				
	Strategies				
	<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	
Player A Strategies	1	-3	-7	1	3
	2	-1	2	-3	1
	3	0	4	2	6
	4	-2	-1	-5	1

Answer: Best strategy for A: 3

Best strategy for B: I

$$V = 0$$

The value  $V = 0$  indicates that the game is a fair one.

10. Solve the following game:

	<b>Player B</b>			
	Strategies			
	<i>I</i>	<i>II</i>	<i>III</i>	
Player A Strategies	1	1	8	2
	2	3	5	6
	3	2	2	1

Answer: Best strategy for A: 2

Best strategy for B: I

$$V = 3$$

The game is favourable to player A

11. Solve the game

		<b>Player B</b>			
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>
Player A	1	4	-1	2	0
	2	-3	-5	-9	-2
	3	2	-8	0	-11

**Answer :  $V = -1$**

12. Solve the game

		<b>Player Y</b>				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player X	1	4	0	1	7	-1
	2	0	-3	-5	-7	5
	3	3	2	3	4	3
	4	-6	4	-1	0	5
	5	0	0	6	0	0

**Answer :  $V = 2$**

13. Solve the game

		<b>Player B</b>				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player A	1	9	3	4	4	2
	2	8	6	8	5	12
	3	10	7	19	18	14
	4	8	6	8	11	6
	5	3	5	16	10	8

**Answer :  $V = 7$**

14. Solve the game

		<b>Player Y</b>					
		17	10	12	5	4	8
		2	5	6	7	6	9
		7	6	9	2	3	1
Player X		10	11	14	8	13	8
		20	18	17	10	15	17
		12	11	15	9	5	11

Answer :  $V = 10$

15. Solve the game

		<b>Player B</b>					
		12	14	8	7	4	9
		2	13	6	7	9	8
		13	6	8	6	3	1
Player A		14	9	10	8	9	6
		20	18	17	11	14	16
		8	12	16	9	6	13

Answer :  $V = 11$

16. Examine whether the following game is fair.

		<b>Player Y</b>			
		6	-4	-3	-2
		3	5	0	8
Player X		7	-2	-6	5

Answer :  $V = 0$ . Therefore, it is a fair game.

### LESSON 3

#### GAMES WITH NO SADDLE POINT

##### LESSON OUTLINE

- The concept of a 2x2 game with no saddle point
- The method of solution

##### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the concept of a 2x2 game with no saddle point
- know the method of solution of a 2x2 game without saddle point
- solve a game with a given payoff matrix
- interpret the results obtained from the payoff matrix

2 x 2 zero-sum game

**When each one of the first player A and the second player B has exactly two strategies, we have a 2 x 2 game.**

Motivating point

**First let us consider an illustrative example.**

Problem 1:

**Examine whether the following 2 x 2 game has a saddle point**

	<b>Player B</b>	
<b>Player A</b>	3	5
	4	2

**Solution:**

First consider the minimum of each row.

Row	Minimum Value
1	3
2	2

$$\text{Maximum of } \{3, 2\} = 3$$

**Next consider the maximum of each column.**

Column	Maximum Value
1	4
2	5

$$\text{Minimum of } \{4, 5\} = 4$$

We see that  $\max \{\text{row minima}\}$  and  $\min \{\text{column maxima}\}$  are not equal. Hence the game has no saddle point.

**Method of solution of a 2x2 zero-sum game without saddle point**

Suppose that a 2x2 game has no saddle point. Suppose the game has the following pay-off matrix.

	<b>Player B</b>	
	<b>Strategy</b>	
<b>Player A Strategy</b>	<i>a</i>	<i>b</i>
	<i>c</i>	<i>d</i>

Since this game has no saddle point, the following condition shall hold:

$$\text{Max}\{\text{Min}\{a,b\}, \text{Min}\{c,d\}\} \neq \text{Min}\{\text{Max}\{a,c\}, \text{Max}\{b,d\}\}$$

In this case, the game is called a mixed game. No strategy of Player A can be called the best strategy for him. Therefore A has to use both of his strategies. Similarly no strategy of Player B can be called the best strategy for him and he has to use both of his strategies.

Let  $p$  be the probability that Player A will use his first strategy. Then the probability that Player A will use his second strategy is  $1-p$ .

If Player B follows his first strategy

Expected value of the pay-off to Player A

$$= \left\{ \begin{array}{l} \text{Expected value of the pay-off to Player A} \\ \text{arising from his first strategy} \end{array} \right\} + \left\{ \begin{array}{l} \text{Expected value of the pay-off to Player A} \\ \text{arising from his second strategy} \end{array} \right\}$$

$$= ap + c(1 - p) \tag{1}$$

In the above equation, note that the expected value is got as the product of the corresponding values of the pay-off and the probability.

If Player B follows his second strategy

$$\left. \begin{array}{l} \text{Expected value of the} \\ \text{pay-off to Player A} \end{array} \right\} = bp + d(1 - p) \tag{2}$$

If the expected values in equations (1) and (2) are different, Player B will prefer the minimum of the two expected values that he has to give to player A. Thus B will have a pure strategy. This contradicts our assumption that the game is a mixed one. Therefore the expected values of the pay-offs to Player A in equations (1) and (2) should be equal. Thus we have the condition

$$\begin{aligned}
 ap + c(1 - p) &= bp + d(1 - p) \\
 ap - bp &= (1 - p)[d - c] \\
 p(a - b) &= (d - c) - p(d - c) \\
 p(a - b) + p(d - c) &= d - c \\
 p(a - b + d - c) &= d - c \\
 p &= \frac{d - c}{(a + d) - (b + c)} \\
 1 - p &= \frac{a + d - b - c - d + c}{(a + d) - (b + c)} \\
 &= \frac{a - b}{(a + d) - (b + c)}
 \end{aligned}$$

$$\left\{ \begin{array}{l} \text{The number of times A} \\ \text{will use first strategy} \end{array} \right\} : \left\{ \begin{array}{l} \text{The number of times A} \\ \text{will use second strategy} \end{array} \right\} = \frac{d - c}{(a + d) - (b + c)} : \frac{a - b}{(a + d) - (b + c)}$$

The expected pay-off to Player A

$$\begin{aligned}
 &= ap + c(1 - p) \\
 &= c + p(a - c) \\
 &= c + \frac{(d - c)(a - c)}{(a + d) - (b + c)} \\
 &= \frac{c\{(a + d) - (b + c)\} + (d - c)(a - c)}{(a + d) - (b + c)} \\
 &= \frac{ac + cd - bc - c^2 + ad - cd - ac + c^2}{(a + d) - (b + c)} \\
 &= \frac{ad - bc}{(a + d) - (b + c)}
 \end{aligned}$$

Therefore, the value V of the game is

$$\frac{ad - bc}{(a + d) - (b + c)}$$

**To find the number of times that B will use his first strategy and second strategy:**

Let the probability that B will use his first strategy be  $r$ . Then the probability that B will use his second strategy is  $1 - r$ .

**When A use his first strategy**

The expected value of loss to Player B with his first strategy =  $ar$

The expected value of loss to Player B with his second strategy =  $b(1 - r)$

Therefore the expected value of loss to B =  $ar + b(1 - r)$  (3)

**When A use his second strategy**

The expected value of loss to Player B with his first strategy =  $cr$



The expected value of loss to Player B with his second strategy =  $d(1-r)$

Therefore the expected value of loss to B =  $cr + d(1-r)$  (4)

If the two expected values are different then it results in a pure game, which is a contradiction.

Therefore the expected values of loss to Player B in equations (3) and (4) should be equal.

Hence we have the condition

$$ar + b(1-r) = cr + d(1-r)$$

$$ar + b - br = cr + d - dr$$

$$ar - br - cr + dr = d - b$$

$$r(a - b - c + d) = d - b$$

$$r = \frac{d - b}{a - b - c + d}$$

$$= \frac{d - b}{(a + d) - (b + c)}$$

**Problem 2:**

Solve the following game

$$X \quad \begin{matrix} & Y \\ \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{matrix}$$

**Solution:**

First consider the row minima.

Row	Minimum Value
1	2
2	1

$$\text{Maximum of } \{2, 1\} = 2$$

**Next consider the maximum of each column.**

Column	Maximum Value
1	4
2	5

$$\text{Minimum of } \{4, 5\} = 4$$

We see that

$$\text{Max \{row minima\} } \neq \text{min \{column maxima\}}$$

So the game has no saddle point. Therefore it is a mixed game.

We have  $a = 2$ ,  $b = 5$ ,  $c = 4$  and  $d = 1$ .

Let  $p$  be the probability that player X will use his first strategy. We have

$$\begin{aligned} p &= \frac{d - c}{(a + d) - (b + c)} \\ &= \frac{1 - 4}{(2 + 1) - (5 + 4)} \\ &= \frac{-3}{3 - 9} \\ &= \frac{-3}{-6} \\ &= \frac{1}{2} \end{aligned}$$

The probability that player X will use his second strategy is  $1 - p = 1 - \frac{1}{2} = \frac{1}{2}$ .

$$\text{Value of the game } V = \frac{ad - bc}{(a + d) - (b + c)} = \frac{2 - 20}{3 - 9} = \frac{-18}{-6} = 3.$$

Let  $r$  be the probability that Player Y will use his first strategy. Then the probability that Y will use his second strategy is  $(1 - r)$ . We have

$$\begin{aligned} r &= \frac{d - b}{(a + d) - (b + c)} \\ &= \frac{1 - 5}{(2 + 1) - (5 + 4)} \\ &= \frac{-4}{3 - 9} \\ &= \frac{-4}{-6} \\ &= \frac{2}{3} \\ 1 - r &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

### Interpretation.

$$p : (1 - p) = \frac{1}{2} : \frac{1}{2}$$

Therefore, out of 2 trials, player X will use his first strategy once and his second strategy once.

$$r : (1-r) = \frac{2}{3} : \frac{1}{3}$$

Therefore, out of 3 trials, player Y will use his first strategy twice and his second strategy once.

### QUESTIONS

1. What is a 2x2 game with no saddle point? Explain.
2. Explain the method of solution of a 2x2 game without saddle point.
3. Solve the following game

$$\begin{array}{c} \text{Y} \\ \text{X} \begin{bmatrix} 12 & 4 \\ 3 & 7 \end{bmatrix} \end{array}$$

$$\text{Answer: } p = \frac{1}{3}, r = \frac{1}{4}, V = 6$$

4. Solve the following game

$$\begin{array}{c} \text{Y} \\ \text{X} \begin{bmatrix} 5 & -4 \\ -9 & 3 \end{bmatrix} \end{array}$$

$$\text{Answer: } p = \frac{4}{7}, r = \frac{1}{3}, V = -1$$

5. Solve the following game

$$\begin{array}{c} \text{Y} \\ \text{X} \begin{bmatrix} 10 & 4 \\ 6 & 8 \end{bmatrix} \end{array}$$

$$\text{Answer: } p = \frac{1}{4}, r = \frac{1}{2}, V = 7$$

6. Solve the following game

$$\begin{array}{c} \text{Y} \\ \text{X} \begin{bmatrix} 20 & 8 \\ -2 & 10 \end{bmatrix} \end{array}$$

$$\text{Answer: } p = \frac{1}{2}, r = \frac{1}{12}, V = 9$$

7. Solve the following game

Y

$$X \begin{bmatrix} 10 & 2 \\ 1 & 5 \end{bmatrix}$$

Answer:  $p = \frac{1}{3}, r = \frac{1}{4}, V = 4$

8. Solve the following game

$$X \begin{matrix} & Y \\ \begin{bmatrix} 12 & 6 \\ 6 & 9 \end{bmatrix} \end{matrix}$$

Answer:  $p = \frac{1}{3}, r = \frac{1}{3}, V = 8$

9. Solve the following game

$$X \begin{matrix} & Y \\ \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix} \end{matrix}$$

Answer:  $p = \frac{1}{2}, r = \frac{1}{2}, V = 9$

10. Solve the following game

$$X \begin{matrix} & Y \\ \begin{bmatrix} 16 & 4 \\ 4 & 8 \end{bmatrix} \end{matrix}$$

Answer:  $p = \frac{1}{4}, r = \frac{1}{4}, V = 7$

11. Solve the following game

$$X \begin{matrix} & Y \\ \begin{bmatrix} -11 & 5 \\ 7 & -9 \end{bmatrix} \end{matrix}$$

Answer:  $p = \frac{1}{2}, r = \frac{7}{16}, V = -2$

12. Solve the following game

$$X \begin{matrix} & Y \\ \begin{bmatrix} -9 & 3 \\ 5 & -7 \end{bmatrix} \end{matrix}$$

Answer:  $p = \frac{1}{2}, r = \frac{5}{12}, V = -2$

## LESSON 4

### THE PRINCIPLE OF DOMINANCE

#### LESSON OUTLINE

- The principle of dominance
- Dividing a game into sub games

#### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the principle of dominance
- solve a game using the principle of dominance
- solve a game by dividing a game into sub games

#### **The principle of dominance**

In the previous lesson, we have discussed the method of solution of a game without a saddle point. While solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game. Such a situation is discussed in the sequel.

In a given pay-off matrix A, we say that the  $i^{\text{th}}$  row dominates the  $k^{\text{th}}$  row if

$$a_{ij} \geq a_{kj} \quad \text{for all } j = 1, 2, \dots, n$$

and

$$a_{ij} > a_{kj} \quad \text{for at least one } j.$$

In such a situation player A will never use the strategy corresponding to  $k^{\text{th}}$  row, because he will gain less for choosing such a strategy.

Similarly, we say the  $p^{\text{th}}$  column in the matrix dominates the  $q^{\text{th}}$  column if

$$a_{ip} \leq a_{iq} \quad \text{for all } i = 1, 2, \dots, m$$

and

$$a_{ip} < a_{iq} \quad \text{for at least one } i.$$

In this case, the player B will lose more by choosing the strategy for the  $q^{\text{th}}$  column than by choosing the strategy for the  $p^{\text{th}}$  column. So he will never use the strategy corresponding to the  $q^{\text{th}}$  column. When dominance of a row ( or a column) in the pay-off matrix occurs, we can delete a row (or a column) from that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

Let us consider an illustrative example involving the phenomenon of dominance in a game.

**Problem 1:**

Solve the game with the following pay-off matrix:

		Player B			
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>
Player A	1	4	2	3	6
	2	3	4	7	5
	3	6	3	5	4

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	2
2	3
3	3

$$\text{Maximum of } \{2, 3, 3\} = 3$$

Next consider the maximum of each column.

Column	Maximum Value
1	6
2	4
3	7
4	6

$$\text{Minimum of } \{6, 4, 7, 6\} = 4$$

The following condition holds:

$$\text{Max \{row minima\} } \neq \text{min \{column maxima\}}$$

Therefore we see that there is no saddle point for the game under consideration.

Compare columns II and III.

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3.

Now we have the reduced game

$$\begin{array}{c} I \quad II \quad IV \\ 1 \begin{bmatrix} 4 & 2 & 6 \end{bmatrix} \\ 2 \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \\ 3 \begin{bmatrix} 6 & 3 & 4 \end{bmatrix} \end{array}$$

For this matrix again, there is no saddle point. Column II dominates column IV. The choice is for player B. So player B will give up his strategy 4

The game reduces to the following:

$$\begin{array}{c} I \quad II \\ 1 \begin{bmatrix} 4 & 2 \end{bmatrix} \\ 2 \begin{bmatrix} 3 & 4 \end{bmatrix} \\ 3 \begin{bmatrix} 6 & 3 \end{bmatrix} \end{array}$$

This matrix has no saddle point.

The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following:

$$\begin{bmatrix} 3 & 4 \\ 6 & 3 \end{bmatrix}$$

Again, there is no saddle point. We have a 2x2 matrix. Take this matrix as  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then we have  $a = 3$ ,  $b = 4$ ,  $c = 6$  and  $d = 3$ . Use the formulae for  $p$ ,  $1-p$ ,  $r$ ,  $1-r$  and  $V$ .

$$\begin{aligned} p &= \frac{d - c}{(a + d) - (b + c)} \\ &= \frac{3 - 6}{(3 + 3) - (6 + 4)} \\ &= \frac{-3}{6 - 10} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4} \\ 1 - p &= 1 - \frac{3}{4} = \frac{1}{4} \end{aligned}$$

$$\begin{aligned}
 r &= \frac{d-b}{(a+d)-(b+c)} \\
 &= \frac{3-4}{(3+3)-(6+4)} \\
 &= \frac{-1}{6-10} \\
 &= \frac{-1}{-4} \\
 &= \frac{1}{4} \\
 1-r &= 1 - \frac{1}{4} = \frac{3}{4}
 \end{aligned}$$

The value of the game

$$\begin{aligned}
 V &= \frac{ad-bc}{(a+d)-(b+c)} \\
 &= \frac{3 \times 3 - 4 \times 6}{-4} \\
 &= \frac{-15}{-4} \\
 &= \frac{15}{4}
 \end{aligned}$$

Thus,  $X = \left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)$  and  $Y = \left(\frac{1}{4}, \frac{3}{4}, 0, 0\right)$  are the optimal strategies.

### Method of convex linear combination

A strategy, say  $s$ , can also be dominated if it is inferior to a convex linear combination of several other pure strategies. In this case if the domination is strict, then the strategy  $s$  can be deleted. If strategy  $s$  dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination will be decided as per the above rules. Let us consider an example to illustrate this case.

### Problem 2:

Solve the game with the following pay-off matrix for firm A:

		Firm B				
		$B_1$	$B_2$	$B_3$	$B_4$	$B_5$
Firm A	$A_1$	4	8	-2	5	6
	$A_2$	4	0	6	8	5
	$A_3$	-2	-6	-4	4	2
	$A_4$	4	-3	5	6	3
	$A_5$	4	-1	5	7	3



Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-2
2	0
3	-6
4	-3
5	-1

Maximum of  $\{-2, 0, -6, -3, -1\} = 0$

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	8
3	6
4	8
5	6

Minimum of  $\{4, 8, 6, 8, 6\} = 4$

Hence,

Maximum of {row minima}  $\neq$  minimum of {column maxima}.

So we see that there is no saddle point. Compare the second row with the fifth row. Each element in the second row exceeds the corresponding element in the fifth row. Therefore,  $A_2$  dominates  $A_5$ . The choice is for firm A. It will retain strategy  $A_2$  and give up strategy  $A_5$ .

Therefore the game reduces to the following.

$$\begin{array}{c}
 B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5 \\
 \begin{array}{l}
 A_1 \\
 A_2 \\
 A_3 \\
 A_4
 \end{array}
 \left[ \begin{array}{ccccc}
 4 & 8 & -2 & 5 & 6 \\
 4 & 0 & 6 & 8 & 5 \\
 -2 & -6 & -4 & 4 & 2 \\
 4 & -3 & 5 & 6 & 3
 \end{array} \right]
 \end{array}$$

Compare the second and fourth rows. We see that  $A_2$  dominates  $A_4$ . So, firm A will retain the strategy  $A_2$  and give up the strategy  $A_4$ . Thus the game reduces to the following:

$$\begin{array}{c} B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5 \\ A_1 \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \end{bmatrix} \\ A_2 \begin{bmatrix} 4 & 0 & 6 & 8 & 5 \end{bmatrix} \\ A_3 \begin{bmatrix} -2 & -6 & -4 & 4 & 2 \end{bmatrix} \end{array}$$

Compare the first and fifth columns. It is observed that  $B_1$  dominates  $B_5$ . The choice is for firm B. It will retain the strategy  $B_1$  and give up the strategy  $B_5$ . Thus the game reduces to the following

$$\begin{array}{c} B_1 \quad B_2 \quad B_3 \quad B_4 \\ A_1 \begin{bmatrix} 4 & 8 & -2 & 5 \end{bmatrix} \\ A_2 \begin{bmatrix} 4 & 0 & 6 & 8 \end{bmatrix} \\ A_3 \begin{bmatrix} -2 & -6 & -4 & 4 \end{bmatrix} \end{array}$$

Compare the first and fourth columns. We notice that  $B_1$  dominates  $B_4$ . So firm B will discard the strategy  $B_4$  and retain the strategy  $B_1$ . Thus the game reduces to the following:

$$\begin{array}{c} B_1 \quad B_2 \quad B_3 \\ A_1 \begin{bmatrix} 4 & 8 & -2 \end{bmatrix} \\ A_2 \begin{bmatrix} 4 & 0 & 6 \end{bmatrix} \\ A_3 \begin{bmatrix} -2 & -6 & -4 \end{bmatrix} \end{array}$$

For this reduced game, we check that there is no saddle point.

Now none of the pure strategies of firms A and B is inferior to any of their other strategies. But, we observe that convex linear combination of the strategies  $B_2$  and  $B_3$  dominates  $B_1$ , i.e. the averages of payoffs due to strategies  $B_2$  and  $B_3$ ,

$$\left\{ \frac{8-2}{2}, \frac{0+6}{2}, \frac{-6-4}{2} \right\} = \{3, 3, -5\}$$

dominate  $B_1$ . Thus  $B_1$  may be omitted from consideration. So we have the reduced matrix

$$\begin{array}{c} B_2 \quad B_3 \\ A_1 \begin{bmatrix} 8 & -2 \end{bmatrix} \\ A_2 \begin{bmatrix} 0 & 6 \end{bmatrix} \\ A_3 \begin{bmatrix} -6 & -4 \end{bmatrix} \end{array}$$

Here, the average of the pay-offs due to strategies  $A_1$  and  $A_2$  of firm A, i.e.

$$\left\{ \frac{8+0}{2}, \frac{-2+6}{2} \right\} = \{4, 2\}$$

dominates the pay-off due to  $A_3$ . So we get a new reduced 2x2 pay-off matrix.

$$\begin{array}{rcc}
 & & \text{Firm B's strategy} \\
 & & B_2 \quad B_3 \\
 \text{Firm A's strategy} & A_1 & \begin{bmatrix} 8 & -2 \end{bmatrix} \\
 & A_2 & \begin{bmatrix} 0 & 6 \end{bmatrix}
 \end{array}$$

We have  $a = 8$ ,  $b = -2$ ,  $c = 0$  and  $d = 6$ .

$$\begin{aligned}
 p &= \frac{d-c}{(a+d)-(b+c)} \\
 &= \frac{6-0}{(6+8)-(-2+0)} \\
 &= \frac{6}{16} \\
 &= \frac{3}{8}
 \end{aligned}$$

$$1-p = 1 - \frac{3}{8} = \frac{5}{8}$$

$$\begin{aligned}
 r &= \frac{d-b}{(a+d)-(b+c)} \\
 &= \frac{6-(-2)}{16} \\
 &= \frac{8}{16} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$1-r = 1 - \frac{1}{2} = \frac{1}{2}$$

Value of the game:

$$\begin{aligned}
 V &= \frac{ad-bc}{(a+d)-(b+c)} \\
 &= \frac{6 \times 8 - 0 \times (-2)}{16} \\
 &= \frac{48}{16} = 3
 \end{aligned}$$

So the optimal strategies are

$$A = \left\{ \frac{3}{8}, \frac{5}{8}, 0, 0, 0 \right\} \text{ and } B = \left\{ 0, \frac{1}{2}, \frac{1}{2}, 0, 0 \right\}.$$

The value of the game = 3. Thus the game is favourable to firm A.

**Problem 3:**

For the game with the following pay-off matrix, determine the saddle point

		Player B			
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>
Player A	1	2	-1	0	-3
	2	1	0	3	2
	3	-3	-2	-1	4

**Solution:**

		<i>Column II</i>	<i>Column III</i>	
1	$\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$		$0 > -1$
2				$3 > 0$
3				$-1 > -2$

The choice is with the player B. He has to choose between strategies II and III. He will lose more in strategy III than in strategy II, irrespective of what strategy is followed by A. So he will drop strategy III and retain strategy II. Now the given game reduces to the following game.

		<i>I</i>	<i>II</i>	<i>IV</i>
1	$\begin{bmatrix} 2 & -1 & -3 \end{bmatrix}$			
2	$\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$			
3	$\begin{bmatrix} -3 & -2 & 4 \end{bmatrix}$			

Consider the rows and columns of this matrix.

Row minimum:

I Row	:	-3	
II Row	:	0	Maximum of $\{-3, 0, -3\} = 0$
III Row	:	-3	

Column maximum:

I Column	:	2	
II Column	:	0	Minimum of $\{2, 0, 4\} = 0$
III Column	:	4	

We see that

$$\text{Maximum of row minimum} = \text{Minimum of column maximum} = 0.$$

So, a saddle point exists for the given game and the value of the game is 0.

**Interpretation:**

No player gains and no player loses. i.e., The game is not favourable to any player. i.e. It is a fair game.

**Problem 4:**

Solve the game

$$\begin{array}{c} \text{Player B} \\ \text{Player A} \begin{bmatrix} 4 & 8 & 6 \\ 6 & 2 & 10 \\ 4 & 5 & 7 \end{bmatrix} \end{array}$$

**Solution:**

First consider the minimum of each row.

Row	Minimum
1	4
2	2
3	4

$$\text{Maximum of } \{4, 2, 4\} = 4$$

Next, consider the maximum of each column.

Column	Maximum
1	6
2	8
3	10

$$\text{Minimum of } \{6, 8, 10\} = 6$$

Since Maximum of { Row Minima } and Minimum of { Column Maxima } are different, it follows that the given game has no saddle point.

Denote the strategies of player A by  $A_1, A_2, A_3$ . Denote the strategies of player B by  $B_1, B_2, B_3$ .

Compare the first and third columns of the given matrix.

$$\begin{array}{cc} B_1 & B_3 \\ \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} & \begin{bmatrix} 6 \\ 10 \\ 7 \end{bmatrix} \end{array}$$

The pay-offs in  $B_3$  are greater than or equal to the corresponding pay-offs in  $B_1$ . The player B has to make a choice between his strategies 1 and 3. He will lose more if he follows

strategy 3 rather than strategy 1. Therefore he will give up strategy 3 and retain strategy 1. Consequently, the given game is transformed into the following game:

$$\begin{array}{cc} & B_1 & B_2 \\ A_1 & \begin{bmatrix} 4 & 8 \end{bmatrix} \\ A_2 & \begin{bmatrix} 6 & 2 \end{bmatrix} \\ A_3 & \begin{bmatrix} 4 & 5 \end{bmatrix} \end{array}$$

Compare the first and third rows of the above matrix.

$$\begin{array}{cc} & B_1 & B_2 \\ A_1 & \begin{bmatrix} 4 & 8 \end{bmatrix} \\ A_3 & \begin{bmatrix} 4 & 5 \end{bmatrix} \end{array}$$

The pay-offs in  $A_1$  are greater than or equal to the corresponding pay-offs in  $A_3$ . The player A has to make a choice between his strategies 1 and 3. He will gain more if he follows strategy 1 rather than strategy 3. Therefore he will retain strategy 1 and give up strategy 3. Now the given game is transformed into the following game.

$$\begin{array}{cc} & B_1 & B_2 \\ A_1 & \begin{bmatrix} 4 & 8 \end{bmatrix} \\ A_2 & \begin{bmatrix} 6 & 2 \end{bmatrix} \end{array}$$

It is a 2x2 game. Consider the row minima.

Row	Minimum
1	4
2	2

$$\text{Maximum of } \{4, 2\} = 4$$

Next, consider the maximum of each column.

Column	Maximum
1	6
2	8

$$\text{Minimum of } \{6, 8\} = 6$$

Maximum {row minima} and Minimum {column maxima} are not equal. Therefore, the reduced game has no saddle point. So, it is a mixed game.

Take  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$ . We have  $a = 4$ ,  $b = 8$ ,  $c = 6$  and  $d = 2$ .

The probability that player A will use his first strategy is  $p$ . This is calculated as

$$\begin{aligned}
 p &= \frac{d - c}{(a + d) - (b + c)} \\
 &= \frac{2 - 6}{(4 + 2) - (8 + 6)} \\
 &= \frac{-4}{6 - 14} \\
 &= \frac{-4}{-8} = \frac{1}{2}
 \end{aligned}$$

The probability that player B will use his first strategy is r. This is calculated as

$$\begin{aligned}
 r &= \frac{d - b}{(a + d) - (b + c)} \\
 &= \frac{2 - 8}{-8} \\
 &= \frac{-6}{-8} \\
 &= \frac{3}{4}
 \end{aligned}$$

Value of the game is V. This is calculated as

$$\begin{aligned}
 V &= \frac{ad - bc}{(a + d) - (b + c)} \\
 &= \frac{4 \times 2 - 8 \times 6}{-8} \\
 &= \frac{8 - 48}{-8} \\
 &= \frac{-40}{-8} = 5
 \end{aligned}$$

### Interpretation

Out of 3 trials, player A will use strategy 1 once and strategy 2 once. Out of 4 trials, player B will use strategy 1 thrice and strategy 2 once. The game is favourable to player A.

### Problem 5: Dividing a game into sub-games

Solve the game with the following pay-off matrix.

		Player B		
		1	2	3
Player A	I	-4	6	3
	II	-3	3	4
	III	2	-3	4

### Solution:

First, consider the row minima.

Row	Minimum
1	-4
2	-3
3	-3

$$\text{Maximum of } \{-4, -3, -3\} = -3$$

Next, consider the column maxima.

Column	Maximum
1	2
2	6
3	4

$$\text{Minimum of } \{2, 6, 4\} = 2$$

We see that Maximum of { row minima}  $\neq$  Minimum of { column maxima}.

So the game has no saddle point. Hence it is a mixed game. Compare the first and third columns.

$$\begin{array}{ccc}
 \text{I Column} & \text{III Column} & \\
 \left[ \begin{array}{c} -4 \\ -3 \\ 2 \end{array} \right] & \left[ \begin{array}{c} 3 \\ 4 \\ 4 \end{array} \right] & \begin{array}{l} -4 \leq 3 \\ -3 \leq 4 \\ 2 \leq 4 \end{array}
 \end{array}$$

We assert that Player B will retain the first strategy and give up the third strategy. We get the following reduced matrix.

$$\left[ \begin{array}{cc} -4 & 6 \\ -3 & 3 \\ 2 & -3 \end{array} \right]$$

We check that it is a game with no saddle point.

### Sub games

Let us consider the 2x2 sub games. They are:

$$\left[ \begin{array}{cc} -4 & 6 \\ -3 & 3 \end{array} \right] \left[ \begin{array}{cc} -4 & 6 \\ 2 & -3 \end{array} \right] \left[ \begin{array}{cc} -3 & 3 \\ 2 & -3 \end{array} \right]$$

First, take the sub game

$$\left[ \begin{array}{cc} -4 & 6 \\ -3 & 3 \end{array} \right]$$



Compare the first and second columns. We see that  $-4 \leq 6$  and  $-3 \leq 3$ . Therefore, the game reduces to  $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$ . Since  $-4 < -3$ , it further reduces to  $-3$ .

Next, consider the sub game

$$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$$

We see that it is a game with no saddle point. Take  $a = -4$ ,  $b = 6$ ,  $c = 2$ ,  $d = -3$ . Then the value of the game is

$$\begin{aligned} V &= \frac{ad - bc}{(a + d) - (b + c)} \\ &= \frac{(-4)(-3) - (6)(2)}{(-4 + 3) - (6 + 2)} \\ &= 0 \end{aligned}$$

Next, take the sub game  $\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$ . In this case we have  $a = -3$ ,  $b = 3$ ,  $c = 2$  and  $d = -3$ . The

value of the game is obtained as

$$\begin{aligned} V &= \frac{ad - bc}{(a + d) - (b + c)} \\ &= \frac{(-3)(-3) - (3)(2)}{(-3 - 3) - (3 + 2)} \\ &= \frac{9 - 6}{-6 - 5} = -\frac{3}{11} \end{aligned}$$

Let us tabulate the results as follows:

Sub game	Value
$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$	-3
$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$	0
$\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$	$-\frac{3}{11}$

The value of 0 will be preferred by the player A. For this value, the first and third strategies of A correspond while the first and second strategies of the player B correspond to the value 0 of the game. So it is a fair game.

**QUESTIONS**

1. Explain the principle of dominance in the theory of games.
2. Explain how a game can be solved through sub games.
3. Solve the following game by the principle of dominance:

		<b>Player B</b>			
		Strategies			
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>
Player A Strategies	1	8	10	9	14
	2	10	11	8	12
	3	13	12	14	13

Answer:  $V = 12$

4. Solve the game by the principle of dominance:

$$\begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix}$$

Answer:  $V = 4$

5. Solve the game with the following pay-off matrix

$$\begin{bmatrix} 6 & 3 & -1 & 0 & -3 \\ 3 & 2 & -4 & 2 & -1 \end{bmatrix}$$

Answer :  $p = \frac{3}{5}, r = \frac{2}{5}, V = -\frac{11}{5}$

6. Solve the game

$$\begin{bmatrix} 8 & 7 & 6 & -1 & 2 \\ 12 & 10 & 12 & 0 & 4 \\ 14 & 6 & 8 & 14 & 16 \end{bmatrix}$$

Answer :  $p = \frac{4}{9}, r = \frac{7}{9}, V = \frac{70}{9}$

LESSON 5

**GRAPHICAL SOLUTION OF A 2x2 GAME WITH NO SADDLE POINT**

LESSON OUTLINE

- The principle of graphical solution
- Numerical example

LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the principle of graphical solution
- derive the equations involving probability and expected value
- solve numerical problems

**Example:** Consider the game with the following pay-off matrix.

**Player B**

$$\text{Player A} \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$$

First consider the row minima.

Row	Minimum
1	2
2	1

$$\text{Maximum of } \{2, 1\} = 2.$$

Next, consider the column maxima.

Column	Maximum
1	4
2	5

$$\text{Minimum of } \{4, 5\} = 4.$$

We see that  $\text{Maximum} \{ \text{row minima} \} \neq \text{Minimum} \{ \text{column maxima} \}$

So, the game has no saddle point. It is a mixed game.

**Equations involving probability and expected value:**

Let  $p$  be the probability that player A will use his first strategy.

Then the probability that A will use his second strategy is  $1-p$ .

Let  $E$  be the expected value of pay-off to player A.

**When B uses his first strategy**

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 2p + 4(1 - p) \\ &= 2p + 4 - 4p \\ &= 4 - 2p \end{aligned} \tag{1}$$

**When B uses his second strategy**

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 5p + 1(1 - p) \\ &= 5p + 1 - p \\ &= 4p + 1 \end{aligned} \tag{2}$$

Consider equations (1) and (2). For plotting the two equations on a graph sheet, get some points on them as follows:

$$E = -2p+4$$

p	0	1	0.5
E	4	2	3

$$E = 4p+1$$

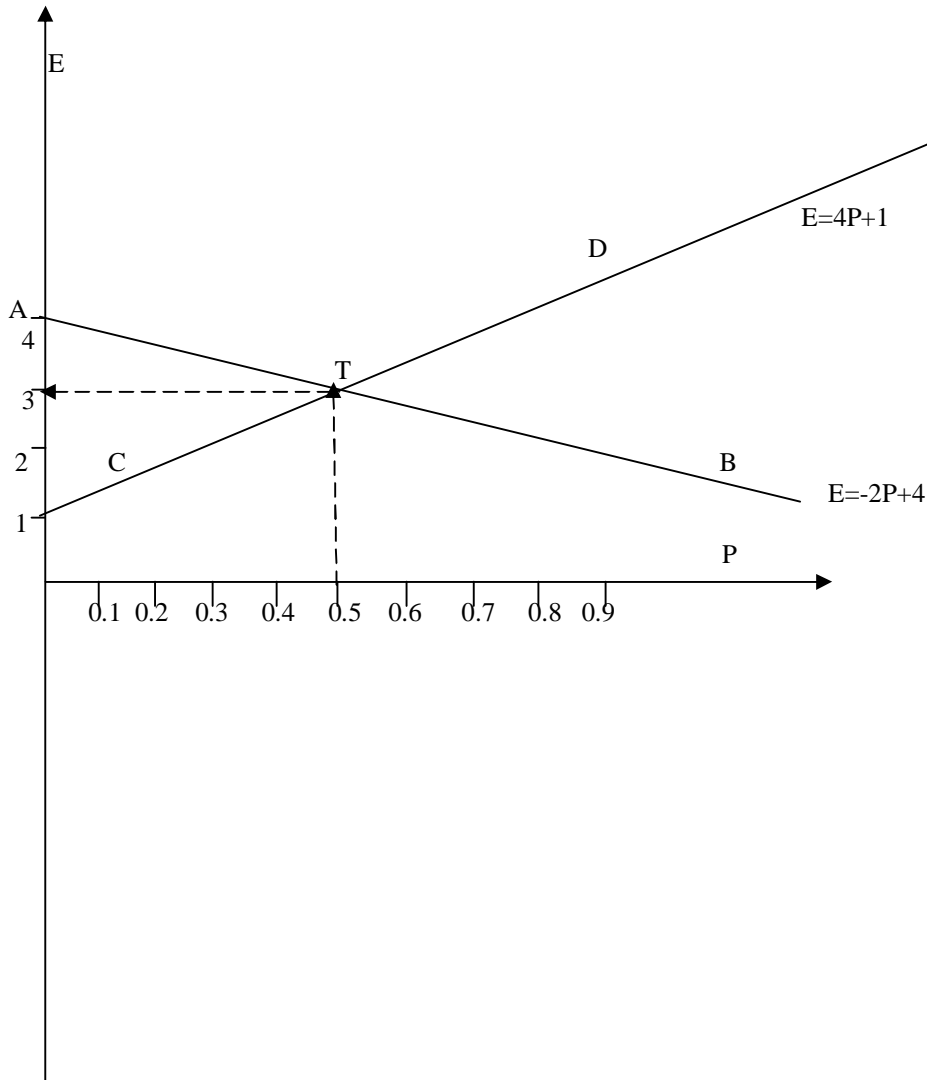
p	0	1	0.5
E	1	5	3

### Graphical solution:

#### Procedure:

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will give the common solution of the two equations (1) and (2). Thus we would obtain the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have  $p = 0.5$  and  $E = 3$ . Therefore, the value V of the game is 3.



**Problem 1:**

Solve the following game by graphical method.

**Player B**

$$\text{Player A} \begin{bmatrix} -18 & 2 \\ 6 & -4 \end{bmatrix}$$

**Solution:**

First consider the row minima.

Row	Minimum
1	- 18
2	- 4

Maximum of  $\{-18, -4\} = -4$ .

Next, consider the column maxima.

Column	Maximum
1	6
2	2

Minimum of  $\{6, 2\} = 2$ .

We see that Maximum { row minima }  $\neq$  Minimum { column maxima }

So, the game has no saddle point. It is a mixed game.

Let  $p$  be the probability that player A will use his first strategy.

Then the probability that A will use his second strategy is  $1-p$ .

#### When B uses his first strategy

The expected value of pay-off to player A is given by

$$\begin{aligned}
 E &= -18p + 6(1-p) \\
 &= -18p + 6 - 6p \\
 &= -24p + 6
 \end{aligned}
 \tag{I}$$

#### When B uses his second strategy

The expected value of pay-off to player A is given by

$$\begin{aligned}
 E &= 2p - 4(1-p) \\
 &= 2p - 4 + 4p \\
 &= 6p - 4
 \end{aligned}
 \tag{II}$$

Consider equations (I) and (II). For plotting the two equations on a graph sheet, get some points on them as follows:

$$E = -24p + 6$$

p	0	1	0.5
E	6	-18	-6

$$E = 6p - 4$$

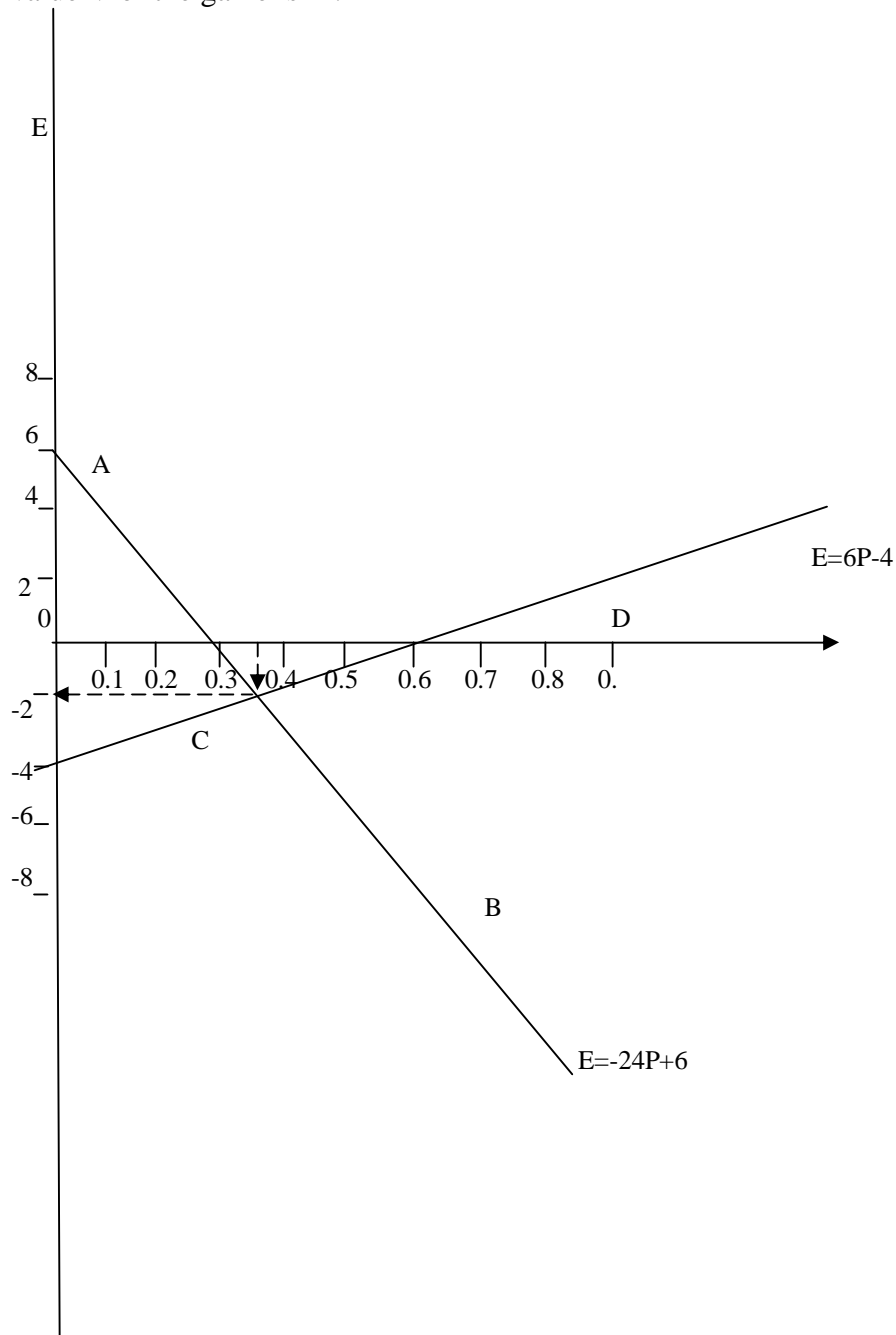
p	0	1	0.5
E	-4	2	-1

#### Graphical solution:

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the

graph. This will provide the common solution of the two equations (1) and (2). Thus we would get the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have  $p = \frac{1}{3}$  and  $E = -2$ . Therefore, the value V of the game is -2.





**1. Explain the method of graphical solution of a 2x2 game.**

**2. Obtain the graphical solution of the game**

$$\begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix}$$

Answer:  $p = \frac{1}{2}, V = 9$

**3. Graphically solve the game**

$$\begin{bmatrix} 4 & 10 \\ 8 & 6 \end{bmatrix}$$

Answer:  $p = \frac{1}{4}, V = 7$

**4. Find the graphical solution of the game**

$$\begin{bmatrix} -12 & 12 \\ 2 & -6 \end{bmatrix}$$

Answer:  $p = \frac{1}{4}, V = -\frac{3}{2}$

**5. Obtain the graphical solution of the game**

$$\begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix}$$

Answer:  $p = \frac{1}{2}, V = 9$

**6. Graphically solve the game**

$$\begin{bmatrix} -3 & -5 \\ -5 & 1 \end{bmatrix}$$

Answer:  $p = \frac{3}{4}, V = -\frac{7}{2}$

## LESSON 6

## 2 x n ZERO-SUM GAMES

## LESSON OUTLINE

- A 2 x n zero-sum game
- Method of solution
- Sub game approach and graphical method
- Numerical example

## LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the concept of a 2 x n zero-sum game
- solve numerical problems

The concept of a 2 x n zero-sum game

**When the first player A has exactly two strategies and the second player B has n (where n is three or more) strategies, there results a 2 x n game. It is also called a rectangular game. Since A has two strategies only, he cannot try to give up any one of them. However, since B has many strategies, he can make out some choice among them. He can retain some of the advantageous strategies and discard some disadvantageous strategies. The intention of B is to give as minimum payoff to A as possible. In other words, B will always try to minimize the loss to himself. Therefore, if some strategies are available to B by which he can minimize the payoff to A, then B will retain such strategies and give such strategies by which the payoff will be very high to A.**

Approaches for 2 x n zero-sum game

**There are two approaches for such games: (1) Sub game approach and (2) Graphical approach.**

#### Sub game approach

The given 2 x n game is divided into 2 x 2 sub games. For this purpose, consider all possible 2 x 2 sub matrices of the payoff matrix of the given game. Solve each sub game and have a list of the values of each sub game. Since B can make out a choice of his strategies, he will discard such of those sub games which result in more payoff to A. On the basis of this consideration, in the long run, he will retain two strategies only and give up the other strategies.

#### Problem

Solve the following game

**Player B**

$$\text{Player A} \begin{bmatrix} 8 & -2 & -6 & 9 \\ 3 & 5 & 10 & 2 \end{bmatrix}$$

**Solution:**

Let us consider all possible 2x2 sub games of the given game. We have the following sub games:

$$1. \begin{bmatrix} 8 & -2 \\ 3 & 5 \end{bmatrix}$$

$$2. \begin{bmatrix} 8 & -6 \\ 3 & 10 \end{bmatrix}$$

$$3. \begin{bmatrix} 8 & 9 \\ 3 & 2 \end{bmatrix}$$

$$4. \begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$$

$$5. \begin{bmatrix} -2 & 9 \\ 5 & 2 \end{bmatrix}$$

$$6. \begin{bmatrix} -6 & 9 \\ 10 & 2 \end{bmatrix}$$

Let E be the expected value of the pay off to player A. Let p be the probability that player A will use his first strategy. Then the probability that he will use his second strategy is 1-p. We form the equations for E in all the sub games as follows:

**Sub game (1)**

$$\text{Equation 1: } E = 8p + 3(1-p) = 5p + 3$$

$$\text{Equation 2: } E = -2p + 5(1-p) = -7p + 5$$

**Sub game (2)**

$$\text{Equation 1: } E = 8p + 3(1-p) = 5p + 3$$

$$\text{Equation 2: } E = -6p + 10(1-p) = -16p + 10$$

**Sub game (3)**

$$\text{Equation 1: } E = 8p + 3(1-p) = 5p + 3$$

$$\text{Equation 2: } E = 9p + 2(1-p) = 7p + 2$$

**Sub game (4)**

$$\text{Equation 1: } E = -2p + 5(1-p) = -7p + 5$$

$$\text{Equation 2: } E = -6p + 10(1-p) = -16p + 10$$

**Sub game (5)**

Equation 1:  $E = -2p + 5(1 - p) = -7p + 5$

Equation 2:  $E = 9p + 2(1 - p) = 7p + 2$

**Sub game (6)**

Equation 1:  $E = -6p + 10(1 - p) = -16p + 10$

Equation 2:  $E = 9p + 2(1 - p) = 7p + 2$

Solve the equations for each sub game. Let us tabulate the results for the various sub games. We have the following:

Sub game	p	Expected value E
1	$\frac{1}{6}$	$\frac{23}{6}$
2	$\frac{1}{3}$	$\frac{14}{3}$
3	$\frac{1}{2}$	$\frac{11}{2}$
4	$\frac{5}{9}$	$\frac{10}{9}$
5	$\frac{3}{14}$	$\frac{7}{2}$
6	$\frac{8}{23}$	$\frac{102}{23}$

**Interpretation:**

Since player A has only 2 strategies, he cannot make any choice on the strategies. On the other hand, player B has 4 strategies. Therefore he can retain any 2 strategies and give up the other 2 strategies. This he will do in such a way that the pay-off to player A is at the minimum. The pay-off to A is the minimum in the case of sub game 4. i.e., the sub game with the matrix  $\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$ .

Therefore, in the long run, player B will retain his strategies 2 and 3 and give up his strategies 1 and 4. In that case, the probability that A will use his first strategy is  $p = \frac{5}{9}$  and the probability that he will use his second strategy is  $1-p = \frac{4}{9}$ . i.e., Out of a total of 9 trials, he will use his first strategy five

times and the second strategy four times. The value of the game is  $\frac{10}{9}$ . The positive sign of V shows that the game is favourable to player A.

**GRAPHICAL SOLUTION:**

Now we consider the graphical method of solution to the given game.

Draw two vertical lines MN and RS. Note that they are parallel to each other. Draw UV perpendicular to MN as well as RS. Take U as the origin on the line MN. Take V as the origin on the line RS.

Mark units on MN and RS with equal scale. The units on the two lines MN and RS are taken as the payoff numbers. The payoffs in the first row of the given matrix are taken along the line MN while the payoffs in the second row are taken along the line RS.

We have to plot the following points: (8, 3), (-2, 5), (-6, 10), (9, 2). The points 8, -2, -6, 9 are marked on MN. The points 3, 5, 10, 2 are marked on RS.

Join a point on MN with the corresponding point on RS by a straight line. For example, join the point 8 on MN with the point 3 on RS. We have 4 such straight lines. They represent the 4 moves of the second player. They intersect in 6 points. Take the lowermost point of intersection of the straight lines. It is called the **Maximin point**. With the help of this point, identify the optimal strategies for the second player. This point corresponds to the points -2 and -6 on MN and 5 and 10 on RS. They correspond to

the sub game with the matrix  $\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$ .

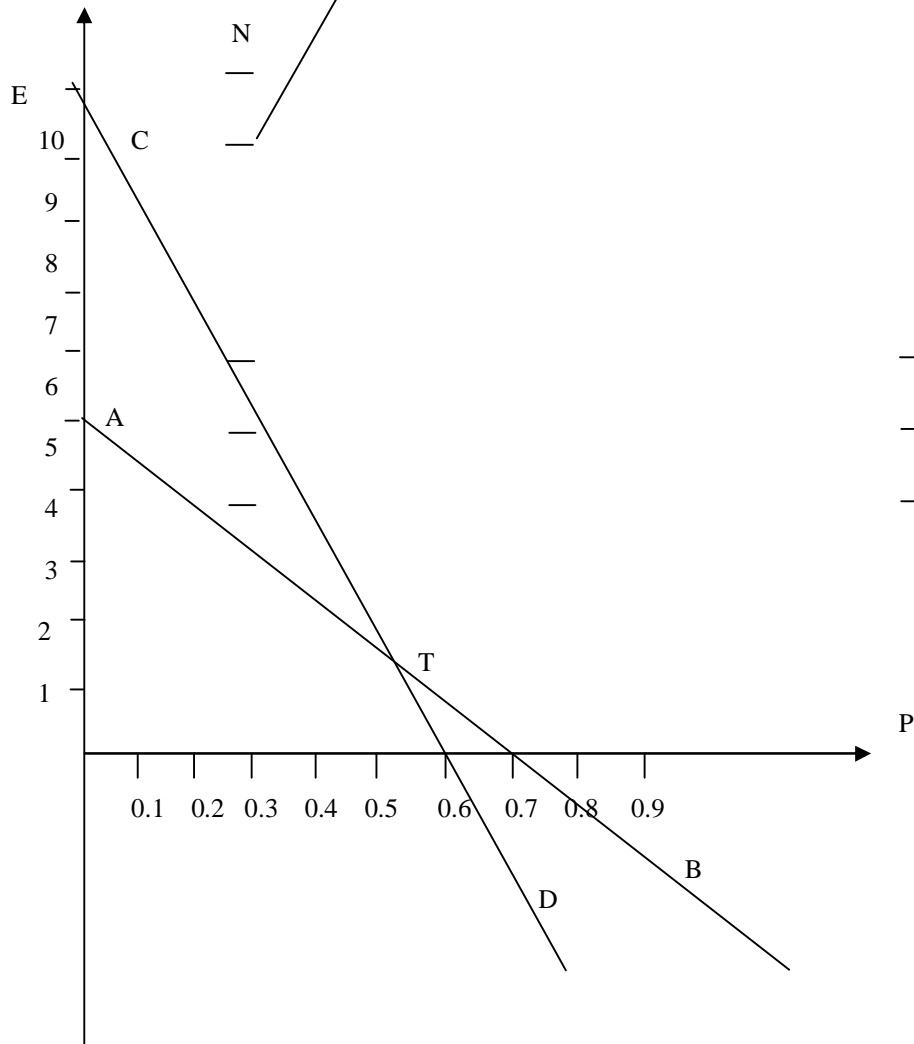
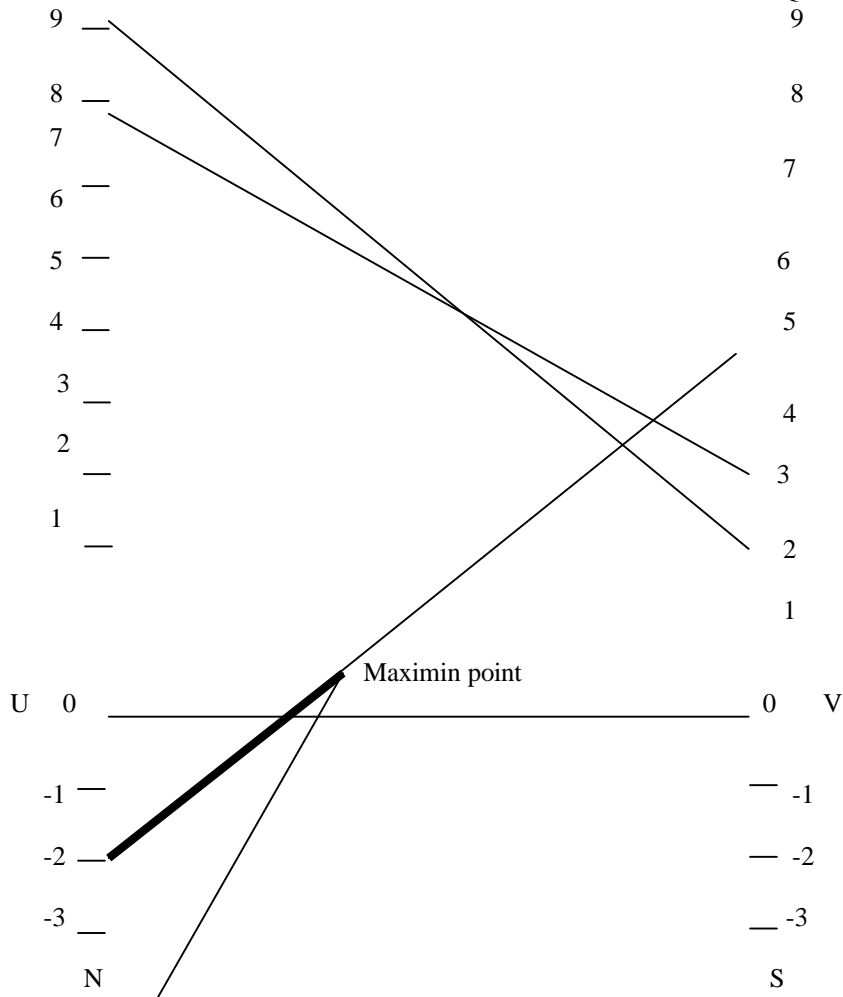
The points -2 and -6 on MN correspond to the second and third strategies of the second player. Therefore, the graphical method implies that, in the long run, the second player will retain his strategies 2 and 3 and give up his strategies 1 and 4.

We graphically solve the sub game with the above matrix. We have to solve the two equations  $E = -7p + 5$  and  $E = -16p + 10$ . Represent the two equations by two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have  $p = \frac{5}{9}$  and  $E =$

$\frac{10}{9}$ . Therefore, the value V of the game is  $\frac{10}{9}$ . We see that the probability that first player will use his

first strategy is  $p = \frac{5}{9}$  and the probability that he will use his second strategy is  $1-p = \frac{4}{9}$ .





$$E = -16P + 10$$

$$E = -7P + 5$$

$$E = -7p + 5$$

p	0	1	0.5
E	5	-2	1.5

$$E = -16p + 10$$

p	0	1	0.5
E	10	-6	2

### QUESTIONS

1. Explain a  $2 \times n$  zero-sum game.
2. Describe the method of solution of a  $2 \times n$  zero-sum game.
3. Solve the following game:

**Player B**

$$\text{Player A} \begin{bmatrix} 10 & 2 & 6 \\ 1 & 5 & 8 \end{bmatrix}$$

$$\text{Answer: } p = \frac{1}{3}, V = 4$$

### LESSON 7

#### m x 2 ZERO-SUM GAMES

#### LESSON OUTLINE

- An  $m \times 2$  zero-sum game
- Method of solution
- Sub game approach and graphical method
- Numerical example

#### LEARNING OBJECTIVES

*After reading this lesson you should be able to*

- understand the concept of an  $m \times 2$  zero-sum game
- solve numerical problems

The concept of an  $m \times 2$  zero-sum game

**When the second player B has exactly two strategies and the first player A has m (where m is three or more) strategies, there results an  $m \times 2$  game. It is also called a rectangular game. Since B has two strategies only, he will find it difficult to discard any one of them. However, since A has**

**more strategies, he will be in a position to make out some choice among them. He can retain some of the most advantageous strategies and give up some other strategies. The motive of A is to get as maximum payoff as possible. Therefore, if some strategies are available to A by which he can get more payoff to himself, then he will retain such strategies and discard some other strategies which result in relatively less payoff.**

Approaches for  $m \times 2$  zero-sum game

**There are two approaches for such games: (1) Sub game approach and (2) Graphical approach.**

### Sub game approach

The given  $m \times 2$  game is divided into  $2 \times 2$  sub games. For this purpose, consider all possible  $2 \times 2$  sub matrices of the payoff matrix of the given game. Solve each sub game and have a list of the values of each sub game. Since A can make out a choice of his strategies, he will be interested in such of those sub games which result in more payoff to himself. On the basis of this consideration, in the long run, he will retain two strategies only and give up the other strategies.

### Problem

Solve the following game:

		<b>Player B</b>	
		Strategies	
		<i>I</i>	<i>II</i>
		Player A Strategies	1
2	-2		10
3	12		4
4	6		5

### Solution:

Let us consider all possible  $2 \times 2$  sub games of the given game. We have the following sub games:

$$7. \begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$$

$$8. \begin{bmatrix} 5 & 8 \\ 12 & 4 \end{bmatrix}$$

$$9. \begin{bmatrix} 5 & 8 \\ 6 & 5 \end{bmatrix}$$

$$10. \begin{bmatrix} -2 & 10 \\ 12 & 4 \end{bmatrix}$$



$$11. \begin{bmatrix} -2 & 10 \\ 6 & 5 \end{bmatrix}$$

$$12. \begin{bmatrix} 12 & 4 \\ 6 & 5 \end{bmatrix}$$

Let  $E$  be the expected value of the payoff to player A. i.e., the loss to player B. Let  $r$  be the probability that player B will use his first strategy. Then the probability that he will use his second strategy is  $1-r$ . We form the equations for  $E$  in all the sub games as follows:

**Sub game (1)**

$$\text{Equation 1: } E = 5r + 8(1-r) = -3r + 8$$

$$\text{Equation 2: } E = -2r + 10(1-r) = -12r + 10$$

**Sub game (2)**

$$\text{Equation 1: } E = 5r + 8(1-r) = -3r + 8$$

$$\text{Equation 2: } E = 12r + 4(1-r) = 8r + 4$$

**Sub game (3)**

$$\text{Equation 1: } E = 5r + 8(1-r) = -3r + 8$$

$$\text{Equation 2: } E = 6r + 5(1-r) = r + 5$$

**Sub game (4)**

$$\text{Equation 1: } E = -2r + 10(1-r) = -12r + 10$$

$$\text{Equation 2: } E = 12r + 4(1-r) = 8r + 4$$

**Sub game (5)**

$$\text{Equation 1: } E = -2r + 10(1-r) = -12r + 10$$

$$\text{Equation 2: } E = 6r + 5(1-r) = r + 5$$

**Sub game (6)**

$$\text{Equation 1: } E = 12r + 4(1-r) = 8r + 4$$

$$\text{Equation 2: } E = 6r + 5(1-r) = r + 5$$

Solve the equations for each 2x2 sub game. Let us tabulate the results for the various sub games. We have the following:

Sub game	R	Expected value E
1	$\frac{2}{9}$	$\frac{22}{3}$

2	$\frac{4}{11}$	$\frac{76}{11}$
3	$\frac{3}{4}$	$\frac{23}{4}$
4	$\frac{3}{10}$	$\frac{32}{5}$
5	$\frac{5}{13}$	$\frac{70}{13}$
6	$\frac{1}{7}$	$\frac{36}{7}$

**Interpretation:**

Since player B has only 2 strategies, he cannot make any choice on his strategies. On the other hand, player A has 4 strategies and so he can retain any 2 strategies and give up the other 2 strategies. Since the choice is with A, he will try to maximize the payoff to himself. The pay-off to A is the maximum in the case of sub game 1. i.e., the sub game with the matrix  $\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$ .

Therefore, player A will retain his strategies 1 and 2 and discard his strategies 3 and 4, in the long run. In that case, the probability that B will use his first strategy is  $r = \frac{2}{9}$  and the probability that he will use his second strategy is  $1-r = \frac{7}{9}$ . i.e., Out of a total of 9 trials, he will use his first strategy two times and the second strategy seven times.

The value of the game is  $\frac{22}{3}$ . The positive sign of V shows that the game is favourable to player A.

**GRAPHICAL SOLUTION:****Now we consider the graphical method of solution to the given game.**

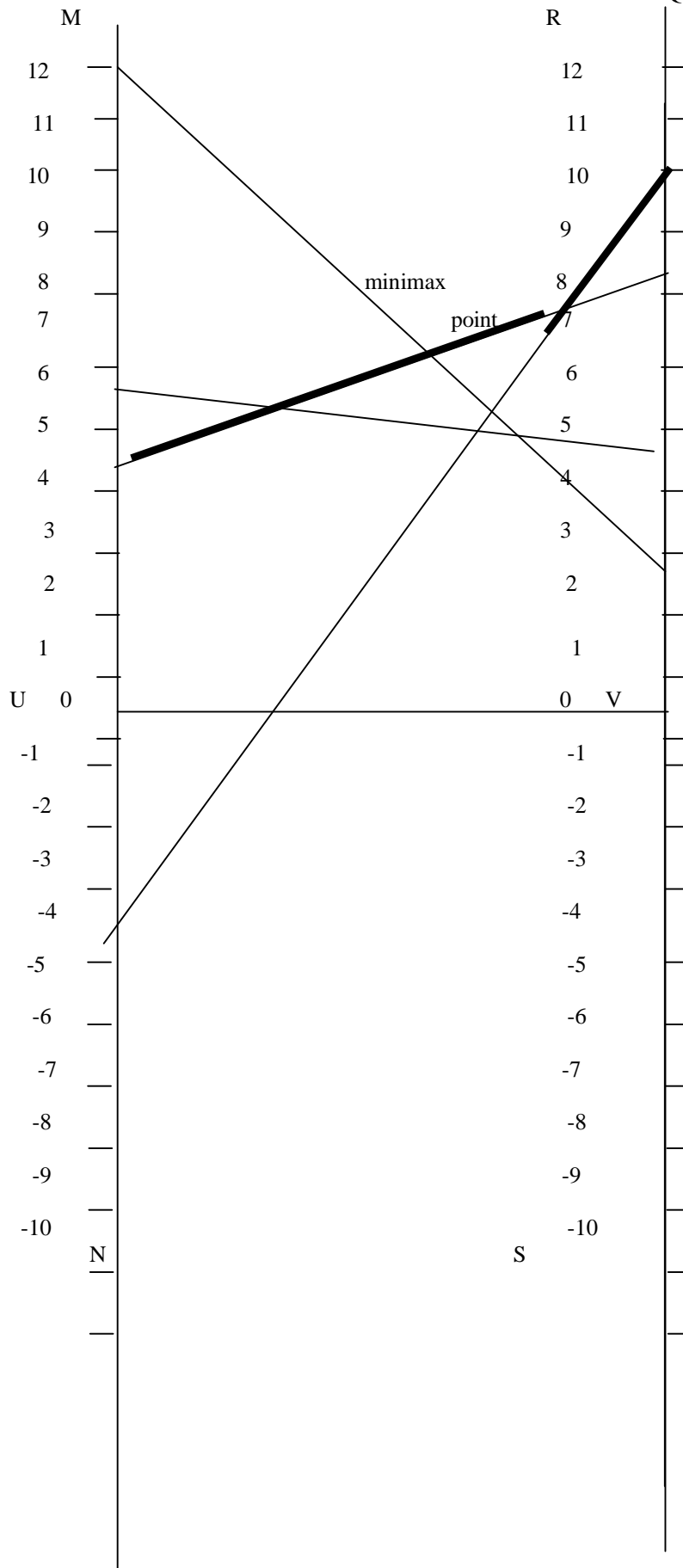
Draw two vertical lines MN and RS. Note that they are parallel to each other. Draw UV perpendicular to MN as well as RS. Take U as the origin on the line MN. Take V as the origin on the line RS.

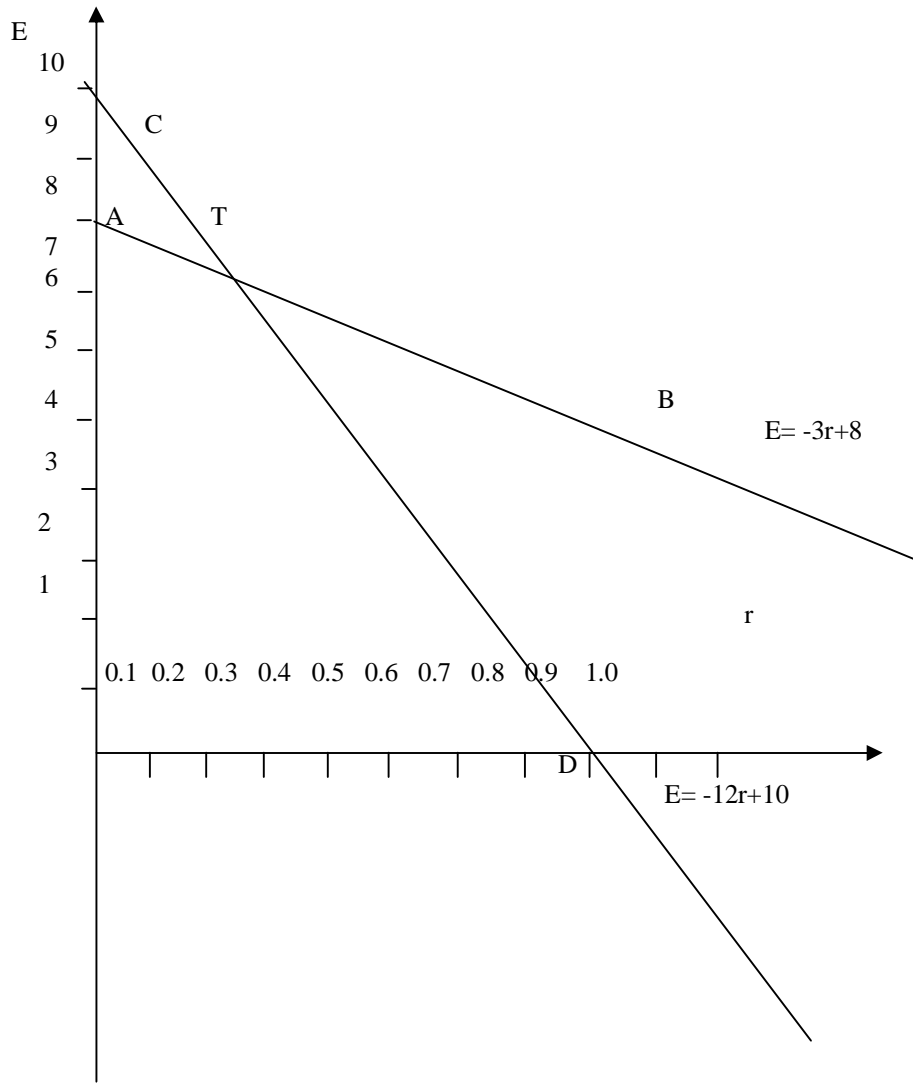
Mark units on MN and RS with equal scale. The units on the two lines MN and RS are taken as the payoff numbers. The payoffs in the first row of the given matrix are taken along the line MN while the payoffs in the second row are taken along the line RS.

We have to plot the following points: (5, 8), (-2, 10), (12, 4), (6, 5). The points 5, -2, 12, 6 are marked on MN. The points 8, 10, 4, 5 are marked on RS.

Join a point on MN with the corresponding point on RS by a straight line. For example, join the point 5 on MN with the point 8 on RS. We have 4 such straight lines. They represent the 4 moves of the first player. They intersect in 6 points. Take the uppermost point of intersection of the straight lines. It is called the **Minimax point**. With the help of this point, identify the optimal strategies for the first player. This point corresponds to the points 5 and -2 on MN and 8 and 10 on RS. They correspond to the sub game with the matrix  $\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$ . The points 5 and -2 on MN correspond to the first and second strategies of the first player. Therefore, the graphical method implies that the first player will retain his strategies 1 and 2 and give up his strategies 3 and 4, in the long run.

We graphically solve the sub game with the above matrix. We have to solve the two equations  $E = -3r + 8$  and  $E = -12r + 10$ . Represent the two equations by two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have  $r = \frac{2}{9}$  and  $E = \frac{22}{3}$ . Therefore, the value V of the game is  $\frac{22}{3}$ . We see that the probability that the second player will use his first strategy is  $r = \frac{2}{9}$  and the probability that he will use his second strategy is  $1-r = \frac{7}{9}$ .





$E = -3r + 8$

$E = -12r + 10$

p	0	1	0.5
E	8	5	6.5

p	0	1	0.5
E	10	-2	4

1. What is an  $m \times 2$  zero-sum game? Explain.
2. How will you solve an  $m \times 2$  zero-sum game? Explain.
3. Solve the following game:

					Player B
					Strategies
			<i>I</i>	<i>II</i>	
		1	20	8	
		2	5	2	
		3	8	12	
Player A Strategies					

Answer:  $r = \frac{1}{4}$ ,  $V = 11$

## LESSON 8

**LINEAR PROGRAMMING APPROACH TO GAME THEORY****LESSON OUTLINE**

4. How to solve a game with LPP?
5. Formulation of LPP
6. Solution by simplex method

**LEARNING OBJECTIVES**

*After reading this lesson you should be able to*

- understand the transformation of a game into LPP
- solve a game by simplex method

## Introduction

When there is neither saddle point nor dominance in a problem of game theory and the payoff matrix is of order 3x3 or higher, the probability and graphical methods cannot be employed. In such a case, linear programming approach may be followed to solve the game.

**Linear programming technique:**

A general approach to solve a game by linear programming technique is presented below. Consider the following  $m \times n$  game:

		<b>Player B</b>		
		$Y_1$	$Y_2$	$Y_n$
Player A	$X_1$	$a_{11}$	$a_{12}$	$a_{1n}$
	$X_2$	$a_{21}$	$a_{22}$	$a_{2n}$
	$X_m$	$a_{m1}$	$a_{m2}$	$a_{mn}$

It is required to determine the optimal strategy for  $A = \{X_1, X_2, \dots, X_m\}$  and  $B = \{Y_1, Y_2, \dots, Y_n\}$ . First we shall determine the optimal strategies of player B.

If player A adopts strategy  $X_1$ , then the expected value of loss to B is

$$a_{11}Y_1 + a_{12}Y_2 + \dots + a_{1n}Y_n \leq V,$$

where V is the value of game. If A adopts strategy  $X_2$ , then the expected value of loss to B is

$$a_{21}Y_1 + a_{22}Y_2 + \dots + a_{2n}Y_n \leq V$$

and so on. Also we have

$$Y_1 + Y_2 + \dots + Y_n = 1$$

and

$$Y_j \geq 0 \text{ for all } j.$$

Without loss of generality, we can assume that  $V > 0$ . Divide each of the above relation by  $V$  and let

$$Y'_j = \frac{Y_j}{V}.$$

Then we have

$$\sum Y'_j = \sum \frac{Y_j}{V} = \frac{1}{V}$$

From this we obtain

$$a_{11}Y'_1 + a_{12}Y'_2 + \dots + a_{1n}Y'_n \leq 1,$$

$$a_{21}Y'_1 + a_{22}Y'_2 + \dots + a_{2n}Y'_n \leq 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + \dots + a_{mn}Y'_n \leq 1$$

and

$$Y'_1 + Y'_2 + \dots + Y'_n = \frac{1}{V}$$

with  $Y'_j \geq 0$  for all  $j$ .

The objective of player B is to minimise the loss to himself. Thus the problem is to minimize  $V$ , or equivalently to maximise  $\frac{1}{V}$ . Therefore, the objective of player B is to maximise the value of  $Y'_1 + Y'_2 + \dots + Y'_n$  subject to the  $m$  linear constraints provided above.

### Statement of the problem:

Maximise:  $Y'_1 + Y'_2 + \dots + Y'_n$ , subject to

$$a_{11}Y'_1 + a_{12}Y'_2 + \dots + a_{1n}Y'_n \leq 1,$$

$$a_{21}Y'_1 + a_{22}Y'_2 + \dots + a_{2n}Y'_n \leq 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + \dots + a_{mn}Y'_n \leq 1$$

$$Y'_1, Y'_2, \dots, Y'_n \geq 0.$$



We can use simplex method to solve the above problem. For this purpose, we have to introduce non-negative slack variables  $s_1, s_2, \dots, s_m$  to each of the inequalities. So the problem can be restated as follows:

**Restatement of the problem:**

Maximise:  $Y'_1 + Y'_2 + \dots + Y'_n + 0s_1 + 0s_2 + \dots + 0s_m$  subject to

$$a_{11}Y'_1 + a_{12}Y'_2 + \dots + a_{1n}Y'_n + s_1 = 1,$$

$$a_{21}Y'_1 + a_{22}Y'_2 + \dots + a_{2n}Y'_n + s_2 = 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + \dots + a_{mn}Y'_n + s_m = 1$$

with  $Y_j = Y'_j V$  for all  $j$  and  $s_1 \geq 0, s_2 \geq 0, \dots, s_m \geq 0$ .

Thus we get the optimal strategy for player B to be  $(Y_1, Y_2, \dots, Y_n)$ .

In a similar manner we can determine the optimal strategy for player A.

**Application:**

We illustrate the method for a 2X2 zero sum game.

Problem 1:

Solve the following game by simplex method for LPP:

**Player B**

Player A	3	6
	5	2

**Solution :**

Row minima :      I row            : 3  
                           II row            : 2  
                           Maximum of {3,2} = 3

Column maxima:    I column        : 5  
                           II column       : 6  
                           Minimum of {5,6} = 5

So, Maximum of {Row minima}  $\neq$  Minimum of {Column maxima}.

Therefore the given game has no saddle point. It is a mixed game. Let us convert the given game into a LPP.

**Problem formulation:**

Let  $V$  denote the value of the game. Let the probability that the player B will use his first strategy be  $r$  and second strategy be  $s$ . Let  $V$  denote the value of the game.

**When A follows his first strategy:**

The expected payoff to A (i.e., the expected loss to B) =  $3r + 6s$ .

This pay-off cannot exceed  $V$ . So we have

$$3r + 6s \leq V \quad (1)$$

**When A follows his second strategy:**

The expected pay-off to A (i.e., expected loss to B) =  $5r + 2s$ .

This cannot exceed  $V$ . Hence we obtain the condition

$$5r + 2s \leq V \quad (2)$$

From (1) and (2) we have

$$3\frac{r}{V} + 6\frac{s}{V} \leq 1$$

and  $5\frac{r}{V} + 2\frac{s}{V} \leq 1$

Substitute  $\frac{r}{V} = x$ ,  $\frac{s}{V} = y$ .

Then we have

$$3x + 6y \leq 1$$

and  $5x + 2y \leq 1$

where  $r$  and  $s$  are connected by the relation

$$r + s = 1.$$

i.e.,  $\frac{r}{V} + \frac{s}{V} = \frac{1}{V}$

i.e.,  $x + y = \frac{1}{V}$

B will try to minimise  $V$ . i.e., He will try to maximise  $\frac{1}{V}$ . Thus we have the following LPP.

$$\text{Maximize } \frac{1}{V} = x + y,$$

subject to the restrictions

$$3x + 6y \leq 1,$$

$$5x + 2y \leq 1,$$

$$x \geq 0, y \geq 0$$

**Solution of LPP:**

Introduce two slack variables  $s_1, s_2$ . Then the problem is transformed into the

following one:

$$\text{Maximize } \frac{1}{V} = x + y + 0.s_1 + 0.s_2$$

subject to the constraints

$$3x + 6y + 1.s_1 + 0.s_2 = 1,$$

$$5x + 2y + 0.s_1 + 1.s_2 = 1,$$

$$x \geq 0, y \geq 0, s_1 \geq 0, s_2 \geq 0$$

Let us note that the above equations can be written in the form of a single matrix equation as

$$A X = B$$

$$\text{where } A = \begin{bmatrix} 3 & 6 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The entries in B are referred to as the b – values. Initially, the basic variables are  $s_1, s_2$ . We have the following simplex tableau:

	x	y	$s_1$	$s_2$	b – value
$s_1$ - row	3	6	1	0	1
$s_2$ - row	5	2	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are  $-1, -1$ . The absolute values are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable  $x$ . We take the new basic variable as  $x$ . Consider the ratio of b-value to  $x$ -value. We have the following ratios:

$$s_1 \text{ - row} \quad : \frac{1}{3}$$

$$s_2 \text{ - row} \quad : \frac{1}{5}$$

$$\text{Minimum of } \left\{ \frac{1}{3}, \frac{1}{5} \right\} = \frac{1}{5}.$$

Hence select  $s_2$  as the leaving variable. Thus the pivotal element is 5. We obtain the following tableau at the end of Iteration No. 1.

	x	y	$s_1$	$s_2$	b – value
$s_1$ - row	0	$\frac{24}{5}$	1	$-\frac{3}{5}$	$\frac{2}{5}$
x - row	1	$\frac{2}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$
Objective function row	0	$-\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$

Now, the negative element in the objective function row is  $-\frac{3}{5}$ . This corresponds to y. We take the new basic variable as y. Consider the ratio of b-value to y-value. We have the following ratios:

$$s_1 \text{ - row} : \frac{\left(\frac{2}{5}\right)}{\left(\frac{24}{5}\right)} = \frac{1}{12}$$

$$x \text{ - row} : \frac{\left(\frac{1}{5}\right)}{\left(\frac{2}{5}\right)} = \frac{1}{2}$$

$$\text{Minimum of } \left\{ \frac{1}{12}, \frac{1}{2} \right\} = \frac{1}{12}$$

Hence select  $s_1$  as the leaving variable. The pivotal element is  $\frac{24}{5}$ . We get the following tableau at the end of Iteration No. 2.

	x	y	$s_1$	$s_2$	b – value
y - row	0	1	$\frac{5}{24}$	$-\frac{1}{8}$	$\frac{1}{12}$
x - row	1	0	$-\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{6}$
Objective function row	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

Since both x and y have been made basic variables, we have reached the stopping condition.

The optimum value of  $\frac{1}{V}$  is  $\frac{1}{4}$ . This is provided by  $x = \frac{1}{6}$  and  $y = \frac{1}{12}$ . Thus the optimum

value of the game is obtained as  $V = 4$ . Using the relations  $\frac{r}{V} = x$ ,  $\frac{s}{V} = y$ , we obtain  $r = \frac{4}{6} = \frac{2}{3}$  and

$$s = \frac{4}{12} = \frac{1}{3}.$$

Problem 2:

Solve the following game:

	Player B
Player A	$\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$

Solution:

The game has no saddle point. It is a mixed game. Let the probability that B will use his first strategy be  $r$ . Let the probability that B will use his second strategy be  $s$ . Let  $V$  be the value of the game.

**When A follows his first strategy:**

The expected payoff to A (i.e., the expected loss to B) =  $2r + 5s$ .

The pay-off to A cannot exceed  $V$ . So we have

$$2r + 5s \leq V \quad (\text{I})$$

**When A follows his second strategy:**

The expected pay-off to A (i.e., expected loss to B) =  $4r + s$ .

The pay-off to A cannot exceed  $V$ . Hence we obtain the condition

$$4r + s \leq V \quad (\text{II})$$

From (I) and (II) we have

$$2\frac{r}{V} + 5\frac{s}{V} \leq 1$$

and  $4\frac{r}{V} + \frac{s}{V} \leq 1$

Substitute

$$\frac{r}{V} = x \text{ and } \frac{s}{V} = y.$$

Thus we have

$$2x + 5y \leq 1$$

and  $4x + y \leq 1$

where  $r$  and  $s$  are connected by the relation

$$r + s = 1.$$

$$\text{i.e., } \frac{r}{V} + \frac{s}{V} = \frac{1}{V}$$

$$\text{i.e., } x + y = \frac{1}{V}$$

The objective of B is to minimise V. i.e., He will try to maximise  $\frac{1}{V}$ .

Thus we are led to the following linear programming problem:

$$\text{Maximize } \frac{1}{V} = x + y$$

subject to the constraints

$$2x + 5y \leq 1,$$

$$4x + y \leq 1,$$

$$x \geq 0, y \geq 0.$$

To solve this linear programming problem, we use simplex method as detailed below.

Introduce two slack variables  $s_1, s_2$ . Then the problem is transformed into the following one:

$$\text{Maximize } \frac{1}{V} = x + y + 0.s_1 + 0.s_2$$

subject to the constraints

$$2x + 5y + 1.s_1 + 0.s_2 = 1,$$

$$4x + y + 0.s_1 + 1.s_2 = 1,$$

$$x \geq 0, y \geq 0, s_1 \geq 0, s_2 \geq 0$$

We have the following simplex tableau:

	x	y	$s_1$	$s_2$	b – value
$s_1$ - row	2	5	1	0	1
$s_2$ - row	4	1	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are  $-1, -1$ . The absolute value are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable  $x$ . We take the new basic variable as  $x$ . Consider the ratio of b-value to  $x$ -value. We have the following ratios:

$$s_1 \text{ - row} \quad : \frac{1}{2}$$

$$s_2 \text{ - row} \quad : \quad \frac{1}{4}$$

$$\text{Minimum of } \left\{ \frac{1}{2}, \frac{1}{4} \right\} = \frac{1}{4}$$

Hence select  $s_2$  as the leaving variable. Thus the pivotal element is 4. We obtain the following tableau at the end of Iteration No. 1.

	x	y	$s_1$	$s_2$	b – value
$s_1$ - row	0	$\frac{9}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$
x - row	1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
Objective function row	0	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$

Now, the negative element in the objective function row is  $-\frac{3}{4}$ . This corresponds to y. We take the new basic variable as y. Consider the ratio of b-value to y-value. We have the following ratios:

$$s_1 \text{ - row} \quad : \quad \frac{\left(\frac{1}{2}\right)}{\left(\frac{9}{2}\right)} = \frac{1}{9}$$

$$x \text{ - row} \quad : \quad \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)} = 1$$

$$\text{Minimum of } \left\{ \frac{1}{9}, 1 \right\} = \frac{1}{9}$$

Hence select  $s_1$  as the leaving variable. The pivotal element is  $\frac{9}{2}$ . We get the following tableau at the end of Iteration No. 2.

	x	y	$s_1$	$s_2$	b – value
y - row	0	1	$\frac{2}{9}$	$-\frac{1}{9}$	$\frac{1}{9}$

$x$ - row	1	0	$-\frac{1}{18}$	$\frac{5}{18}$	$\frac{2}{9}$
Objective function row	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Since both  $x$  and  $y$  have been made basic variables, we have reached the stopping condition.

The optimum value of  $\frac{1}{V}$  is  $\frac{1}{3}$ .

This is provided by  $x = \frac{2}{9}$  and  $y = \frac{1}{9}$ . Thus the optimum value of the game is got as  $V = 3$ .

Using the relations  $\frac{r}{V} = x$ ,  $\frac{s}{V} = y$ , we obtain  $r = \frac{6}{9} = \frac{2}{3}$  and  $s = \frac{3}{9} = \frac{1}{3}$ .

Problem 3:

Solve the following game by simplex method for LPP:

**Player B**

Player A	$-48$	$2$
	$6$	$-4$

**Solution :**

Row minima : I row : -48  
 II row : -4  
 Maximum of  $\{-48, -4\} = -4$

Column maxima: I column : 6  
 II column : 2  
 Minimum of  $\{6, 2\} = 2$

So, Maximum of {Row minima}  $\neq$  Minimum of {Column maxima}.

Therefore the given game has no saddle point. It is a mixed game. Let us convert the given game into a LPP.

**Problem formulation:**

Let  $V$  denote the value of the game. Let the probability that the player B will use his first strategy be  $r$  and second strategy be  $s$ . Let  $V$  denote the value of the game.

**When A follows his first strategy:**

The expected payoff to A (i.e., the expected loss to B) =  $-48r + 2s$ .

This pay-off cannot exceed  $V$ . So we have



$$-48r + 2s \leq V$$

**When A follows his second strategy:**

The expected pay-off to A (i.e., expected loss to B) =  $6r - 4s$ .

This cannot exceed V. Hence we obtain the condition

$$6r - 4s \leq V \quad (2)'$$

From (1)' and (2)' we have

$$-48\frac{r}{V} + 2\frac{s}{V} \leq 1$$

and

$$6\frac{r}{V} - 4\frac{s}{V} \leq 1$$

Substitute  $\frac{r}{V} = x$ ,  $\frac{s}{V} = y$ .

Thus we have

$$-48x + 2y \leq 1$$

$$\text{and } 6x - 4y \leq 1$$

where  $r$  and  $s$  are connected by the relation

$$r + s = 1.$$

$$\text{i.e., } \frac{r}{V} + \frac{s}{V} = \frac{1}{V}$$

$$\text{i.e., } x + y = \frac{1}{V}$$

B will try to minimise V. i.e., He will try to maximise  $\frac{1}{V}$ . Thus we have the following LPP.

$$\text{Maximize } \frac{1}{V} = x + y,$$

subject to the restrictions

$$-48x + 2y \leq 1,$$

$$6x - 4y \leq 1,$$

$$x \geq 0, y \geq 0$$

**Solution of LPP:**

Introduce two slack variables  $s_1, s_2$ . Then the problem is transformed into the following one:

$$\text{Maximize } \frac{1}{V} = x + y + 0.s_1 + 0.s_2$$

subject to the constraints

$$\begin{aligned} -48x + 2y + 1.s_1 + 0.s_2 &= 1, \\ 6x - 4y + 0.s_1 + 1.s_2 &= 1, \\ x \geq 0, y \geq 0, s_1 \geq 0, s_2 &\geq 0 \end{aligned}$$

Initially, the basic variables are  $s_1, s_2$ . We have the following simplex tableau:

	x	y	$s_1$	$s_2$	b – value
$s_1$ - row	-48	2	1	0	1
$s_2$ - row	6	-4	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are  $-1, -1$ . The absolute value are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable  $x$ . We take the new basic variable as  $x$ . Consider the ratio of b-value to  $x$ -value. We have the following ratios:

$$s_1 \text{ - row : } -\frac{1}{48}$$

$$s_2 \text{ - row: } \frac{1}{6}$$

$$\text{Minimum of } \left\{ -\frac{1}{48}, \frac{1}{6} \right\} = -\frac{1}{48}.$$

Hence select  $s_1$  as the leaving variable. Thus the pivotal element is  $-48$ . We obtain the following tableau at the end of Iteration No. 1.

	x	y	$s_1$	$s_2$	b – value
$x$ - row	1	$-\frac{1}{24}$	$-\frac{1}{48}$	0	$-\frac{1}{48}$
$s_2$ - row	0	$-\frac{15}{4}$	$\frac{1}{8}$	1	$\frac{9}{8}$
Objective function row	0	$-\frac{25}{24}$	$-\frac{1}{48}$	0	$-\frac{1}{48}$

Now, the negative element in the objective function row is  $-\frac{25}{24}$ . This corresponds to  $y$ . We take the new basic variable as  $y$ . Consider the ratio of b-value to  $y$ -value. We have the following ratios:

$$x\text{-row} : \frac{\left(-\frac{1}{48}\right)}{\left(-\frac{1}{24}\right)} = \frac{1}{2}$$

$$s_2\text{-row} : \frac{\left(\frac{9}{82}\right)}{\left(-\frac{15}{4}\right)} = -\frac{3}{10}$$

$$\text{Minimum of } \left\{\frac{1}{2}, -\frac{3}{10}\right\} = -\frac{3}{10}$$

Hence select  $s_2$  as the leaving variable. The pivotal element is  $-\frac{15}{4}$ . We get the following tableau at the end of Iteration No. 2.

	x	y	$s_1$	$s_2$	b – value
x - row	1	0	$-\frac{1}{480}$	$-\frac{1}{90}$	$-\frac{1}{30}$
y - row	0	1	$-\frac{1}{30}$	$-\frac{4}{15}$	$-\frac{3}{10}$
Objective function row	0	0	$-\frac{1}{18}$	$-\frac{5}{18}$	$-\frac{1}{3}$

Since both  $x$  and  $y$  have been made basic variables, we have reached the stopping condition.

The optimum value of  $\frac{1}{V}$  is  $-\frac{1}{3}$ . This is provided by  $x = -\frac{1}{30}$  and  $y = -\frac{3}{10}$ .

Thus the optimum value of the game is got as  $V = -3$ . Using the relations  $\frac{r}{V} = x$ ,  $\frac{s}{V} = y$ , we obtain

$$r = \frac{1}{10} \text{ and } s = \frac{9}{10}.$$

**Transform the following game into an LPP:**

$$\begin{pmatrix} 1 & 8 & 3 \\ 6 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

**Solution:**

We have to determine the optimal strategy for player B. Using the entries of the given matrix, we obtain the inequalities

$$\begin{aligned} r + 8s + 3t &\leq V, \\ 6r + 4s + 5t &\leq V, \\ s + 2t &\leq V \end{aligned}$$

Dividing by V, we get

$$\begin{aligned} \frac{r}{V} + 8\frac{s}{V} + 3\frac{t}{V} &\leq 1, \\ 6\frac{r}{V} + 4\frac{s}{V} + 5\frac{t}{V} &\leq 1, \\ \frac{s}{V} + 2\frac{t}{V} &\leq 1 \end{aligned}$$

subject to the condition

$$r + s + t = V.$$

Consequently, we have

$$\frac{r}{V} + \frac{s}{V} + \frac{t}{V} = \frac{1}{V}.$$

Substitute

$$\frac{r}{V} = x, \frac{s}{V} = y, \frac{t}{V} = w.$$

Then we have the relations

$$\begin{aligned} x + 8y + 3w &\leq 1, \\ 6x + 4y + 5w &\leq 1, \\ y + 2w &\leq 1. \end{aligned}$$

We have to minimise V. i.e, We have to maximise  $\frac{1}{V} = \frac{r}{V} + \frac{s}{V} + \frac{t}{V}$ . i.e, We have to maximise  $x + y + w$ .

Thus, the given game is transformed into the following equivalent LPP:

$$\text{maximise } x + y + w$$

subject to the restrictions

$$\begin{aligned}x + 8y + 3w &\leq 1, \\6x + 4y + 5w &\leq 1, \\y + 2w &\leq 1, \\x \geq 0, y \geq 0, w &\geq 0.\end{aligned}$$

## QUESTIONS

1. Explain how a game theory problem can be solved as an LPP.
2. Transform the game

$$\begin{array}{cc} Y_1 & Y_2 \\ X_1 \begin{bmatrix} a_{11} & a_{12} \\ X_2 \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \end{bmatrix}\end{array}$$

into an LPP.

3. Using simplex method for LPP, solve the following game:

$$\begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\text{Answer: } r = \frac{1}{2}, s = \frac{1}{2}, V = 4$$

4. Solve the following game with LPP approach:

$$\begin{bmatrix} 10 & 6 \\ 4 & 8 \end{bmatrix}$$

$$\text{Answer: } r = \frac{1}{4}, s = \frac{3}{4}, V = 7$$

**LESSON 9****GOAL PROGRAMMING FORMULATION****LESSON OUTLINE**

7. Introduction to goal programming
8. Formulation of goal programming problems

**LEARNING OBJECTIVES**

*After reading this lesson you should be able to*

- understand the importance of goal programming
- formulate goal programming problems

**Introduction**

Generally speaking, the objective of a business organization is to maximize profits and to minimize expenditure, loss and wastage. However, a company may not always attempt at profit maximization only. At times, a necessity may arise to pay attention to other objectives also. We describe some such situations in the sequel.

A manufacturing organization may like to ensure uninterrupted supply of its products even if it means additional expenditure for the procurement of raw materials or personal delivery of goods during truckers' strike, etc. with the objective of assuring the good will of the customers.

A company may be interested in the full utilization of the capacity of the machines and therefore mechanics may be recruited for attending to break downs of the machines even though the occurrence of such break downs may be very rare.

A company, driven by social consciousness, may spend a portion of its profits on the maintenance of trees, parks, public roads, etc. to ensure the safety of the environment, with the objective earning the support of the society.

Another organization may have the objective of establishing brand name by providing high quality products to the consumers and for this purpose it may introduce rigorous measures of quality checks even though it may involve an increased expenditure.

While all the sales persons in a company are formally trained and highly experienced, the management may still pursue a policy to depute them for periodical training in reputed institutes so as to maximize their capability, without minding the extra expenditure incurred for their training.

A travel agency may be interested to ensure customer satisfaction of the highest order and as a consequence it may come forward to operate bus services even to remote places at the normal rates, so as to retain the customers in its fold.

A bank may offer services beyond normal working hours or on holidays even if it means payment of overtime to the staff, in order to adhere to the policy of customer satisfaction on priority basis.

A business organization may accord priority for the welfare of the employees and so a major part of the earnings may be apportioned on employee welfare measures.

A garment designer would like to be always known for the latest fashion and hence may spend more money on fashion design but sell the products at the normal rates, so as to earn the maximum reputation.

A newspaper may be interested in earning the unique distinction of 'Reporter of Remote Rural Areas' and so it may spend more money on journalists and advanced technology for communication.

The above typical instances go to show that the top level management of a business organization may embark upon different goals in addition to profit maximization. Such goals may be necessitated by external events or through internal discussion. At times, one such goal may be in conflict with another goal.

'Goal programming' seeks to deal with the process of decision making in a situation of multiple goals set forth by a business organization. A management may accord equal priority to different goals or sometimes a hierarchy of goals may be prescribed on their importance. One has to strive to achieve the goals in accordance with the priorities specified by the management. Sometimes the goals may be classified as **higher level goals** and **lower level goals** as perceived by the management and one would be interested in first achieving the higher order goals and afterwards considering lower order goals.

Some of the goals that may be preferred by a business organization are : maximum customer satisfaction, maximum good will of the customers, maximum utilization of the machine capacity, maximum reliability of the products, maximum support of the society, maximum utilization of the work force, maximum welfare of the employees, etc.

Since different goals of an organization are based on different units, the goal programming has a multi-dimensional objective function. This is in contrast with a linear programming problem in which the objective function is uni-dimensional.

Given a goal of an organization, one has to determine the conditions under which there will be under-achievement and over-achievement of the goal. The ideal situation will be the one with neither under-achievement nor over-achievement of the goal.

### **Formulation of Goal Programming Problems**

In the sequel, we consider illustrative situations so as to explain the process of problem formulation in goal programming.

#### **Notations**

If there is a single goal, we have the following notations:

Let  $D_u$  denote the under-achievement of the goal.

Let  $D_o$  denote the over-achievement of the goal.

If there are two goals, we have the following notations:

Denote the under-achievement and the over-achievement of one goal by  $D_{u1}$  and  $D_{o1}$  respectively.

Denote the under-achievement and the over-achievement of another goal by  $D_{u2}$  and  $D_{o2}$  respectively.

### **PROBLEM 1:**

Alpha company is known for the manufacture of tables and chairs. There is a profit of Rs. 200 per table and Rs. 80 per chair. Production of a table requires 5 hours of assembly and 3 hours in finishing. In order to produce a chair, the requirements are 3 hours of assembly and 2 hours of finishing. The company has 105 hours of assembly time and 65 hours of finishing. The company manager is

interested to find out the optimal production of tables and chairs so as to have a maximum profit of Rs. 4000. Formulate a goal programming problem for this situation.

**Solution:**

The manager is interested not only in the maximization of profit but he has also fixed a target of Rs. 4000 as profit. Thus, the problem involves a single goal of achieving the specified amount of profit.

Let  $D_u$  denote the under achievement of the target profit and let  $D_o$  be the over achievement.

The objective in the given situation is to minimize under achievement. Let  $Z$  be the objective function. Then the problem is the minimization of  $Z = D_u$ .

**Formulation of the constraints:**

Let the number of tables to be produced be  $x$  and let the number of chairs to be produced be  $Y$ .

Profit from  $x$  tables = Rs.  $200x$

Profit from  $y$  chairs = Rs.  $80y$

The total profit = Profit from  $x$  tables and  $y$  chairs  
 + under achievement of the profit target  
 – over achievement of the profit target

So we have the relationship  $200x + 80y + D_u - D_o = 4000$ .

**Assembly time:**

To produce  $x$  tables, the requirement of assembly time =  $5x$  hours. To produce  $y$  chairs, the requirement is  $3y$  hours. So, the total requirement is  $5x + 3y$  hours. But the available time for assembly is 105 hours. Therefore constraint

$$5x + 3y \leq 105$$

must be fulfilled.

**Finishing time:**

To produce  $x$  tables, the requirement of finishing time =  $3x$ . To produce  $y$  chairs, the requirement is  $2y$ . So, the total requirement is  $3x + 2y$ . But the availability is 65 hours. Hence we have the restriction

$$3x + 2y \leq 65.$$

Non-negativity restrictions:

The number of tables and chairs produced, the under achievement of the profit target and the over achievement cannot be negative. Thus we have the restrictions

$$x \geq 0, y \geq 0, D_u \geq 0, D_o \geq 0.$$



**Statement of the problem:**

$$\text{Minimize } Z = D_u$$

subject to the constraints

$$200x + 80y + D_u - D_o = 4000$$

$$5x + 3y \leq 105$$

$$3x + 2y \leq 65$$

$$x \geq 0, y \geq 0, D_u \geq 0, D_o \geq 0$$

**Problem 2:**

Sweet Bakery Ltd. produces two recipes A and B. Both recipes are made of two food stuffs I and II. Production of one Kg of A requires 7 units of food stuff I and 4 units of food stuff II whereas for producing one Kg of B, 4 units of food stuff I and 3 units of food stuff II are required. The company has 145 units of food stuff I and 90 units of food stuff II. The profit per Kg of A is Rs. 120 while that of B is Rs. 90. The manager wants to earn a maximum profit of Rs. 2700 and to fulfil the demand of 12 Kgs of A. Formulate a goal programming problem for this situation.

**Solution:**

The management has two goals.

1. To reach a profit of Rs. 2700
2. Production of 12 Kgs of recipe A.

Let  $D_{up}$  denote the under achievement of the profit target.

Let  $D_{op}$  denote the over achievement of the profit target.

Let  $D_{uA}$  denote the under achievement of the production target for recipe A.

Let  $D_{oA}$  denote the over achievement of the production target for recipe A.

The objective in this problem is to minimize the under achievement of the profit target and to minimize the under achievement of the production target for recipe A.

Let  $Z$  be the objective function. Then the problem is the minimization of

$$Z = D_{up} + D_{uA}.$$

**Constraints**

Suppose the company has to produce  $x$  kgs of recipe A and  $y$  kgs of recipe B in order to achieve the two goals.

**Condition on profit:**

Profit from  $x$  kgs of A =  $120x$

Profit from  $y$  kgs of B =  $90y$

$$\begin{aligned} \text{The total profit} &= \text{Profit from } x \text{ kgs of A} + \text{Profit from } y \text{ kgs of B} \\ &+ \text{under achievement of the profit target} \\ &- \text{over achievement of the profit target} \\ &= 120x + 90y + D_{up} - D_{op} \end{aligned}$$

Thus we have the restriction

$$120x + 90y + D_{up} - D_{op} = 2700.$$

Constraint for food stuff I:

$$7x + 4y \leq 145$$

Constraint for food stuff II:

$$4x + 3y \leq 90$$

Non-negativity restrictions:

$$x, y, D_{up}, D_{op}, D_{uA}, D_{oA} \geq 0$$

**Condition on recipe A:**

$$\begin{aligned} \text{The target production of A} &= \text{optimal production of A} \\ &+ \text{under achievement in production target of A} \\ &- \text{over achievement of the production target of A.} \end{aligned}$$

Thus we have the condition

$$x + D_{uA} - D_{oA} = 12$$

**Statement of the problem:**

$$\text{Minimize } Z = D_{up} + D_{uA}$$

subject to the constraints

$$120x + 90y + D_{up} - D_{op} = 2700$$

$$x + D_{uA} - D_{oA} = 12$$

$$7x + 4y \leq 145$$

$$4x + 3y \leq 90$$

$$x, y, D_{up}, D_{op}, D_{uA}, D_{oA} \geq 0$$

**QUESTIONS**

1. Explain the necessity of a goal programming.

2. Describe some instances of goal programming.
3. Explain the formulation of a goal programming problem.

## UNIT – V

**Lesson Outline**

- Introduction
- Terminologies of Queueing System
- Empirical Queueing Models
- Simulation – Introduction
- Types of Simulation
- Major Steps of Simulation
- Replacement and Maintenance Analysis – Introduction
- Types of Maintenance
- Types of Replacement Problem
- Determination of Economic life of an Asset

**Learning Objectives**

After reading this lesson you should be able to

- Understand the nature and scope of Queueing System.
- Queueing models and the solution to queueing model problems.
- Importance of Simulation
- Need for Replacement and maintenance
- Solution to problems involving economic life of an Asset.

## 5-1 – Queueing Theory

### 5.1.1 Introduction:

A flow of customers from finite or infinite population towards the service facility forms a **queue (waiting line)** an account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, **waiting time** is required either for the service facilities or for the customers arrival. In general, the **queueing system** consists of one or more queues and one or more servers and operates under a set of procedures. Depending upon the server status, the incoming customer either waits at the queue or gets the turn to be served. If the server is free at the time of arrival of a customer, the customer can directly enter into the counter for getting service and then leave the system. In this process, over a period of time, the system may experience “Customer waiting” and /or “Server idle time”

### 5.1.2 Queueing System:

A queueing system can be completely described by

- (1) the input (arrival pattern)
- (2) the service mechanism (service pattern)
- (3) The queue discipline and
- (4) Customer's behaviour

### 5.1.3. The input (arrival pattern)

The input described the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random manner which is not possible for prediction. Thus the arrival pattern can be described in terms of probabilities and consequently the probability distribution for **inter-arrival** times (the time between two successive arrivals) must be defined. We deal with those Queueing system in which the customers arrive in poisson process. The mean arrival rate is denoted by  $\lambda$ .

### 5.1.4 The Service Mechanism:-

This means the arrangement of service facility to serve customers. If there is infinite number of servers, then all the customers are served instantaneously or arrival and there will be no queue. If the number of servers is finite then the customers are served according to a specific order with service time a constant or a random variable. Distribution of service time follows 'Exponential distribution' defined by

$$f(t) = \lambda e^{-\lambda t}, \quad t > 0$$

The mean Service rate is  $E(t) = 1/\lambda$

### 5.1.5 Queueing Discipline:-

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are

1. First come first served – (FCFS)
2. First in first out – (FIFO)
3. Last in first out – (LIFO)
4. Selection for service in random order (SIRO)

### 5.1.6 Customer's behaviour

1. Generally, it is assumed that the customers arrive into the system one by one. But in some cases, customers may arrive in groups. Such arrival is called **Bulk arrival**.
2. If there is more than one queue, the customers from one queue may be tempted to join another queue because of its smaller size. This behaviour of customers is known as **jockeying**.
3. If the queue length appears very large to a customer, he/she may not join the queue. This property is known as **Balking** of customers.
4. Sometimes, a customer who is already in a queue will leave the queue in anticipation of longer waiting line. This kind of departure is known as **reneging**.

### 5.1.7 List of Variables

The list of variable used in queueing models is give below:

$n$  - No of customers in the system

$C$  - No of servers in the system

$P_n(t)$  – Probability of having  $n$  customers in the system at time  $t$ .

$P_n$  - Steady state probability of having customers in the system

$P_0$  - Probability of having zero customer in the system

$L_q$  - Average number of customers waiting in the queue.

$L_s$  - Average number of customers waiting in the system (in the queue and in the service counters)

$W_q$  - Average waiting time of customers in the queue.

$W_s$  - Average waiting time of customers in the system (in the queue and in the service counters)

$\delta$  - Arrival rate of customers

$\mu$  - Service rate of server

$\phi$  - Utilization factor of the server

$\delta_{\text{eff}}$  - Effective rate of arrival of customers

$M$  - Poisson distribution

$N$  - Maximum numbers of customers permitted in the system. Also, it denotes the size of the calling source of the customers.

$GD$  - General discipline for service. This may be first in first – serve (FIFS), last-in-first serve (LIFS) random order (Ro) etc.

**5.1.8 Traffic intensity (or utilization factor)**

An important measure of a simple queue is its traffic intensity given by

$$\text{Traffic intensity } \phi = \frac{\text{Mean arrival time}}{\text{Mean service time}} = \frac{\delta}{\mu} \quad (< 1)$$

and the unit of traffic intensity is Erlang

**5.1.9 Classification of Queuing models**

Generally, queuing models can be classified into six categories using Kendall's notation with six parameters to define a model. The parameters of this notation are

P- Arrival rate distribution ie probability law for the arrival /inter – arrival time.

Q - Service rate distribution, ie probability law according to which the customers are being served.

- R - Number of Servers (ie number of service stations)
- X - Service discipline
- Y - Maximum number of customers permitted in the system.
- Z - Size of the calling source of the customers.

A queuing model with the above parameters is written as (P/Q/R : X/Y/Z)

**5.1.10 Model 1 : (M/M/1) : (GD/∞/∞) Model**

In this model

- (i) the arrival rate follows poisson (M) distribution.
- (ii) Service rate follows poisson distribution (M)
- (iii) Number of servers is 1
- (iv) Service discipline is general discipline (ie GD)
- (v) Maximum number of customers permitted in the system is infinite (∞)
- (vi) Size of the calling source is infinite (∞)

The steady state equations to obtain, P<sub>n</sub> the probability of having customers in the system and the values for L<sub>s</sub>, L<sub>q</sub>, W<sub>s</sub> and W<sub>q</sub> are given below.

$$n = 0, 1, 2, \dots, \infty \text{ where } \phi = \frac{\delta}{\mu} < 1$$

L<sub>s</sub> – Average number of customers waiting in the system (ie waiting in the queue and in the service station)

$P_n = \phi^n (1 - \phi)$
$L_s = \frac{\phi}{1 - \phi}$

$$L_q = L_s - \frac{\delta}{\mu}$$

$$= \frac{\phi}{1 - \phi} - \phi$$

$$= \frac{\phi - (1 - \phi)\phi}{1 - \phi}$$

$$\begin{aligned} \text{Average number of customers in the system} &= W_s \\ &= \frac{L_s}{\delta} = \frac{\phi}{(1 - \phi)\delta} \\ &= \frac{\phi}{1 - \phi} \times \frac{1}{\mu\phi} = \frac{1}{\mu(1 - \phi)} \end{aligned}$$

$$= \frac{1}{\mu - \mu\phi} \quad (\text{Since } \delta = \mu\phi)$$

$$\frac{1}{\mu - \mu\phi}$$

$$= \frac{1}{\mu - \delta}$$

$$W_s = \frac{1}{\mu - \delta}$$

$W_q$  = Average waiting time of customers in the queue.

$$\begin{aligned} &= L_q / \delta = [1 / \delta] [\phi^2 / [1 - \phi]] \\ &= 1 / \mu\phi [\phi^2 / [1 - \phi]] \\ &= \frac{\phi}{\mu - \mu\phi} \quad \text{Since } \mu\phi = \delta \end{aligned}$$

$$W_q = \frac{\phi}{\mu - \delta}$$

Example 1:

The arrival rate of customers at a banking counter follows a poisson distribution with a mean of 30 per hours. The service rate of the counter clerk also follows poisson distribution with mean of 45 per hour.

- What is the probability of having zero customer in the system ?
- What is the probability of having 8 customer in the system ?
- What is the probability of having 12 customer in the system ?
- Find  $L_s$ ,  $L_q$ ,  $W_s$  and  $W_q$

### Solution

Given arrival rate follows poisson distribution with mean =30

$$\therefore \delta = 30 \text{ per hour}$$

Given service rate follows poisson distribution with mean = 45

$$\therefore \mu = 45 \text{ Per hour}$$

$$\begin{aligned} \therefore \text{Utilization factor } \phi &= \delta / \mu \\ &= 30 / 45 \\ &= 2 / 3 \\ &= 0.67 \end{aligned}$$

- The probability of having zero customer in the system

$$\begin{aligned} P_0 &= \phi^0 (1 - \phi) \\ &= 1 - \phi \\ &= 1 - 0.67 \end{aligned}$$



$$\begin{aligned}
 &= 0.33 \\
 \text{b) The probability of having 8 customers in the system} \\
 P_8 &= \phi^8 (1 - \phi) \\
 &= (0.67)^8 (1 - 0.67) \\
 &= 0.0406 \times 0.33 \\
 &= 0.0134
 \end{aligned}$$

Probability of having 12 customers in the system is

$$\begin{aligned}
 P_{12} &= \phi^{12} (1 - \phi) \\
 &= (0.67)^{12} (1 - 0.67) \\
 &= 0.0082 \times 0.33 = \mathbf{0.002706}
 \end{aligned}$$

$$\begin{aligned}
 L_s &= \frac{\phi}{1 - \phi} = \frac{0.67}{1 - 0.67} \\
 &= \frac{0.67}{0.33} = 2.03 \\
 &= \mathbf{2 \text{ customers}}
 \end{aligned}$$

$$\begin{aligned}
 L_q &= \frac{\phi^2}{1 - \phi} = \frac{(0.67)^2}{1 - 0.67} = \frac{0.4489}{0.33} \\
 &= \mathbf{1 \text{ Customer}}
 \end{aligned}$$

$$\begin{aligned}
 W_s &= \frac{1}{\mu - \delta} = \frac{1}{45 - 30} = \frac{1}{15} \\
 &= \mathbf{0.0666 \text{ hour}}
 \end{aligned}$$

$$\begin{aligned}
 W_q &= \frac{\phi}{\mu - \delta} = \frac{0.67}{45 - 30} = \frac{0.67}{15} \\
 &= \mathbf{0.4467 \text{ hour}}
 \end{aligned}$$

Example 2 :

At one-man barber shop, customers arrive according to poisson dist with mean arrival rate of 5 per hour and the hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:

- (i) Average number of customers in the shop and the average numbers waiting for a haircut.
- (ii) The percentage of time arrival can walk in straight without having to wait.
- (iii) The percentage of customers who have to wait before getting into the barber's chair.

Solution:-

$$\begin{aligned}
 \text{Given mean arrival of customer } \delta &= 5/60 = 1/12 \\
 \text{and mean time for server } \mu &= 1/10 \\
 \therefore \phi = \delta / \mu &= [1/12] \times 10 = 10/12 \\
 &= \mathbf{0.833}
 \end{aligned}$$

(i) Average number of customers in the system (numbers in the queue and in the service station)

$$\begin{aligned}
 L_s &= \phi / 1 - \phi = 0.83 / 1 - 0.83 \\
 &= 0.83 / 0.17 \\
 &= 4.88 \\
 &= \mathbf{5 \text{ Customers}}
 \end{aligned}$$

(ii) The percentage of time arrival can walk straight into barber's chair without waiting is

$$\begin{aligned}
 \text{Service utilization} &= \phi \% \\
 &= \delta / \mu \% \\
 &= 0.833 \times 100 \\
 &= \mathbf{83.3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & \text{The percentage of customers who have to wait before getting into the barber's chair} = (1-\phi)\% \\
 & (1-0.833)\% = 0.167 \times 100 \\
 & = \mathbf{16.7\%}
 \end{aligned}$$

**Example 3 :**

Vehicles are passing through a toll gate at the rate of 70 per hour. The average time to pass through the gate is 45 seconds. The arrival rate and service rate follow poisson distribution. There is a complaint that the vehicles wait for a long duration. The authorities are willing to install one more gate to reduce the average time to pass through the toll gate to 35 seconds if the idle time of the toll gate is less than 9% and the average queue length at the gate is more than 8 vehicle, check whether the installation of the second gate is justified?

**Solutions:-**

$$\begin{aligned}
 \text{Arrival rate of vehicles at the toll gate } \delta &= 70 \text{ per hour} \\
 \text{Time taken to pass through the gate} &= 45 \text{ Seconds} \\
 \text{Service rate } \mu &= \frac{1 \text{ hours}}{45 \text{ seconds}} \\
 &= \frac{3600}{45} = 80 \\
 &= \mathbf{80 \text{ Vehicles per hour}}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Utilization factor } \phi &= \delta/\mu \\
 &= 70 / 80 \\
 &= \mathbf{0.875}
 \end{aligned}$$

$$\begin{aligned}
 \text{(a) Waiting no. of vehicles in the queue is } L_q & \\
 L_q &= \phi^2 / (1 - \phi) = \frac{(0.875)^2}{1-0.875} \\
 &= \frac{0.7656}{0.125} \\
 &= \mathbf{6.125} \\
 &= \mathbf{6 \text{ Vehicles}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) Revised time taken to pass through the gate} &= 30 \text{ seconds} \\
 \therefore \text{The new service rate after installation of an} & \\
 \text{additional gate} &= 1 \text{ hour} / 35 \text{ Seconds} = 3600/35 \\
 &= \mathbf{102.68 \text{ Vehicles / hour}} \\
 \therefore \text{Utilization factor } \phi &= \delta / \mu = 70 / 102.86 \\
 &= \mathbf{0.681}
 \end{aligned}$$

$$\begin{aligned}
 \text{Percentage of idle time of the gate} &= (1-\phi)\% \\
 &= (1-0.681)\% \\
 &= 0.319\% \\
 &= 31.9 \\
 &= \mathbf{32\%}
 \end{aligned}$$

This idle time is not less than 9% which is expected.

Therefore the installation of the second gate is not justified since the average waiting number of vehicles in the queue is more than 8 but the idle time is not less than 32%. Hence idle time is far greater than the number. of vehicles waiting in the queue.

**5.1.11 Second Model (M/M/C) : (GD/∞/∞)Model**

The parameters of this model are as follows:

- (i) Arrival rate follows poisson distribution
- (ii) Service rate follows poisson distribution
- (iii) No of servers is C'.
- (iv) Service discipline is general discipline.

(v) Maximum number of customers permitted in the system is infinite

Then the steady state equation to obtain the probability of having n customers in the system is

$$P_n = \frac{\phi^n P_0}{n!}, \quad 0 \leq n \leq C$$

$$= \frac{\phi^n P_0}{C^{n-c} C!} \quad \text{for } n > c \text{ Where } \phi / c < 1$$

Where  $[\delta / \mu c] < 1$  as  $\phi = \delta / \mu$

C-1

$$\therefore P_0 = \left\{ \sum_{n=0}^{C-1} \frac{\phi^n}{n!} + \frac{\phi^c}{c! [1 - \phi/c]} \right\}^{-1}$$

where  $c! = 1 \times 2 \times 3 \times \dots$  upto C

$$L_q = \left[ \frac{\phi^{c+1}}{c-1! (c-\phi)} \right] \times P_0$$

$$= (c\phi P_c) / (c-\phi)^2$$

$$L_s = L_q + \phi \quad \text{and} \quad W_s = W_q + 1 / \mu$$

$$W_q = L_q / \delta$$

Under special conditions  $P_0 = 1 - \phi$  and  $L_q = \phi^{C+1} / c^2$  Where  $\phi < 1$  and

$$P_0 = (C-\phi) (c-1)! / c^c$$

and  $L_q = \phi / (c-\phi)$ , where  $\phi / c < 1$

**Example 1:**

At a central warehouse, vehicles are at the rate of 24 per hour and the arrival rate follows poisson distribution. The unloading time of the vehicles follows exponential distribution and the unloading rate is 18 vehicles per hour. There are 4 unloading crews. Find

- (i)  $P_0$  and  $P_3$
- (ii)  $L_q, L_s, W_q$  and  $W_s$

**Solution:**

Arrival rate  $\delta = 24$  per hour  
 Unloading rate  $\mu = 18$  Per hour  
 No. of unloading crews  $C=4$   
 $\phi = \delta / \mu = 24 / 18 = 1.33$

C-1

$$(i) P_0 = \left\{ \sum_{n=0}^{C-1} \frac{\phi^n}{n!} + \frac{\phi^c}{c! [1 - \phi/c]} \right\}^{-1}$$

$$= \left\{ \sum_{n=0}^3 \frac{(1.33)^n}{n!} + \frac{(1.33)^4}{4! [1 - (1.33)/4]} \right\}^{-1}$$

$$= \left\{ \frac{(1.33)^0}{0!} + \frac{(1.33)^1}{1!} + \frac{(1.33)^2}{2!} + \frac{(1.33)^3}{3!} + \frac{(1.33)^4}{24! [1 - (1.33)/4]} \right\}^{-1}$$

$$= [1 + 1.33 + 0.88 + 0.39 + 3.129/16.62]^{-1}$$

$$= [3.60 + 0.19]^{-1} = [3.79]^{-1}$$

$$= \mathbf{0.264}$$

We know  $P_n = (\phi^n / n!) P_0$  for  $0 \leq n \leq c$

$$\therefore P_3 = (\phi^3 / 3!) P_0 \quad \text{Since } 0 \leq 3 \leq 4$$

$$= \frac{[(1.33)^3 / 6] \times 0.264}{}$$

$$= 2.353 \times 0.044$$

$$= \mathbf{0.1035}$$

(ii)  $L_q = \frac{\phi^{C+1} \times P_0}{(C-1)! (C-\phi)^2}$

$$= \frac{(1.33)^5 \times 0.264}{3! \times (4-1.33)^2}$$

$$= \frac{(4.1616) \times 0.264}{6 \times (2.77)^2}$$

$$\begin{aligned}
 &= \frac{(4.1616) X}{46.0374} \quad 0.264 \\
 &= 1.099 / 46.0374 \\
 &= 0.0239 \\
 &= \mathbf{0.0239 \text{ Vehicles}} \\
 L_s &= L_q + \phi &= 0.0239 + 1.33 \\
 & &= \mathbf{1.3539 \text{ Vehicles}} \\
 W_q &= L_q / \delta &= 0.0239 / 24 \\
 & &= \mathbf{0.000996 \text{ hrs}} \\
 W_s &= W_q + 1 / \mu &= 0.000996 + 1/18 \\
 & &= 0.000996 + 0.055555 \\
 & &= \mathbf{0.056551 \text{ hours.}}
 \end{aligned}$$

### Example 2 :-

A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if the people arrive in poisson fashion at the rate of 10 per hour

- What is the probability of having to wait for service?
- What is the expected percentage of idle time for each girl?
- If a customer has to wait, what is the expected length of his waiting time?

### Solution:-

$$P_0 = \frac{C-1}{\left\{ \sum_{n=0}^{C-1} \frac{\phi^n}{n!} + \frac{\phi^C}{C! [1 - \phi/C]} \right\}^{-1}}$$

Where  $\phi = \delta / \mu$   $\therefore$  given arrival rate = 10 per hour

$$\delta = 10 / 60 = 1 / 6 \text{ per minute}$$

Service rate = 4 minutes

$$\therefore \mu = 1 / 4 \text{ person per minute}$$

$$\text{Hence } \phi = \delta / \mu = (1/6) \times 4 = 2/3 = \mathbf{0.67}$$

$$P_0 = \frac{1}{\left\{ \sum_{n=0}^1 \frac{\phi^n}{n!} + \frac{(0.67)^2}{2! [1 - 0.67/2]} \right\}^{-1}}$$

$$\begin{aligned}
 &= [1 + (\phi / 1!) ] + 0.4489 / (2 - 0.67)]^{-1} \\
 &= [1 + 0.67 + 0.4489 / (1.33)]^{-1} \\
 &= [1 + 0.67 + 0.34]^{-1} \\
 &= [2.01]^{-1} \\
 &= \mathbf{1 / 2}
 \end{aligned}$$

The Probability of having to wait for the service is

$$\begin{aligned}
 \mathbf{P(w > 0)} &= \frac{\phi^c}{C! [1 - \phi/C]} \times P_0 \\
 &= \frac{0.67^2 \times (1/2)}{2! [1 - 0.67/2]} \\
 &= 0.4489 / 2.66 \\
 &= \mathbf{0.168}
 \end{aligned}$$

- The probability of idle time for each girl is  
 $= 1 - P(w > 0)$

$$= 1 - 1/3$$

$$= 2/3$$

∴ Percentage of time the service remains idle = 67% approximately

c) The expected length of waiting time ( $w/w > 0$ )

$$= 1 / (c \mu - \delta)$$

$$= 1 / [(1/2) - (1/6)]$$

$$= \mathbf{3 \text{ minutes}}$$

### Examples 3 :

A petrol station has two pumps. The service time follows the exponential distribution with mean 4 minutes and cars arrive for service in a poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What proportion of time the pump remains idle?

**Solution: Given C=2 The arrival rate = 10 cars per hour.**

$$\therefore \delta = 10 / 60 = 1 / 6 \text{ car per minute}$$

Service rate = 4 minute per cars.

$$\text{Ie } \mu = 1/4 \text{ car per minute.}$$

$$\begin{aligned} \phi &= \delta / \mu \\ &= (1/6) / (1/4) \\ &= 2 / 3 \\ &= \mathbf{0.67} \end{aligned}$$

Proportion of time the pumps remain busy

$$= \phi / c = 0.67 / 2$$

$$= 0.33$$

$$= 1 / 3$$

∴ The proportion of time, the pumps remain idle

$$= 1 - \text{proportion of the pumps remain busy}$$

$$= 1 - 1/3 = 2/3$$

$$P_0 = \frac{C-1}{\{[\sum_{n=0}^{C-1} \phi^n / n!] + \phi^c / (c! [1 - \phi/c])\}^{-1}}$$

$$= [(0.67)^0 / 0!] + (0.67)^1 / 1! + (0.67)^2 / 2! [1 - (0.67/2)]^{-1}$$

$$= [1 + 0.67 + 0.4489 / (1.33)]^{-1}$$

$$= [1 + 0.67 + 0.33]^{-1}$$

$$= [2]^{-1}$$

$$= \mathbf{1/2}$$

Probability that a customer has to wait for service

$$= p [w > 0]$$

$$= \phi^c \times P_0 = (0.67)^2 \times 1/2$$

$$= \frac{[c [1 - \phi / c]]}{0.4489} = \frac{[2! [1 - 0.67/2]]}{0.4489}$$

$$= \frac{1.33 \times 2}{0.4489} = \frac{2.66}{0.4489}$$

$$= \mathbf{0.1688}$$

## 5.2 Simulation :

Simulation is an experiment conducted on a model of some system to collect necessary information on the behaviour of that system.

### 5.2.1 Introduction :

**The representation of reality in some physical form or in some form of Mathematical equations are called Simulations .**

Simulations are imitation of reality.

**For example :**

1. Children cycling park with various signals and crossing is a simulation of a road traffic system
2. Planetarium
3. Testing an air craft model in a wind tunnel.

### **5.2.2. Need for simulation :**

Consider an example of the queueing system, namely the reservation system of a transport corporation. The elements of the system are booking counters (servers) and waiting customers (queue). Generally the arrival rate of customers follow a Poisson distribution and the service time follows exponential distribution. Then the queueing model (M/M/1) :  $(\lambda/\mu/\infty/\infty)$  can be used to find the standard results. But in reality, the following combinations of distributions may exist.

1. Arrival rate does not follow Poisson distribution, but the service rate follows an exponential distribution.
2. Arrival rate follows a Poisson distribution and the service rate does not follow exponential distribution.
3. Arrival rate does not follow Poisson distribution and the service time also does not follow exponential distribution. In each of the above cases, the standard model (M/M/1) :  $(\lambda/\mu/\infty/\infty)$  cannot be used. The last resort to find the solution for such a queueing problem is to use simulation.

### **5.2.3. Some advantage of simulation :**

1. Simulation is Mathematically less complicated
2. Simulation is flexible
3. It can be modified to suit the changing environments.
4. It can be used for training purpose
5. It may be less expensive and less time consuming in a quite a few real world situations.

### **5.2.4. Some Limitations of Simulation :**

1. Quantification or Enlarging of the variables may be difficult.
2. Large number of variables make simulations unwieldy and more difficult.
3. Simulation may not yield optimum or accurate results.
4. Simulation are most expensive and time consuming model.
5. We cannot rely too much on the results obtained from simulation models.

### **5.2.5. Steps in simulation :**

1. Identify the measure of effectiveness.
2. Decide the variables which influence the measure of effectiveness and choose those variables, which affects the measure of effectiveness significantly.
3. Determine the probability distribution for each variable in step 2 and construct the cumulative probability distribution.
4. Choose an appropriate set of random numbers.
5. Consider each random number as decimal value of the cumulative probability distribution.
6. Use the simulated values so generated into the formula derived from the measure of effectiveness.
7. Repeat steps 5 and 6 until the sample is large enough to arrive at a satisfactory and reliable decision.

### **5.2.6. Uses of Simulation**

Simulation is used for solving

1. Inventory Problem
2. Queueing Problem
3. Training Programmes etc.

**Example :**

Customers arrive at a milk booth for the required service. Assume that inter – arrival and service time are constants and given by 1.5 and 4 minutes respectively. Simulate the system by hand computations for 14 minutes.

- (i) What is the waiting time per customer?
- (ii) What is the percentage idle time for the facility?  
(Assume that the system starts at t = 0)

**Solution :**

**First customer** arrives at the service center at t = 0  
∴ His departure time after getting service = 0 + 4 = 4 minutes.

**Second customer** arrives at time t = 1.5 minutes  
∴ **he has to wait = 4 – 1.5 = 2.5 minutes.**

**Third customer** arrives at time t = 3 minutes  
∴ he has to wait for = 8-3 = 5 minutes

**Fourth customer** arrives at time t = 4.5 minutes and he has to wait for 12 – 4.5 = 7.5 minutes.

During this 4.5 minutes, the first customer leaves in 4 minutes after getting service and the second customer is getting service.

**Fifth customer** arrives at t = 6 minutes  
∴ he has to wait 14 – 6 = 8 minutes

**Sixth customer** arrives at t = 7.5 minutes  
∴ he has to wait 14 – 7.5 = 6.5 minutes

**Seventh customer** arrives at t = 9 minutes  
∴ he has to wait 14 – 9 = 5 minutes

During this 9 minutes the second customer leaves the service in 8<sup>th</sup> minute and third person is to get service in 9<sup>th</sup> minute.

**Eighth customer** arrives at t = 10.5 minutes  
∴ he has to wait 14 – 10.5 = 3.5 minutes

**Ninth customer** arrives at t = 12 minutes  
∴ he has to wait 14 – 12 = 2 minutes

But by 12<sup>th</sup> minute the third customer leaves the Service

**10<sup>th</sup> Customer** arrives at t = 13.5 minutes  
∴ he has to wait 14-13.5 = 0.5 minute

From this simulation table it is clear that

(i) Average waiting time for 10 customers

$$= \frac{2.5+5+7.5+8+6.5+5.0+3.5+2+0.5}{10}$$

$$= \frac{40.5}{10} = 4.05$$

(ii) Average waiting time for 9 customers who are in waiting for service  $\frac{40.5}{9} = 4.5$  minutes.

But the average service time is 4 minutes which is less than the average waiting time, the percentage of idle time for service = 0%

**Exercise :**

1. The arrival rate of customers at a banking counter follows a poisson distribution with a mean of 45 per hours. The service rate of the counter clerk also poisson distribution with a mean of 60 per hours.

- (a) What is the probability of having Zero customer in the system (P<sub>0</sub>).
- (b) What is the probability of having 5 customer in the system (P<sub>5</sub>).
- (c) What is the probability of having 10 customer in the system (P<sub>10</sub>).
- (d) Find Ls, Lq, Ws and Wq

2. Vehicles pass through a toll gate at a rate of 90 per hour. The average time to pass through the gate is 36 seconds. The arrival rate and service rate follow poisson distribution. There is a complaint the vehicles wait for long duration. The authorities are willing to install one more gate to reduce the average time to pass through the toll gate to 30 seconds if the idle time of the toll gate is less than 10% and the average queue length at the gate is more than 5 vehicles. Vehicle whether the installation of second gate is justified?

3. At a central ware house, vehicles arrive at the rate of 24 per hours and the arrival rate follows poisson distribution. The unloading time of the vehicles follows exponential distribution and the unloading rate is 18 vehicles per hour. There are 4 unloading crews. Find the following.

- a)  $P_0$  and  $P_3$
- b)  $L_q$ ,  $L_s$ ,  $W_q$  and  $W_s$

4. Explain Queneing Discipline

5. Describe the Queueing models (M/M/1) : (GD/ $\infty$  / $\infty$ )  
and (M/M/C) : (GD/  $\infty$  /  $\infty$  )

6. Cars arrive at a drive-in restaurant with mean arrival rate of 30 cars per hors and the service rate of the cars is 22 per hors. The arrival rate and the service rate follow poisson distribution. The number parking space for cars is only 5. Find the standard results.

**(Ans  $L_q = 2.38$  cars,  $L_s = 3.3133$  Cars,  $W_q = 0.116$  hors and  $W_s = 0.1615$  hors)**

7. In a harbour, ship arrive with a mean rate of 24 per week. The harbour has 3 docks to handle unloading and loading of ships. The service rate of individual dock is 12 per week. The arrival rate and the service rate follow poisson distribution. At any point of time, the maximum No. of ships permitted in the harbour is 8. Find  $P_0$ ,  $L_q$ ,  $L_s$ ,  $W_q$ ,  $W_s$

**(Ans  $P_0 = 0.1998$ ,  $L_q = 1.0371$  ships,  $L_s = 2.9671$  ships,  $W_q = 0.04478$  week and  $W_s = 0.1281$  week)**

8. Define simulation and its advantages.

9. Discuss the steps of simulation.

### **5.3. Replacement models**

#### **5.3.1. Introduction:**

The replacement problems are concerned with the situations that arise when some items such as men, machines and usable things etc need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

If a firm wants to survive the competition it has to decide on whether to replace the out dated equipment or to retain it, by taking the cost of maintenance and operation into account. There are two basic reasons for considering the replacement of an equipment.

They are



- (i) Physical impairment or malfunctioning of various parts.
- (ii) Obsolescence of the equipment.

The physical impairment refers only to changes in the physical condition of the equipment itself. This will lead to decline in the value of service rendered by the equipment, increased operating cost of the equipments, increased maintenance cost of the equipment or the combination of these costs. Obsolescence is caused due to improvement in the existing Tools and machinery mainly when the technology becomes advanced therefore, it becomes uneconomical to continue production with the same equipment under any of the above situations. Hence the equipments are to be periodically replaced. Some times, the capacity of existing facilities may be inadequate to meet the current demand. Under such cases, the following two alternatives will be considered.

1. Replacement of the existing equipment with a new one
2. Argument the existing one with an additional equipments.

### **5.3.2 Type of Maintenance**

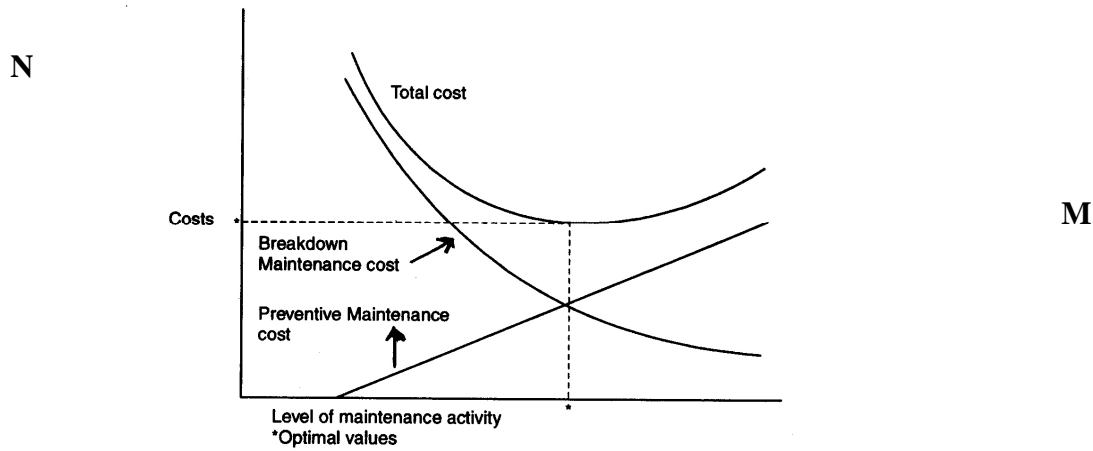
Maintenance activity can be classified into two types

- i) Preventive Maintenance
- ii) Breakdown Maintenance

Preventive maintenance (PN) is the periodical inspection and service which are aimed to detect potential failures and perform minor adjustments a requires which will prevent major operating problem in future. Breakdown maintenance is the repair which is generally done after the equipment breaks down. It is offer an emergency which will have an associated penalty in terms of increasing the cost of maintenance and downtime cost of equipment, Preventive maintenance will reduce such costs up-to a certain extent . Beyond that the cost of preventive maintenance will be more when compared to the cost of the breakdown maintenance.

Total cost = Preventive maintenance cost + Breakdown maintenance cost.

This total cost will go on decreasing up-to P with an increase in the level of maintenance up-to a point, beyond which the total cost will start increasing from P. The level of maintenance corresponding to the minimum total cost at P is the Optional level of maintenance this concept is illustrated in the follows diagram



*The points M and N denote optimal level of maintenance and optimal cost respectively*

**5.3.3 Types of replacement problem**

The replacement problem can be classified into two categories.

- i) Replacement of assets that deteriorate with time (replacement due to gradual failure, due to wear and tear of the components of the machines) This can be further classified into the following types.
  - a) Determination of economic type of an asset.
  - b) Replacement of an existing asset with a new asset.
- ii) simple probabilistic model for assets which will fail completely (replacement due to sudden failure).

**5.3.4. Determination of Economic Life of an asset**

Any asset will have the following cost components

- i) Capital recovery cost (average first cost), Computed from the first cost (Purchase price) of the asset.
- ii) Average operating and maintenance cost.
- iii) Total cost which is the sum of capital recovery cost (average first cost) and average operating and maintenance cost.

**A typical shape of each of the above cost with respect to life of the asset is shown below**

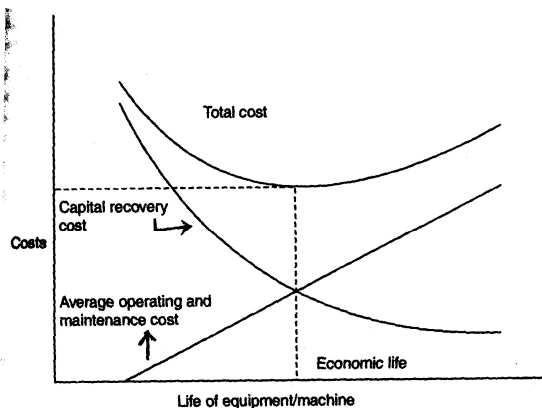


Fig. 13.2 Chart showing economic life.

From figure, when the life of the machine increases, it is clear that the capital recovery cost (average first cost) goes on decreasing and the average operating and maintenance cost goes on increasing. From the beginning the total cost goes on decreasing upto a particular life of the asset and then it starts increasing. The point P where the total cost is in the minimum is called the Economic life of the asset. To solve problems under replacement, we consider the basics of interest formula.

Present worth factor denoted by (P/F, i,n). If an amount P is invested now with amount earning interest at the rate i per year, then the future sum (F) accumulated after n years can be obtained.

- P - Principal sum at year Zero
- F - Future sum of P at the end of the n<sup>th</sup> year
- i - Annual interest rate
- n - Number of interest periods.

Then the formula for future sum  $F = P ( 1 + i ) ^ n$

$$P = F/(1 + i)^n = Fx (\text{present worth factor})$$

If A is the annual equivalent amount which occurs at the end of every year from year one through n years is given by

$$\begin{aligned} A &= \frac{P \times i (1 + i)^n}{(1 + i)^n - 1} \\ &= P ( A / P, i, n ) \\ &= P \times \text{equal payment series capital recovery factor} \end{aligned}$$

**Example:**

A firm is considering replacement of an equipment whose first cost is Rs. 1750 and the scrap value is negligible at any year. Based on experience, it is found that maintenance cost is zero during the first year and it increases by Rs. 100 every year thereafter.

- (i) When should be the equipment replaced if
  - a) i = 0%
  - b) i = 12%

**Solution :**

Given the first cost = Rs 1750 and the maintenance cost is Rs. Zero during the first years and then increases by Rs. 100 every year thereafter. Then the following table shows the calculation.

**Calculations to determine Economic life**

**(a) First cost Rs. 1750                      Interest rate = 0%**

End of year (n)	Maintenance cost at end of year	Summation of maintenance	Average cost of maintenance through the	Average first cost if replaced at the given	Average total cost through the given year
1	0	0	0	1750	1750
2	100	100	50	875	1750 + 100 = 1850
3	200	300	100	583	1750 + 200 + 100 = 2050
4	300	600	150	438	1750 + 300 + 200 + 100 = 2350
5	400	1000	200	350	1750 + 400 + 300 + 200 + 100 = 2750
6	500	1500	250	300	1750 + 500 + 400 + 300 + 200 + 100 = 3250
7	600	2100	300	263	1750 + 600 + 500 + 400 + 300 + 200 + 100 = 3850
8	700	2800	350	238	1750 + 700 + 600 + 500 + 400 + 300 + 200 + 100 = 4550
9	800	3600	400	217	1750 + 800 + 700 + 600 + 500 + 400 + 300 + 200 + 100 = 5350
10	900	4500	450	200	1750 + 900 + 800 + 700 + 600 + 500 + 400 + 300 + 200 + 100 = 6250

		<b>Cost</b>	<b>given year</b>	<b>year and</b>	
<b>A</b>	<b>B (Rs)</b>	<b>C (Rs)</b>	<b>D (in Rs)</b>	<b>E (Rs)</b>	<b>F (Rs)</b>
		<b>C = <math>\Sigma B</math></b>	<b>C/A</b>	<b><math>\frac{1750}{A}</math></b>	<b>D + E</b>
1	0	0	0	1750	1750
2	100	100	50	875	925
3	200	300	100	583	683
4	300	600	150	438	588
5	400	1000	200	350	550
<b>6</b>	<b>500</b>	<b>1500</b>	<b>250</b>	<b>292</b>	<b>542</b>
7	600	2100	300	250	550
8	700	2800	350	219	569

The value corresponding to any end-of-year (n) in Column F represents the average total cost of using the equipment till the end of that particular year.

In this problem, the average total cost decreases till the end of the year 6 and then it increases. Hence the optimal replacement period is 6 years i.e. the economic life of the equipment is 6 years.

**(e) When interest rate  $i = 12\%$**

When the interest rate is more than 0% the steps to get the economic life are summarized in the following table.

**Calculation to determine Economic life**

**First Cost = Rs. 1750      Interest rate = 12%**

<b>End of year (n)</b>	<b>Maintenance cost at end of years</b>	<b>(P/F, 12%, n)</b>	<b>Present worth as beginning of years of maintenance costs</b>	<b>Summation of present worth of maintenance costs through the given year</b>	<b>Present simulator maintenance cost and first cost</b>	<b><math>(A/P, 12\%, n) = \frac{i(1+i)^n}{(1+i)^n - 1}</math></b>	<b>Annual equipment total cost through the given year</b>
<b>A</b>	<b>B</b>	<b>C</b>	<b>D</b>	<b>E</b>	<b>F</b>	<b>G</b>	<b>H</b>
	<b>B (iR)</b>	<b><math>C = \frac{1}{(1+12/100)^n}</math></b>	<b>BxC</b>	<b><math>\Sigma D</math></b>	<b>E+ Rs. 1750</b>		<b>FxG</b>
1	0	0.8929	0	0	1750	1.1200	1960
2	100	0.7972	79.72	79.72	1829.72	0.5917	1082.6
3	200	0.7118	142.36	222.08	1972.08	0.4163	820.9
4	300	0.6355	190.65	412.73	2162.73	0.3292	711.9
5	400	0.5674	226.96	639.69	2389.69	0.2774	662.9
6	500	0.5066	253.30	892.99	2642.99	0.2432	642.7
7	600	0.4524	271.44	1164.43	2914.430	0.2191	638.5
8	700	0.4039	282.73	1447.16	3197.16	0.2013	680.7

Identify the end of year for which the annual equivalent total cost is minimum in column. In this problem the annual equivalent total cost is minimum at the end of year hence the economics life of the equipment is 7 years.

### 5.3.5. Simple probabilistic model for items which completely fail

Electronic items like bulbs, resistors, tube lights etc. generally fail all of a sudden, instead of gradual failure. The sudden failure of the item results in complete breakdown of the system. The system may contain a collection of such items or just an item like a single tube-light. Hence we use some replacement policy for such items which would minimize the possibility of complete breakdown. The following are the replacement policies which are applicable in these cases.

#### i) **Individual replacement policy :**

Under this policy, each item is replaced immediately after failure.

#### ii) **Group replacement policy :**

Under group replacement policy, a decision is made with regard the replacement at what equal intervals, all the item are to be replaced simultaneously with a provision to replace the items individually which fail during the fixed group replacement period.

Among the two types of replacement policies, we have to decide which replacement policy we have to follow. Whether individual replacement policy is better than group replacement policy. With regard to economic point of view. To decide this, each of the replacement policy is calculated and the most economic one is selected for implementation.

#### **Exercise :**

1. List and explain different types of maintenance
2. Discuss the reasons for maintenance.
3. Distinguish between breakdown maintenance and preventive maintenance.
4. Distinguish between individual and group replacement policies.
5. A firm is considering replacement of an equipment whose first cost is Rs.4000 and the scrap value is negligible at the end of any year. Based on experience, it has been found that the maintenance cost is zero during the first year and it is Rs.1000 for the second year. It increase by Rs.300 every years thereafter.

a) When should the equipment be replace if  $i = 0\%$

b) When should the equipment be replace if  $i = 12\%$

**Ans . a) 5 years**

**b) 5 years**

6. A company is planning to replace an equipment whose first cost is Rs.1,00,000. The operating and maintenance cost of the equipment during its first year of operation is Rs.10,000 and it increases by Rs. 2,000 every year thereafter. The release value of the equipment at the end of the

first year of its operation is Rs.65,000 and it decreases by Rs.10,000 every year thereafter. Find the economic life of the equipment by assuming the interest rate as 12%.

**[Ans : Economic life = 13 years and the corresponding annual equivalent cost = Rs. 34,510]**

7. The following table gives the operation cost, maintenance cost and salvage value at the end of every year of machine whose purchase value is Rs. 12,000. Find the economic life of the machine assuming.
- The interest rate as 0%
  - The interest rate as 15%

End year	of	Operation cost at the end of year (Rs)	Maintenance cost at the end of year (Rs)	Salvage value at the end of year (Rs)
1		2000	2500	8000
2		3000	3000	7000
3		4000	3500	6000
4		5000	4000	5000
5		6000	4500	4000
6		7000	5000	3000
7		8000	5500	2000
8		9000	6000	1000

**Ans :**

- Economic life of the machine = 2 years
- Economic life of the machine = 2 years

